# Homological Methods Applied to the Derived Series of Groups

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## 0. Introduction

0.1. Let $R$ be a non-trivial commutative ring with unit and let $G$ be a group. We say $G$ lies in the class $\mathbf{E}(R)$ if the $G$-trivial module $R$ has an $RG$-projective resolution

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow R$$

for which the map

$$\mathbf{1}_R \otimes \partial_2 : R \otimes_{RG} P_2 \to R \otimes_{RG} P_1$$

is *injective*. We say $G$ lies in the class $\mathbf{E}$, or that $G$ is an $\mathbf{E}$-group, if $G$ lies in $\mathbf{E}(R)$ for every $R$. The purpose of this paper is to investigate the derived and the lower central series of $\mathbf{E}$-groups.

The second homology group with coefficients in $R$, $H_2(G, R)$, vanishes for a group belonging to $\mathbf{E}(R)$. In general, the converse is false, but it does hold for groups $G$ whose cohomological dimension $cd_R G$ is at most two.

An $\mathbf{E}$-group $G$ can be characterized by the following two properties (see Lemma 2.3):
- $G$ lies in $\mathbf{E}(\mathbf{Z})$
- The abelianization $G_{ab}$ of $G$ is torsion-free.

From this it is clear that all free groups and all knot groups are $\mathbf{E}$-groups.

0.2. The motivation for studying $\mathbf{E}$-groups comes from three fields: from the theory of knot groups, from the theory of poly-nilpotent groups, and from the theory of parafree groups. We begin our discussion by presenting the relevant facts about *knot groups.*

The multiplicator $H_2(G, \mathbf{Z})$ of a knot group $G$ is zero, as is the multiplicator $H_2(G', \mathbf{Z})$ of the derived group of a knot group ([26, p. 156], [24, p. 198, Corollary (3.1)]). The question arises whether this is true for the multiplicators of the higher derived groups $G^{(\alpha)}$ of $G$.

The abelianization $G_{ab}$ of a knot group $G$ is free cyclic. The abelianization $G'_{ab}$ of the derived knot group is torsion-free [4, p. 349, Theorem (1.3)] (but in general not free abelian). Again one may ask whether the abelianizations of the higher derived groups $G^{(\alpha)}$ are torsion-free.

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The answer to both questions is affirmative. It follows immediately from part (i) of the following result on closure properties of the class **E**.

THEOREM A. *The class* **E** *has the following closure properties:*

(i) *Every derived group* $G^{(\alpha)}$ *($\alpha$ any ordinal) of an* **E**-*group* $G$ *is an* **E**-*group.*

(ii) *Every term* $G_\beta (1 \leqslant \beta \leqslant \omega)$ *of the lower central series of an* **E**-*group* $G$ *is an* **E**-*group.*

(iii) *If* $G$ *is an* **E**-*group then the quotient* $G/(\bigcap_\alpha G^{(\alpha)})$ *is an* **E**-*group whose cohomological dimension is at most two.*

We note that statements (i) and (ii) imply that every term of the iterated lower central series of an **E**-group is an **E**-group. In particular, the multiplicator of every such term is trivial.

*Remark.* In the present paper, we shall not investigate consequences of statement (iii). Here we merely point out that a preliminary discussion may be found in [22, p. 41, Section 6.4]. There it is shown that the derived length of an **E**-group is 0, 1, 2, a limit ordinal $\lambda$, or $\lambda+1$.

0.3. A second motivation for investigating **E**-groups stems from *subgroup theorems.* Denote the lower central series of a group $G$ by $G_j$ $(j=1, 2, \ldots)$. By definition, $G_1$ coincides with $G$. Let $\mathbf{k}$ be the symbol (1) or an $s$-tuple of natural numbers greater or equal to two,

$$\mathbf{k}=(k_1, \ldots, k_s) \quad (s \geqslant 1, k_i \geqslant 2). \tag{0.1}$$

The terms $\{G_\mathbf{k}\}$ of the iterated lower central series of $G$ are defined by recursion on $s$ as follows:

$$\begin{aligned} G_{(k_1)} &= G_{k_1} \\ G_{(k_1, \ldots, k_s)} &= (G_{(k_1, \ldots, k_{s-1})})_{k_s}. \end{aligned}$$

The following result on subgroups of free poly-nilpotent groups is well-known (see e.g. [15, p. 117, 42.35] (and [15, p. 76, 26.33])).

(*) Let $T$ be a subset of a free group $F$ and let $\mathbf{k}$ be an $s$-tuple as in (0.1). If $T$ is independent modulo $F_2$ then $T$ freely generates in $F/F_\mathbf{k}$ a free poly-nilpotent subgroup.

One might ask for conditions on a (not necessarily free) group $G$ which ensure that a statement analogous to (*) holds, but where the free group $F$ is replaced by $G$. The following result indicates a step in that direction.

(**) Let $T$ be a subset of a group $G$ whose multiplicator $H_2(G, \mathbf{Q})$ is zero. (Here $\mathbf{Q}$ denotes the additive group of the rationals.) If $T$ is independent modulo $G_2$ then, for every $j$ $(2 \leqslant j < \omega)$, the set $T$ freely generates in $G/G_j$ a free nilpotent subgroup.

(The above statement can be deduced from a result of J. Stallings [17, p. 180, Theorem 7.3] (see [22, p. 69, Satz 8.1]).)

We shall prove the following combination of (*) and (**).

THEOREM B. *Let $T$ be a subset of an* $\mathbf{E}(\mathbf{Q})$*-group $G$ and let* $\mathbf{k}$ *be an s-tuple as in* (0.1). *If $T$ is independent modulo $G_2$ then $T$ freely generates in $G/G_{\mathbf{k}}$ a free poly-nilpotent subgroup.*

0.4. The third motivation comes from the work of G. Baumslag on *parafree groups* ([1], [2]). We say a group $G$ is (absolutely) parafree if $G$ is residually nilpotent and if there exists a group homomorphism $\varphi: F \to G$ from a free group into $G$ which induces, for every $j$, an isomorphism $\varphi: F/F_j \simeq G/G_j$ $(2 \leqslant j < \omega)$.

*Remark.* Our definition of parafreeness is not quite the same as Baumslag's definition, but is equivalent to it. The proof of the equivalence is implicit in an argument given in Baumslag's second paper [2, p. 522, Proof of Theorem 4.1]. (See also U. Stammbach [21, pp. 162–164, Proposition 4.1, Proposition 4.3]).

Of course, every free group is parafree. The problem is to find non-free, parafree groups. As shown by G. Baumslag, there are plenty of them. Our contribution to this problem is the following result.

THEOREM C. *Let $V$ and $W$ be non-trivial elements of the free group $\hat{F}$ on $y_1$ and $y_2$. Let $G$ denote the group*

$$\langle x, y_1, y_2, y_3 : x[V, x]([W, y_3])^{\delta}\rangle \qquad (\delta = \pm 1),$$

*and let $\varphi: F \to G$ be the obvious map from the free group on $y_1, y_2, y_3$ into $G$. Then the following statements are true:*

(i) *$\varphi: F/F_j \simeq G/G_j$ is isomorphic for every $j$ $(2 \leqslant j < \omega)$.*

(ii) *$\varphi: F/F_{(2,j)} \simeq G/G_{(2,j)}$ is isomorphic for every $j (2 \leqslant j < \omega)$.*

(iii) *If $V$ is an element of $\hat{F}_{\mathbf{k}}$ then $\varphi: F/F_{(\mathbf{k},j)} \simeq G/G_{(\mathbf{k},j)}$ is isomorphic for every $j$ $(2 \leqslant j < \omega)$.*

(iv) *If $W$ is an element outside $\hat{F}_2$ then $G$ is an extension of a free group by a free cyclic group. Moreover, $G$ is residually nilpotent.*

(v) *If $V$ is an element of $\hat{F}_2$ then $G$ is not free.*

0.5. The paper consists of five sections. In the first one, we set up the basic machinery used to prove results on the classes $\mathbf{E}(R)$. For this we introduce the classes $\mathbf{D}(R)$. We say the group $G$ lies in the class $\mathbf{D}(R)$ if any map between $RG$-projective modules, whose image under the functor

$$R \otimes_{RG} - : {}_{RG}\mathcal{M}od \to {}_{R}\mathcal{M}od$$

is injective, is itself injective. We say $G$ lies in $\mathbf{D}$ if $G$ lies in every $\mathbf{D}(R)$. We cite some results on the classes $\mathbf{D}(R)$ ($R$ arbitrary):

(i) The free cyclic group lies in $\mathbf{D}(R)$.
(ii) Any subgroup $U<G$ of a group $G$ lying in $\mathbf{D}(R)$ lies in $\mathbf{D}(R)$.
(iii) Any product $\prod_j G_j$ of groups $G_j$ lying in $\mathbf{D}(R)$ lies in $\mathbf{D}(R)$.
(iv) If $G$ has a transfinite descending subnormal series

$$G=G_0 \rhd G_1 \rhd G_2 \rhd \cdots G_\omega \rhd G_{\omega+1} \rhd \cdots G_\alpha = e$$

all whose factors lie in $\mathbf{D}(R)$, then $G$ lies in $\mathbf{D}(R)$.
(v) Any direct limit of groups lying in $\mathbf{D}(R)$ lies in $\mathbf{D}(R)$.

