Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 49 (1974)

Artikel: Inner Illumination of Convex Polytopes
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DOI: https://doi.org/10.5169/seals-37979

Inner Illumination of Convex Polytopes

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Dedicated to Professor H. Hadwiger on his Sixty-fifth Birthday.

1. Introduction

The notion of an n-polytope which is illuminated by its vertices is due to H. Hadwiger [4], who continued earlier work by P. S. Soltan [6] and B. Grünbaum [3]. An n-polytope P is said to be illuminated by its vertices, if for each vertex x of P there is another vertex y of P such that the line segment joining x and y meets the interior of P. Dually, P may be called facet-disjoint, if each facet of P has an empty intersection with some other facet of P. Set $k(n) := \min\{f^{n-1}(P) : P$ is a facet-disjoint n polytope$\} = \min\{f^0(P) : P$ is an n-polytope illuminated by its vertices$\}$. In [4], H. Hadwiger asked whether $k(n)$ equals $2n$, for all dimensions n. Easy considerations show that this is the case for all $n \leqslant 4$, and that, in these dimensions, the crosspolytopes are the only n-polytopes with $2n$ vertices which are illuminated by them. Several geometrists, myself included, have tried hard to prove the corresponding statement in higher dimensions. Here we determine the numbers $k(n)$. It turns out that $k(n) = 2n$, for all $n \leqslant 7$, whereas, for large n, the situation changes drastically, the approximate value of $k(n)$ being $n + 2\sqrt{n}$. The problem treated here was first discussed at a seminar which H. Hadwiger held in summer 1970. I would like to express my gratitude to him and, with him, to all those whose conversations have encouraged me to think about inner illumination.

2. Notation

Those geometric terms for which we don't give a definition here shall be understood as in the book [2] by B. Grünbaum. It is only when dealing with polyhedral complexes that our notation differs slightly from Grünbaum's. We find it convenient to introduce a unique (-1)-dimensional polytope, namely the empty set $\emptyset$. The boundary complex of a polytope P shall be denoted by ∂P. We set $\partial\emptyset := \emptyset$, whereas, for $\dim P \geqslant 0$, the boundary complex of P is understood in the usual way.

DEFINITION 1. A polyhedral complex in the n-dimensional Euclidean space E^n is a finite collection C of convex polytopes $P \subset E^n$ such that

(1) for each $P \in C$, the boundary complex ∂P is a subset of C,

(2) whenever P, Q are elements of C, we have

$P \cap Q \in (\{P\} \cup \partial P) \cap (\{Q\} \cup \partial Q)$.

Let C be a polyhedral complex in E^n. We define the star, the antistar and the link of an element $x \in C$ in the usual way. Namely, for $x \in C$ we set

$$\begin{aligned} \operatorname{st}(x, C) &:= \{y \in C\colon \text{there is an element } z \in C \text{ such that } x \cup y \subset z\}, \\ \operatorname{ast}(x, C) &:= \{y \in C\colon x \cap y = \emptyset\}, \\ \operatorname{link}(x, C) &:= \operatorname{st}(x, C) \cap \operatorname{ast}(x, C). \end{aligned}$$

If C is a polyhedral complex in E^n we set, for each integer i, $\Delta^i C := \{x \in C\colon \dim x = i\}$, and $f^i C := \operatorname{card} \Delta^i C$. If there is no risk of confusion, we use the same letters for a polytope and for its boundary complex. For example, if P is a polytope and x an element of its boundary complex ∂P, we often write $f^i P$ and $\operatorname{link}(x, P)$, instead of $f^i \partial P$ and $\operatorname{link}(x, \partial P)$.

3. Illuminated Polytopes

If x and y are points in E, we denote the line segment joining them by $[x, y] := = \operatorname{conv}\{x, y\}$.

DEFINITION 2. We say that an n-polytope $P \subset E^n$ is illuminated (by the set of its vertices through its interior), if for each $x \in \Delta^0 P$ there exists a vertex $y \in \Delta^0 P$ such that $[x, y] \cap \operatorname{int} P \neq \emptyset$.

