On the genus of finite CW–H-spaces

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The study of the Grothendieck groups of certain stable categories (initiated in [1] and [2] and pursued in [3]) led to the study of two phenomenas: That of non cancellation [4] and that of the genus of a space [7]. Their inter-relations were observed and their study led to the discovery of new finite CW-H-spaces. Our main reference in this study of the genus of an H-space should be [7] but many of the facts studied here are closely related to those studied in [1], [4] and [6]. Here we use the notion of genus with regard to p-equivalence in the sense of Serre ([5]) rather than that of localization which are equivalent for H-spaces (see [7]).

0. Notations, Definitions and Summary of Results

As usual we denote by PA and QA the modules of primitives and indecomposables in a Hopf algebra A.

We denote by **P** the set of all primes. All spaces considered are of the homotopy type of simply connected CW complexes of finite type, all graded modules, according-ly, are finitely generated in each dimension.

Let $p \in \mathbf{P}$. A homomorphism $\varphi : G \to G'$ between two finitely generated abelian groups is said to be a p-epimorphism (p-monomorphism) if $\operatorname{coker} \varphi$ ($\ker \varphi$) is a finite group with order prime to p. If φ is both p-epimorphism and p-monomorphism then it is said to be p-isomorphism. If $\mathbf{P_1} \subset \mathbf{P}$ then φ is said to be a $\mathbf{P_1}$-epimorphism ($\mathbf{P_1}$-monomorphism or $\mathbf{P_1}$-isomorphism) if it is a p-epimorphism (p-monomorphism or p-isomorphism) for every $p \in \mathbf{P_1}$.

A map $\varphi : Y \to X$ is said to be a $\mathbf{P_1}$ equivalence iff $\pi_*(\varphi)$ (equivalenty $H_*(\varphi, Z)$, $H^*(\varphi, Z)$) is a $\mathbf{P_1}$ isomorphism. By the genus of the space X we mean the set $G(X)$ of homotopy classes of spaces Y which are p-equivalent to X for every $p \in \mathbf{P}$.

If X is a CW complex the homotopy (or Postnikov) approximation of X in $\dim \leqslant m$ is a pair $(Ht_m(X), \tau_m)$ where $\tau_m : X \to Ht_m(X)$ is such that $\pi_k(\tau_m)$ is an isomorphism for $k \leqslant m$ and $\pi_k(Ht_m(X)) = 0$ for $k > m$. Similarly, the homology (or Moore) approximation of X is a pair $(Hl_m(X), \tau'_m)$ where $\tau'_m : Hl_m(X) \to X$ yields an isomorphism $H_k(Hl_m(X), Z) \to H_k(X, Z)$ for $k \leqslant m$ and $H_k(Hl_m(X), Z) = 0$ for $k > m$.

Though we work in the category of CW complexes and continuous maps by a commutative diagram we mean commutative up to homotopy.

A commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_2} & Y_2 \\ \downarrow{h_2} & & \downarrow{h_1} \\ Y_1 & \xrightarrow{f_1} & X_0 \end{array}$$

(0.1)

is said to be a fiber square induced by f_1, h_1 if up to homotopy one has

$$X = \{y_1, y_2, \varphi \in Y_1 \times Y_2 \times PX_0 \mid f_1(y_1) = \varphi(0), h_1(y_2) = \varphi(1)\}.$$

$$h_2 = p_1 \mid X, \qquad f_2 = p_2 \mid X$$

where

$$p_i : Y_1 \times Y_2 \times PX_0 \to Y_i \quad \text{are the projections.}$$

(0.1) is said to be a fiber square in dim $\leq m$ iff it becomes a fiber square after applying Ht_m to the entire diagram. If X is a finite dimensional H-space then $H^*(X, Q) = A(\tilde{x}_{n_1}, \tilde{x}_{n_2}, ..., \tilde{x}_{n_r})$, $\tilde{x}_{n_i} \in H^{n_i}(X, Q)$ $n_i \leq n_{i+1}$. X is then said to be of type $(n_1, n_2, ..., n_r)$. Given a vector $\bar{n} = (n_1, n_2, ..., n_r)$ of natural numbers we write $K(Z, \bar{n}) = \prod_{i=1}^{r} K(Z, n_i)$, $S^{\bar{n}} = \prod_{i=1}^{r} S^{\bar{n}_i}$.

In this study we consider $G(X)$ for a finite dimensional H-space. Using exactly the same (or dual) methods one can obtain similar (or dual) results for H-spaces with finitely many non-vanishing homotopy groups (or for finite dimensional co-H-spaces).

To state results of this study concerning the structure of $G(X)$ one needs few notations: If A and B are matrices (over any ring R) denote by $A*B$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Let X be an H-space of type $(n_1, n_2, ..., n_r)$. Hom $^o_Z(QH^*(X, Z)/\text{torsion}$, $QH^*(X, Z)/\text{torsion})$ can be identified with the set of all matrices (over Z) of the form $A_{s_1}*A_{s_2}*...*A_{s_t}$, A_{s_i} being an $s_i \times s_i$ matrix, $s_1, s_2, ... s_t$ uniquely determined by $n_1, n_2, ... n_r$. Denote all such matrices by $\mathcal{M}(s_1, s_2, ..., s_t; Z)$. Given an integer t let $\mathcal{A} = \mathcal{A}_t$ be the set of all matrices in $\mathcal{M}(s_1, s_2, ..., s_t; Z)$ which are invertible mod Z_t. Then one has:

Theorem (essentially 2.4). There exists an integer t $(= \bar{t}_k$ in 2.4) and a correspondence $\xi : \mathcal{A} = \mathcal{A}_t \to G(X)$ described as follows: Fix a map $h_0 : X \to K(Z, \bar{n})$ so that $QH^*(h_0, Z)/\text{torsion}$ is an isomorphism. Given $A \in \mathcal{A}$ let $f_0 = f_0(A) : K(Z, \bar{n}) \to K(Z, \bar{n})$

be given by $H^*(f_0, Z) \iota_{n_i} = \sum_j a_{ij} \iota_{n_j} \ A = (a_{ij})$. If

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{f_1} & X \\
\downarrow{\tilde{h}_1} & & \downarrow{h_0} \\
K(Z, \tilde{n}) & \xrightarrow{f_0} & K(Z, \tilde{n})
\end{array}
$$

is a fiber square then

(a) $[Hl_k(\tilde{Y})] = [Y] = \xi(A) \in G(X) \quad (k = \dim X)$

(b) $QH^*(h_1, Z)/\text{torsion}$ is an isomorphism where $h_1 : Y \to K(Z, \tilde{n})$ is induced by $\tilde{h} : h_1 = \tilde{h}_1 \tau'$.

Corollary (2.6) ξ is onto.

If $\tilde{\alpha} : \mathcal{M}(s_1, s_2, ..., s_l; Z) \to \mathcal{M}(s_1, s_2, ..., s_l; Z_t)$ is the reduction it induces homomorphism

$$
\bar{\alpha} : GL(s_1, ..., s_l; Z) \to GL(s_1, s_2, ..., s_l, Z_t)
$$

of invertible matrices and $(\text{coker} \, \bar{\alpha}) = [(Z_t^*)/\{\pm 1\}]^l$ where Z_t^* are the units in Z_t. Note that $\mathcal{A} = \tilde{\alpha}^{-1} \, GL(s_1, s_2, ..., s_l; Z_t)$.

Structure Theorem (2.7) ξ factors through $\text{coker} \, \bar{\alpha}$ to obtain a correspondence $\bar{\xi} : [Z_t^*/\{\pm 1\}]^l \to G(X)$ which is onto.

The correspondence is given by

$$
\bar{\xi}(d_1, d_2, ..., d_l) = \xi(\bar{\xi}_{d_1} * I_{d_2} * \cdots * I_{d_l})
$$

where I_{d_i} is the $s_i \times s_i$ matrix given by

$$
I_{d_i} = \begin{pmatrix} I & 0 \\ 0 & d_i \end{pmatrix}
$$

One also obtains:

H-*structure theorem* (2.10, *compare with* [6] *and* [7] *lemma* 1.4):
Let X be a finite CW complex. If for every prime p there exists an H-space $X(p)$ and a p-equivalence $f_p : X \to X(p)$ and if $H^*(X(p), Q)$ are all isomorphic as Hopf algebras then X is an H-space.