We remark that by (i), (iv) and (v) every torsion-free nilpotent group belongs to **D**. This is, however, not true for an arbitrary torsion-free supersolvable group (see Subsection 1.5).

In the second section we introduce the classes $\mathbf{E}(R)$. The connection between $\mathbf{D}(R)$ and $\mathbf{E}(R)$ is given by

LEMMA 2.1. *Let $G$ be an extension of the form $N \lhd G \twoheadrightarrow Q$. If $Q$ lies in $\mathbf{D}(R)$, and if $G$ belongs to $\mathbf{E}(R)$, then $N$ belongs to $\mathbf{E}(R)$.*

In the remainder of Section 2 Theorem A is proved.

The third section contains the proof of Theorem B. The proof uses properties of $\mathbf{D}(\mathbf{Q})$-groups and the result of J. Stallings [17] mentioned above.

In the fourth section we provide a large number of examples of finitely presented $\mathbf{E}(\mathbf{Z})$-groups. We say a group $G$ lies in $\mathbf{M}(\mathbf{Z})$ if $G$ has a presentation

$$\langle z_1, \ldots, z_M : r_1, \ldots, z_N \rangle$$

such that the difference $M-N$ equals the torsion-free rank of the abelianized group $G_{ab}$. We say $G$ lies in **M** if $G$ lies in $\mathbf{M}(\mathbf{Z})$ and if $G_{ab}$ is torsion-free. The class **M** was studied by W. Magnus in 1939 ([13], see also U. Stammbach [19]). We shall show that $\mathbf{M}(\mathbf{Z})$ is a subclass of $\mathbf{E}(\mathbf{Z})$ (Proposition 4.1).

The fifth and last section is devoted to the proof of Theorem C. For the statements (i) through (iv) we rely heavily on results obtained in the preceding sections of the paper, especially on Theorem B and on an isomorphism criterion proved in Subsection 4.3. The question whether $G$ is free or not can be decided by Whitehead's algorithm [25].

0.6. The theory of **E**-groups grew out of an attempt to prove, by homological methods, results on the derived series of a given group. This program was suggested to me by Professor U. Stammbach who also proposed a theorem of G. Baumslag [2] (see Theorem D, Subsection 5.3) as a test result. As Theorem C illustrates, the theory succeeds in the test direction. Moreover, it generalizes results on knot groups. I would, however, like to emphasize that it deals with only a very narrow class of groups; for example, few non-abelian soluble groups belong to **E** (cf. [22, pp. 38–41,

§6.3]). This contrasts with the results of J. Stallings [17] and U. Stammbach [18] on the lower central series.

Many of the results of this paper go back to my thesis [22]. In this article they are reformulated and new results have been added. My work has been expedited by many people to some of whom I am particularily indebted. U. Stammbach not only suggested the problem but also supervised and encouraged my work on its solution. G. Baumslag, K. W. Gruenberg, P. M. Neumann and J. E. Roseblade, through their comments on various aspects of the theory, indicated new directions. Miss R. Boller transformed my manuscript into a carefully typed preprint. To these and others who have assisted me I express my gratitude.

## 1. The Classes $\mathbf{D}(R)$ and the Class $\mathbf{D}$

### 1.1. *The Basic Definitions*

Throughout this paper, $R$ shall denote a non-trivial commutative ring with unit, $G$ a group, $RG$ the group algebra, and $\varepsilon: RG \twoheadrightarrow R$ the standard augmentation, given by $\sum r_i g_i \mapsto \sum r_i$. The ring $R$ shall be viewed as a (trivial) $RG$-module via $\varepsilon$. The symbol $j < \omega$ shall mean that $j$ is a non-negative integer; $\omega$ denoting the first limit ordinal.

We say $G$ lies in the class $\mathbf{D}(R)$ if any map between $RG$-projective (left $RG$-) modules whose image under the functor

$$R \otimes_{RG} \text{-} : {}_{RG}\mathcal{M}od \to {}_{R}\mathcal{M}od$$

is *injective*, is itself *injective*.

*Remark.* In a picturesque way we can say that $G$ belongs to $\mathbf{D}(R)$ if, and only if, the functor $R \otimes_{RG}$- detects injective mappings between projective modules.

We say $G$ lies in the class $\mathbf{D}$ if $G$ lies in $\mathbf{D}(R)$ for every $R$.

### 1.2. *Residually Nilpotent Modules*

We shall denote the kernel of $\varepsilon: RG \twoheadrightarrow R$ by $I$. It is called the augmentation ideal. Its powers $\{I^j \mid I^0 = RG\}_{j<\omega}$ induce, for every left $RG$-module $A$, a filtration $\{I^j A\}_{j<\omega}$ defined by

$$I^0 A = A$$
$$I^j A = \{a \in A : a = \sum (1-g_{i1}) \ldots (1-g_{ij})\, a_i\} \qquad (1 \leqslant j).$$

Note that every $RG$-module homomorphism $\eta: A \to B$ is compatible with the filtrations $\{I^j A\}$ and $\{I^j B\}$.

DEFINITION. A left $RG$-module $A$ is called *residually nilpotent* if the intersection $\bigcap_{j<\omega} I^j A$ reduces to 0. (The $R$-algebra $RG$ is called residually nilpotent if the left $RG$-module $RG$ is residually nilpotent.)

We associate with the filtration $\{I^jA\}_{j<\omega}$ the graduated $G$-trivial $RG$-module $\operatorname{gr} A$ defined by

$$\operatorname{gr} A = \{I^jA/I^{j+1}A\}_{j<\omega}.$$

The proof of the following lemma is straightforward and is omitted.

LEMMA 1.1. *Any submodule of an arbitrary product of residually nilpotent modules is again residually nilpotent. In particular, any $RG$-projective module is residually nilpotent provided the group algebra $RG$ is residually nilpotent.*

The next lemma will be used in Proposition 1.3 and again in Lemma 1.10.

LEMMA 1.2. *Suppose the following conditions are fulfilled:*

(i) *$RG$ is residually nilpotent.*

(ii) *$\eta: A \to B$ is a map between $RG$-projective modules.*

(iii) *$\mathbf{1}_{(\operatorname{gr} RG)_j} \otimes \eta: I^j/I^{j+1} \otimes_{RG} A \to I^j/I^{j+1} \otimes_{RG} B$ is injective for every $j<\omega$.*

*Then $\eta$ itself is injective.*

*Proof.* We shall first exhibit a natural transformation

$$\tau: (\operatorname{gr} RG) \otimes_{RG} A \to \operatorname{gr} A$$

which is an isomorphism whenever $A$ is $RG$-flat.

For any $j<\omega$ and any $RG$-module $A$, the diagram of canonical mappings

$$\begin{array}{ccccccc} I^{j+1} \otimes_{RG} A & \to & I^j \otimes_{RG} A & \to & (\operatorname{gr} RG)_j \otimes_{RG} A & \to & 0 \\ \downarrow \sigma_{j+1} & & \downarrow \sigma_j & & \downarrow \tau_j & & \\ 0 \to I^{j+1}A & \to & I^jA & \to & (\operatorname{gr} A)_j & \to & 0 \end{array}$$

is commutative. Its rows are exact and the vertical maps are onto. If $A$ is $RG$-flat, $\sigma_j$ is moreover injective since the top map in the commutative square

$$\begin{array}{ccc} I^j \otimes_{RG} A & \to & RG \otimes_{RG} A \\ \downarrow \sigma_j & & \cong \downarrow \sigma_0 \\ I^jA & \subsetneq & A \end{array}$$

is injective. Thus $\tau_j: (\operatorname{gr} RG)_j \otimes_{RG} A \to (\operatorname{gr} A)_j$ is bijective provided $A$ is $RG$-flat.

Now every $RG$-projective module is $RG$-flat. In the presence of condition (ii), the condition (iii) can therefore be replaced by

(iv) $(\operatorname{gr}\eta)_j: I^jA/I^{j+1}A \to I^jB/I^{j+1}B$ is injective for every $j<\omega$. But using Lemma 1.1, the conditions (i), (ii) and (iv) are easily seen to imply the injectivity of $\eta$. This proves Lemma 1.2.

### 1.3. *The Non-Triviality of the Class* **D**

PROPOSITION 1.3. *The free cyclic group belongs to the class* **D**.

*Proof.* Let $R$ be an arbitrary non-trivial commutative ring with unit, and let $C=(c)$ be free cyclic. Our first objective is to prove that $RC$ is residually nilpotent.