Equivalently, P is illuminated if the set $\Delta^0 P$ of its vertices is not contained in the star $\operatorname{st}(x, P) := \operatorname{st}(x, \partial P)$ of any vertex $x \in \Delta^0 P$.

DEFINITION 3. For $n \geqslant 1$, set $k(n) := \min\{f^0 P : P \subset E^n$ is an illuminated n-polytope$\}$ and $K(n) := \{P : P \subset E^n$ is an illuminated n-polytope with $f^0 P = k(n)\}$.

If α is a real number we denote by $\langle\alpha\rangle$ the smallest integer which is not smaller than α. For $n \geqslant 1$, set $\{\sqrt{n}\} := \langle(\sqrt{4n+1} - 1)/2\rangle$ and $\kappa(n) := \min\{2n, n + \{\sqrt{n}\} + \langle n/\{\sqrt{n}\}\rangle + 1\}$. The purpose of this paper is to prove the following result.

THEOREM 1. *For each positive integer n, the equation $k(n) = \kappa(n)$ holds.*

By duality this theorem is equivalent to the following statement.

COROLLARY 1. *Let $P \subset E^n$ be an n-polytope such that, given any facet $x \in \Delta^{n-1} P$, there exists $y \in \Delta^{n-1} P$ with $x \cap y = \emptyset$. Then $f^{n-1} P \geqslant \kappa(n)$. Furthermore, there are n-polytopes for which equality holds.*

It is easy to see that $K(n)$ always contains simplicial polytopes. On the other hand we don't know whether all elements of $K(n)$ must be simplicial.

4. Blocks

In this section we want to prove that $k(n) \leqslant \kappa(n)$.

DEFINITION 4. A simplicial n-polytope P is called a block of order $k \geqslant 2$ if there is a set $X \subset \Delta^{n-1}P$ of cardinality k, such that $\cap X = \emptyset$ and $\Delta^0 P \subset \cup X$.

The set X is called a fundamental system for the block P. Recall that a vertex x of a polytope P is called r-valent in P, if there are exactly r edges of P issueing from x.

DEFINITION 5. A simplicial n-polytope P is called an enlightened block of order $k \geqslant 2$ if there is a set $X \subset \Delta^0 P$ of cardinality k with the following properties:

(3) each element of X is n-valent in P,

(4) $Q := \operatorname{conv}(\Delta^0 P \sim X)$ is an n-dimensional block,

(5) $Y := \{\operatorname{conv} \Delta^0 \operatorname{link}(x, P) : x \in X\}$ is a fundamental system for Q.

The set $X \subset \Delta^0 P$ is called an enlightening set for P. Clearly, P arises from Q by adding pyramids above the facets of Y.

LEMMA 1. *Let $P \subset E^n$ be an n-dimensional enlightened block. Then $f^0 P \geqslant \kappa(n)$.*

Proof. Assume that P is of order $k+1 \geqslant 2$, and let X, with $\operatorname{card} X = k+1$, be an enlightening set for P. Notice that $k + \langle n/k \rangle \geqslant \{\sqrt{n}\} + \langle n/\{\sqrt{n}\}\rangle$. The polytope $Q := \operatorname{conv}(\Delta^0 P \sim X)$ is an n-dimensional block of order $k+1$, and since $f^0 P = f^0 Q + k + 1$, it suffices to prove $f^0 Q \geqslant n + \langle n/k \rangle$. Let Y, with $\operatorname{card} Y = k+1$, be a fundamental system for Q, and consider a facet $y \in Y$. For each $z \in Y \sim \{y\}$, set $\alpha(z) := \Delta^0 y \sim \Delta^0(z \cap y)$. $\cap\, y = \emptyset$ implies $\cup \{\alpha(z) : z \in Y \sim \{y\}\} = \Delta^0 y$, hence there is a facet z_0 in $Y \sim \{y\}$ such that $\operatorname{card} \alpha(z_0) \geqslant \langle n/k \rangle$, or $\operatorname{card}(\Delta^0 z_0 \cup \Delta^0 y) \geqslant n + \langle n/k \rangle$, and the proof of Lemma 1 is completed.

