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Spines of Topological Manifolds

ERIK KJAER PEDERSEN

In this paper we prove that a closed 2-connected topological manifold has a PL-spine, i.e. there is a locally tamely embedded complex such that a regular neighborhood of this complex is the manifold with a disc deleted (dimension is assumed to be at least 6). This “spine method” together with the relative edition of regular neighborhoods of complexes in topological manifolds [5] makes it easy to use general position arguments in topological manifolds. This will be used in a forthcoming paper to extend various embedding theorems to the topological category.

The methods we use are *PL*-approximation theorems due to Cernavskii, Connally, Miller, Rushing... as quoted in [5] theorem 2 and blocktransversality for *PL* complexes and *PL* submanifolds as was first considered by C. Morlet [4] and later extended by D. Stone [6].

DEFINITION 1. A spine of a topological manifold M with $\partial M \neq \emptyset$ is a locally tamely embedded complex $K \subset M$ so that K is a strong deformation retract of M and $K \subset M$ is a simple homotopy equivalence. In case $\partial M = \emptyset$ by the spine of M we mean a spine of M with a disc deleted.

THEOREM 2. Let $(M, \partial_- M, \partial_+ M)$ be a triad of topological manifolds $\dim(M) = m$, and assume $m \geq 6$ and

$$\Pi_j(M, \partial_+ M) = 0 \quad \text{for } j < m - r, \quad r \leq m - 3.$$

Further assume there is a *PL*-complex P locally tamely embedded in the interior of $\partial_+ M$, $\dim(P) = p$ and $m - p \geq 4$. Then there is a complex K of dimension $\max(p + 1, r, 2)$, locally tamely embedded in M such that

$$K \cap \partial_+ M = P$$

$$K \cap \partial_- M = K'$$

K' a subcomplex of K , K' has a neighborhood of the form $K' \times I$ in K and

$$\partial_- M \cup K \subset M$$

is a strong deformation retract and a simple homotopy equivalence.

Theorem 2 has an immediate corollary:

COROLLARY 3. *Let M be a closed topological manifold, $\dim(M) \geq 6$ and $\pi_j(M) = 0$ for $j \leq r$, $r > 2$. Then M has a spine of dimension $m - r$.*

Proof. Let $\bar{M} = M - (\text{interior of a disc})$. Put $\partial_- M = \emptyset$, $\partial_+ M = S^{m-1}$, $P = \emptyset$, and apply Theorem 2.

Proof of Theorem 2. First let us consider the case where $P = \emptyset$. Put $k = \max(r, 2)$. According to Kirby and Siebenmann [3] M has a handlebodydecomposition relative to $\partial_- M$ with no handles of dimension greater than k : Kirby and Siebenmann prove that $(M, \partial_- M)$ has a handlebodydecomposition, and one can then cancel handles to get a minimal handledecomposition. Because of problems with torsion one needs at least 1- and 2-handles.

We filter M by the handlefiltration

$$\partial_- M \times I = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

where M_{i+1} is obtained from M_i by adjoining a single handle, no handles of dimension greater than $m - 3$. The proof will be by downwards induction on the statement:

There is a locally tamely embedded complex

$$K_i \subset \overline{M - M_i} \quad \dim(K_i) \leq k$$

such that

$$K'_i = K_i \cap \partial_+ M_i$$

is contained in the interior of $\partial_+ M_i$, K'_i has a neighborhood in K_i of the form $K'_i \times [0, 1]$ and $M_i \cup K_i$ is a simple strong deformation retract of M .