Every element $x\neq 0$ of $RC$ can be written in the form

$$x=c^N(r_0+r_1c+\cdots+r_gc^g) \quad (r_0\neq 0, r_g\neq 0). \tag{1.1}$$

As we require $r_0\neq 0$ and $r_g\neq 0$, the integer $N$, the non-negative integer $g$ and all the $r_i\in R$ $(1\leqslant i\leqslant g)$ are uniquely determined. Suppose now that $x$ is in $I^j$. Because the $RG$-module $I^j$ is generated by $(1-c)^j$, $x$ has a representation

$$x=\tilde{x}(1-c)^j, \quad \text{where}$$
$$\tilde{x}=c^{\tilde{N}}(\tilde{r}_0+\tilde{r}_1c+\cdots+\tilde{r}_{\tilde{g}}c^{\tilde{g}}) \quad (\tilde{r}_0\neq 0, \tilde{r}_{\tilde{g}}\neq 0).$$

It results the representation

$$x=c^{\tilde{N}}(\tilde{r}_0+\tilde{s}_1c+\cdots+\tilde{s}_{\tilde{g}+j-1}c^{\tilde{g}+j-1}+(-1)^j\,\tilde{r}_{\tilde{g}}c^{\tilde{g}+j}). \tag{1.2}$$

Since $\tilde{r}_0$ as well as $(-1)^j\,\tilde{r}_{\tilde{g}}$ are different from zero, the representations (1.1) and (1.2) are identical. In particular, $g=\tilde{g}+j$, so that $j\leqslant g$. This proves $RC$ to be residually nilpotent.

*Remark.* (1.2) shows also that $(1-c)^j$ is not a zero-divisor. Therefore $I^j$ is an $RC$-free, cyclic module and $(\operatorname{gr} RC)_j=I^j/I^{j+1}$ is $RC$-isomorphic with $RC/I$, i.e. with $R$.

Now let $\eta: A\to B$ be any map between $RC$-projective modules, for which $\mathbf{1}_R\otimes_C\eta$ is injective. By the preceding remark, the map $\mathbf{1}_{(\operatorname{gr} RC)_j}\otimes_C\eta$ is, for every $j<\omega$, naturally isomorphic with $\mathbf{1}_R\otimes_C\eta$ and thus injective. $RC$ being residually nilpotent, it follows from Lemma 1.2 that $\eta$ itself is injective. As $\eta$ was arbitrary, this means that $C$ belongs to $\mathbf{D}(R)$.

### 1.4. *Some Closure Properties of* $\mathbf{D}(R)$

DEFINITION. A system $\{U_j\}_{j\in J}$ of subgroups of a group $G$ is called *inverse* if there exists for every pair $(j',j'')\in J\times J$ an index $j\in J$ such that $U_j\subseteq U_{j'}\cap U_{j''}$. Note that the inverse limit in $\mathcal{G}\imath$ of such a system is canonically isomorphic with the intersection $\bigcap_{j\in J} U_j$.

In Proposition 1.5 the following technical result on inverse systems of subgroups shall be needed.

LEMMA 1.4. *Suppose $\{U_j\}_{j\in J}$ is an inverse system of subgroups of a group $G$. Denote its intersection by $U$. Let $\eta: A\to B$ be a map from an $RG$-projective module into* *an arbitrary RG-module such that all the induced maps* $\mathbf{1}_R\otimes_{U_j}\eta$ *are injective. Then* $\mathbf{1}_R\otimes_U\eta$ *is injective.*

*Proof.* (For an arbitrary $RG$-module $M$, we denote by

$$\iota_M: R\otimes_U M \to \prod_j (R\otimes_{U_j} M)$$

the canonical map given by the maps $r\otimes_U m \mapsto r\otimes_{U_j} m$ and by the universal property of the product.) In the commutative square

$$\begin{tikzcd} R\otimes_U A \arrow[r, "\mathbf{1}_R\otimes_U \eta"'] \arrow[d, "\iota_A"] & R\otimes_U B \arrow[d, "\iota_B"] \\ \prod_j (R\otimes_{U_j} A) \arrow[r, "\Pi(\mathbf{1}_R\otimes_{U_j}\eta)"'] & \prod_j (R\otimes_{U_j} B) \end{tikzcd}$$

the bottom map is by hypothesis injective. We shall show that the left-hand map $\iota_A$ is also injective, thus proving the assertion.

Assume first that $A$ is the group algebra $RG$. Select a right transversal $\{T_s\}_{s\in G/U}$ of $G$ in $U$. Then every element $x\neq 0$ of $R\otimes_U RG$ has a unique representation of the form

$$x=\sum r_s(x)\otimes_U T_s.$$

Since $x\neq 0$, there exists an $s\in G/U$ for which $r_s(x)\neq 0$. As the support $\operatorname{supp}(x)$ $(=\{s\in G/U: r_s(x)\neq 0\})$ is finite, there exists an index $j=j(x)$ in $J$ such that for every pair $(s, s')$ of different indices out of $\operatorname{supp}(x)$ the element $T_s\cdot T_{s'}^{-1}$ avoids $U_j$. This means that the set $\{T_s\}_{s\in\operatorname{supp}(x)}$ can be enlarged to a right transversal of $U_j$ in $G$, and this in term implies that the image of $x$ in $R\otimes_{U_j} RG$ is different from zero. Hence $\iota_{RG}$ is injective.

Next note that for an arbitrary coproduct $\coprod_k M_k$ of $RG$-modules the diagram of canonical mappings

$$\begin{tikzcd} \coprod_k (R\otimes_U M_k) \arrow[r, "\coprod_k \iota_{M_k}"'] \arrow[d, "="] & \coprod_k \prod_j (R\otimes_{U_j} M_k) \arrow[d, tail, "\sigma"] \\ \coprod_k (R\otimes_U M_k) \arrow[r] \arrow[d, "\cong"] & \prod_j \coprod_k (R\otimes_{U_j} M_k) \arrow[d, "\cong"] \\ R\otimes_U (\coprod_k M_k) \arrow[r, "\iota_{\coprod_k M_k}"'] & \prod_j R\otimes_{U_j} (\coprod_k M_k) \end{tikzcd} \tag{1.3}$$

is commutative and that $\sigma$ is injective. It follows therefore from (1.3) and from the injectivity of $\iota_{RG}$ that $\iota_A$ is injective for every $RG$-free module $A$. One further application of (1.3) shows that $\iota_A$ is injective for every direct summand $A$ of an arbitrary $RG$-free module, i.e. for every $RG$-projective module.

PROPOSITION 1.5. *The class* $\mathbf{D}(R)$ *satisfies the following closure properties:*

(i) $\mathrm{s}\mathbf{D}(R)=\mathbf{D}(R)$, *i.e. any subgroup* $U<G$ *of a group* $G$ *lying in* $\mathbf{D}(R)$ *lies in* $\mathbf{D}(R)$.

(ii) $\grave{\mathrm{E}}\mathbf{D}(R)=\mathbf{D}(R)$, *i.e. any group* $G$ *having a transfinite descending subnormal series*

$$G=G_0 \rhd G_1 \rhd G_2 \rhd \cdots G_\omega \rhd G_{\omega+1} \rhd \cdots G_\alpha = e$$

*all whose factors* $G_\beta/G_{\beta+1}$ $(\beta<\alpha)$ *lie in* $\mathbf{D}(R)$, *lies in* $\mathbf{D}(R)$.

(iii) $\Pi\,\mathbf{D}(R)=\mathbf{D}(R)$, *i.e. any product* $\prod_j G_j$ *of groups lying in* $\mathbf{D}(R)$ *lies in* $\mathbf{D}(R)$.

*Proof.* As the three operations s, $\grave{\mathrm{E}}$ and $\Pi$ are closure operations, we need only prove the inclusions "$\subseteq$". (i) $\mathrm{s}\mathbf{D}(R)\subseteq\mathbf{D}(R)$. Suppose $U<G$ and $G$ belonging to $\mathbf{D}(R)$. Let $\eta: A\to B$ be a map between $RU$-projective modules, for which $\mathbf{1}_R\otimes_U\eta$ is injective. The change-of-rings functor

$$RG\otimes_U\text{-}: {}_{RU}\mathcal{M}od \to {}_{RG}\mathcal{M}od$$

sends $RU$-projective modules to $RG$-projective modules. The map $\mathbf{1}_R\otimes_G(\mathbf{1}_{RG}\otimes_U\eta)$ is naturally isomorphic with $\mathbf{1}_R\otimes_U\eta$. Therefore

$$\bar{\eta}=\mathbf{1}_{RG}\otimes_U\eta$$

is a map between $RG$-projective modules, for which $\mathbf{1}_R\otimes_G\bar{\eta}$ is injective. Because $G$ lies by hypothesis in $\mathbf{D}(R)$ the map $\bar{\eta}$ itself is injective which in term implies that $\eta$ is injective.

(ii) $\grave{\mathrm{E}}\mathbf{D}(R)\subseteq\mathbf{D}(R)$. Suppose $G$ has a transfinite subnormal series

$$G=G_0 \rhd G_1 \rhd G_2 \rhd \cdots G_\omega \rhd G_{\omega+1} \rhd \cdots G_\alpha = e$$

all whose factors $G_\beta/G_{\beta+1}$ $(\beta<\alpha)$ belong to $\mathbf{D}(R)$. Let $\eta: A\to B$ be a map between $RG$-projective modules, for which $\mathbf{1}_R\otimes_G\eta$ is injective. Using transfinite induction, we shall show that all the maps $\mathbf{1}_R\otimes_{G_\beta}\eta$ $(\beta\leqslant\alpha)$ are injective which clearly implies that $\eta$ is injective.