LEMMA 2. *Assume $n \geqslant 8$. There is an n-dimensional enlightened block $P \subset E^n$ such that $f^0 P = \kappa(n)$*

Proof. For $n \geqslant 8$ we have $\kappa(n) = n + \{\sqrt{n}\} + \langle n/\{\sqrt{n}\}\rangle + 1$. If k and l are positive integers, we set $A(k, l) := \{x \in \mathbf{Z} : l \leqslant x \leqslant l + k - 1\}$. To abbreviate our notation, set $p := \{\sqrt{n}\}$, $q := \langle n/\{\sqrt{n}\}\rangle$. Consider the moment curve $\varphi : \mathbf{R} \to E^n$ defined by $\varphi(t) := (t, t^2, \ldots, t^n)$. $Q := \operatorname{conv} \varphi A(n+p, 1)$ is a cyclic n-polytope with $n+p$ vertices. For $j \in \mathbf{Z}$, $1 \leqslant j \leqslant q$, we set $x_j := \operatorname{conv} \varphi(A(n+p, 1) \sim A(p, (j-1)p+1))$, and, further $x_{q+1} := \operatorname{conv} \varphi(A(n+p, 1) \sim A(p, n+1))$. By Gale's evenness condition, each member of $X := \{x_l : 1 \leqslant l < q+1\}$ is a facet of Q. Furthermore

(6) $\operatorname{card} X = q + 1$

(7) $\Delta^0 Q \subset \cup X$

(8) $\cap X = \emptyset$.

Hence Q is an n-dimensional block of order $q+1$, and X is a fundamental system for Q. By adding a pyramid above each facet of X we obtain an n-dimensional enlightened block P with $f^0P=\kappa(n)$, which proves Lemma 2.

PROPOSITION 1. For each integer $n\geqslant 1$, we have $k(n)\leqslant\kappa(n)$.

Proof. For $n\leqslant 7$ we have $\kappa(n)=2n$, and Proposition 1 immediately follows from the observation that the n-dimensional crosspolytope is always illuminated. For $n\geqslant 8$ our proposition is a corollary of lemma 2.

5. Simple Lights

In this and the next two sections we collect the material which we need to prove $k(n)\geqslant\kappa(n)$.

For $n\geqslant 2$, the n-dimensional crosspolytopes are illuminated, whereas the n-simplices are not. This gives us the trivial estimate $n+2\leqslant k(n)\leqslant 2n$, for all $n\geqslant 2$.

Here we want to show that, under certain circumstances, there is an enlightened block in the set $K(n)$ of minimal illuminated n-polytopes. We obtain this result by pulling a vertex of some element $P\in K(n)$. Such pulling processes have been useful in many geometric situations, see [1] or [5], for example.

DEFINITION 6. Let $P\subset E^n$ be an illuminated n-polytope and x a vertex of P. We say that $Y\subset\Delta^0P$ lies opposite to x in P, if

(9) for all $y\in Y$, $[y, x]\cap\operatorname{int}P\neq\emptyset$,

(10) for each $u\in U:=\Delta^0P\sim(\{x\}\cup Y)$, there is an element $v\in U$ such that $[u, v]\cap\operatorname{int}P\neq\emptyset$.

DEFINITION 7. Let $P\subset E^n$ be an illuminated n-polytope, and x a vertex of P. We set $\gamma(x, P):=\max\{\operatorname{card}Y: Y\subset\Delta^0P, \text{ and } Y \text{ lies opposite to } x \text{ in } P\}$.

PROPOSITION 2. *Let $P\in K(n)$ be a minimal illuminated n-polytope, and assume that there is a vertex $x\in\Delta^0P$ such that $\gamma(x, P)\geqslant 2$. Then there exists a simplicial polytope $Q\in K(n)$, which has an n-valent vertex.*

Proof. If $P\subset E^n$ is an illuminated n-polytope with $\gamma(x, P)\geqslant 2$, for some $x\in\Delta^0P$, then each polytope combinatorially equivalent to P, and each polytope Q with $f^0Q=f^0P$, whose vertices are sufficiently close to those of P, has the same property. This remark allows us to make the following assumptions about P.

