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## Classification Theorems for Quadratic Forms over Fields

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### 1. Introduction

In the study of quadratic form theory, the Classification Problem has always occupied a unique and central role. Namely, given a field  $F^3$ ), what are the basic invariants which classify (the isometry classes of) quadratic forms over  $F$ ? The question in this generality has so far defied an answer, as no one has been able to exhibit a complete set of natural invariants which work for *all* fields. However, for *specific* classes of fields, the Classification Problem has been solved in various specific ways. Thus, one way to treat the Classification Problem is to ask the following slightly different question: which are the fields whose quadratic forms are classified by a *prescribed* set of invariants? Let us record from the literature some known answers to this alternative question, in order to lead up to and motivate the main result in this note.

**CLASSIFICATION THEOREM 1'** (Triviality). *Quadratic forms over  $F$  are classified by "dim" iff  $F$  is quadratically closed, iff  $IF=0$ .*

Here,  $IF$  denotes the ideal of even dimensional forms in the Witt ring,  $W(F)$ .

**CLASSIFICATION THEOREM 1** (Sylvester-Pfister Law). *Quadratic forms over  $F$  are classified by "dim" and the total signature (i.e. the totality of signatures with respect to all orderings of  $F$ ) iff  $F$  is pythagorean, iff  $IF$  is torsion-free.*

The following is also easy to see:

**CLASSIFICATION THEOREM 2'**. *The following are equivalent:*

- (1) *Quadratic forms over  $F$  are classified by "dim" and "det".*
- (2)  $I^2F=0$ .
- (3) *All  $F$ -quaternion algebras split.*
- (4) *If an  $F$ -quaternion algebra splits over a quadratic extension of  $F$ , then it splits over  $F$ .*
- (5) *All binary forms  $\langle 1, a \rangle$  ( $a \in F$ ) are universal.*

**EXAMPLES** (for which the above statements hold): finite fields; algebraic extensions of  $\mathbf{C}(x)$ ; the power series field  $\mathbf{C}((x))$ .

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<sup>3)</sup> All fields have characteristic different from 2 in this paper.

**CLASSIFICATION THEOREM 2.** *The following are equivalent:*

- (1) *Quadratic forms over  $F$  are classified by “dim”, “det” and the total signature.*
- (2)  *$I^2F$  is torsion-free.*
- (3) *If an  $F$ -quaternion algebra splits in every real closure of  $F$ , then it splits over  $F$ .*
- (4) *If an  $F$ -quaternion algebra splits over some  $F(\sqrt{w})$  where  $w$  is totally positive, then it splits over  $F$ .*
- (5) *All binary forms  $\langle 1, a \rangle$  represent all totally positive elements of  $F$ .*

Proofs of these equivalences are covered by [2, Cor. 2.9] and [6, Theorem E]. The latter contains also further statements equivalent to each of the above.

**EXAMPLES.** Algebraic extensions of  $\mathbf{R}(x)$ ; any formally real field with square classes  $\{\pm 1, \pm 2\}$ .

**CLASSIFICATION THEOREM 3' [5, Theorem 3.11].** *The following are equivalent:*

- (1) *Quadratic forms over  $F$  are classified by “dim”, “det” and the Hasse invariant.*
- (2)  *$I^3F=0$ .*
- (3) *All  $F$ -Cayley algebras split.*
- (4) *If an  $F$ -Cayley algebra splits over a quadratic extension of  $F$ , then it splits over  $F$ .*
- (5) *All quaternionic norm forms  $\langle 1, a, b, ab \rangle$  ( $a, b \in F$ ) are universal.*

**EXAMPLES.** Algebraic extensions of  $\mathbf{C}(x, y)$ ;  $p$ -adic fields; non-formally real global fields;  $\mathbf{C}((t_1))$  ( $((t_2))$ ).