*Inductive step.* Suppose $\beta<\alpha$. By the induction hypothesis, $\mathbf{1}_R\otimes_{G_\beta}\eta$ is injective. $\eta$ may also be viewed as a map between $R(G_\beta)$-projective modules. $G_\beta$ is an extension of $G_{\beta+1}$ by $G_\beta/G_{\beta+1}$. So

$$\bar{\eta}=\mathbf{1}_R\otimes_{G_{\beta+1}}\eta$$

is a map between $R(G_\beta/G_{\beta+1})$-projective modules, for which $\mathbf{1}_R\otimes_{G_\beta/G_{\beta+1}}\bar{\eta}$, being naturally isomorphic with $\mathbf{1}_R\otimes_{G_\beta}\eta$, is injective. As $G_\beta/G_{\beta+1}$ lies in $\mathbf{D}(R)$ we deduce that $\bar{\eta}$ is injective.

*Limiting step.* Suppose $\lambda\leqslant\alpha$ is a limit ordinal. By the induction hypothesis, all the maps $\mathbf{1}_R\otimes_{G_\beta}\eta$ $(\beta<\lambda)$ are injective. Since, from the definition of a descending subnormal system, $G_\lambda=\bigcap_{\beta<\lambda}G_\beta$, and since the chain of subgroups

$$G=G_0 \rhd G_1 \rhd G_2 \rhd \cdots G_\lambda$$

forms an inverse system we can infer from Lemma 1.4 that $\mathbf{1}_R \otimes_{G_\lambda} \eta$ is injective.

(iii) $\Pi\, \mathbf{D}(R) \subseteq \mathbf{D}(R)$. Suppose $G$ is an arbitrary product $\prod_{j \in J} G_j$ of groups $G_j$ belonging to $\mathbf{D}(R)$. Consider for any finite subset $F \subset J$ the normal subgroup

$$N_F = \{g \in \prod_j G_j : j \in F \Rightarrow g(j) = e\}.$$

These normal subgroups form an inverse system with trivial intersection. Moreover, the quotients $G/N_F$ ($\cong \prod_{j \in F} G_j$), being finite products of groups belonging to $\mathbf{D}(R)$, lie by the already established closure property "$\mathbf{\dot{E}D}(R) = \mathbf{D}(R)$" in $\mathbf{D}(R)$.

Now let $\eta : A \to B$ be a map between $RG$-projective modules, for which $\mathbf{1}_R \otimes_G \eta$ is injective. Because for every finite $F \subset J$ the group $G/N_F$ lies in $\mathbf{D}(R)$, it follows that all the maps $\mathbf{1}_R \otimes_{N_F} \eta$ are injective. But $\bigcap_{F \subset J} N_F = e$, so that we can infer from Lemma 1.4 that $\eta$ itself is injective.

So far in showing a group $G$ to lie in $\mathbf{D}(R)$ we always considered maps $\eta : A \to B$ between arbitrary $RG$-projective modules. It is, however, sufficient to test all maps $\eta : A \to B$ between *finitely generated, RG-free modules*. This is the content of the next lemma, which will be crucial in the proof of Proposition 1.7.

LEMMA 1.6. *The following statements are equivalent*:

(i) $G \in \mathbf{D}(R)$.

(ii) *Any map $\eta : A \to B$ between RG-free modules, for which $\mathbf{1}_R \otimes_G \eta$ is injective, is itself injective.*

(iii) *Any map $\eta : A \to B$ between finitely generated, RG-free modules, for which $\mathbf{1}_R \otimes_G \eta$ is injective, is itself injective.*

*Proof.* The implication "(i)$\Rightarrow$(iii)" is evident.

(ii)$\Rightarrow$(i). For every $RG$-projective module $C$ we select a (distinguished) complement $\bar{C}$. We define the construction $\rightsquigarrow$ associating to every map $\eta$ between projective modules a map $\tilde{\eta}$ between free modules, by setting

$$\eta : A \to B \rightsquigarrow \tilde{\eta} : \begin{array}{l} A \oplus \bar{A} \to B \oplus \bar{B} \oplus A \oplus \bar{A} \\ a \oplus \bar{a} \mapsto a^\eta \oplus 0 \oplus 0 \oplus \bar{a} \end{array}.$$

Plainly, $\eta$ is injective if (and only if) $\tilde{\eta}$ is injective, and $\mathbf{1}_R \otimes_G \tilde{\eta}$ is injective if (and only if) $\mathbf{1}_R \otimes_G \eta$ is injective. The claim follows immediately from this remark.

(ii)$\Rightarrow$(iii). This implication depends upon the fact that every $RG$-free module is the union of its finitely generated, free, direct summands. The details are omitted.

PROPOSITION 1.7. $\mathbf{L}\mathbf{D}(R) = \mathbf{D}(R)$, *i.e. if $\mathcal{J}$ is a directed system, and $\mathbf{G}$: $\mathcal{J} \to \mathcal{G}r$ a functor such that, for every index $j$ in $\mathcal{J}$ the group $\mathbf{G}(j)$ belongs to $\mathbf{D}(R)$, then the direct limit $\varinjlim \mathbf{G}$ belongs to $\mathbf{D}(R)$.*

*Proof.* As L is a closure operation, we need only prove the inclusion "$\mathbf{L}\mathbf{D}(R) \subseteq \subseteq \mathbf{D}(R)$".

Denote the group $\varinjlim \mathbf{G}$ by $G$. By Lemma 1.6 it is sufficient to test a map $\eta: A \to B$ between finitely generated $RG$-free modules, for which $\mathbf{1}_R \otimes_G \eta$ is injective. If we choose in $A$ an $RG$-basis $\{a_1, \ldots, a_M\}$ and in $B$ an $RG$-basis $\{b_1, \ldots, b_N\}$, the map $\eta$ can be described in these bases by an $M \times N$-matrix $H$. In $H$ occur only finitely many elements, say $g_1, \ldots, g_F$, of $\varinjlim \mathbf{G}$. So there exists an index $m$ in $\mathscr{J}$ such that each $g_k$ $(1 \leqslant k \leqslant F)$ has a preimage in $\mathbf{G}(m)$. It need not be unique. Fix for every $k$ $(1 \leqslant k \leqslant F)$ a preimage and call it $g_k(m)$. Then the matrix $H$ has a corresponding pointwise preimage $H(m)$ under the ringhomomorphism $R(\mathbf{G}(m)) \to R(\varinjlim \mathbf{G})$. We may suppose that $m$ is the minimal element of $\mathscr{J}$, so that there exists, for every index $j$ in $\mathscr{J}$, a map $\mathbf{G}(m \to j): \mathbf{G}(m) \to \mathbf{G}(j)$.

Define next, for every index $j$ in $\mathscr{J}$, a mapping

$$\boldsymbol{\eta}(j): \mathbf{A}(j) \to \mathbf{B}(j)$$

between finitely generated $R(\mathbf{G}(j))$-free modules as is shown below:

$\mathbf{A}(j)$ is $R(\mathbf{G}(j))$-free on $a_1(j), \ldots, a_M(j)$.
$\mathbf{B}(j)$ is $R(\mathbf{G}(j))$-free on $b_1(j), \ldots, b_N(j)$.
$\boldsymbol{\eta}(j)$ is given in the bases $\{a_s(j)\}$ and $\{b_t(j)\}$ by the matrix $H(j)$ which is by definition the pointwise image of $H(m)$ under the ring homomorphism $R(\mathbf{G}(m \to j)): R(\mathbf{G}(m)) \to R(\mathbf{G}(j))$.

The functor $\mathbf{G}: \mathscr{J} \to \mathscr{G}r$ gives rise to a functor $\mathbf{A}: \mathscr{J} \to \mathscr{A}b$. It maps $j$ onto $\mathbf{A}(j)$ considered as an abelian group. If $j \to j'$ is a morphism of $\mathscr{J}$ the induced homomorphism $\mathbf{A}(j \to j'): \mathbf{A}(j) \to \mathbf{A}(j')$ is given by

$$g_s(j) \cdot a_s(j) \mapsto g'_s(j') \cdot a_s(j') \qquad (1 \leqslant s \leqslant M),$$

$g'_s(j')$ denoting the image of $g_s(j)$ under the group homomorphism $\mathbf{G}(j \to j')$, and by $R$-linearity. Similarily, $\mathbf{G}$ gives rise to a functor $\mathbf{B}: \mathscr{J} \to \mathscr{A}b$. Thirdly, the functor $\mathbf{G}$ induces a functor

$$\boldsymbol{\eta}: \mathscr{J} \to (\mathscr{A}b \downarrow \mathscr{A}b).$$

It maps $j$ onto $\boldsymbol{\eta}(j)$ considered as a homomorphism between abelian groups. If $j \to j'$ is a morphism of $\mathscr{J}$ the square

$$\begin{array}{ccc} \mathbf{A}(j) & \xrightarrow{\boldsymbol{\eta}(j)} & \mathbf{B}(j) \\ \downarrow{\scriptstyle \mathbf{A}(j \to j')} & & \downarrow{\scriptstyle \mathbf{B}(j \to j')} \\ \mathbf{A}(j') & \xrightarrow[\boldsymbol{\eta}(j')]{} & \mathbf{B}(j') \end{array}$$

is commutative. So we are allowed to define $\boldsymbol{\eta}(j\to j')$ as the pair $\{\mathbf{A}(j\to j'), \mathbf{B}(j\to j')\}$. The colimit $\varinjlim \boldsymbol{\eta}$ exists and coincides with $\eta$ when $\eta$ is viewed as a map between abelian groups.