(11) P is simplicial.

(12) There are a vertex $x\in\Delta^0P$, a set $Y\subset\Delta^0P$ which lies opposite to x in P, elements y and $z\neq y$ in Y and a hyperplane H separating x from the remaining vertices of P such that $\{y, z\}\subset[(\operatorname{relint}(H\cap P))+\operatorname{pos}\{y-x\}]$.

To see (12), choose a vertex x of P with $\gamma(x, P) \geqslant 2$, let $Y \subset \Delta^0 P$ be a set of cardinality at least 2 which lies opposite to x in P, and y, z two different elements of Y. If H is an arbitrary hyperplane strictly separating x from the remaining vertices of P, set $L := H \cap \operatorname{conv}\{x, y, z\}$. By the choice of Y we have $L \subset \operatorname{relint}(H \cap P)$. Let R be the ray $R := x + \operatorname{pos}\{x - y\}$ issueing from x. There is a point $x' \neq x$ on R such that $H \cap \operatorname{conv}\{x', y, z\} \subset \operatorname{relint}(H \cap P)$. Let H' be the hyperplane which is parallel to H and contains x'. There is a P-admissible projective transformation π of E^n, which sends H' to infinity, such that πP has the property required by (12). Since πP, being combinatorially equivalent to P, shares all the other relevant properties with P, we may assume, without lack of generality, that P itself satisfies (12).

By moving the vertices of P a little we can reach that the following additional conditions hold

(13) $\Delta^0 P$ is a set in general position, and the vertex x is the origin of E^n.

(14) Whenever g_1 and g_2 are different facets of P, none of which contains one of the points x, y, then $\operatorname{aff}(g_1) \cap \operatorname{lin}\{y\} \neq \operatorname{aff}(g_2) \cap \operatorname{lin}\{y\}$.

By (12) and by the fact that x is the origin of E^n, we find a number $\lambda > 1$ such that, with $u := \lambda y$, the relation $z \in \operatorname{int} \operatorname{conv}((\Delta^0 P \sim \{y\}) \cup \{u\})$ holds. For each number $\tau \in I := [0, 1]$ we set $y_\tau := \tau u + (1 - \tau) y$ and $P_\tau := \operatorname{conv}((\Delta^0 P \sim \{y\}) \cup \{y_\tau\})$.

Define $I' := \{\tau \in I : \text{there is no } g \in \Delta^{n-1} P \text{ such that } y_\tau \in \operatorname{aff}(g)\}$. We may assume $1 \in I'$. I' is the disjoint union of a finite set $\mathfrak{A}$ of intervals, which are all open in I. Let $\leqslant$ be the ordering of $\mathfrak{A}$ which is induced by the natural ordering of I. By (13), P_τ is a simplicial n-polytope, for each $\tau \in I'$. For $\tau \in I'$ set $A_\tau := \operatorname{ast}(y_\tau, P_\tau)$. We have $A_\tau \subset \partial P$, and each of the sets $\bigcup A_\tau$, $\tau \in I'$, is a polyhedral $(n-1)$-ball, containing the vertex $x \in \Delta^0 P$ in its interior. If τ and τ' are contained in the same interval of $\mathfrak{A}$, then $A_\tau = A_{\tau'}$, and the polytopes P_τ, $P_{\tau'}$ are combinatorially equivalent.

If $\tau < \tau'$, and τ, τ' are contained in successive intervals of $\mathfrak{A}$, then there is a facet $g \in \Delta^{n-1} A_\tau$ such that $A_{\tau'}$ is the complex generated by $\Delta^{n-1} A_\tau \sim \{g\}$. This easily follows from (14).