A Product Theorem (2.11). Let $A, A' \in \mathcal{A}$. Then

$$
\xi(A) \times \xi(A') = \xi(I) \times \xi(A \cdot A') = X \times \xi(A \cdot A').
$$

Consequently if $A_i, B_i \in \mathcal{A}$ and

$$
A_1 \cdot A_2 \cdots \cdot A_m = B_1 \cdot B_2 \cdots \cdot B_m
$$

then

$$\xi(A_1) \times \xi(A_2) \cdots \times \xi(A_m) \approx \xi(B_1) \times \xi(B_2) \cdots \times \xi(B_m).$$

Theorem (2.12). *If* $[Y] \in G(X)$ *then*

$$Y^{\varphi(t)/2} \approx X^{\varphi(t)/2}$$

where φ is the Euler function. Finally, one obtains

A non cancellation theorem (3.5 *and compare with* [1], [2] *and* [7], p. 83). *If* $[Y] \in G(X)$ *then* $Y \times S^{\tilde{n}} \approx X \times S^{\tilde{n}}$.

1. Some Algebraic Lemmas

The following are well known simple facts concerning finitely generated abelian groups:

1.1. LEMMA: *Let G and G' be finitely generated abelian groups. If for every prime p there exists a p isomorphism $\varphi_p : G \to G'$ then G and G' are isomorphic.*

1.2. LEMMA: *Suppose $G \approx G'$. Then $\varphi : G \to G'$ is a p-isomorphism if and only if $\varphi \otimes Z_p$ is an isomorphism.*

1.3. COROLLARY. *Let $\{\varphi_p \,|\, p \in \mathbf{P}_1\}$ be a finite family, $\varphi_p : G \to G' \approx G$ a p-isomorphism. Put $a = \prod_{p \in \mathbf{P}_1} p$. Then $\varphi = \sum_{p \in \mathbf{P}_1} (a \cdot p^{-1}) \varphi_p$ is a $\mathbf{P}_1$ isomorphism.*

Proof. For every $p_0 \in P_1$ $\varphi \otimes Z_{p_0} = a p_0^{-1}(\varphi_{p_0} \otimes Z_{p_0})$ is an isomorphism. Hence φ is a $\mathbf{P}_1$ isomorphism.

1.4. COROLLARY. *If $[Y] \in G(X)$ then*
(a) $H^*(Y) \approx H^*(X)$ *and* $\pi(X) \approx \pi(Y)$.
(b) *Let $f : Y \to X$. If $\pi(f) \otimes Z_p$ is an isomorphism for all $p \in \mathbf{P}_1$ then f is a $\mathbf{P}_1$ equivalence.*

Throughout this chapter let X be a finite CW-H-space.

1.5. PROPOSITION. *If $[Y] \in G(X)$ then there exists a partition $\mathbf{P} = \mathbf{P}_1 \cup \mathbf{P}_2$ and two maps $f_i : Y \to X$ $i = 1, 2$ so that f_i is a $\mathbf{P}_i$ equivalence.*

Proof. Choose an arbitrary prime $q \in \mathbf{P}$. Let $f_2 : Y \to X$ be a q-equivalence. Then there exists a set $\mathbf{P}_2$ of primes so that $\mathbf{P}_1 = \mathbf{P} - \mathbf{P}_2$ is finite and f_2 is a $\mathbf{P}_2$ equivalence. For every $p \in \mathbf{P}_1$ let f_p be a p-equivalence. Let a be the product of all primes in $\mathbf{P}_1$. If $f_1 : Y \to X$ is given by $[f_1] = \prod_{p \in \mathbf{P}_1} [f_p]^{a/p}$ (where the product represents a product in

the algebraic loop $[Y, X]$ with an arbitrary bracketing) then by 1.3 $\pi_n(f_1)=$
$=\sum_{p \in \mathbf{P}_1}(ap^{-1})\pi_n(f_p)$ is a $\mathbf{P}_1$ isomorphism and f_1 is a $\mathbf{P}_1$ equivalence.

One notices that $\mathbf{P}_1$ may be replaced by any larger (but finite) set of primes.

1.6. LEMMA. *Given a commutative diagram*

$$
\begin{array}{ccc}
T & \xrightarrow{h_2} & Y_1 \\
\downarrow{h_1} & & \downarrow{f_1} \\
Y_2 & \xrightarrow{f_2} & X_0
\end{array}
$$
(1.6.1)

If for $i=1, 2$ h_i *and* f_i *are* $\mathbf{P}_i$ *equivalences in* $dim \leqslant m$, $\mathbf{P}=\mathbf{P}_1 \cup \mathbf{P}_2$ *then* (1.6.1) *is a fiber square in* $dim \leqslant m$.

Proof. Form the fiber square of f_1 and f_2

$$
\begin{array}{ccc}
Z' & \xrightarrow{h'_2} & X \\
\downarrow{h'_1} & & \downarrow{f_1} \\
Y & \xrightarrow{f_2} & X_0
\end{array}
$$

then there exists $\alpha: Z \to Z'$ with $h_i \sim h'_i \alpha$ $i=1, 2$. f_i being $\mathbf{P}_i$ equivalence in $dim \leqslant m$ implies that h'_i is a $\mathbf{P}_i$ equivalence and as h_i is a $\mathbf{P}_i$ equivalence so is α. Now, α being $\mathbf{P}_1$ and $\mathbf{P}_2$ equivalence in $dim \leqslant m$ is a homotopy equivalence in that range.

Let R be a ring. If $M=\{M_n \mid 0 \leqslant n \leqslant l\}$ is a graded module over R, M_n-free of rank s_n then $\widetilde{\mathcal{M}}(R)=\mathrm{Hom}^{\circ}_R(M, M)$ can be represented by the set of matrices of the form $A_1 * A_2 * \cdots * \mathcal{A}_l$ where A_n is $s_n \times s_n$ matrix over R and $B_1 * B_2$ is the matrix of the form

$$
\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}
$$

Denote this set by $\mathcal{M}(s_1, s_2, ..., s_l; R)$. Similarly define $GL(s_1, s_2, ..., s_l; R)$ and $SL(s_1, s_2, ..., s_l; R)$ as the invertible and determinant one matrices in $\mathcal{M}(s_1, s_2, ..., s_l; R)$. The product $*g$ in $M(s_1, s_2, ..., s_l; R)$ given by

$$
(A_1 * A_2 * \cdots * A_l) * g (\hat{A}_1 * \hat{A}_2 * \cdots * \hat{A}_l) = (A_1 * \hat{A}_1) * (A_2 * \hat{A}_2) * \cdots * (A_l * \hat{A}_l)
$$

corresponds to the natural homeomorphism $\mathrm{Hom}^{\circ}(M, M) \otimes \mathrm{Hom}^{\circ}(M, M) \to$ $\to \mathrm{Hom}^{\circ}(M \otimes M, M \otimes M)$. Let X be of type $(n_1, n_2, ..., n_r)$. Define $s_1, s_2, ..., s_l$ by the relation

$$
n_{s_1+s_2+\cdots+s_m} < n_{s_1+s_2+\cdots+s_m+1} = n_{s_1+s_2+\cdots+s_m+s_m+1} \quad 0 \leqslant m < l
$$
(1.7)

Put $M_m = QH^n_{s_1+s_2+\cdots+s_m}(X, Z)/\mathrm{torsion}$ and one has

$$
\mathrm{Hom}^{\circ}[QH^*(X, Z)/\mathrm{torsion}, QH^*(X, Z)/\mathrm{torsion}] = \mathcal{M}(s_1, s_2, ..., s_l; Z).
$$

Let t be an integer. Put

$$\alpha : GL(n, Z) \to GL(n, Z_t)$$

Then one has

1.8. LEMMA. $GL(n, Z_t)/\mathrm{im}\,\alpha = Z_t^*/\{\pm 1\}$ *where* Z_t^* *are the units in* Z_t. *The homomorphism* $GL(n, Z_t) \to GL(n, Z_t)/\mathrm{im}\,\alpha$ *is given by the determinant.*

Let $\tilde{\alpha} : \mathcal{M}(s_1, s_2, ..., s_l; Z) \to \mathcal{M}(s_1, ..., s_l; Z_t)$ induce

$$\bar{\alpha} : GL(s_1, s_2, ..., s_l; Z) \to GL(s_1, s_2, ..., s_l; Z_t)$$

then

$$GL(s_1, s_2, ..., s_l; Z_t)/\mathrm{im}\,\bar{\alpha} \approx [Z_t^*/\{\pm 1\}]^l.$$

2. The Structure of $G(X)$

Throughout this chapter let (X, μ) be an H-space of type $\bar{n} = (n_1, n_2, ..., n_r)$. To this one associate the sequence $(s_1, s_2, ..., s_l)$ defined by (1.7). Let $\dim X \le k$. Given an integer m, $m \ge k$ let $t_m = t_m(G(X))$ be the order of the finite group

$$\sum_{n \le m} (\ker \sigma_n + \mathrm{coker}\,\sigma_n)$$

where $\sigma_n : \pi_n(X) \to PH_n(X)/\text{torsion}$ is the Hurewicz-Serre homomorphism.