Now consider an arbitrary map $\boldsymbol{\eta}(j)$. By construction the matrices $H(j)^{\varepsilon}$ and $H^{\varepsilon}$, $\varepsilon$ denoting the (pointwise applied) appropriate augmentation, are identical. But $\mathbf{1}_R\otimes_G \eta$, described by $H^{\varepsilon}$, is by hypothesis injective. So, for every index $j$ in $\mathscr{J}$ the map $\mathbf{1}_R\otimes_{\mathbf{G}(j)}\boldsymbol{\eta}(j)$, described by $H(j)^{\varepsilon}$, is injective. Since every $\mathbf{G}(j)$ belongs to $\mathbf{D}(R)$, every $\boldsymbol{\eta}(j)$ is injective. It follows that $\varinjlim \boldsymbol{\eta}$, which underlies $\eta$, is injective. This establishes the claim.

### 1.5. *Concluding Remarks*

By way of illustration, we state explicitly some easy consequences of Proposition 1.3, Proposition 1.5 and Proposition 1.7.

COROLLARY 1.8. *The following statements are true:*

(i) *Every poly-free-cyclic group belongs to* $\mathbf{D}$.
(ii) *Every torsion-free nilpotent group belongs to* $\mathbf{D}$.
(iii) *Every free group belongs to* $\mathbf{D}$.

We conclude the investigation of the classes $\mathbf{D}(R)$ by establishing a necessary condition for a group to lie in $\mathbf{D}(S)$, where now $S$ is an integral domain of characteristic zero. We first recall the definition of a (locally) indicable group (G. Higman [9, p. 241ff.], cf. [8, pp. 61–62, §4.5]). A group $G$ is called *indicable* if every non-trivial, finitely generated subgroup $U<G$ can be mapped onto the free cyclic group. Note that one requires precisely that the abelianization $U_{ab}$ of any finitely generated subgroup $U$ is infinite, or equivalently, that $U_{ab}$ has a free cyclic, direct summand. The announced condition then reads as follows.

PROPOSITION 1.9. *If $S$ is a non-trivial integral domain of characteristic zero, then the class $\mathbf{D}(S)$ is contained in the class of indicable groups.*

*Proof.* Suppose $G$ belongs to $\mathbf{D}(S)$ and let $U<G$ be a finitely generated subgroup of $G$. By Proposition 1.5 $U$ belongs to $\mathbf{D}(S)$. The augmentation ideal $I(SU)$ is a finitely generated $SU$-module. So the $G$-trivial module $S$ has an $SU$-free resolution

$$\begin{tikzcd} & & I(SU) \arrow[dl] & & & \\ S & SU \arrow[l, two heads] & & F_1 \arrow[ll, "\partial_1"'] \arrow[ul, two heads] & F_2 \arrow[l, "\partial_2"'] & \cdots \arrow[l] \end{tikzcd}$$

in which $F_1$ is finitely generated. Let $A$ be any left $SU$-module. The cohomology groups $H^*(U, A)$ with coefficients in $A$ may be computed from the bottom row in the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_U(I(SU), A) \arrow[dr] & & \\
\mathrm{Hom}_U(SU, A) \arrow[ur] \arrow[rr, "\mathrm{Hom}(\partial_1, 1_A)"'] \arrow[d, "\cong"] & & \mathrm{Hom}_U(F_1, A) \arrow[r, "\mathrm{Hom}(\partial_2, 1_A)"'] \arrow[d, "\cong"] & \mathrm{Hom}_U(F_2, A) \arrow[d, "\cong"] \\
A \arrow[rr] & & \coprod_J A \arrow[r] & \prod_I A
\end{tikzcd} \tag{1.4}$$

By construction $J$ is finite so that we have a canonical isomorphism $\coprod_J A \xrightarrow{\sim} \prod_J A$.

Assume $U_{ab}$ is finite. We claim that $U$ must be trivial. If $U_{ab}$ is finite, then $\mathrm{Hom}_{\mathbf{Z}}(U_{ab}, S) = H^1(U, S)$ vanishes, the characteristic of $S$ being zero. Now $\mathrm{Hom}(\partial_1, \mathbf{1}_S)$ is easily seen to be the zero map. Hence $H^1(U, S)=0$ implies that $\mathrm{Hom}(\partial_2, \mathbf{1}_S)$ is injective. But by hypothesis $S$ is an integral domain. So it follows from linear algebra that there exists a finite subset $I'$ such that the composition

$$\mathbf{1}_S \otimes_U \eta : \coprod_J S \xrightarrow[\mathrm{Hom}(\partial_2, \mathbf{1}_S)]{} \prod_I S \xrightarrow[p]{} \prod_{I'} S \xleftarrow{\sim} \coprod_{I'} S$$

is injective. Here $p$ denotes the obvious projection. Thus the map

$$\eta : \coprod_J SU \xrightarrow[\mathrm{Hom}(\partial_2, \mathbf{1}_{SU})]{} \prod_I SU \xrightarrow[p]{} \prod_{I'} SU \xleftarrow{\sim} \coprod_{I'} SU$$

is a homomorphism between $SU$-free modules for which $\mathbf{1}_S \otimes_U \eta$ is injective. But $U$ lies in $\mathbf{D}(S)$. Thus $\eta$ and, consequently, $\mathrm{Hom}(\partial_2, \mathbf{1}_{SU})$ are injective. This forces $\mathrm{Hom}(\partial_1, \mathbf{1}_{SU})$ to be the zero map. Now use (1.4) to compute $H^0(U, SU)$. One obtains that $(SU)^U = SU$. This means that $SU$ is a $U$-trivial module. As by hypothesis $S$ is non-trivial, $U$ must be trivial. This establishes the claim.

*Remarks.* A poly-cyclic group is indicable if and only if it is poly-free-cyclic. A nilpotent group is indicable if and only if it is locally poly-free-cyclic, i.e. if it is torsion-free. This shows that the first two statements in Corollary 1.8 cannot be improved. We also remind the reader that a torsion-free poly-cyclic group need not be poly-free-cyclic. Standard counterexamples are the groups

$$G(k) = \langle a, b, t : a^t = a^{-1}, b^t = b^{-1}, [a, b] = t^k \rangle \quad (k \in \mathbf{Z}) \tag{1.5}$$

found by G. Zappa [27] and K. A. Hirsch [11]. If $k \neq 0$ is even, $G(k)$ is a poly-cyclic (even supersolvable), infinite group. It is an extension of a torsion-free nilpotent group by a cyclic group of order two. (As we shall see, this implies that $G(k)$ lies in $\mathbf{D}(\mathbf{Z}_2)$.) Moreover, if $k$ is a multiple of four, $G(k)$ is torsion-free. However, $G(k)$ is not indicable, $G(k)_{ab}$ being finite ($k \neq 0$).

A finitely generated perfect group does not lie in **D**, but this need not be true if the group is infinitely generated. For example, the group

$$\langle (x_h)_{h\in\mathbf{Z}} : x_h = [x_{h+1}, x_{h+2}]\rangle$$

is perfect, but locally free. So it belongs to **D**.

We add a word on the group algebra $SG$ of a group belonging to $\mathbf{D}(S)$, where $S$ is any integral domain. If the characteristic of $S$ is zero $G$ is indicable and so $SG$ has no zero-divisors (G. Higman [9]). If the characteristic is $p$ the group algebra $SG$ may have zero-divisors which, however, are bound to be elements of the augmentation ideal. The situation is illustrated by Lemma 1.10. The lemma, moreover, shows that the torsion-free groups given by (1.5) lie in $\mathbf{D}(\mathbf{Z}_2)$, although they do not belong to $\mathbf{D}(\mathbf{Z})$.

LEMMA 1.10. *A finite group belongs to* $\mathbf{D}(\mathbf{Z}_p)$ *if and only if it is a p-group* (*p denoting a prime*).

*Proof.* Suppose the finite group $G$ belongs to $\mathbf{D}(\mathbf{Z}_p)$. Take an element $g\in G$ of order a prime, say $q$, and consider the homomorphism

$$\eta:\mathbf{Z}_p(g)\to\mathbf{Z}_p(g),$$

given by $x\mapsto(1+g+g^2+\cdots+g^{q-1})\cdot x$. It is not injective. Since the subgroup $(g)<G$ belongs also to $\mathbf{D}(\mathbf{Z}_p)$, the map $1_{\mathbf{Z}_p}\otimes_{(g)}\eta$ cannot be injective either. Therefore $q=p$.