Note that Theorems 1', 2' are implicitly addressed to non-formally real fields, while Theorems 1, 2 are respectively their generalizations to *arbitrary* fields. This strikes a resonant note to the papers [3, 4, 6], where it is demonstrated that many things said about non-real fields can be appropriately generalized to arbitrary fields. In this perspective, one is naturally led to conjecture that Theorem 3' can be superceded by the following much broader statement:

**CLASSIFICATION THEOREM 3.** *The following are equivalent:*

- (1) *Quadratic forms over  $F$  are classified by “dim”, “det”, Hasse invariant and the total signature.*
- (2)  *$I^3F$  is torsion-free.*
- (3) *If an  $F$ -Cayley algebra splits in every real closure of  $F$ , then it splits over  $F$ .*
- (4) *If an  $F$ -Cayley algebra splits over some  $F(\sqrt{w})$  where  $w$  is totally positive, then it splits over  $F$ .*
- (5) *All quaternionic norm forms  $\langle 1, a, b, ab \rangle$  ( $a, b \in F$ ) represent all totally positive elements of  $F$ .*

EXAMPLES. Algebraic extensions of  $\mathbf{R}(x, y)$ ; global fields.

The purpose of this note is to render a proof of Theorem 3. Though we restrict ourselves to fields throughout the paper, it seems reasonable to expect that the same theorem works essentially over semi-local rings. This observation is supported by the work of Mandelberg [9], where the non-dyadic semi-local analog of Theorem 3' has already been obtained (under mild restrictions). On the other hand, Sah [12, Theorem 3] has established the analog of Theorem 3' for fields of characteristic 2.

A word about notations. For a field  $F$ , write  $\dot{F}=F-\{0\}$ , and  $\sigma(F)=$ the set of totally positive elements (=non-zero sums of squares, by Artin-Schreier). For  $a_i \in \dot{F}$ ,  $\langle a_1, \dots, a_n \rangle$  denotes the “ $n$ -fold Pfister form”  $\bigotimes_{i=1}^n \langle 1, a_i \rangle$ . The ideal power  $I^n F$  is additively generated by all  $n$ -fold Pfister forms in  $W(F)$ . If a Pfister form  $\varphi$  lies in  $W_t(F)$ , we say that  $\varphi$  is a *torsion* Pfister form. Standard facts about quadratic forms can be found in [8].

## 2. Auxiliary Results

For convenience of the reader, we shall recall here a few results from our earlier work [2, 3, 5], to be used in the sequel.

**PROPOSITION 1** [5, Section 3]. *Let  $K=F(\sqrt{a})$  be a quadratic extension of  $F$ . Let  $s: K \rightarrow F$  be the  $F$ -linear functional defined by  $s(1)=0$ ,  $s(\sqrt{a})=1$ . Let  $s_*: W(K) \rightarrow W(F)$  be the transfer map induced by  $s$ , and  $r^*: W(F) \rightarrow W(K)$  be the functorial map. Then,*

(1) *We have a zero sequence*

$$0 \rightarrow \langle -a \rangle \cdot I^{n-1} F \rightarrow I^n F \xrightarrow{r^*} I^n K \xrightarrow{s_*} I^n F \quad \text{for all } n \geq 0.$$

(By definition,  $I^{-1} F = I^0 F = W(F)$ .)

- (2) *The above sequence is exact for  $n=0, 1, 2$ .*
- (3) *The above sequence is exact for  $n=3$ , except possibly at the term  $I^3 K$ .*
- (4) *If  $\gamma \in I^3 K$  is 8-dimensional and  $s_*(\gamma)=0$ , then there exists  $q \in I^3 F$  such that  $r^*(q)=\gamma$ .*

**PROPOSITION 2** [3, Cor. 2.3]. *Suppose  $\sigma$  is a  $2n$ -dimensional form such that  $2\sigma=0 \in W(F)$ . Then  $\sigma \cong \perp_{i=1}^n \langle a_i \rangle \langle -w_i \rangle$  for suitable  $a_i \in \dot{F}$ , and  $w_i$  which are sums of two squares.*

**COROLLARY 1.** *If  $\sigma$  is a Pfister form, then  $2\sigma=0$  iff  $\sigma \cong \langle -w, \dots \rangle$  where  $w$  is a sum of two squares.*

*Proof.* “If” is clear. Assuming  $2\sigma=0$ , we have  $\sigma \cong \langle a \rangle \langle 1, -w \rangle \perp \dots$ , where  $a \in \dot{F}$

and  $w=b^2+c^2 \neq 0$ . Using standard facts about Pfister forms,  $\sigma \cong \langle a \rangle \cdot \sigma \cong \langle 1, -w, \dots \rangle \cong \langle -w, \dots \rangle$ . Q.E.D.<sup>4)</sup>.