Another interpretation for t_m can be given by the following

2.1. LEMMA. *Let* $h_0 : X \to K(Z, \bar{n}) = \prod_i K(Z, n_i)$ *satisfy:* $x_{n_i} = H^*(h_0, Z) \iota_{n_i}$, $1 \le i \le r$, *represent a basis for* $QH^*(X, Z)/\text{torsion}$. *Let* $F = \text{fiber}\,h_0$. *Then* $\mathrm{order}\,\pi_n(F) = \mathrm{order}(\ker\sigma_n + \mathrm{coker}\,\sigma_{n+1})$. *Hence,* $t_m = \mathrm{order}\,\sum_{n \le m} \pi_n(F)$.
(Note that $\mathrm{coker}\,\sigma_{m+1} = PH_{m+1}(X, Z)/\text{torsion} = 0$).

Proof. One has the following commutative diagram

$$
\begin{array}{ccc}
\pi_n(X) & \xrightarrow{\pi_n(h_0)} & \pi_n(K(Z, \bar{n})) \\
\downarrow{\scriptstyle\sigma_n(X)} & & \approx \downarrow{\scriptstyle\sigma_n(K)} \\
PH_n(X, Z)/\text{torsion} & \xrightarrow[\approx]{h_{0*}} & PH_n(K(Z, \bar{n}), Z)/\text{torsion}
\end{array}
$$

As $QH^*(h_0, Z)/\text{torsion}$ is an isomorphism so is its dual h_{0*}. Hence, $\ker\sigma_n = \ker\pi_n(h_0)$ and $\mathrm{coker}\,\sigma_n = \mathrm{coker}\,\pi_n(h_0)$ and obviously $\mathrm{order}\,\pi_n(F) = \mathrm{order}[\ker\pi_n(h_0) + \mathrm{coker}\,\pi_{n+1}(h_0)]$.

2.2. PROPOSITION. *Given a set* $\{x_{n_1}, x_{n_2}, ..., x_{n_r}\}$, $x_{n_i} \in H^{n_i}(X, Z)/\text{torsion}$ *which*

reduces to a basis for $QH^*(X, Z)$/torsion *and given maps* $g: Y \to \tilde{Y}, f: Y \to X$, *(Y and* $\tilde{Y}$
arbitrary CW *complexes of* $\dim \leqslant m$) *so that* $\mathrm{im}[\tilde{H}(g, Z)$/torsion] *is an ideal in*
$\tilde{H}(Y, Z)$/torsion. *For any set* $\{z_{n_1}, ..., z_{n_r}\}$, $z_{n_i} \in H^{n_i}$ $(\tilde{Y}, Z)$/torsion *there exist maps*
$\tilde{f}_i: \tilde{Y} \to X, i = 1, 2, ..., r$ *with* torsion $\pi_n(\tilde{Y}) \subset \ker \pi_n(\tilde{f}_i)$ *for* $n \leqslant m$ *so that*

$$[H^*(\tilde{f}, Z)/\text{torsion} - H^*(f, Z)/\text{torsion}] \, x_{n_i} = t_m[(H^*(g, Z)/\text{torsion}) \, z_{n_i}] \quad (2.2.1)$$

where $\tilde{f}$ *is given by*

$$[\tilde{f}] = [\tilde{f}_r \circ g] \, ([\tilde{f}_{r-1} \circ g] \cdots ([\tilde{f}_2 \circ g] \, ([\tilde{f}_1 \circ g] \cdot [f]) \cdots))$$

(the products taken in $[Y, X]$).

   Proof. Let $X_m = Ht_m(X)$. As $\dim Y \leqslant m$, $\dim \tilde{Y} \leqslant m$ and $m \geqslant k$ one can replace all
maps in $[Y, X]$ and $[\tilde{Y}, X]$ by the corresponding maps in $[Y, X_m]$ and $[\tilde{Y}, X_m]$. Con-
sider x_{n_i} being in H^{n_i} (X_m, Z)/torsion and it suffices to prove relation (2.2.1) for X_m
instead of X. By 2.1 $t_m = \mathrm{order} \, \pi_*(F_m)$ (F_m the fiber of $X_m \to K(Z, \bar{n})$). Hence, $t_m[1]$
$([1] \in [K[Z, \bar{n}), K(Z, \bar{n})])$ lifts to $h_1: K(Z, \bar{n}) \to X$, $[h_0 \circ h_1] = t_m[1]$. We construct $\tilde{f}_i$
inductively. Suppose $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_{i-1}$ where constructed so that if

$$[\tilde{f}_{i-1}] = [\tilde{f}_{i-1} \circ g] \, ([\tilde{f}_{i-2} \circ g] \, (\cdots ([\tilde{f}_1 \circ g] \cdot [f])))$$

then

$$[H^*(\tilde{f}_{i-1}, Z)/\text{torsion} - H^*(f, Z)/\text{torsion}] \, x_{n_j} = \begin{cases} t_m(H^*(g, Z)/\text{torsion}) \, z_{n_j} & j < i \\ t_m(H^*(g, Z)/\text{torsion}) \, \tilde{z}_{n_j} & j \geqslant i \end{cases}$$

(there is no problem in starting the induction by starting with $\tilde{f}_0: \tilde{Y} \to X$ being the
constant map and $\tilde{z}_{n_j} = 0$). Let $\tilde{f}_i$ be the composition

$$\tilde{Y} \xrightarrow{z_{n_i} - \tilde{z}_{n_i}} K(Z, n_i) \xrightarrow{\text{inj}} K(Z, \bar{n}) \xrightarrow{h_1} X.$$

Then torsion $\pi(\tilde{Y}) \subset \ker \pi(\tilde{f}_i)$. Let $(\ )^*$ denote $H^*(\ , Z)$/torsion. If $[\tilde{f}_i] =$
$= [\tilde{f}_i \circ g] \, [\tilde{f}_{i-j}]$ then

$$\tilde{f}_i^* x_{n_j} = \Delta^*(\tilde{f}_i \circ g \times \tilde{f}_{i-1})^* \mu^* x_{n_j} = \sum_{s=0}^{s_0} (\tilde{f}_i \circ g)^* x'_{n_j, s} \cdot \tilde{f}_{i-1}^* x''_{n_j, s}$$

where $x'_{n_j, s}$ and $x''_{n_j, s}$ are given by $\mu^* x_{n_j} = \sum_{s=0}^{s_0} x'_{n_j, s} \otimes x''_{n_j, s}$ $(x'_{n_j, 0} = x_{n_j} = x''_{n_j, s_0}$,
$x''_{n_j, 0} = 1 = x'_{n_j, s_0}$ and for $0 < s < s_0$, $0 < \dim x'_{n_j, s} < n_j$, $0 < \dim x''_{n_j, s} < n_j)$.
   As $H^n(\tilde{f}_i \circ g, Z)$/torsion$= 0$ for $n < n_i$ and $\tilde{f}_i^* x_{n_i} = \tilde{f}_i^* h_0^* 1_{n_i} = t_m(z_{n_i} - \tilde{z}_{n_i})$ one has
$\tilde{f}_i^* x_{n_j} = \tilde{f}_{i-1}^* x_{n_j}$ if $j < i$,

$$\tilde{f}_i^* x_{n_i} = \tilde{f}_{i-1}^* x_{n_i} + t_m g^*(z_{n_i} - \tilde{z}_{n_i})$$

and hence for $j \leqslant i$

$$(\hat{f}_i^* - f^*) x_{n_j} = (\hat{f}_i^* - \hat{f}_{i-1}^*) x_{n_j} + (\hat{f}_{i-1}^* - f^*) x_{n_j} = t_m g^* z_n$$

For $j > i$ (as $\hat{f}_i^* x_{n_j} = 0$ for $j \neq i$) one has

$$\alpha_j = (\hat{f}_i^* - \hat{f}_{i-1}^*) x_{n_j} = \sum_{s=1}^{s_0-1} \hat{f}_{i-1}^* x_{n_j,s}'' (\tilde{f}_i \circ g)^* x_{n_j,s}'$$

and as $\overline{(\tilde{f}_i \circ g)^*} \subset t_m \overline{(\operatorname{im} g^*)}$ and as $\overline{\operatorname{im} g^*}$ is an ideal $\alpha_j \in t_m \overline{\operatorname{im} g^*}$ and hence for $i > j$ $(\hat{f}_i^* - f^*) x_{n_j} = (\hat{f}_i^* - \hat{f}_{i-1}^*) x_{n_j} + t_m g^* \tilde{z}_{n_j} = t_m g^* \tilde{z}_{n_j}'$ and 2.2 follows.