Conversely, suppose $G$ is a finite $p$-group. Let $\eta: A\to B$ be a map between $\mathbf{Z}_pG$-projective modules, for which $1_{\mathbf{Z}_p}\otimes_G\eta$ is injective. $\mathbf{Z}_pG$ is a nilpotent module (cf. [3, p. 681, Theorem 9]) and the quotients $I^j/I^{j+1}$ $(j<\omega)$, being $\mathbf{Z}_p$-vector spaces, are $\mathbf{Z}_p$-flat. So we can deduce from Lemma 1.2 that $\eta$ itself is injective.

## 2. The Proof of Theorem A

### 2.1. *The Definitions of the Classes* $\mathbf{E}(R)$ *and of the Class* $\mathbf{E}$

As in the first section, $R$ shall always denote a non-trivial commutative ring with unit.

We say a group $G$ lies in the class $\mathbf{E}(R)$ if the $G$-trivial module $R$ has an $RG$-projective resolution

$$\cdots\to P_3\to P_2\xrightarrow{\partial_2}P_1\to P_0\twoheadrightarrow R$$

such that the image of the *second differential* $\partial_2$ under the functor

$$R\otimes_{RG}-:{}_{RG}\mathcal{M}od\to{}_{R}\mathcal{M}od$$

is *injective.*

We say $G$ lies in $\mathbf{E}$ if $G$ lies in $\mathbf{E}(R)$ for every $R$.

*Remark.* If $G$ belongs to $\mathbf{E}(R)$ then $H_2(G, R)=0$. The converse, although false in general, holds for groups of cohomological dimension at most two.

### 2.2. *Some Properties of* $\mathbf{E}(R)$

We shall first give a lemma connecting the classes $\mathbf{D}(R)$ and $\mathbf{E}(R)$. This lemma will be crucial in the proof of Theorem A.

LEMMA 2.1. *Suppose the group $G$ is an extension of the form $N \lhd G \twoheadrightarrow Q$. If $Q$ lies in $\mathbf{D}(R)$ and if $G$ belongs to $\mathbf{E}(R)$, then $N$ belongs to $\mathbf{E}(R)$.*

*Proof.* Suppose $G$ belongs to $\mathbf{E}(R)$. Let $\mathbf{P}_* \twoheadrightarrow R$ be an $RG$-projective resolution of $R$ for which $\mathbf{1}_R \otimes_G \partial_2$ is injective. Consider

$$R \otimes_N \mathbf{P}_*.$$

It is a complex of $RQ$-projective modules for which $\mathbf{1}_R \otimes_Q (\mathbf{1}_R \otimes_N \partial_2)$ is injective. By hypothesis $Q$ lies in $\mathbf{D}(R)$. So $\mathbf{1}_R \otimes_N \partial_2$ is injective. Because $\mathbf{P}_* \twoheadrightarrow R$ is also an $RN$-projective resolution of $R$, we have shown that $N$ belongs to $\mathbf{E}(R)$.

For future reference we state explicitly the obvious

COROLLARY 2.2. *A group $G$ which belongs both to $\mathbf{D}(R)$ and to $\mathbf{E}(R)$ is of cohomological dimension $cd_R G$ at most two.*

The particular importance of the class $\mathbf{E}(\mathbf{Z})$ is shown by the following lemma 2.3. This lemma also characterizes the class $\mathbf{E}$.

LEMMA 2.3. *The following statements are true for any group $G$:*

(i) $G \in \mathbf{E}(\mathbf{Z}) \Rightarrow (G \in \mathbf{E}(R) \Leftrightarrow H_2(G, R)=0)$.

(ii) $G \in \mathbf{E} \quad \Leftrightarrow G \in \mathbf{E}(\mathbf{Z})$ *and $G/G'$ is torsion-free.*

*Proof.* Suppose $G \in \mathbf{E}(\mathbf{Z})$ and $H_2(G, R)=0$. Let

$$\cdots \to P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow \mathbf{Z}$$

be a $G$-projective resolution of $\mathbf{Z}$ for which $\mathbf{1}_\mathbf{Z} \otimes_G \partial_2$ is injective. Then $\mathbf{1}_\mathbf{Z} \otimes_G \partial_3$ is the zero map. Tensoring the above complex with $R \otimes_\mathbf{Z}$ – one gets an $RG$-projective resolution of $R$. Because the map $\mathbf{1}_R \otimes_{RG} (\mathbf{1}_R \otimes_\mathbf{Z} \partial_3)$ is also zero, $H_2(G, R)$ is trivial if and only if $\mathbf{1}_R \otimes_{RG} (\mathbf{1}_R \otimes_\mathbf{Z} \partial_2)$ is injective. This proves one half of (i). The converse is obvious. Part (ii) then follows from statement (i) and the universal coefficient theorem.

### 2.3. *Additional Properties of* $\mathbf{E}$

We recall the definition of the *transfinite series of derived groups* $\{G^{(\alpha)}\}$ of a group

$G$. It is given by

$$G^{(0)}=G$$
$$G^{(\alpha+1)}=[G^{(\alpha)}, G^{(\alpha)}]$$
$$G^{(\lambda)}=\bigcap_{\alpha<\lambda} G^{(\alpha)} \qquad (\lambda \text{ a limit ordinal}).$$

We now state a fundamental closure property of the class **E** (Part (i) of Theorem A).

PROPOSITION 2.4. *Any derived group $G^{(\alpha)}$ of an* **E**-*group $G$ is again an* **E**-*group.*

*Proof.* We use transfinite induction on $\alpha$. The claim holds obviously for $\alpha=0$. Now let $\alpha>0$ be an ordinal such that for every $\beta<\alpha$ the group belongs to **E**. Consider $G/G^{(\alpha)}$. It has a transfinite descending normal series (sc. $\{G^{(\beta)}/G^{(\alpha)}\}$) all whose factors are torsion-free abelian (see Lemma 2.3). Therefore $G/G^{(\alpha)}$ lies for every $R$ in $\mathbf{D}(R)$ (see Proposition 1.5 and Corollary 1.8). Lemma 2.1 then implies that $G^{(\alpha)}$ belongs to $\mathbf{E}(R)$ for every $R$ and hence, by definition, to **E**. This establishes the claim.

We also recall the definition of the *lower central series* $\{G_\beta\}_{1\leqslant\beta\leqslant\omega}$. It is given by

$$G_1=G$$
$$G_{j+1}=[G_j, G] \qquad (1\leqslant j<\omega)$$
$$G_\omega=\bigcap_{j<\omega} G_j$$

The next proposition deals with the second assertion of Theorem A:

PROPOSITION 2.5. *Any term $G_\beta$ $(1\leqslant\beta\leqslant\omega)$ of the lower central series of an* **E**-*group $G$ is again an* **E**-*group.*

*Proof.* Suppose $G$ belongs to **E**. As $G/G_2$ is torsion-free and $H_2(G, \mathbf{Z})$ is trivial, all the quotients $G_j/G_{j+1}$ $(1\leqslant j<\omega)$ are torsion-free [23]. It follows that $G/G_\beta$ $(1\leqslant\beta\leqslant\omega)$ lies in **D** for every $\beta$ which, by Lemma 2.1, yields the claim.

An immediate consequence of Proposition 2.4 and Proposition 2.5 is

COROLLARY 2.6. *For any* **E**-*group $G$ the following implications are true:*

$$\text{(i) } \alpha \text{ any ordinal} \Rightarrow \begin{cases} H_2(G^{(\alpha)}, \mathbf{Z}) & \textit{is trivial, and} \\ G^{(\alpha)}/G^{(\alpha+1)} & \textit{is torsion-free.} \end{cases}$$

$$\text{(ii) } 1\leqslant\beta\leqslant\omega \Rightarrow \begin{cases} H_2(G_\beta, \mathbf{Z}) & \textit{is trivial, and} \\ G_\beta/[G_\beta, G_\beta] & \textit{is torsion-free.} \end{cases}$$

We are left with the third claim of Theorem A. For convenience we restate it as

PROPOSITION 2.7. *If $G$ is an* **E**-*group then $G/(\bigcap_\alpha G^{(\alpha)})$ is an* **E**-*group whose cohomological dimension is at most two.*

*Proof.* Let $\delta$ denote the smallest ordinal for which $G^{(\delta)}=G^{(\delta+1)}$. Then $G^{(\delta)}= =\bigcap_\alpha G^{(\alpha)}$. Suppose that

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \to P_0 \twoheadrightarrow \mathbf{Z}$$

is a $G$-projective resolution for which $\mathbf{1_Z}\otimes_G \partial_2$ is injective. We assert that

$$0 \to \mathbf{Z}\otimes_{G^{(\delta)}} P_2 \xrightarrow{\tilde{\partial}_2} \mathbf{Z}\otimes_{G^{(\delta)}} P_1 \to \mathbf{Z}\otimes_{G^{(\delta)}} P_0 \twoheadrightarrow \mathbf{Z}$$

is a $G/G^{(\delta)}$-projective resolution of $\mathbf{Z}$. The modules $\mathbf{Z}\otimes_{G^{(\delta)}} P_i$ $(i=2, 1, 0)$ are $G/G^{(\delta)}$-projective. The map $\mathbf{1_Z}\otimes_{G/G^{(\delta)}} \tilde{\partial}_2$, being naturally isomorphic with $\mathbf{1_Z}\otimes_G \partial_2$, is injective and $G/G^{(\delta)}$ lies in $\mathbf{D}$. Therefore $\tilde{\partial}_2$ is injective. The homology at $\mathbf{Z}\otimes_{G^{(\delta)}} P_1$ equals $H_1(G^{(\delta)}, \mathbf{Z})=G^{(\delta)}/G^{(\delta+1)}=0$. Since $\mathbf{Z}\otimes_{G^{(\delta)}}$ – is a right exact functor, the complex is also exact at $\mathbf{Z}\otimes_{G^{(\delta)}} P_0$ and at $\mathbf{Z}$. Thus the assertion is established. It follows that the cohomological dimension of $G/G^{(\delta)}$ is at most two. As mentioned above, $\mathbf{1_Z}\otimes_{G/G^{(\delta)}} \tilde{\partial}_2$ is injective. This shows that $G/G^{(\delta)}$ is an $\mathbf{E}$-group.