**PROPOSITION 3** [2, Theorem 2.8]. *Every element in  $I^2 F \cap W_t(F)$  is a sum of forms  $\langle\langle a, -w \rangle\rangle$ , where  $a \in \dot{F}$  and  $w \in \sigma(F)$ .*

We shall now prove some lemmas.

**LEMMA 1.** *Let  $n \geq 1$ . Suppose there are no anisotropic  $n$ -fold Pfister form  $\varphi$  satisfying  $2\varphi=0 \in W(F)$ . Then there are no anisotropic torsion  $m$ -fold Pfister form for any  $m \geq n$ .*

*Proof.* Suppose  $\gamma$  is an  $m$ -fold Pfister form ( $m \geq n$ ) such that  $2^{t+1}\gamma=0$  but  $2^t\gamma \neq 0$ . Consider the  $(m+t)$ -fold Pfister form  $2^t\gamma$  which is killed by 2. According to Corollary 1,  $2^t\gamma \cong \langle\langle -w, x_2, \dots, x_{m+t} \rangle\rangle$  ( $w=b^2+c^2 \neq 0, x_i \in \dot{F}$ ). But by hypothesis  $\langle\langle -w, x_2, \dots, x_n \rangle\rangle = 0$  since it is killed by 2. Thus  $2^t\gamma=0$ , a contradiction.

**COROLLARY 2** (Pfister: see [8, p. 300]). *Let  $r \geq 1$ . If any  $r$ -fold Pfister form represents any non-zero sum of 2 squares, then any  $r$ -fold Pfister form represents all of  $\sigma(F)$ .*

*Proof.* Apply Lemma 1 with  $n=m=r+1$ , using again Corollary 1.

**COROLLARY 3.** *In Theorem 3, we have  $(3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow$ :*

(A) *There are no anisotropic torsion 3-fold Pfister forms (over  $F$ ).*

*Proof.* A Cayley algebra splits iff its 3-fold Pfister norm form is hyperbolic [7, p. 371]. Thus,  $(3) \Leftrightarrow (A)$  follows from Pfister's Local-Global Principle [11]. By Corollary 1 and Lemma 1, we have  $(A) \Leftrightarrow (5)$ . Considering the form  $\langle\langle a, b, -w \rangle\rangle$  ( $w \in \sigma(F)$ ), we get  $(4) \Leftrightarrow (5)$ .

**LEMMA 2.** *Let  $K=F(\sqrt{a})$  be a quadratic extension of  $F$ . Then  $I^2 K$  coincides with the  $\dot{F}$ -module  $J$  in  $W(K)$  generated by  $\langle\langle e, z \rangle\rangle$  where  $e \in \dot{F}$ ,  $z \in \dot{K}$ .*

*Proof.* Let  $x, y \in \dot{K}$  and  $b \in \dot{F}$ . From the equation  $\langle\langle xb, y \rangle\rangle = \langle b \rangle \langle\langle x, y \rangle\rangle + \langle\langle -b, y \rangle\rangle \in W(K)$ , it follows that  $\langle\langle x, y \rangle\rangle \in J \Rightarrow \langle\langle xb, y \rangle\rangle \in J$ . Thus, we need only show that  $\varphi = \langle\langle c + \sqrt{a}, d - \sqrt{a} \rangle\rangle \in J$ , where  $c, d \in F$ . If  $c = -d$ ,  $\varphi$  is hyperbolic. If  $c \neq -d$ ,  $\varphi \cong \langle\langle c+d, (c+\sqrt{a})(d-\sqrt{a}) \rangle\rangle \in J$ . Q.E.D.<sup>5)</sup>.

**LEMMA 3.** *Let  $K=F(\sqrt{w})$  be a quadratic extension of  $F$ , where  $w \in \sigma(F)$ . Assume that Property (A) (see Corollary 3) holds for  $F$ . Then it also holds for  $K$ .*

<sup>4)</sup> Here is a proof which avoids Prop. 2. Write  $\sigma = \langle 1 \rangle \perp \sigma'$ . Since  $\sigma \cong \langle -1 \rangle \sigma$ ,  $\sigma$  represents  $-1$ , and hence  $\sigma'$  represents some  $-w = -(b^2 + c^2) \neq 0$ . We then have  $\sigma \cong \langle\langle -w, \dots \rangle\rangle$ .