Let $\tilde{t}_m$ be the smallest integer divisible by t_m and by every torsion prime in $H^*(X, Z)$. Let $\mathbf{P}_m = \{ p \mid p \mid \tilde{t}_m \}$ and $\widetilde{\mathbf{P}}_m = \mathbf{P} - \mathbf{P}_m$. Applying 2.2 one obtains:

2.3. PROPOSITION. *Let* $[Y] \in G(X)$. *Given a* $\mathbf{P}_k$ *equivalence* $f : Y \to X$, *a set of integral classes* $x_{n_1}, x_{n_2}, \ldots, x_{n_r}$ *representing a basis for* $QH^*(X, Z)/\text{torsion}$ *and a set* $\tilde{z}_{n_1}, \tilde{z}_{n_2}, \ldots, \tilde{z}_{n_r}, \tilde{z}_{n_i} \in H^{n_i}(Y, Z)$, *there exists a* $\mathbf{P}_k$ *equivalence* $\hat{f} : Y \to X$ *with*

$$[H^*(\hat{f}, Z) - H^*(f, Z)] x_{n_j} = \tilde{t}_k \tilde{z}_{n_j} + \theta_j$$

where θ_j *is a torsion element.*

Proof. Apply 2.2 for $Y = \tilde{Y}$, $g = 1$, $z_{n_j} = \tilde{t}_k / t_k \, \tilde{z}_{n_j}$. Then one has only to show that $\hat{f} = f$ is a $\mathbf{P}_k$ equivalence. (Note that if $H^*(X, Z)$ is a torsion free $t_k = \tilde{t}_k$ and then $H^*(\hat{f}, Z) \otimes Z_{t_k} = H^*(f, Z) \otimes Z_{t_k}$ implies that $\hat{f}$ is a $\mathbf{P}_k$ equivalence). By 2.2 torsion $\pi(\hat{f}_i) = 0$ and hence torsion $\pi(\hat{f}) = $ torsion $\pi(f)$.

Consider the following diagram:

$$
\begin{array}{ccc}
\pi(Y)/\text{torsion} & \xrightarrow[\pi(f)/\text{torsion}]{\pi(\hat{f})/\text{torsion}} & \pi(X)/\text{torsion} \\
\downarrow{\scriptstyle \tilde{\sigma}(Y)} & & \downarrow{\scriptstyle \tilde{\sigma}(X)} \\
PH_*(Y)/\text{torsion} & \xrightarrow[f_*]{\hat{f}_* = f_*} & PH_*(X)/\text{torsion}
\end{array}
$$

($\tilde{\sigma}$ – the Hurewicz-Serre monomorphism).

Now all groups are isomorphic and $\det[\tilde{\sigma}(Y)] = \det[\tilde{\sigma}(X)]$

$$H^*(\hat{f}, Z)/\text{torsion} \otimes Z_{t_k} = H^*(f, Z)/\text{torsion} \otimes Z_{t_k}$$

It follows that $H^*(\hat{f}, Z)$ and its dual $\hat{f}_*$ are $\mathbf{P}_k$ isomorphisms. Now if G is free abelain a homomorphism $\alpha : G \to G$ is a $\mathbf{P}_k$ isomorphism if and only if $(\det \alpha, \tilde{t}_k) = 1$. Now,

$$(\det \hat{f}_*)(\det \tilde{\sigma}(Y)) = (\det \tilde{\sigma}(X))(\det \pi(\hat{f})/\text{torsion})$$

implies $(\det \hat{f}_*) = \det(\pi(\hat{f})/\text{torsion})$ and hence $\pi(\hat{f})/\text{torsion}$ is a $\mathbf{P}_k$ isomorphism.

As torsion $\pi(\hat{f}) = $ torsion $\pi(f)$ and $\pi(\hat{f})/$torsion are $\mathbf{P}_k$ isomorphisms so is $\pi(\hat{f})$ and 2.3 follows. One can construct elements in $G(X)$ as follows:

2.4. PROPOSITION. *Let* $h_0: X \to K(Z, \bar{n})$ *realize a basis for* $QH^*(X, Z)/$torsion *i.e.*: $x_{n_i} = H^*(h_0, Z) \iota_{n_i}$ *reduces to a basis for* $QH^*(X, Z)/$torsion. *Given any matrix* $A = (a_{ij}) \in \mathcal{M}(s_1, s_2, ..., s_l; Z)$ *with* $(\det A, \bar{\iota}_k) = 1$ *let* $f_0: K(Z, \bar{n}) \to K(Z, \bar{n})$ *be given by* $H^*(f_0, Z) \iota_{n_i} = \sum_j a_{ij} \iota_{n_j}$. *Let* $Y = Y(A) = Hl_k(\tilde{Y})$ *where*

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{f}_1} & X \\
\downarrow{\tilde{h}_1} & & \downarrow{h_0} \\
K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n})
\end{array}
$$

is a fiber square induced by h_0, f_0. *Then:*

(a) $\xi(A) = [Y] \in G(X)$

(b) *The set* $\{y_{n_i} = H^{n_i}(h_1, Z) \iota_{n_i}\}$ *reduces to a basis for* $QH^*(Y, Z)/$torsion, *where* $h_1: Y \to K(Z, \bar{n})$ *is the composition*

$$
H_{l_k}(\tilde{Y}) = Y \xrightarrow{\tau'_k} \tilde{Y} \xrightarrow{\tilde{h}_1} K(Z, \bar{n}).
$$

Proof. By 2.1 h_0 is a $\tilde{\mathbf{P}}_k = \mathbf{P} - \mathbf{P}_k$ equivalence in $\dim \leqslant k$ hence so is $\tilde{h}_1$. It follows that in $\dim \leqslant k$ $Y \approx \tilde{Y} \approx_{\tilde{\mathbf{P}}_k} K(Z, \bar{n}) \approx_{\tilde{\mathbf{P}}_k} X$. Consequently, $Y \approx_{\tilde{\mathbf{P}}_k} X$ in $\dim \leqslant k$ but both being of $\dim \leqslant k$ it follows that $Y \approx_{\tilde{\mathbf{P}}_k} X$. Now, $(\det A, \bar{\iota}_k) = 1$ implies that $A = \pi(f_0)$ is a $\mathbf{P}_k$ isomorphism and hence f_0 (and consequently $\tilde{f}_1$) are $\mathbf{P}_k$ equivalences. Again in $\dim \leqslant k$ $Y \approx \tilde{Y}$ and $Y \approx_{\tilde{\mathbf{P}}_k} X$ and (a) follows. To prove (b) one notices first that in $\dim \leqslant k$ one can consider

$$
\begin{array}{ccc}
Y & \xrightarrow{f_1} & X \\
\downarrow{h_1} & & \downarrow{h_0} \\
K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n})
\end{array}
$$

as a fiber square. Further, if $[Y] \in G(X)$ then $\operatorname{coker}\sigma_n(Y) \approx \operatorname{coker}\sigma_n(X)$ and $\ker\sigma_n(Y) = \ker\sigma_n(X)$ (as all these groups are finite they involve only a finite set $\mathbf{P}_1$ of primes and by the procedure of 1.5 there exists a $\mathbf{P}_1$ equivalence $Y \to X$ which will yield the desired isomorphisms). One obtains the following commutative diagrams for $n \leqslant k$

$$
\begin{array}{c}
\pi_n(F) \\
\nearrow \quad \searrow \\
\pi_n(Y) \xrightarrow{\pi_n(f_1)} \pi_n(X) \\
j_1 \nearrow \quad \downarrow{\pi_n(h_1)} \qquad \downarrow{\pi_n(h_0)} \quad \searrow{\varphi_1} \\
\pi_n(\tilde{F}) \qquad\qquad\qquad\qquad \pi_{n-1}(\tilde{F}) \\
\searrow{j_0} \qquad \nearrow \quad \searrow \quad \nearrow{\varphi_0} \\
\pi_n(K(Z, \bar{n})) \xrightarrow{\pi_n(f_0)} \pi_n(K(Z, \bar{n})) \\
\searrow \qquad \nearrow \\
\pi_{n-1}(F)
\end{array}
$$

where $\tilde{F}=$ fiber $f_0=$ fiber $f_1 \approx \prod_i K(Z_{\lambda_i}, n_i-1)$ where the matrix $\operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$ is equivalent to A. As $j_0=0$ and as the order of $\pi_n(\tilde{F})$ is prime to that of $\pi_n(F)\,\pi_n(j_1)=0$ and $\pi_n(f_1)$ is a monomorphism. As φ_1 factors through a free group $\varphi_1(\operatorname{tor}\pi_n(X))=0$ and one has a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & \pi_n(Y)/\text{torsion} & \xrightarrow{f_{1\#}} & \pi_n(X)/\text{torsion} & \xrightarrow{\tilde{\phi}_1} & \pi_{n-1}(\tilde{F}) & \to & 0 \\
 & & \downarrow{\sigma_n(Y)} & & \downarrow{\sigma_n(X)} & & \downarrow{\sigma'} & & \\
0 & \to & PH_n(Y)/\text{torsion} & \xrightarrow{f_{1*}} & PH_n(X)/\text{torsion} & \xrightarrow{\varphi} & \operatorname{coker} f_1^* & & \\
 & & \downarrow{h_{1*}} & & \downarrow{h_{0*}} & & \downarrow{h'} & & \\
0 & \to & PH_n(K(Z,\bar{n}),Z)/\text{torsion} & \xrightarrow{f_{0*}} & PH_n(K(Z,\bar{n}),Z)/\text{torsion} & \xrightarrow{\tilde{\phi}_0} & \pi_{n-1}(\tilde{F}) & & \\
 & & \sigma_n(K)\uparrow\approx & & \approx\uparrow\sigma_n(K) & & \nearrow\varphi_0 & & \\
0 & \to & \pi_n(K(Z,\bar{n})) & \xrightarrow{\pi_n(f_0)} & \pi_n(K(Z,\bar{n})) & & & &
\end{array}
$$