## 3. The Proof of Theorem B

### 3.1 *The Crucial Lemma*

LEMMA 3.1. *Let $\varphi: F \to G$ be a map from a free group into a group of $\mathbf{E}(R)$ for which*

$$\varphi(R): H_1(F, R) \to H_1(G, R)$$

*is injective. Suppose $Q$ is a quotient of $G$ lying in $\mathbf{D}(R)$. Then*

$$\varphi(RQ): H_1(F, RQ) \to H_1(G, RQ)$$

*is injective.*

*Proof.* Let $\cdots \to 0 \to I(RF) \overset{\mu}{\rightarrowtail} RF \overset{\varepsilon}{\twoheadrightarrow} R$ be the $RF$-free augmentation resolution of $R$, and let $\mathbf{P}_* \twoheadrightarrow R$ be an $RG$-projective resolution of $R$ for which $\mathbf{1}_R\otimes_G \partial_2$ is injective. View $\mathbf{P}_* \twoheadrightarrow R$ as an acyclic complex of $RF$-modules via $\varphi$. There exists a chain map $\boldsymbol{\tau}_*$ yielding the commutative diagram

$$\begin{array}{ccccccc}
\cdots \to & 0 & \to & I(RF) & \xrightarrow{\mu} & RF & \twoheadrightarrow R \\
& \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} & \downarrow{=} \\
\cdots \to & P_2 & \xrightarrow[\partial_2]{} & P_1 & \xrightarrow[\partial_1]{} & P_0 & \twoheadrightarrow R
\end{array}$$

of $RF$-modules. Note that $\tau_1$ factors as

$$\begin{tikzcd}
I(RF) \arrow[dd, "\tau_1"'] \arrow[dr] & \\
 & RG\otimes_F I(RF) \arrow[dl, "\sigma_1"] \\
P_1 &
\end{tikzcd}$$

since $P_1$ is an $RG$-module.

The map $\varphi: F\to G$ induces a natural transformation $\varphi(-)$ between the functors

$$H_1(F,-): \mathcal{M}od_{RG} \to \mathcal{M}od_R$$

and

$$H_1(G,-): \mathcal{M}od_{RG} \to \mathcal{M}od_R .$$

This natural transformation $\varphi(-)$ is defined by the composition $\mathrm{p}_A\circ(\tau_1)'_*\circ\mathbf{1}$ in the diagram

$$\begin{tikzcd}
H_1(F,A) \arrow[d, "="] & & & \\
\ker(\mathbf{1}_A\otimes_F\mu) \arrow[r, tail, "i_A"'] \arrow[dd, dashed, "(\tau_1)'_*"] & A\otimes_F I(RF) \arrow[rr, "\mathbf{1}_A\otimes\mu"'] \arrow[dd, "(\tau_1)_*"] \arrow[dr, "\cong"'] & & A\otimes_F RF \arrow[dd, "(\tau_0)_*"] \\
 & & A\otimes_G(RG\otimes_F I(RF)) \arrow[dl, "\mathbf{1}_A\otimes\sigma_1"] & \\
\ker(\mathbf{1}_A\otimes_G\partial_1) \arrow[r, tail, "j_A"'] \arrow[d, "\mathrm{p}_A"] & A\otimes_G P_1 \arrow[rr, "\mathbf{1}_A\otimes\partial_1"'] & & A\otimes_G P_0 \\
H_1(G,A) & & &
\end{tikzcd}$$

The injectivity of $\varphi(A)$ can therefore be characterized by (3.1).

$$\varphi(A) \ \text{ is injective} \Leftrightarrow \begin{array}{l}\text{(i) } (\tau_1)'_* \ \text{ is injective.}\\ \text{(ii) } \operatorname{im}(j_A\circ(\tau_1)'_*)\cap\operatorname{im}(\mathbf{1}_A\otimes_G\partial_2)=0.\end{array} \tag{3.1}$$

If $A$ is the $G$-trivial module $R$, the embedding $i_R$ is onto and (3.1) may be rewritten as

$$\varphi(R) \ \text{ is injective} \Leftrightarrow \begin{array}{l}\text{(i) } \mathbf{1}_R\otimes_G\sigma_1 \ \text{ is injective.}\\ \text{(ii) } \operatorname{im}(\mathbf{1}_R\otimes_G\sigma_1)\cap\operatorname{im}(\mathbf{1}_R\otimes_G\partial_2)=0.\end{array} \tag{3.2}$$

Now define the auxiliary map

$$\eta=\sigma_1\oplus\partial_2:(RG\otimes_F I(RF))\oplus P_2\to P_1 .$$

The induced map $\mathbf{1}_R \otimes_G \eta$ is injective because $\mathbf{1}_R \otimes_G \partial_2$ is injective by the choice of $\mathbf{P}_* \twoheadrightarrow R$, and because (i) and (ii) hold by the hypothesis on $\varphi(R)$. Consider $\mathbf{1}_{RQ} \otimes_G \eta$. It is a map between $RQ$-projective modules, whose image under the functor $\mathbf{1}_R \otimes_Q \text{-} : {}_{RQ}\mathcal{M}od \to {}_R\mathcal{M}od$ is injective. As $Q$ lies by assumption in $\mathbf{D}(R)$, we infer that $\mathbf{1}_{RQ} \otimes_G \eta$ is injective. In particular, $\varphi(RQ)$ is injective, as we set out to prove.

### 3.2. *The Iterated Lower Central Series*

DEFINITION. Let $\mathbf{K}$ denote the set consisting of the one-tuple $(1)$ and of all the ordered, finite tuples $(k_1, \ldots, k_s)$ of natural numbers $k_i \geqslant 2$ $(s \geqslant 1)$. The number $s$ is called the length of the tuple. For an arbitrary group $G$, the terms of the *iterated lower central series* $\{G_{\mathbf{k}}\}$ are defined by recursion on the length of $\mathbf{k}$ and given by

$$G_{(1)} = G$$
$$G_{(k_1)} = G_{k_1} \qquad (= \underbrace{[G, \ldots [G, G] \ldots]}_{k_1})$$
$$G_{(k_1, \ldots, k_{s-1}, k_s)} = (G_{(k_1, \ldots, k_{s-1})})_{k_s}.$$

Note that all the terms $\{G_{\mathbf{k}}\}$ are fully invariant subgroups of $G$.

In the proof of Proposition 3.2 we shall need an auxiliary, iterated descending central series $\{{}_{\mathbf{k}}G\}$. It is defined by

$${}_{(1)}G = G$$
$${}_{(k+1)}G = \ker\left({}_{(k)}G \underset{\text{can}}{\twoheadrightarrow} ({}_{(k)}G/[{}_{(k)}G, G]) \otimes_{\mathbf{Z}} \mathbf{Q}\right)$$
$${}_{(k_1, \ldots, k_s)}G = {}_{(k_s)}\left({}_{(k_1, \ldots, k_{s-1})}G\right).$$

Note that every term ${}_{\mathbf{k}}G$ is a fully invariant subgroup of $G$, and that $\{{}_{\mathbf{k}}G\}$ is the most rapidly descending iterated central series all whose quotients

$$G/{}_{(2)}G,$$
$${}_{(k_1, \ldots, k_s)}G/{}_{(k_1, \ldots, k_s, 2)}G$$

and

$${}_{(k_1, \ldots, k_s)}G/{}_{(k_1, \ldots, k_s+1)}G$$

are torsion-free.

We are now ready to prove Theorem B which we restate, in a slightly different formulation, as

PROPOSITION 3.2. *Let $\varphi: F \to G$ be a map from a free group into an $\mathbf{E}(\mathbf{Q})$-group for which*

$$\varphi: F/F_2 \to G/G_2$$

*is injective. Then*

$$\varphi: F/F_{\mathbf{k}} \to G/G_{\mathbf{k}}$$

*is injective for every* $\mathbf{k} \in \mathbf{K}$.