<sup>5)</sup> As observed by Mandelberg, this Lemma together with Frobenius reciprocity yields a quick inductive proof of the inclusion  $s_*(I^n K) \subset I^n F$  asserted in Prop. 1(1).

*Proof.* Let  $\gamma = \langle\langle x, y, -z \rangle\rangle$ , where  $x, y \in \dot{K}$  and  $z$  is a sum of 2 squares in  $\dot{K}$ . It suffices to show that  $\gamma$  must be hyperbolic. Let  $r^*, s_*$  be as in Proposition 1 (with  $a=w$ ).

*Step 1.* We claim that  $s_*(\gamma) = 0 \in W(F)$ . Since  $s_*: W(K) \rightarrow W(F)$  is a  $\dot{F}$ -module homomorphism (by Frobenius reciprocity), we may assume that  $x \in \dot{F}$ , by Lemma 2. Consequently,  $s_*(\gamma) = s_* \langle\langle x, y, -z \rangle\rangle = \langle\langle x \rangle\rangle \cdot s_*(\langle\langle y, -z \rangle\rangle)$ . The latter lies in  $\langle\langle x \rangle\rangle \cdot (I^2 F \cap W_t(F))$ , which is zero in view of Proposition 3 and the hypothesis on  $F$ .

*Step 2.* Since  $s_*(\gamma) = 0$ , there exists an anisotropic form  $q \in I^3 F$  such that  $r^*(q) = \gamma$  (Proposition 1 (4)). *In the following, assume that  $\gamma$  is anisotropic.* We may then write  $q \cong f \perp \langle\langle -w \rangle\rangle \cdot g$ , where  $f, g$  are forms over  $F$ ,  $\dim f = 8$  (see [8, p. 200]). If  $\dim g = 1$ , then  $\dim q = 10$ , and  $q \in I^3 F \Rightarrow q$  is isotropic [11, Case 5 on p. 123], a contradiction. If  $\dim g \geq 2$ ,  $\langle\langle -w \rangle\rangle \cdot g$  contains a subform  $\langle b \rangle \langle\langle -w, c \rangle\rangle$ , which is universal by hypothesis and hence  $q$  is isotropic, again a contradiction. Consequently,  $g$  is the zero form, and  $\dim q = 8$ . This means that  $q \cong \langle b_1 \rangle \langle\langle b_2, b_3, b_4 \rangle\rangle$  where  $b_i \in \dot{F}$  [11, Case 4 on p. 123].

*Step 3.* Our hypothesis for  $F$  implies that  $\langle\langle -w \rangle\rangle \cdot I^2 F = 0$ . By Proposition 1 (3), it follows that  $r^*: I^3 F \rightarrow I^3 K$  is injective. Since  $r^*(q) = \gamma$  is torsion,  $q$  must be torsion too. But then  $\langle\langle b_2, b_3, b_4 \rangle\rangle$  is an anisotropic torsion Pfister form – a final contradiction.

Q.E.D.

### 3. Proof of Theorem 3

We are now ready to complete the proof of Theorem 3. In view of Corollary 3, we need only show  $(1) \Rightarrow (A) \Rightarrow (2) \Rightarrow (1)$ .

$(1) \Rightarrow (A)$ . Consider  $\langle\langle a, b, -w \rangle\rangle$  ( $w \in \sigma(F)$ ) and  $\langle\langle 1, 1, -1 \rangle\rangle$ . These both have dimension 8, determinant 1, trivial Hasse invariant, and zero total signature. Hence (1) implies that  $\langle\langle a, b, -w \rangle\rangle$  is hyperbolic.

$(A) \Rightarrow (2)$ . Suppose  $F$  satisfies (A), but there exists a nonzero anisotropic form  $\sigma \in I^3 F$  with  $2\sigma = 0$ . We may suppose  $F$  to have been chosen such that  $\dim \sigma = 2n$  is as small as possible. By Proposition 2,  $\sigma \cong \perp_{i=1}^n \langle a_i \rangle \langle\langle -w_i \rangle\rangle$ ,  $a_i \in \dot{F}$ ,  $w_i \in \sigma(F)$ . Let  $K = F(\sqrt{w_1})$ . This field also satisfies (A), by Lemma 3. Since the anisotropic part of  $\sigma$  over  $K$  is  $< 2n$ , the form  $\sigma$  must become hyperbolic over  $K$ , by the choice of  $n$ . According to Proposition 1 (3) and the Property (A), we get  $\sigma \in \langle\langle -w_1 \rangle\rangle \cdot I^2 F = 0$ , a contradiction.