$\tilde{\sigma}_n(Y)$ and $\tilde{\sigma}_n(X)$ are monomorphisms.

$\operatorname{Coker}\tilde{\sigma}_n(Y)\approx\operatorname{coker}\sigma_n(Y)\approx\operatorname{coker}\sigma_n(X)\approx\operatorname{coker}\tilde{\sigma}_n(X)$ are $\mathbf{P}_k$ torsion groups and as f_1 is a $\mathbf{P}_k$ equivalence $f_{1\#}$ and f_{1*} are $\mathbf{P}_k$ isomorphisms and they induce a $\mathbf{P}_k$ isomorphism $\operatorname{coker}\tilde{\sigma}_n(Y)\to\operatorname{coker}\tilde{\sigma}_n(X)$ and hence an isomorphism. It follows that the left upper square in the last diagram is a push out diagram and σ' and consequently h' are isomorphisms. As h_{0*} is an isomorphism so is h_{1*} and so is its dual $QH^*(h_1, Z)/$ torsion.

Again let $\tilde{\alpha}:\mathcal{M}(s_1, s_2, ..., s_l; Z)\to\mathcal{M}(s_1, \quad s_2, ..., \quad s; \quad Z_{\bar{\imath}_k})$ and let $\mathcal{A}= =\tilde{\alpha}^{-1}\times GL(s_1, s_2, ..., s_l; Z_{\bar{\imath}_k})$. 2.4(a) defines a correspondence

$$\xi:\mathcal{A}\to G(X). \qquad \xi(I)=[X].$$

2.5. PROPOSITION. *Let* $[Y]\in G(X)$. *Let* $f:Y\to X$ *be a* $\mathbf{P}_k$ *equivalence and let* $A\in\mathcal{A}$ *represent* $QH^*(f, Z)/$*torsion. Then*

$$[Y]=\xi(A).$$

Proof. Suppose A represents $QH^*(f, Z)/$torsion with respect to bases represented by $x_{n_1}, ..., x_{n_r}$, $x_{n_i}\in H^{n_i}(X, Z)$ and $y'_{n_1}, ..., y'_{n_r}$, $y'_{n_i}\in H^{n_i}(Y, Z)$. We shall replace f by a $\mathbf{P}_k$ equivalence $\hat{f}$, $QH^*(\hat{f}, Z)/$torsion$=QH^*(f, Z)/$torsion and $y'_{n_1}, ..., y'_{n_2}$ by $y_{n_1}, ..., y_{n_r}$ representing the same basis as $y'_{n_1}, ..., y'_{n_r}$ so that if $h_0:X\to K(Z,\bar{n})$, $h_1:Y\to K(Z,\bar{n}), f_0:K(Z,\bar{n})\to K(Z,\bar{n})$ are given by

$$H^*(h_0, Z)\,\iota_{n_i}=x_{n_i}, \qquad H^*(h_1, Z)\,\iota_{n_i}=y_{n_i}$$
$$H^*(f_0, Z)\,\iota_{n_i}=\sum_j a_{ij}\cdot\iota_{n_j} \qquad A=(a_{ij}),$$

then

$$h_0\circ\hat{f}\sim f_0\circ h_1.$$

In the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow{h_1} & & \downarrow{h_0} \\
K(Z, \bar{n}) & \xrightarrow{f_0} & K(Z, \bar{n})
\end{array}
$$

f_0 and f are $\widetilde{\mathbf{P}}_k$ equivalences and as $QH(h_i, Z)/$torsion $i = 0, 1$ are isomorphisms by 2.1 they are $\mathbf{P}_k$ equivalences in $\dim \leqslant k$. Hence, by 1.6 the last diagram represents a fiber square in $\dim \leqslant k$ and hence $[Y] = \xi(A)$. Equivalently one has to construct $\hat{f}$ and find y_{n_i} so that

$$
H^*(\hat{f}, Z)\, x_{n_i} = \sum a_{ij} y_{n_j}.
$$

First note that without a loss of generality one may assume A to be diagonal for otherwise $D = E_1 A E_2$ where E_i are invertible and D-diagonal, $D = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$. Then if E_2^{-1} applied on $y_{n_1}, ..., y_{n_r}$ yields the basis $\bar{y}_{n_1}, \bar{y}_{n_2}, ..., \bar{y}_{n_r}$ and E_1 applied on $x_{n_1}, ..., x_{n_r}$ yields a basis $\bar{x}_{n_1}, ..., \bar{x}_{n_r}$ then

$$
H^*(f, Z)\,(\bar{x}_{n_1}, ..., \bar{x}_{n_r}) = E_1 A E_2 E_2^{-1} (y_{n_1}, y_{n_2}, ..., y_{n_r}) = D(\bar{y}_{n_1}, ..., \bar{y}_{n_r})
$$

So $H^*(f, Z)\, x_{n_i} = \sum_j a_{ij} y_{n_j}$ if and only if $H^*(f, Z)\, \bar{x}_{n_i} = \lambda_i \bar{y}_{n_i}$. So assume $A = D = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$. Hence, it is given that

$$
H^*(f, Z)\, x_{n_i} = \lambda_i y'_{n_i} + d_i + \theta_i,
$$

d_i-decomposable and θ_i a torsion element. Now $(\det A, \bar{t}_k) = 1$ implies $(\lambda_i, \bar{t}_k) = 1$. Let $1 + a_i \cdot \bar{t}_k = b \cdot \lambda_i$. Now apply 2.3 for $\tilde{z}_{n_j} = a_j \cdot d_j$. $\hat{f}$ then will satisfy

$$
H^*(\hat{f}, Z)\, x_{n_i} = H^*(f, Z)\, x_{n_i} + \bar{t}_k a_i d_i + \hat{\theta}_i = \lambda_i y'_{n_i} + (1 + \bar{t}_k a_i)\, d_i + \hat{\theta}_i + \theta_i.
$$

As $(\mathrm{order}\ (\theta_i + \hat{\theta}_i), \lambda_i) = 1$ $\theta_i + \hat{\theta}_i = \lambda_i \cdot \theta'_i$ and $H^*(f_1, Z)\, x_{n_i} = \lambda_i (y'_{n_i} + b_i \cdot d_i + \theta'_i)$. $y_{n_i} = y'_{n_i} + b_i d_i + \theta'_i$ are the desired classes.

2.6. COROLLARY. $\xi: A \to G(X)$ of 2.4(a) is onto.

Proof. If $[Y] \in G(X)$ then by 1.5 there exists a $\mathbf{P}_k$ equivalence $f: Y \to X$ and by 2.5 $[Y] = \xi(A)$ for some $A \in \mathscr{A}$.

