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# Spines of Topological Manifolds

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In this paper we prove that a closed 2-connected topological manifold has a PL-spine, i.e. there is a locally tamely embedded complex such that a regular neighborhood of this complex is the manifold with a disc deleted (dimension is assumed to be at least 6). This “spine method” together with the relative edition of regular neighborhoods of complexes in topological manifolds [5] makes it easy to use general position arguments in topological manifolds. This will be used in a forthcoming paper to extend various embedding theorems to the topological category.

The methods we use are *PL*-approximation theorems due to Cernavskii, Connally, Miller, Rushing... as quoted in [5] theorem 2 and blocktransversality for *PL* complexes and *PL* submanifolds as was first considered by C. Morlet [4] and later extended by D. Stone [6].

**DEFINITION 1.** A spine of a topological manifold  $M$  with  $\partial M \neq \emptyset$  is a locally tamely embedded complex  $K \subset M$  so that  $K$  is a strong deformation retract of  $M$  and  $K \subset M$  is a simple homotopy equivalence. In case  $\partial M = \emptyset$  by the spine of  $M$  we mean a spine of  $M$  with a disc deleted.

**THEOREM 2.** Let  $(M, \partial_- M, \partial_+ M)$  be a triad of topological manifolds  $\dim(M) = m$ , and assume  $m \geq 6$  and

$$\Pi_j(M, \partial_+ M) = 0 \quad \text{for } j < m - r, \quad r \leq m - 3.$$

Further assume there is a *PL*-complex  $P$  locally tamely embedded in the interior of  $\partial_+ M$ ,  $\dim(P) = p$  and  $m - p \geq 4$ . Then there is a complex  $K$  of dimension  $\max(p + 1, r, 2)$ , locally tamely embedded in  $M$  such that

$$K \cap \partial_+ M = P$$

$$K \cap \partial_- M = K'$$

$K'$  a subcomplex of  $K$ ,  $K'$  has a neighborhood of the form  $K' \times I$  in  $K$  and

$$\partial_- M \cup K \subset M$$

is a strong deformation retract and a simple homotopy equivalence.

Theorem 2 has an immediate corollary:

**COROLLARY 3.** *Let  $M$  be a closed topological manifold,  $\dim(M) \geq 6$  and  $\pi_j(M) = 0$  for  $j \leq r$ ,  $r > 2$ . Then  $M$  has a spine of dimension  $m - r$ .*

*Proof.* Let  $\bar{M} = M - (\text{interior of a disc})$ . Put  $\partial_- M = \emptyset$ ,  $\partial_+ M = S^{m-1}$ ,  $P = \emptyset$ , and apply Theorem 2.

*Proof of Theorem 2.* First let us consider the case where  $P = \emptyset$ . Put  $k = \max(r, 2)$ . According to Kirby and Siebenmann [3]  $M$  has a handlebodydecomposition relative to  $\partial_- M$  with no handles of dimension greater than  $k$ : Kirby and Siebenmann prove that  $(M, \partial_- M)$  has a handlebodydecomposition, and one can then cancel handles to get a minimal handledecomposition. Because of problems with torsion one needs at least 1- and 2-handles.

We filter  $M$  by the handlefiltration

$$\partial_- M \times I = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

where  $M_{i+1}$  is obtained from  $M_i$  by adjoining a single handle, no handles of dimension greater than  $m - 3$ . The proof will be by downwards induction on the statement:

There is a locally tamely embedded complex

$$K_i \subset \overline{M - M_i} \quad \dim(K_i) \leq k$$

such that

$$K'_i = K_i \cap \partial_+ M_i$$

is contained in the interior of  $\partial_+ M_i$ ,  $K'_i$  has a neighborhood in  $K_i$  of the form  $K'_i \times [0, 1]$  and  $M_i \cup K_i$  is a simple strong deformation retract of  $M$ .