By $z \in \operatorname{int} P_1$ we find $f^0 P_1 < f^0 P$. Let $K \in \mathfrak{A}$ be the first interval with the property that $f^0 P_\tau < f^0 P$, for the numbers $\tau \in K$. By (14), $f^0 P_\tau = f^0 P - 1$, for all $\tau \in K$. Let $v \in \Delta^0 P \sim \{y\}$ be the vertex which does not belong to $\Delta^0 P_\tau$, for $\tau \in K$, and set $H := (\operatorname{pos}\{v\}) \sim \{x\}$. If we choose $\tau \in K$ arbitrarily, there is a facet $g \in \Delta^{n-1} P_\tau$ with $y_\tau \in \Delta^0 g$ such that $H \cap bdP_\tau$ is a point w of relint g. Choose $\varepsilon > 0$ such that $w(\varepsilon) := w + \varepsilon v$ is beyond g, with respect to P_τ and beneath all remaining facets of P_τ. Notice that $P \subset P_\tau$. The simplicial polytope $Q := \operatorname{conv}(P_\tau \cup \{w(\varepsilon)\})$ belongs to $K(n)$, and $w(\varepsilon)$ is an n-valent vertex of Q, as required by Proposition 2.

PROPOSITION 3. *Assume that for an integer $n \geqslant 3$ there is a simplicial polytope $P \in K(n)$ which has an n-valent vertex. Then $K(n)$ contains an enlightened block.*

Proof. For a simplicial polytope $P \in K(n)$, let $\Sigma(P)$ be the set of n-valent vertices of P, and $\sigma(P)$ their number. $\alpha:=\max\{\sigma(P):P\in K(n), P$ simplicial$\}$ satisfies the relation $1\leqslant\alpha\leqslant 2n$. Let $P\in K(n)$ be a simplicial polytope with $\sigma(P)=\alpha$. We may assume (15) $\Delta^0 P$ is a set in general position.

If P is not an enlightened block, we easily derive that the set $L:=\bigcap\{\Delta^0 \operatorname{link}(x, P): x\in\Sigma(P)\}$ is not empty. We choose $p\in\Sigma(P)$ arbitrarily and find $L\subset\Delta^0 \operatorname{link}(p, P)$. Consider the set $C:=\{z\in\Delta^0 P:[z, u]\cap\operatorname{int}P=\emptyset$, for all $u\in\Delta^0 P, u\neq p\}$. If C is empty, let y be an arbitrary vertex of the n-polytope $Q:=\operatorname{conv}(\Delta^0 P\sim\{p\})$. Since $C=\emptyset$, there is an element $z\in\Delta^0 Q$ with $[y, z]\cap\operatorname{int}P\neq\emptyset$. Since $n\geqslant 3$, we easily conclude $[y, z]\cap\cap\operatorname{int}Q\neq\emptyset$, and Q is illuminated by its vertices, contradicting the fact that $P\in K(n)$.

Hence C is not empty. We choose $x\in L$ and $y\in C$ arbitrarily. By the definitions of L and C we find

(16] $[x, y]\in\Delta^1 P$,

(17) $x\in\bigcap\{\operatorname{link}(u, P):u\in\Sigma(P)\sim\{p\}\}$.

We may assume

(18) x is the origin of E^n,

(19) whenever g_1 and g_2 are different facets of P, none of which contains one of the points x, y, then $\operatorname{aff}(g_1)\cap\operatorname{lin}\{y\}\neq\operatorname{aff}(g_2)\cap\operatorname{lin}\{y\}$.

We choose $z\in\Delta^0 P$ such that $[x, z]\cap\operatorname{int}P\neq\emptyset$ and set $R:=\operatorname{lin}\{y, z\}\cap\operatorname{conv}\Delta^0 P\sim\sim\{p\})$, where p is the vertex of P mentioned below (15). R is a 2-polytope with $\{x, y, z\}\subset\Delta^0 R$. Let $a\in\Delta^0 R$ be such that $a\neq y$, $a\in\operatorname{link}(x, R)$, and $b\in\Delta^0 R$ such that $b\neq x$, $b\in\operatorname{link}(a, R)$.