It is easy to start the induction, we let K_{s-1} be the core of the last handle. Then clearly $M_{s-1} \cup K_{s-1}$ is a simple strong deformation retract of $M = M_s$, so assume the statement for $i + 1$. Now

$$M_{i+1} = M_i \cup_{S^{j-1} \times D^{m-j}} D^j \times D^{m-j}$$

for some $j \leq k$. Let

$$\bar{E} = D^j \times D^{m-j} \cap \partial M_{i+1} = D^j \times S^{m-j-1}$$

take an outside collar $\partial \bar{E} \times [0, 2]$ of $\partial \bar{E}$ in $\partial_+ M_{i+1}$ and let

$$E_1 = \bar{E} \cup \partial \bar{E} \times [0, 1], \quad E_2 = \bar{E} \cup \partial \bar{E} \times [0, 2].$$

\bar{E} has a *PL* structure being a codimension 0 submanifold of the boundary of $D^j \times D^{m-j}$, and we can extend this *PL* structure to E_1 and E_2 using the collar. K'_{i+1} is of codi-

mension more than 3 in $\partial_+ M_{i+1}$, so by [1], see e.g. [5] Theorem 2, since $\dim \partial_+ M_{i+1} \geq 5$, there is an ambient ε -isotopy of E_2 fixing ∂E_2 that moves K'_{i+1} to be *PL* embedded in E_2 except in a neighborhood of ∂E_2 which can be assumed small. So we may assume, since this can be taken to be the restriction of an ambient isotopy of M , that $K'_{i+1} \cap E_1 \subset E_1$ is *PL*. Using [4] we can isotop K'_{i+1} further by a small ambient isotopy so that K'_{i+1} intersects $\partial \bar{E} = S^{j-1} \times S^{m-j-1}$ blocktransversally. Assume this done, and denote

$$Z = K'_{i+1} \cap \partial \bar{E}.$$

Since the normal blockbundle of $\partial \bar{E}$ in E_1 is a trivial one dimensional bundle we obtain that $\partial \bar{E}$ has a neighborhood in E of the form $\partial \bar{E} \times (-1, 1)$ and

$$\partial \bar{E} \times (-1, 1) \cap K'_{i+1} = Z \times (-1, 1)$$

since it is the restriction of the trivial blockbundle to Z , by blocktransversality. Z is a *PL* subcomplex of $S^{j-1} \times S^{m-j-1}$, which is the boundary of $S^{j-1} \times D^{m-j}$, of dimension $m-1-r$, so of codimension at least 3. By [2] Theorem 5.2 there is a subcomplex Z' of $S^{j-1} \times D^{m-j}$ of dimension $\min(\dim(Z)+1, j)$ so that

$$Z' \cap S^{j-1} \times S^{m-j-1} = Z$$

and $S^{j-1} \times D^{m-j}$ simplicially collapses to Z' ($S^{j-1} \times D^{m-j}$ is the mapping cylinder of the projection $S^{j-1} \times S^{m-j-1} \rightarrow S^{j-1}$, so take Z' to be the mapping cylinder of the restriction to Z). Using [2] lemma 2.20 this implies that

$$S^{j-1} \times D^{m-j} \times I$$

simplicially collapses to

$$S^{j-1} \times D^{m-j} \times 0 \cup Z' \times I \cup S^{j-1} \times D^{m-j} \times 1$$

so taking $S^{j-1} \times D^{m-j} \times I$ to be a collar of $S^{j-1} \times D^{m-j}$ in $D^j \times D^{m-j}$ we see that if we define D to be

$$D = \overline{D^j \times D^{m-j} - S^{j-1} \times D^{m-j} \times I}$$

there is a simple strong deformation retract of $M_{i+1} \cup K_{i+1}$ to $M_i \cup Z' \times I \cup K_{i+1} \cup D$. However D is a disc, and $Z' \times 1 \cup K_{i+1} \cap \partial D$ is of codimension bigger than 3 in ∂D , so we may as before assume it is *PL*-embedded and D now simplicially collapses to the cone of $Z' \times 1 \cup K_{i+1} \cap \partial D$ thus finishing the induction step. It is clear by construction that K'_i has a product neighborhood in K_i .

In case $P \neq \emptyset$ the proof is the same except we have to go through the motions of the induction step in the initial step of the induction too.

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