It is easy to start the induction, we let  $K_{s-1}$  be the core of the last handle. Then clearly  $M_{s-1} \cup K_{s-1}$  is a simple strong deformation retract of  $M = M_s$ , so assume the statement for  $i + 1$ . Now

$$M_{i+1} = M_i \cup_{S^{j-1} \times D^{m-j}} D^j \times D^{m-j}$$

for some  $j \leq k$ . Let

$$\bar{E} = D^j \times D^{m-j} \cap \partial M_{i+1} = D^j \times S^{m-j-1}$$

take an outside collar  $\partial \bar{E} \times [0, 2]$  of  $\partial \bar{E}$  in  $\partial_+ M_{i+1}$  and let

$$E_1 = \bar{E} \cup \partial \bar{E} \times [0, 1], \quad E_2 = \bar{E} \cup \partial \bar{E} \times [0, 2].$$

$\bar{E}$  has a *PL* structure being a codimension 0 submanifold of the boundary of  $D^j \times D^{m-j}$ , and we can extend this *PL* structure to  $E_1$  and  $E_2$  using the collar.  $K'_{i+1}$  is of codi-

mension more than 3 in  $\partial_+ M_{i+1}$ , so by [1], see e.g. [5] Theorem 2, since  $\dim \partial_+ M_{i+1} \geq 5$ , there is an ambient  $\varepsilon$ -isotopy of  $E_2$  fixing  $\partial E_2$  that moves  $K'_{i+1}$  to be *PL* embedded in  $E_2$  except in a neighborhood of  $\partial E_2$  which can be assumed small. So we may assume, since this can be taken to be the restriction of an ambient isotopy of  $M$ , that  $K'_{i+1} \cap E_1 \subset E_1$  is *PL*. Using [4] we can isotop  $K'_{i+1}$  further by a small ambient isotopy so that  $K'_{i+1}$  intersects  $\partial \bar{E} = S^{j-1} \times S^{m-j-1}$  blocktransversally. Assume this done, and denote

$$Z = K'_{i+1} \cap \partial \bar{E}.$$

Since the normal blockbundle of  $\partial \bar{E}$  in  $E_1$  is a trivial one dimensional bundle we obtain that  $\partial \bar{E}$  has a neighborhood in  $E$  of the form  $\partial \bar{E} \times (-1, 1)$  and

$$\partial \bar{E} \times (-1, 1) \cap K'_{i+1} = Z \times (-1, 1)$$

since it is the restriction of the trivial blockbundle to  $Z$ , by blocktransversality.  $Z$  is a *PL* subcomplex of  $S^{j-1} \times S^{m-j-1}$ , which is the boundary of  $S^{j-1} \times D^{m-j}$ , of dimension  $m-1-r$ , so of codimension at least 3. By [2] Theorem 5.2 there is a subcomplex  $Z'$  of  $S^{j-1} \times D^{m-j}$  of dimension  $\min(\dim(Z)+1, j)$  so that

$$Z' \cap S^{j-1} \times S^{m-j-1} = Z$$

and  $S^{j-1} \times D^{m-j}$  simplicially collapses to  $Z'$  ( $S^{j-1} \times D^{m-j}$  is the mapping cylinder of the projection  $S^{j-1} \times S^{m-j-1} \rightarrow S^{j-1}$ , so take  $Z'$  to be the mapping cylinder of the restriction to  $Z$ ). Using [2] lemma 2.20 this implies that

$$S^{j-1} \times D^{m-j} \times I$$

simplicially collapses to

$$S^{j-1} \times D^{m-j} \times 0 \cup Z' \times I \cup S^{j-1} \times D^{m-j} \times 1$$

so taking  $S^{j-1} \times D^{m-j} \times I$  to be a collar of  $S^{j-1} \times D^{m-j}$  in  $D^j \times D^{m-j}$  we see that if we define  $D$  to be

$$D = \overline{D^j \times D^{m-j} - S^{j-1} \times D^{m-j} \times I}$$

there is a simple strong deformation retract of  $M_{i+1} \cup K_{i+1}$  to  $M_i \cup Z' \times I \cup K_{i+1} \cup D$ . However  $D$  is a disc, and  $Z' \times 1 \cup K_{i+1} \cap \partial D$  is of codimension bigger than 3 in  $\partial D$ , so we may as before assume it is *PL*-embedded and  $D$  now simplicially collapses to the cone of  $Z' \times 1 \cup K_{i+1} \cap \partial D$  thus finishing the induction step. It is clear by construction that  $K'_i$  has a product neighborhood in  $K_i$ .

In case  $P \neq \emptyset$  the proof is the same except we have to go through the motions of the induction step in the initial step of the induction too.

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