Now that we know  $(A) \Leftrightarrow (2)$ , we may restate Lemma 3 as:

**COROLLARY 4.** *Let  $K = F(\sqrt{w})$ ,  $w \in \sigma(F)$ . Then,  $I^3 F$  is torsion-free  $\Rightarrow I^3 K$  is torsion-free.*

*Remark.* The same implication, of course, holds for  $I^2$ . The proof is immediate from the  $I^2$ -exact sequence in Proposition 1 (2).

It still remains to ascertain one last implication:  $(2) \Rightarrow (1)$ , for the conclusion of

the proof of Theorem 3. To do this, we first make some observations about invariants. For a regular quadratic form  $q$  over  $F$ , the *Clifford invariant*,  $\Gamma(q)$ , is given by the class of the Clifford algebra of  $q$  in the Brauer-Wall group  $BW(F)$  (see [8, p. 115]). It is well-known that  $\Gamma(q)$  contains exactly the same information as the aggregate of “dim. mod. 2”, “det” and the Hasse invariant [8, p. 120–123]. Thus, we have nothing to lose in working with  $\Gamma$ . On the other hand, the single invariant  $\Gamma$  is much nicer to work with, because it is *well-defined* on  $W(F)$ , and is a *homomorphism* into  $BW(F)$ . Working with  $\Gamma$  in general avoids many unpleasant calculations. We shall now prove

**PROPOSITION 4.** *Suppose  $I^3F$  is torsion-free, and  $q$  is a form such that  $q \in W_t(F)$ . Then  $\Gamma(q)=1 \Rightarrow q$  is hyperbolic.*

*Proof.* Assume that  $\Gamma(q)=1$ , but  $q$  is non-hyperbolic. Since  $\ker(\Gamma: W(F) \rightarrow \rightarrow BW(F)) \subset I^2F$ , we have  $q \in I^2F \cap W_t(F)$ , so we can write  $q = \sum_{i=1}^n \langle\langle a_i, -w_i \rangle\rangle$ ,  $a_i \in F$ ,  $w_i \in \sigma(F)$ . We may suppose  $F$ ,  $q$  to have been chosen such that  $n$  is as small as possible. Let  $K=F(\sqrt{w_1})$ . Since  $q = \sum_{i=2}^n \langle\langle a_i, -w_i \rangle\rangle$  in  $W(K)$ , and  $I^3K$  is still torsion-free (Corollary 4!),  $q$  must become hyperbolic over  $K$ , by the choice of  $n$ . According to Proposition 1 (2),  $q = \langle\langle -w_1 \rangle\rangle \cdot \langle b_1, \dots, b_{2r} \rangle \in W(F)$ , for suitable  $b_i \in F$ . Thus,

$$\begin{aligned} q &= \langle b_1 \rangle \langle\langle -w_1, b_1 b_{r+1} \rangle\rangle + \dots + \langle b_r \rangle \langle\langle -w_1, b_r b_{2r} \rangle\rangle \\ &\equiv \langle\langle -w_1, b_1 b_{r+1} \rangle\rangle + \dots + \langle\langle -w_1, b_r b_{2r} \rangle\rangle \\ &\equiv \langle\langle -w_1, (-1)^{r+1} b_1 \dots b_{2r} \rangle\rangle \pmod{I^3F}. \end{aligned}$$

Since  $\Gamma(q)=1$  and  $\Gamma(I^3F)=1$  (see [8, p. 117]), we see that  $\Gamma(\langle\langle -w_1, (-1)^{r+1} b_1 \dots b_{2r} \rangle\rangle)=1$ . This means that  $\langle\langle -w_1, (-1)^{r+1} b_1 \dots b_{2r} \rangle\rangle$  is hyperbolic (see [8, p. 116]). Thus,  $q \in I^3F$ . But then  $q \in I^3F \cap W_t(F)=0$ , a contradiction. Q.E.D.