We may suppose that

(20) $\operatorname{aff}\{a, b\}\cap\operatorname{pos}\{y\}\neq\emptyset$. Namely, if (20) is not fulfilled for the polytope P, we subject Q to an appropriate projective transformation. We choose a point $u\in\operatorname{pos}\{y\}$ such that $\operatorname{aff}\{a, b\}\cap\operatorname{pos}\{y\}\subset[x, u]$. We can assume

(21) $[p, u]\cap\operatorname{relint}\operatorname{conv}\operatorname{link}(p, P)\neq\emptyset$.

If this is not fulfilled for P, we may bring p closer to the hyperplane $\operatorname{aff}\Delta^0 \operatorname{link}(p, P)$. For each number $\tau\in I:=[0, 1]$ we set $y_\tau:=\tau u+(1-\tau)y$ and $P_\tau:=\operatorname{conv}((\Delta^0 P\sim\sim\{y\})\bigcup\{y_\tau\})$. Define $I':=\{\tau\in I:$ there is no $g\in\Delta^{n-1}P$ such that $y_\tau\in\operatorname{aff}(g)\}$. We may assume $1\in I'$. I' is the disjoint union of a finite set $\mathfrak{A}$ of intervals, which are all open in I. Let $\leqslant$ be the ordering of $\mathfrak{A}$ which is induced by the natural ordering of I. By (19), P_τ is a simplicial n-polytope, for each $\tau\in I'$.

By (21), each complex $\operatorname{st}(x, P_\tau)$, $\tau\in I'$, is isomorphic to $\operatorname{st}(x, P)$ under an isomorphism which maps y_τ into y and leaves the remaining vertices fixed. If τ and τ' belong to the same interval of $\mathfrak{A}$, then P_τ and $P_{\tau'}$ are combinatorially isomorphic. If $\tau<\tau'$ and τ, τ' are contained in successive intervals of $\mathfrak{A}$, then the relation between P_τ and $P_{\tau'}$ is similar to the one described at the corresponding stage in the proof of proposition 2.

By (20) we find $z\notin\Delta^0 P_1$ and hence $f^0 P_1<f^0 P$. Let $K\in\mathfrak{A}$ be the first interval with the property that $f^0 P_\tau<f^0 P$, for the numbers $\tau\in K$. By (19), $f^0 P_\tau=f^0 P-1$ for all

$\tau \in K$. Let $v \in \Delta^0 P \sim \{y\}$ be the vertex which does not belong to $\Delta^0 P_\tau$, for $\tau \in K$, and set $H := (\mathrm{pos}\{v\}) \sim \{x\}$. If we choose $\tau \in K$ arbitrarily, there is a facet $g \in \Delta^{n-1} P_\tau$ with $y_\tau \in \Delta^0 g$ such that $H \cap bdP_\tau$ is a point w of relint g. Choose $\varepsilon > 0$ such that $w(\varepsilon) := := w + \varepsilon v$ is beyond g. with respect to P_τ and beneath all remaining facets of P_τ. Notice that $P \subset P_\tau$ and $p \notin \mathrm{link}(w(\varepsilon), Q)$ where we have set $Q := \mathrm{conv}(P_\tau \cup \{w(\varepsilon)\})$. The polytope Q belongs to $K(n)$, and we have $\sigma(Q) \geqslant \sigma(P) + 1$ contradicting the maximality of $\sigma(P)$. Hence P must be an enlightened block and Proposition 3 is proved.

6. Antipodal Systems of Sets

Let C be a set, and $\mathfrak{C}$ a finite set of nonvoid subsets of C. For each $x \in \mathfrak{C}$ we set
$\alpha(x, \mathfrak{C}) := \{y \in \mathfrak{C} : x \cap y = \emptyset\}$,
$\beta(x, \mathfrak{C}) := \{y \in \mathfrak{C} : y \cap z \neq \emptyset, \quad \text{for all} \quad z \in \mathfrak{C} \sim \{x\}\}$.

DEFINITION 8. The collection $\mathfrak{C}$ is called antipodal. if $\alpha(x, \mathfrak{C}) \neq \emptyset$, for all $x \in \mathfrak{C}$.