It is quite obvious from the definition of ξ in 2.4 that if $B \in GL(s_1, s_2, ..., s_l; Z)$ then $\xi(BA) = \xi(A)$. A simple application of 2.3 and 2.5 shows also that for any $C \in \mathscr{A}(s_1, s_2, ..., s_l; Z)$ $\xi(A + \bar{t}_k C) = \xi(A)$. Hence if,

$$
\bar{\alpha}: GL(s_1, s_2, ..., s_l; Z) \to GL(s_1, s_2, ..., s_l; Z_{\bar{t}_k})
$$

then ξ factors through $\mathrm{coker}\,\bar{\alpha} = (Z^*_{\bar{t}_k}/\{\pm 1\})^l$ (see 1.8). Hence, one obtains the following

2.7. A STRUCTURE THEOREM for $G(X)$. *There exists a correspondence* $\xi:(Z_{t_k}^*/\{\pm 1\})^l \to G(X)$ *which is onto.* ξ *is given by*

$$\xi(d_1, d_2, ..., d_l) = \xi(I_{d_1} * I_{d_2} * \cdots * I_{d_l}),$$

$d_i \in Z_{t_k}^*$, *where* I_{d_i} *is the* $s_i \times s_i$ *matrix given by*

$$I(d_i, s_i) = I_{d_i} = \begin{pmatrix} I & 0 \\ 0 & d_i \end{pmatrix}$$

By (2.4) it follows that $\tilde{Y} \approx \text{fiber}(\tilde{g}: X \to K_\lambda)$ for some map $\tilde{g}$ where $K_\lambda = \prod_{i=1}^r K(Z_{\lambda_i}, n_i)$ and the λ_i's are defined by the fact that A is equivalent to $\text{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$. Moreover, if $\varrho_{\lambda_i}: H^*(\ , Z) \to H^*(\ , Z_{\lambda_i})$ is the reduction then $H^*(\tilde{g}, Z_{\lambda_i}) \bar{\iota}_{n_i} = \varrho_{\lambda_i} x_{n_i}$ where $\bar{\iota}_{n_i} \in H_{n_i}(K(Z_{\lambda_i}, n_i), Z_{\lambda_i})$ is the fundamental class and $x_{n_1}, x_{n_2}, ..., x_{n_r}$ are integral classes representing a basis for $QH^*(X, Z)/\text{torsion}$. Note that given $[Y] \in G(X)$ by 1.5 there exists a $\mathbf{P}_{2k}$ equivalence $f: Y \to X$ and by 2.5 $[Y] = \xi(A)$, $(\det A, \bar{\iota}_{2k}) = 1$.

2.8. PROPOSITION. *Let* $m = 2k$. *Given integers* $\lambda_1, \lambda_2, ..., \lambda_r$ *prime to* $\bar{\iota}_m$ *and elements* $x_{n_1}, x_{n_2}, ..., x_{n_r} \in H^*(X, Z)$ *representing a basis for* $QH^*(X, Z)/\text{torsion}$, *there exists an H-structure* μ_0 *for* X *with respect to which* $\varrho_{\lambda_i} x_{n_i}$ *are primitive.*

Proof. As all λ_i are prime to all torsion primes of $H^*(X, Z)$ $H^*(X, Z_{\lambda_i}) \approx$ $\approx H^*(X, Z) \otimes Z_{\lambda_i}$. Replacing each λ_i by the product $\lambda = \lambda_1 \lambda_2 \cdots \lambda_r$ suffices to prove the theorem for $\lambda_1 = \lambda_2 = \cdots = \lambda_r = \lambda$.

Denote $(\ )^* = H^*(\ , Z)/\text{torsion}$. Let $\tilde{x}_{n_i}$ be the image of x_{n_i} in $H^*(X, Z)/\text{torsion}$. Given any H-structure μ of X one has

$$\mu^* \tilde{x}_{n_i} = \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{x}_{n_i} + \sum_{s=1}^a \tilde{x}'_{n_i, s} \otimes \tilde{x}''_{n_i, s}.$$

Put $\omega_i = \bar{\mu}^* \tilde{x}_{n_i} = \mu^* \tilde{x}_{n_i} - \tilde{x}_{n_i} \otimes 1 - 1 \otimes \tilde{x}_{n_i} = \sum_{s=1}^a x'_{n,s} \otimes x''_{n,s}$ then $\omega_i = \Lambda^* \tilde{\omega}_i$ where $\tilde{\omega}_i \in H^*(X \wedge X, Z)/\text{torsion}$ and $\Lambda: X \times X \to X \wedge X$ is the reduction. As $\dim X \times X \leqslant m$, $\dim X \wedge X \leqslant m$ one can apply 2.2 for

$$Y = X \times X, \quad \tilde{Y} = X \wedge X, \quad f = \mu, \quad g = \Lambda.$$

(and note that $\overline{\text{im} \Lambda^*}$ is an ideal) and $z_{n_i} = k \tilde{\omega}_i$ where k is an integer satisfying $1 + \bar{\iota}_m k = b\lambda$. If $\tilde{f} = \mu_0$ then, as each $\tilde{f}_i$ of 2.2 factors through $X \wedge X$, $\tilde{f}_i \mid X \vee X = *$ and $\mu_0 \mid X \vee X = \mu \mid X \vee X$. Hence μ_0 is an H-structure. Now,

$$\mu_0^* \tilde{x}_{n_i} = \mu_0^* \tilde{x}_{n_i} + \bar{\iota}_m k \omega_i = \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{x}_{n_i} + \omega_i + \bar{\iota}_m k \omega_i$$
$$= \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{x}_{n_i} + \lambda b \omega_i.$$

As $\varrho_\lambda x_{n_i} = \tilde{\varrho}_\lambda \tilde{x}_{n_i}$ $(\tilde{\varrho}_\lambda : H^*(X, Z)/\text{torsion} \to H^*(X, Z_\lambda))$

$$H^*(\mu_0, Z_\lambda) \varrho_\lambda x_{n_i} = H^*(\mu_0, Z_\lambda) \tilde{\varrho}_\lambda \tilde{x}_{n_i}$$

$$= \tilde{\varrho}_\lambda \tilde{x}_{n_i} \otimes 1 + 1 \otimes \tilde{\varrho}_\lambda \tilde{x}_{n_i} = \varrho_\lambda x_{n_i} \otimes 1 + 1 \otimes \varrho_\lambda x_{n_i}$$

and $\varrho_\lambda x_{n_i}$ are primitive.

2.9. COROLLARY. *If* $[Y] \in G(X)$. *Then* Y *is an* H-*space and there exists a* P_{2k} *equivalence* $f_1 : Y \to X$ *which is an* H-*map with respect to some* H-*structure of* X.

Proof. By the remarks preceding 2.8 one may replace k by $m = 2k$ in 2.7 and then $\tilde{f}_1 : \tilde{Y} \to X$ is the fiber of a map $\tilde{g} : X \to K_\lambda$ $K_\lambda = \prod_i K(\lambda_i, n_i)$ $(\lambda_i, \tilde{t}_{2k}) = 1$. By 2.8 choosing the H-structure of X properly $\tilde{g}$ is an H-map and so $\tilde{Y}$ and $\tilde{f}_1$ become an H-space and an H-map. As $Hl_{2k}(\tilde{Y}) = Hl_k(\tilde{Y}) = Y$ Y and $\tau' : Y \to \tilde{Y}$ admit H-structure and so does $f_1 = \tilde{f}_1 \circ \tau'$.

2.10. COROLLARY. (See [6] and [7]). *Let* X *be a finite* CW *complex. If for every prime* p *there exist an* H-*space* $X(p)$ *and a* p-*equivalence* $f_q : X \to X(p)$ *and if all* $H^*(X(p), Q), p \in P$, *are isomorphic as Hopf algebras then* X *admits an* H-*structure.*

Proof. Let $H^*(X, Q) \approx H^*(X(p), Q) \approx A(_{n_1}, x_{n_2}, \ldots, x_{n_r})$. As $X \approx (S^{\bar{n}} = S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}$ for almost all primes one may assume that the number of different spaces $X(p)$ and maps f_q is finite. Now, for every $p, q \in \mathbf{P}$ $H^*(X(p), Q) \approx H^*(X(q), Q)$. Hence $K(Z, \bar{n}) = \prod_{i=1}^r K(Z, n_i)$ admits an H-structure and there exist Q equivalences $h_p : X(p) \to K(Z, \bar{n})$ which are H-maps. Moreover, using the procedure of [8] one may replace $X(p)$ by $X'(p)$ so that h_p decomposes into two H-maps $- X(p) \overset{h'_p}{\to} X'(p) \overset{h''_p}{\to} K(Z, \bar{n})$ where h'_p is a p-equivalence and h''_p is a $\mathbf{P} - \{p\}$ equivalence. The pullback $X' = \prod_{h''_p} X'(p)$ is then an H-map, and $X' \approx_p X'(p) \approx_p X$. Hence $[Y] \in G(X')$ and apply 2.9.