Using Pfister's Local-Global Principle, we obtain:

**COROLLARY 5.** *Suppose  $I^3F$  is torsion-free. Let  $s_\alpha: W(F) \rightarrow W(F_\alpha) \cong \mathbb{Z}$  be the “signature maps”, where  $\{F_\alpha\}$  are a complete family of real closures of  $F$ . Then*

$$(\Gamma, \prod_\alpha s_\alpha): W(F) \rightarrow BW(F) \oplus \prod_\alpha W(F_\alpha)$$

*is a monomorphism. In particular, quadratic forms over  $F$  are classified by “dim”, the Clifford invariant and the total signature.*

Since “dim”, “det” and the Hasse invariant together determine  $\Gamma$  as observed before, Corollary 5 provides the implication (2)  $\Rightarrow$  (1) in Theorem 3. *The proof of Theorem 3 is now complete.*

We shall now make some remarks about Theorem 3.

**Remark 1.** The statements (1) to (5) in Theorem 3 are also equivalent to each of the following: (6) *Quadratic forms over  $F$  are classified by “dim” and Milnor's total*

*Stiefel-Whitney class*  $w$  in [10]. (7) *Quadratic forms over  $F$  are classified by “dim” and Delzant’s total Stiefel-Whitney class*  $\tilde{w}$  in [1]. (Note:  $w$  takes its value in the algebraic  $k$ -groups of Milnor, while  $\tilde{w}$  takes its value in the Galois cohomology of  $F$ ). In fact, (6)  $\Leftrightarrow$  (2) has been shown in [2, Theorem 2.15]. (7)  $\Rightarrow$  (6) is trivial since  $\tilde{w}$  is a “specialization” of  $w$ . To see that (1)  $\Rightarrow$  (7), suppose  $\varphi$  and  $\sigma$  have the same “dim” and the same  $\tilde{w} = (\tilde{w}_i)$ . For  $i = 1, 2$ , this says that  $\varphi$  and  $\sigma$  have the same “det” and the same Hasse invariant. But they also have the same total signature, since  $\varphi - \sigma \in W_t(F)$  by [13, Cor. 6.2]. Therefore,  $\varphi \cong \sigma$  by (1).

*Remark 2.* Suppose  $F, F'$  are fields for which there exists a ring isomorphism  $g: W(F) \cong W(F')$ . Then, if the statements in Theorem 3 apply to  $F$ , they will likewise apply to  $F'$ . This is because  $IF$  is the unique maximal ideal in  $W(F)$  containing 2, which implies that  $g(I^3 F) = I^3 F'$ .

*Remark 3.* The “hereditary” property in Corollary 4 is peculiar to quadratic extensions of the type  $F(\sqrt{w}) \supset F$  ( $w \in \sigma(F)$ ). In fact, let  $F_1$  be a pythagorean field which has a non-pythagorean algebraic extension  $E_1 = F_1(\alpha)$ . Let  $F_2 = F_1((x))$ ,  $F_3 = F_2((y))$ , and  $E_i = F_i(\alpha)$ . Then, for  $i = 1, 2, 3$ ,  $F_i$  satisfies Theorem i, but  $E_i$  does not. (If  $F_1$  is *formally real* pythagorean, then so are  $F_2$  and  $F_3$  and they even satisfy Theorem 1.)

*Remark 4.* A number of other properties also share the “hereditary” feature of Corollary 4, under quadratic extensions of the type  $K = F(\sqrt{w})$  ( $w \in \sigma(F)$ ). For example, it can be shown that, *if every totally positive element of  $F$  is a sum of  $2^n$  squares, then the same holds for  $K$* . If a field satisfies the statements of Theorem 3, then, in particular,  $w \in \sigma(F) \Rightarrow \langle\langle 1, 1, -w \rangle\rangle$  is hyperbolic  $\Rightarrow w$  is a sum of four squares. However, this latter property (though “hereditary” in the above sense) does not imply the statements in Theorem 3. For example, every totally positive element in  $L = \mathbb{Q}((t))$  is a sum of four squares (see [8, p. 315]), but  $I^3 L$  is *not* torsion-free (e.g.  $\langle\langle 1, -3, t \rangle\rangle$  is an anisotropic torsion Pfister form over  $L$ ).