DEFINITION 9. The collection $\mathfrak{C}$ is called primitive, if $\mathfrak{C}$ is antipodal and if, further, $\mathfrak{C} = \{x\} \cup \beta(x, C)$ for some $x \in \mathfrak{C}$.

DEFINITION 10. The collection $\mathfrak{C}$ is called free, if the elements of $\mathfrak{C}$ are pairwise disjoint.

PROPOSITION 4. *Let $\mathfrak{C}$ be an antipodal collection of sets. $\mathfrak{C}$ is a disjoint union of collections, each of which is either primitive or free.*

Proof. We proceed by induction on $\mathrm{card}\,\mathfrak{C}$. The case $\mathrm{card}\,\mathfrak{C} \leqslant 2$ is trivial. We assume $\mathrm{card}\,\mathfrak{C} \geqslant 3$ and distinguish two cases.

A. There is a set $x \in \mathfrak{C}$ such that $\beta(x, \mathfrak{C}) \neq \emptyset$. We set $\mathfrak{A} := \{x\} \cup \beta(x, \mathfrak{C})$ and $\mathfrak{B} := \mathfrak{C} \sim \mathfrak{A}$. Clearly, $\mathfrak{A}$ is primitive. We may assume $\mathfrak{B} \neq \emptyset$ and have to show that $\mathfrak{B}$ is antipodal. Given $y \in \mathfrak{B}$, there is an element $z \in \mathfrak{C}$ such that $y \cap z = \emptyset$. Since $y \notin \beta(x, \mathfrak{C})$, we may assume $z \neq x$, and by the definition of $\beta(x, \mathfrak{C})$ z does not belong to $\beta(x, \mathfrak{C})$, hence z belongs to $\mathfrak{B}$, and $\mathfrak{B}$ is antipodal.

B. $\beta(x, \mathfrak{C}) = \emptyset$, for all $x \in \mathfrak{C}$. We choose $x_1 \in \mathfrak{C}$ and $x_2 \in \alpha(x_1, \mathfrak{C})$. We may suppose that there is $x_3 \in \mathfrak{C} \sim \{x_1, x_2\}$ which has a nonvoid intersection with all elements in $\mathfrak{C} \sim \{x_1, x_2\}$. Since $\beta(x_1, \mathfrak{C}) = \beta(x_2, \mathfrak{C}) = \emptyset$, we conclude $x_1 \cap x_3 = x_2 \cap x_3 = \emptyset$. We may assume that there exists $x \in \mathfrak{C} \sim \{x_1, x_3\}$ which has a nonvoid intersection with all elements $\mathfrak{C} \sim \{x_1, x_3\}$. In the case $x \neq x_2$ we would have $x \in \beta(x_1, \mathfrak{C}) \neq \emptyset$.

Hence, if we set $\mathfrak{A} := \{x_1, x_2, x_3\}$ and $\mathfrak{B} := \mathfrak{C} \sim \mathfrak{A}$, we have

(22) $y \cap x_2 \neq \emptyset, \quad \text{for all} \quad y \in \mathfrak{B}$.

Similarly,

(23) $y \cap x_1 \neq \emptyset, \quad \text{for all} \quad y \in \mathfrak{B}$.

Since each element of $\mathfrak{B}$ has a nonvoid intersection with x_3, too, we conclude that $\mathfrak{B}$ is either empty or an antipodal system of sets. Because $\mathfrak{A}$ is free, our proposition follows.

7. Scattered Sets in Complexes

DEFINITION 11. Let C be a polyhedral complex. A set $x \subset \bigcup C$ is called scattered of order k in C, if x is the union of k sets $x_i \subset \bigcup C$, $1 \leqslant i \leqslant k$, each of which is the disjoint union of finitely many cells of C.