Let $A, \hat{A} \in \mathscr{A} \subset M(s_1, s_2, \ldots, s_l; Z)$. Then

$$\bar{\alpha}A \equiv \bar{\alpha}(I(d_1, s_1) * I(d_2, s_2) * \cdots * I(d_l, s_l)) \bmod \operatorname{im} \bar{\alpha}.$$

$$\bar{\alpha}\hat{A} \equiv \bar{\alpha}(I(\hat{d}_1, s_1) * I(\hat{d}_2, s_2) * \cdots * I(\hat{d}_l, s_l)) \bmod \operatorname{im} \bar{\alpha}$$

(If $A = A_1 * A_2 * \cdots * A_l$ then $d_i = \det A_i \in Z_{t_k}^*$ and similarly for $\hat{A}_i$ and $\hat{d}_i$). It can be easily seen that

$$\xi(A) \times \xi(\hat{A}) = \xi(A *_g \hat{A}) \in G(X \times X)$$

As $I(d_i, s_i) * I(\hat{d}_i, s_i) \equiv I * I(d_i \cdot \hat{d}_i, s_i) = I(d_i \cdot \hat{d}_i, 2s_i) \bmod \operatorname{im} \bar{\alpha}$ one has $\tilde{\alpha}(A *_g \hat{A} \equiv \equiv \tilde{\alpha}(I *_g A\hat{A}) \bmod \operatorname{im} \bar{\alpha}$ and therefore

$$\xi(A) \times \xi(\hat{A}) = \xi(A *_g \hat{A}) = \xi(I *_g A\hat{A}) = \xi(I) \times \xi(A \cdot \hat{A}).$$

We thus proved

2.11. A PRODUCT THEOREM. *Let* $A, \hat{A} \in \mathscr{A}$. *Then* $\xi(A) \times \xi(\hat{A}) = X \times \xi(A \cdot \hat{A})$ *and consequently if* $A_i, B_i \in \mathscr{A}$ *and if* $A_1 \cdot A_2 \cdot \cdots \cdot A_n$ *and* $B_1 \cdot B_2 \cdot \cdots \cdot B_m$ *have the same image in* $GL(s_1, s_2, \ldots, s_l : Z_{i_k})/\text{im}\,\bar{\alpha}$ *then*

$$\xi(A_1) \times \xi(A_2) \times \cdots \times \xi(A_m) = \xi(B_1) \times \xi(B_2) \times \cdots \times \xi(B_m).$$

If φ is the Euler function then as $I(d_i, s_i)^{\varphi(i_k)/2} \equiv I \mod \text{im}\,\bar{\alpha}$ one obtains (see [7], p. 82):

2.12. COROLLARY. *For every* $[Y] \in G(X)$ $Y^{\varphi(i_k)/2} \approx X^{\varphi(i_k)/2}$.
Similarly one obtains

2.13. COROLLARY. *Let* Y *be a finite* CW H-*space if* $QH^n(Y, Z)/\text{torsion} \neq 0$ *whenever* $QH^n(X, Z)/\text{torsion} \neq 0$ *then* $G(X \times Y) = \{X\} \times G(Y)$.

3. Genus and non-Cancellation

3.1. DEFINITION. (See [5]). Let K be a finite CW complex, $\mathbf{P}_1$ – a set of primes. K is said to be $\mathbf{P}_1$ universal if for every prime $q \notin \mathbf{P}_1$ there exists a $\mathbf{P}_1$ equivalence $f_q : K \to K$ so that $\bar{H}^*(f_q, Z_q) = 0$.

A space Y is said to be an H_0 space if $H^*(Y, Q)$ is a free (commutative and associative) algebra. If $H^*(Y, Q) = \Lambda(x_{n_1}, x_{n_2}, \ldots, x_{n_r})$ then Y is said to be of type $n_1, n_2, \ldots, n_r$. The following is a generalization of [7] Lemma 1.5:

3.2. PROPOSITION. *Let* $\mathbf{P} = \mathbf{P}_1 \cup \mathbf{P}_2$. *Let* X, Y_1, Y_2 *be finite* CW *complexes* $f : X \to Y_1 \times Y_2$. *If* X *is an* H_0-*space*, $Y_i - \mathbf{P}_i$ *universal* $i = 1, 2$ *and if* $p_i \circ f$ *is a* $\mathbf{P}_i$ *equivalence where* $p_i : Y_1 \times Y_2 \to Y_i$, $i = 1, 2$, *is the projection then* f *admits a homotopy left inverse.*

Proof. We may assume that $\mathbf{P}_1 \cap \mathbf{P}_2 = \emptyset$. Let $h_0 : X \to X_0 = K(Z, \bar{n})$ be a rational equivalence. Put $m = \max(n_r, \dim Y_1 \times Y_2)$. Factor $\hat{h}_0 : Ht_m(X) \to X_0$ by maps $h_{i,i-1} : X_i \to X_{i-1}$, $0 < i \leq s$ $X_s = Ht_m(X)$, $h_{i,i-1}$ being the fiber of a map $k_i : X_{i-1} \to K(Z_{p_i}, m_i)$ $p_i \in \mathbf{P}$. Let $\alpha_i : Y_1 \times Y_2 \to Y_1 \times Y_2$ be given as follows: If $p_i \notin \mathbf{P}_1$ let $g_i' : Y_1 \to Y_1$ be a $\mathbf{P}_1$ equivalence with $\bar{H}^*(g_i', Z_{pi}) = 0$. If $p_i \notin \mathbf{P}_2$ let $g_i'' : Y_2 \to Y_2$ be defined similarly. Let

$$\alpha_i = \begin{cases} g_i' \times 1 & \text{if } p_i \notin \mathbf{P}_1 \\ 1 \times g_i'' & \text{if } p_i \notin \mathbf{P}_2 \end{cases}$$

suffices to find a left inverse for $\hat{f} = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_s \circ f$ and one has $p_i \circ \hat{f}$ is a $\mathbf{P}_i$ equivalence. Let $y_{n_i} = H^*(h_0, Z)\,\iota_{n_1}, (\iota_{n_i}, \ldots, \iota_{n_r}) \in H^*(K(Z, \bar{n}), Z)$ the "fundamental vector". As $p_1 \circ f$ is a $\mathbf{P}_1$ equivalence $\lambda_1 y_{n_i} \in \text{im}\, H^*(p_1 \circ \hat{f}, Z)$ λ_1 prime to $\mathbf{P}_1$. Similarly

$\lambda_2 y_{n_i} \in \operatorname{im} H^*(p_2 \circ f, Z)$. As $(\lambda_1, \lambda_2) = 1$ $\lambda_1 y_{n_i}$, $\lambda_2 y_{n_i} \in \operatorname{im} H^*(f, Z)$ implies $y_{n_i} \in \operatorname{im} H^*(f, Z)$. Hence, h_0 factors through $Y_1 \times Y_2$, $h_0 = r_0 \circ f$, $r_0 : Y_1 \times Y_2 \to$ $\to K(Z, \bar{n}) = X_0$. Suppose one obtains inductively the following commutative diagram:

$$X \xrightarrow{f} Y_1 \times Y_2 \xrightarrow{\alpha_s} \cdots Y_1 \times Y_2 \xrightarrow{\alpha_n} Y_1 \times Y_2$$
$$\tau_m \downarrow \qquad\qquad\qquad\qquad \downarrow r_{n-1}$$
$$Ht_m(X) = X_s \xrightarrow{h_s, s-1} X_{s-1} \longrightarrow \cdots X_n \xrightarrow{h_n, n-1} X_{n-1} \xrightarrow{k_n} K(Z_{p_n}, m_n)$$

Without loss of generality suppose $p_n \notin \mathbf{P}_1$. Then $H^*(\alpha_n, Z_{p_i}) = H^*(g_n', Z_{p_i}) \otimes 1 = \varepsilon \otimes 1$ (where $\varepsilon : H^* \to Z_p \to H^*$ is induced by augmentation). Hence, one obtains

$$Y_1 \times Y_2 \xrightarrow{\quad p_2 \quad} Y_2$$
$$\downarrow \alpha_n \qquad\qquad \downarrow \bar{r}$$
$$Y_1 \times Y_2 \xrightarrow{r_{n-1}} X_{n-1} \xrightarrow{k_n} K(Z_{p_n}, m_n)$$

$p_2 \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$ is a $\mathbf{P}_2$-equivalence, and as $k_n \circ h_{n,n-1} \sim *$, $* \sim k_n \circ r_{n-1} \circ \alpha_n \circ \alpha_{n+1} \circ \cdots \circ$ $\circ \cdots \circ \alpha_s \circ f \sim \bar{r} \circ p_2 \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$. It follows that $\bar{r} \sim *$, $k_n \circ r_{n-1} \circ \alpha_n \sim *$ and $r_n' : Y_1 \times Y_2 \to$ $\to X_n$ exists so that $h_{n,n-1} \circ r_n' \sim r_{n-1} \circ \alpha_n$. Comparing $\varphi_1 = r_n' \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$ and $\varphi_2 = h_{n+1,n} \circ h_{n+2,n+1} \circ \cdots \circ h_{s,s-1} \circ \tau_m$ there exists $\omega : X \to K(Z_{p_n}, m_n - 1)$ so that $\omega * \varphi_1 \sim$ $\sim \varphi_2$ (where $*$ denotes the action of $[X, K(Z_{p_n}, m_n - 1)]$ on $[X, X_n]$ induced by the principal fibration $K(Z_{p_n}, m_n - 1) \to X_n \to X_{n-1}$). But $\alpha_{n+1} \circ \alpha_{n+2} \circ \cdots \circ \alpha_s \circ f$ is a $\mathbf{P}_2$ equimorphism, hence, ω factors as $\tilde{\omega} \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f$, $\tilde{\omega} : Y_1 \times Y_2 \to K(Z_{p_n}, m_n - 1)$.

