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A Criterion for a Meromorphic Function to be Normal

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A meromorphic function f defined in the unit disc D is said to be a *normal* function if

$$\sup\{(1-|z|^2)f^{\#}(z):|z|<1\}<\infty,$$

where

$$f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$$

is the spherical derivative of f (see [2]). The purpose of this paper is to answer the following question of Pommerenke [6, Problem 3.2, p. 357]: if $M>0$ is given, does there exist a finite set E such that if f is meromorphic in D then the condition that $(1-|z|^2)f^{\#}(z)\leqq M$ for each $z\in f^{-1}(E)$ implies that f is a normal function? Pommerenke noted that if M is sufficiently small then the answer is affirmative. We prove here that the answer is affirmative for all non-negative M and, in fact, that the set E can be chosen to be any set consisting of five complex numbers, finite or infinite, where the set E is independent of the number M. We also comment on the sharpness of the number "five" in our result, showing that "five" cannot be replaced by "three" and that there are at least some cases in which "five" cannot be replaced by "four."

We make use of the following theorem of Lohwater and Pommerenke.

THEOREM. *A non-constant function f meromorphic in D is not a normal function if and only if there exist sequences $\{z_n\}$ and $\{p_n\}$ with $z_n\in D$, $|z_n|\to 1$, $p_n>0$, $p_n/(1-|z_n|)\to 0$, such that the sequence $\{f(z_n+p_nt)\}$ converges locally uniformly to a non-constant function $g(t)$ meromorphic in the complex plane* [3, Theorem 1].

(We note that Lohwater and Pommerenke stated this theorem a bit differently, but the result they proved contains the theorem as stated here.)

The main result of this paper is the following theorem.

THEOREM 1. *Let E be any set consisting of five complex numbers, finite or infinite. If f is a meromorphic function in D such that*

$$\sup\{(1-|z|^2)f^{\#}(z):z\in f^{-1}(E)\}<\infty,$$

then f is a normal function.

Proof. The statement of the theorem is equivalent to the following statement: if f is a meromorphic function in D such that f is not a normal function, then for each complex number λ, with at most four exceptions,

$$\sup\{(1-|z|^2)f^{\#}(z): z\in f^{-1}(\lambda)\}=\infty.$$

It is this version of the theorem that we will prove.

Suppose that f is a meromorphic function in D which is not a normal function. By the theorem of Lohwater and Pommerenke there exist sequences $\{z_n\}$ and $\{p_n\}$ with $z_n \in D$, $|z_n|\to 1$, $p_n>0$, $p_n/(1-|z_n|)\to 0$, and a non-constant function g meromorphic in the complex plane such that the sequence of functions $\{g_n\}$ converges locally uniformly to g, where $g_n(t)=f(z_n+p_n t)$ for each positive integer n. Let λ be any complex number, finite or infinite, for which the equation $g(t)=\lambda$ has a solution t_0 which is not a multiple solution, that is, $g^{\#}(t_0)\neq 0$. By a theorem of Hurwitz [2, Theorem 14.3.4, p. 205], in each neighborhood of t_0 all but a finite number of the functions g_n assume the value λ. Thus there exists a sequence of points $\{t_n\}$ such that $t_n\to t_0$ and $g_n(t_n)=\lambda$ for n sufficiently large. Also, since the convergence of the sequence of functions $\{g_n\}$ to g is locally uniform, we have that $g_n^{\#}(t_n)\to g^{\#}(t_0)$. Letting $s_n=z_n+p_n t_n$, we get that $g_n^{\#}(t_n)=p_n f^{\#}(s_n)$ so that

$$\begin{aligned} f^{\#}(s_n)(1-|s_n|) &= g_n^{\#}(t_n)(1-|s_n|)/p_n \\ &= g_n^{\#}(t_n)((1-|z_n|)/p_n)((1-|s_n|)/(1-|z_n|)). \end{aligned}$$

Letting $n\to\infty$, we have that $g_n^{\#}(t_n)\to g^{\#}(t_0)$, $(1-|z_n|)/p_n\to\infty$, and $(1-|s_n|)/(1-|z_n|)\to 1$ so that $f^{\#}(s_n)(1-|s_n|)\to\infty$, or equivalently, $(1-|s_n|^2)f^{\#}(s_n)\to\infty$.

We have now shown that if the equation $g(t)=\lambda$ has a solution which is not a multiple solution, then

$$\sup\{(1-|z|^2)f^{\#}(z): z\in f^{-1}(\lambda)\}=\infty.$$

However there can be at most four values λ for which all solutions to the equation $g(t)=\lambda$ are multiple solutions (see [1, Corollary 3, p. 231] or [4, p. 284]). Thus the proof of the theorem is completed.