Notice that the empty set is always a cell of the polyhedral complex C. We don't worry about the fact, that $x \subset \bigcup C$ may be scattered of different orders k and $l \neq k$. By $H_i(x)$ we denote the i-th singular homology group of the space x, with integer coefficients. We have $H_i(\emptyset)=0$, for all $i \geqslant 0$. Our next proposition easily follows from the exactness of the Mayer-Vietoris sequence for excisive couples, as it is described, for example, in the book [7].

PROPOSITION 5. *Let C be a polyhedral complex, and $x \subset \bigcup C$ a set, which is scattered of order k in C. Then $H_i(x)=0$, for all $i \geqslant k$.*

Proof. We proceed by induction on k. The case $k=1$ is trivial. For $k \geqslant 2$, assume that $x = \bigcup \{x_i : 1 \leqslant i \leqslant k\}$ where each x_i is a disjoint union of cells of C. Set $y := \bigcup \{x_i : 2 \leqslant i \leqslant k\}$ and $z := x_1 \cap y$. By the inductive assumption we have $H_i(y) = H_i(z) = 0$, for each $i \geqslant k-1$. Further, the sequence $\ldots \xrightarrow{\partial} H_i(z) \to H_i(y) \oplus H_i(x_1) \to H_i(x) \xrightarrow{\partial} H_{i-1}(z) \to \cdots$ is exact. For $i \geqslant k$ we have $H_{i-1}(z) = H_i(z) = 0$ hence $H_i(x)$ is isomorphic to $H_i(y) \oplus H_i(x_1) = 0$, which implies the desired result.

Now we are able to derive our principal result.

8. The Main Theorem

Proof of Theorem 1. Theorem 1 clearly holds for all $n \leqslant 2$. So we may assume $n \geqslant 3$, for the rest of this section.

A. For all $n \geqslant 3$, $k(n) \leqslant \kappa(n)$. See Proposition 1.

B. For all $n \geqslant 3$, $k(n) \geqslant \kappa(n)$. We distinguish two cases.

B1. Assume that $K(n)$ contains a polytope P with $\gamma(x, P) \geqslant 2$, for some vertex $x \in \Delta^0 P$. By Proposition 2 and Proposition 3, $K(n)$ contains an enlightened block Q. Lemma 1 shows $k(n) = f^0 Q \geqslant \kappa(n)$.

B2. Assume that $K(n)$ contains no polytope as described above under $B1$. Choose an element P in $K(n)$, let $\hat{P}$ be its dual polytope, and $\varphi : (\partial P \sim \{\emptyset\}) \to (\partial \hat{P} \sim \{\emptyset\})$ the antiisomorphism which assigns to each $x \in \partial P$, $x \neq \emptyset$ its dual face $\varphi x \in \partial \hat{P}$.

Since P is illuminated, the set $\Delta^{n-1} \hat{P}$ is an antipodal collection of sets. By Proposition 4 there is a set $\mathfrak{A}$ of pairwise disjoint collections of sets such that $\bigcup \mathfrak{A} = \Delta^{n-1} \hat{P}$

and such that each member of $\mathfrak{A}$ is either primitive or free. Consider an element $A \in \mathfrak{A}$. If A were free, with $\operatorname{card} A \geqslant 3$, we would have $\gamma(\varphi^{-1}y, P) \geqslant 2$, for each facet $y \in A$, contradicting our assumption $B2$ about $K(n)$. Hence

(24) $A = \{x\} \cup \beta(x, A)$, for each $A \in \mathfrak{A}$ and some $x \in A$, where, again by $\gamma(\varphi^{-1}x, P) \leqslant 1$, $\beta(x, A)$ consists of a single facet of $\hat{P}$.

If we had $f^0 P < 2n$, this would imply $f^{n-1}\hat{P} < 2n$, and by (24) $bd\hat{P}$ would be scattered in $\partial\hat{P}$ of some order $k \leqslant n-1$. By proposition 5, we could conclude $H_{n-1}(bd\hat{P}) = 0$, contradicting the fact that $bd\hat{P}$ is a polyhedral $(n-1)$-sphere. Hence $f^0 P \geqslant 2n \geqslant \kappa(n)$, and our theorem is proved.

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Received July 25, 1973