## Appendix: Similarity Factors and a Theorem of Dieudonné

For a quadratic form  $q$  of dimension  $n$  over  $F$ , let  $d_{\pm}(q)$  denote  $(-1)^{n(n-1)/2} \cdot \det(q)$  (the “signed determinant”), and let  $s(q)$  denote the Hasse invariant of  $q$ . Also, let  $D(q)$  denote the nonzero values of  $F$  represented by  $q$ , and let  $G(q)$  denote the group of similarity factors of  $q$  (i.e.  $G(q) = \{a \in F : a \cdot q \cong q\}$ ).

**LEMMA 4.** *If  $\dim q = n = 2r$ , and  $a \in F$ , then  $s(a \cdot q) = s(q)$  iff  $a \in D\langle 1, -d_{\pm}(q) \rangle$ . In particular,  $G(q) \subset D\langle 1, -d_{\pm}(q) \rangle$ .*

*Proof.* From [8, p. 140, Ex. 8],  $s(a \cdot q)$  and  $s(q)$  differ by a quaternion algebra  $(a, (-1)^{n(n-1)/2} \cdot d^{n-1}/F)$ , where  $d = \det(q)$ . Since  $n = 2r$ , this quaternion algebra is  $(a, d_{\pm}(q)/F)$ , which splits iff  $a \in D\langle 1, -d_{\pm}(q) \rangle$ . Q.E.D.

In general,  $b \in D\langle 1, -d_{\pm}(q) \rangle$  need not imply  $b \in G(q)$ . For  $b$  to be in  $G(q)$ , there

exists at least one other obvious necessary condition, namely,  $b$  must be positive in every ordering of  $F$  at which  $q$  is non-hyperbolic. Thus, if we write  $G'(q)$  for the group

$$\{b \in D \langle 1, -d_{\pm}(q) \rangle : b > 0 \text{ in every ordering of } F \text{ at which } q \text{ is non-hyperbolic}\},$$

we have an inclusion  $G(q) \subset G'(q)$ , for all even dimensional forms  $q$ .

**THEOREM.** *The conditions (1) through (5) in Classification Theorem 3 are also equivalent to each of the following:*

- (8)  $G(q) = G'(q)$  for all even dimensional forms  $q$  over  $F$ .
- (9)  $G(q) = G'(q)$  for all torsion 2-fold Pfister forms  $q$  over  $F$ .
- (10) Torsion 2-fold Pfister forms over  $F$  are universal.

*Proof.* (1)  $\Rightarrow$  (8). If  $b \in G'(q)$ , then,  $q$  and  $b \cdot q$  have the same “dim”, “det”, Hasse invariant (by Lemma 4), and the same total signature (by inspection). Thus,  $b \cdot q \cong q$  by (1).

(8)  $\Rightarrow$  (9) is obvious.

(9)  $\Rightarrow$  (10). If  $q$  is a torsion 2-fold Pfister form, the group  $G'(q)$  clearly coincides with  $\dot{F}$ . Thus, (9) implies that  $G(q) = \dot{F}$ , i.e.,  $q$  is universal.

To complete the proof, we shall show that (10) implies the Condition (A) in Corollary 3. By Lemma 1, it is sufficient to show that any 3-fold Pfister form  $\varphi$  satisfying  $2\varphi = 0 \in W(F)$  is isotropic. By Corollary 1,  $\varphi \cong \langle\langle -w, x, y \rangle\rangle$ , where  $w$  is a sum of two squares, and  $x, y \in \dot{F}$ . By (10),  $\langle\langle -w, x \rangle\rangle$  is universal, so  $\varphi$  is isotropic.

Q.E.D.

Since global fields satisfy the condition (1) (by the Hasse-Minkowski Theorem), we obtain:

**COROLLARY 6.** *If  $F$  is a global field, then  $G(q) = G'(q)$  for any even dimensional form  $q$ .*

This result is a theorem of Dieudonné [14, Théorème 3]. However, our proof ((1)  $\Rightarrow$  (8) above) is a drastic simplification of Dieudonné’s long arguments in [14] (which, incidentally, also use the Hasse-Minkowski Theorem). Actually, Dieudonné’s proof in [14] seems to contain a gap (in the middle of p. 402), as pointed out by Dan Shapiro. We would like to thank Dan Shapiro who called our attention to Dieudonné’s paper [14], and collaborated in this appendix.

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