Replacing r_n' by $\tilde{\omega} * r_n' = r_n$ one obtains $r_n : Y_1 \times Y_2 \to X_n$ and

$$r_n \circ \alpha_{n+1} \circ \cdots \alpha_s \circ f = (\tilde{\omega} * r_n') \circ \alpha_{n+1} \circ \cdots \alpha_0 \sim$$
$$\sim (\tilde{\omega} \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f) * (r_n' \circ \alpha_{n+1} \circ \cdots \circ \alpha_s \circ f) \sim$$
$$\sim \omega * \varphi_1 \sim \varphi_2 = h_{n+1,n} \circ h_{n+2,n+1} \circ \cdots h_{s,s-1} \circ \tau_m.$$

The final step of lifting $r_s : Y_1 \times Y_2 \to X_s = Ht_m(X)$ to $r : Y_1 \times Y_2 \to X$ with $r \circ f \sim 1$ is automatic as $\dim X \leqslant m$, $\dim Y_1 \times Y_2 \leqslant m$ and hence $[X, \tau_m] : [X, X] \xrightarrow{\approx} [X, X_s]$, $[Y_1 \times Y_2, \tau_m] : [Y_1 \times Y_2, X] \xrightarrow{\approx} [Y_1 \times Y_2, X_s]$ and $\tau_m = [X, \tau_m]$ [1].

3.3. LEMMA. *Let*

$$X \xrightarrow{h_2} Y_1$$
$$h_1 \downarrow \qquad \downarrow f_1$$
$$Y_2 \xrightarrow{f_2} X_0$$

be a fiber square. If X_0 is an H-space then $X \xrightarrow{(h_1 \times h_2) \circ \Delta} Y_1 \times Y_2 \xrightarrow{[f_2][f_1]^{-1}} X_0$ is a (quasi) fibration.

Proof. Denote $\mu_{X_0}(x, x')$ by $x \cdot x'$ and if $c : X \to X$ is the homotopy inverse put

$c(x) = x^{-1}$. Now,

$$X = \{y_1, y_2, \varphi \in Y_1 \times Y_2 \times PX_0 \mid f_1(y_1) = \varphi(0), \varphi(1) = f_2(y_2)\}$$

and

$$F = \{y_1, y_2, \varphi \in Y_1 \times Y_2 \times \mathscr{L}X_0 \mid f_2(y_2) \cdot f_1(y_1)^{-1} = \varphi(1)\}$$

is the fiber of $[f][f_1]^{-1}$. Let $\alpha: X \to F$ be given by $\alpha(y_1, y_2, \varphi) = y_1, y_2, a(f_1(y_1)) + \varphi \cdot f_1(y_1)^{-1}$ where $a(z)$ is the homotopy connecting the base point with $z \cdot z^{-1}$. The obvious maps $X \to Y_1 \times Y_2$ and $F \to Y_1 \times Y_2$ are (Hurewicz) fibrations with the same fiber ΩX_0 and $\alpha \mid \Omega X_0 = 1$. Hence, α is a homotopy equivalence.

Combining 1.6, 3.2 and 3.3 one gets

3.4. THEOREM. *Given a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{h_2} & Y_1 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_1} \\ Y_2 & \longrightarrow & X_0 \end{array}$$

Suppose f_i, h_i are $\mathbf{P}_i$ equivalences in $\dim \leqslant n$, $i = 1, 2$, $\mathbf{P}_1 \cup \mathbf{P}_2 = \mathbf{P}$. If Y_1 is $\mathbf{P}_i$ universal $i = 1, 2$, X is an H_0 space and X_0 is an H-space then

$$Y_1 \times Y_2 \approx X \times X_0 \quad \text{in} \quad \dim \leqslant n.$$
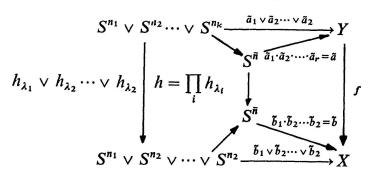
Proof. By 1.6 3.3.1 is a fiber square. By 3.3 one has a fibration

$$X \xrightarrow{(h_1 \times h_2)\varDelta} Y_1 \times Y_2 \to X_0$$

where h_i are $\mathbf{P}_i$ equivalences and one can apply 3.2 to find a left inverse for $(h_1 \times h_2)\varDelta$ and 3.3 follows.

Let X be again an H-space of type $(n_1, n_2, \ldots, n_r)$ and of dimension $\leqslant k$. The set of primes $\mathbf{P}_{2k}$ contains all primes involved in $H_*(X, Z)$, $\pi_n(X)$ $n \leqslant 2k$ and $\operatorname{coker}(\pi_*(X)/\text{torsion} \to PH_*(X, Z)/\text{torsion})$. One can easily see that every map $S^{\bar{n}} \to X$ yielding an isomorphism of $\pi(\ )/\text{torsion}$ is a $\widetilde{\mathbf{P}}_k = \mathbf{P} - \mathbf{P}_k$ equivalence. (Hence $\mathbf{P}_k$ contains only regular primes). Let $[Y] \in G(X)$ and let $f: Y \to X$ be a $\mathbf{P}_{2k}$ equivalence which is an H-map (see 2.9). Diagonalizing $\pi(f)/\text{torsion}$ one obtains bases $a_1, \ldots, a_r \in \pi(Y)/\text{torsion}$ and $b_1, \ldots, b_r \in \pi(X)/\text{torsion}$ so that $(\pi(f)/\text{torsion}) a_1 = \lambda_i b_i$, λ_i prime to $\mathbf{P}_{2k}$. If $\tilde{a}_i \in \pi_{n_i}(Y)$, $\tilde{b}_i \in \pi_{n_i}(X)$ represent a_i and b_i respectively then $\pi(f)\ \tilde{a}_i = \lambda_i \tilde{b}_i + \theta_i$, θ_i - a torsion element. But λ_i is prime to the torsion primes in $\pi_n(X)$ $n \leqslant 2k$ hence to the order of θ_i and $\theta_i = \lambda_i \tilde{\theta}_i$. Replacing $\tilde{b}_i$ by $\tilde{b}_i + \tilde{\theta}_i$ if necessary one may assume $\pi(f)\ \tilde{a}_i =$

$=\lambda_i \tilde{b}_i$. Hence one obtains the following commutative diagram:

As f is an H-map one can add $h = \prod_i h_{\lambda_i}$ to the above diagram to obtain a commutative diagram:

$$S^{\bar{n}} \xrightarrow{\tilde{a}} Y$$
$$\downarrow h \qquad \downarrow f$$
$$S^{\bar{n}} \xrightarrow{\tilde{b}} X$$

Now f and h are $\mathbf{P}_k$ equivalences while $\tilde{a}$ and $\tilde{b}$ are $\mathbf{P}_k$-equivalences and one can apply 3.4 to get the following:

3.5. A NON CANCELATION THEOREM. (Compare with [1], [2] and [7], p. 83). *If* $[Y] \in G(X)$ *then* $Y \times S^{\bar{n}} \approx X \times S^{\bar{n}}$.

REFERENCES

[1] FREYD, P., *Stable homotopy*. Proceeding of the conference on categorical algebra (La Jolla, 1965). Springer 1966.

[2] ——, *Stable homotopy II*. AMS, Proceedings of symposia in pure mathematics, Vol. XVII, 1970, pp. 161–183.

[3] HILTON, P. J., *On the Grothendieck group of compact polyhedra*, Fund. Math. *61* (1967), pp. 199–214.

[4] HILTON, P. J. and ROITBERG, J., *On principal S^3 bundles over spheres*, Ann. of Math. *90* (1969), pp. 91–107.

[5] MIMURA, M. and TODA, H., *On p-equivalences and p-universal spaces*, Comment. Math. Helv. *46* (1971), pp. 87–97.

[6] MISLIN, G., *H-spaces* mod p. *H-spaces conference*, Neuchâtel 1970, Springer lecture notes 196.

[7] ——, *The Genus of an H-space*, Symp. on algebraic topology, Seattle 1971, Springer lecture notes 249, pp. 75–83.

[8] ZADBRODSKY, A., *Homotopy associativity and finite CW complexes*, Topology *9* (1970), pp. 121–128.

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