Theorem 1 gives a sufficient condition for a meromorphic function to be a normal function. It follows from the definition of a normal function that this condition is also necessary. Thus we can state a slightly stronger form of Theorem 1 as follows:

THEOREM 2. *If f is a meromorphic function in D, then f is a normal function if and only if there exist five distinct values $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ such that*

$$\sup\{(1-|z|^2)f^{\#}(z): z\in D, \quad f(z)\in\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\}\}<\infty.$$

We remark that Theorems 1 and 2 are similar to the "five island" condition for a meromorphic function to be a normal function (see [5, Corollary, p. 89]). In fact, special cases of Theorem 1 can be proved from the "five island" condition. However, the "five island" condition is sufficient for a meromorphic function to be a normal function, but it is not a necessary condition.

We now deal with the sharpness of Theorem 1. It is not known if Theorem 1 is sharp in the sense that there may be a particular four point set E which yields the same result. However, we show here that there are some four point sets for which Theorem 1 fails.

THEOREM 3. *There exists a non-normal meromorphic function f in D and a set F consisting of four points such that $f^{\#}(z)=0$ for each $z \in f^{-1}(F)$.*

Proof. Let p be a positive real number and let q be a complex number with positive imaginary part. Let $\wp(z)$ be the Weierstrass $\wp$-function with primitive periods p and q. Let H denote the upper half plane, and let $e_1=\wp(p/2)$, $e_2=\wp(q/2)$, and $e_3=\wp((p+q)/2)$. The equation $\wp(z)=\lambda$ has only multiple solutions for $\lambda \in \{e_1, e_2, e_3, \infty\}$ (see, for example, [1, Chapter 13]). Thus $\wp^{\#}(z)=0$ for $\wp(z) \in \in \{e_1, e_2, e_3, \infty\}$. Let $g(z)=i(1+z)/(1-z)$ and let $f(z)=\wp(g(z))$ for $z \in D$. Since the function g maps the unit disc D onto H, with $g(1)=\infty$, we can choose a sequence $\{x_n\}$ of positive real numbers with $0<x_n<1$ and $x_n \to 1$ such that $\{\wp(g(x_n))\}$ is bounded and $\{\wp'(g(x_n))\}$ is bounded away from both 0 and ∞. (We need simply choose the points x_n such that the points $g(x_n)$ stay bounded away from all points of the form $(np+mq)/2$ with n and m integers.) Then $\wp^{\#}(g(x_n))$ is bounded away from 0 so that

$$(1-|x_n|^2)f^{\#}(x_n)=2\wp^{\#}(g(x_n))\,(1+|x_n|)/(1-|x_n|)$$

tends to ∞ as $n \to \infty$. Thus f is not a normal function. But the same calculation gives $f^{\#}(z)=\wp^{\#}(g(z))=0$ for $f(z) \in \{e_1, e_2, e_3, \infty\}$. Setting $F=\{e_1, e_2, e_3, \infty\}$, the theorem is proved.

Next we show that Theorem 1 fails if we replace the five point set E with a set containing only three points.

THEOREM 4. *Let E be any set consisting of three complex numbers. Then there exists a non-normal meromorphic function g in D such that $g^{\#}(z)=0$ whenever $g(z) \in E$.*

Proof. Let f be the function used in the proof of Theorem 3, let $E=\{a, b, c\}$ and let $L(z)$ be the linear fractional transformation for which $L(e_1)=a$, $L(e_2)=b$, and $L(e_3)=c$. Then $g(z)=L(f(z))$ is not a normal function, but $g^{\#}(z)=L^{\#}(f(z)) \cdot |f'(z)|=0$ whenever $f(z) \in \{e_1, e_2, e_3\}$, that is whenever $g(z) \in E$.

We remark that the function g constructed in the proof of Theorem 4 actually satisfies $g^{\#}(z)=0$ whenever $g(z) \in \{a, b, c, L(\infty)\}$. Thus we have shown that if E is any set consisting of three points, then there is a four point set E' containing E such that Theorem 1 fails for the four point set E'. We finally remark that the choice of this fourth point appears to be dictated by the possible values of e_1, e_2, and e_3 in the Weierstrass $\wp$ function, and they must satisfy the condition $e_1+e_2+e_3=0$. Thus it appears that a different method must be used to settle the question whether Theorem 4 is valid if E is taken to be an arbitrary four point set.

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