*Proof.* We shall first prove the following implication:

$$\begin{array}{ll} \text{If} \quad \varphi: F/_{(2)}F \to G/_{(2)}G & \text{is injective then} \\ \qquad \varphi: F/_{\mathbf{k}}F \to G/_{\mathbf{k}}G & \text{is injective for every } \mathbf{k} \in \mathbf{K}. \end{array} \tag{3.3}$$

We use induction on the length of $\mathbf{k}$. The group $G$ lies in $\mathbf{E}(\mathbf{Q})$ so that $H_2(G, \mathbf{Q})$ is trivial. Therefore (3.3) follows for $s=1$ from a result of J. Stallings ([17, p. 180, Theorem 7.3], cf. [22, p. 69, Satz 8.1]). Consider now $(\mathbf{k}, k_{s+1})=(k_1, \ldots, k_s, k_{s+1})$, $s \geqslant 1$. By induction hypothesis

$$\varphi: F/_{\mathbf{k}}F \to G/_{\mathbf{k}}G \tag{3.4}$$

is injective. It follows that $\varphi_1$ in the composition

$$H_1(F, \mathbf{Q}(F/_{\mathbf{k}}F)) \xrightarrow{\varphi_1} H_1(F, \mathbf{Q}(G/_{\mathbf{k}}G)) \xrightarrow[\varphi(\mathbf{Q}(G/G_{\mathbf{k}}))]{} H_1(G, \mathbf{Q}(G/_{\mathbf{k}}G))$$

is injective. So check the hypothesis of Lemma 3.1. The map

$$\varphi(\mathbf{Q}): H_1(F, \mathbf{Q}) \to H_1(G, \mathbf{Q})$$

is injective since $\varphi: F/_{(2)}F \to G/_{(2)}G$ is injective, since $\mathbf{Q}$ is $\mathbf{Z}$-flat, and since

$$F/_{(2)}F \otimes_{\mathbf{Z}} \mathbf{Q} \to G/_{(2)}G \otimes_{\mathbf{Z}} \mathbf{Q}$$

is naturally isomorphic with $\varphi(\mathbf{Q})$. The group $G/_{\mathbf{k}}G$ is poly- (torsion-free nilpotent), and so belongs to $\mathbf{D} \subseteq \mathbf{D}(\mathbf{Q})$. We can now infer from Lemma 3.1 that $\varphi(\mathbf{Q}(G/_{\mathbf{k}}G))$ in the above composition is injective. As ${}_{\mathbf{k}}F/_{(\mathbf{k},2)}F$ is torsion-free, it follows that

$$\varphi: {}_{\mathbf{k}}F/_{(\mathbf{k},2)}F \to {}_{\mathbf{k}}G/_{(\mathbf{k},2)}G$$

is injective. Because $G/_{\mathbf{k}}G$ lies in $\mathbf{D}(\mathbf{Q})$, and because $G$ belongs by the hypothesis of Proposition 3.2 to $\mathbf{E}(\mathbf{Q})$, the group ${}_{\mathbf{k}}G$ belongs to $\mathbf{E}(\mathbf{Q})$ so that $H_2({}_{\mathbf{k}}G, \mathbf{Q})=0$. It follows from Theorem 7.3 in [17] that for every $j$ $(2 \leqslant j < \omega)$

$$\varphi: {}_{\mathbf{k}}F/_{(\mathbf{k},j)}F \to {}_{\mathbf{k}}G/_{(\mathbf{k},j)}G \tag{3.5}$$

is injective. From (3.4) and (3.5) one finally deduces that

$$\varphi: F/_{(\mathbf{k},k_{s+1})}F \to G/_{(\mathbf{k},k_{s+1})}$$

is injective. This completes the proof of claim (3.3).

Now note that for a free group $F$ the terms ${}_{\mathbf{k}}F$ and $F_{\mathbf{k}}$ coincide for every $\mathbf{k} \in \mathbf{K}$. From this fact and the auxiliary result (3.3), Proposition 3.2 is readily deduced (see the proof of Satz 8.1 in [22]).

## 4. The Classes $\mathbf{M}(\mathbf{Z})$ and $\mathbf{M}$

### 4.1. *Definition and Elementary Properties of* $\mathbf{M}(\mathbf{Z})$ *and of* $\mathbf{M}$

We say a group $G$ lies in $\mathbf{M}(\mathbf{Z})$ if $G$ is finitely presentable and if the deficiency $\operatorname{def} G$ of $G$ equals the torsion-free rank of the abelianized group $G_{ab}$.

We recall the definition of $\operatorname{def} G$: The deficiency $\operatorname{def}\mathscr{P}$ of a finite presentation $\mathscr{P}=\langle z_1, \ldots, z_M : r_1, \ldots, r_N\rangle$ is the difference $M-N$. The deficiency of a finitely presentable group $G$ is $\sup\{\operatorname{def}\mathscr{P} : \mathscr{P}$ presents $G\}$. As is well-known, $\operatorname{def} G$ is at most equal to $rk(G_{ab})$ (see also formula (4.7)).

We say a group $G$ lies in $\mathbf{M}$ if $G$ lies in $\mathbf{M}(\mathbf{Z})$ and if $G_{ab}$ is torsion-free.

*Remark.* The class $\mathbf{M}$ has been investigated by W. Magnus [13] in 1939. We cite two of his results:

HILFSSATZ 1. [13, p. 310]. *Any group belonging to* $\mathbf{M}$ *has a presentation of the form*

$$\langle x_1, \ldots, x_t, y_1, \ldots, y_d : x_1 C_1(x_m, y_n), \ldots, x_t C_t(x_m, y_n)\rangle, \tag{4.1}$$

*where* $C_i(x_m, y_n)$ $(1 \leqslant i \leqslant t)$ *has zero exponent sum on all generators.*

HILFSSATZ 2. [13, p. 311]. *If* $G$ *has a presentation of the form* (4.1), *then the images of* $y_1, \ldots, y_d$ *in* $G/G_j$ *freely generate* $G/G_j$ *for every* $j$ $(2 \leqslant j < \omega)$. *Thus* $G/G_j$ *is free nilpotent of rank* $d$ *and of class* $j-1$. *Moreover, the images of* $y_1, \ldots, y_d$ *in* $G$ *freely generate a free subgroup of* $G$.

For a modern account of Hilfssatz 1, see [14, Section 3.3]; for Hilfssatz 2, see [14, pp. 351–353, Theorem 5.14 and Corollary 5.14.1] or [19, p. 133, Korollar 1].

### 4.2. *Around Lyndon's Resolution*

The relevance of the class $\mathbf{M}(\mathbf{Z})$ for our investigation stems from the fact that $\mathbf{M}(\mathbf{Z})$ is a subclass of $\mathbf{E}(\mathbf{Z})$, as we shall show in this subsection.

Let $R \lhd F \overset{\pi}{\twoheadrightarrow} G$ be an extension of groups. There exists an associated short exact sequence

$$R_{ab} \overset{\kappa}{\rightarrowtail} \mathbf{Z}G \otimes_F IF \overset{\nu}{\twoheadrightarrow} IG \tag{4.2}$$

of left $G$-modules (see e.g. [10, VI. 6] for details). The $G$-module structure of $R_{ab}$ in (4.2) is induced by conjugation.

Suppose $F$ is a free group. Then the extension $R \lhd F \overset{\pi}{\twoheadrightarrow} G$ is called a free presentation of $G$. Splicing the sequence (4.2) with $IG \overset{\mu}{\rightarrowtail} \mathbf{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbf{Z}$, one obtains the exact sequence

$$0 \to R_{ab} \overset{\kappa}{\to} \mathbf{Z}G \otimes_F IF \xrightarrow{\mu\circ\nu} \mathbf{Z}G \overset{\varepsilon}{\to} \mathbf{Z} \to 0. \tag{4.3}$$

(Note that the module $\mathbf{Z}G\otimes_F IF$ is $G$-free because $IF$ is $F$-free.) Choosing a set $\{r_i\}_{i\in I}\subset R$ whose normal closure in $F$ is $R$, one gets a corresponding surjection

$$\beta\colon \coprod_{i\in I}(\mathbf{Z}G)_i \twoheadrightarrow R_{ab}, \qquad 1_i\mapsto r_i[R, R].$$

Again, one may splice $\beta$ with (4.3) obtaining

$$\coprod_{i\in I}(\mathbf{Z}G)_i \xrightarrow{\kappa\circ\beta} \mathbf{Z}G\otimes_F IF \xrightarrow{\mu\circ\nu} \mathbf{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbf{Z}. \tag{4.4}$$

This exact sequence is the beginning of a $G$-free resolution of $\mathbf{Z}$ found by R. C. Lyndon [12, p. 656, Lemma 5.1].

Now apply the functor

$$\mathbf{Z}\otimes_G\text{-}\colon {}_G\mathscr{M}od \to \mathscr{A}b$$

to (4.2). There results the exact sequence

$$0\to \ker(1_{\mathbf{Z}}\otimes_G\kappa)\to R/[F, R] \xrightarrow{1_{\mathbf{Z}}\otimes_G\kappa} IF/IF^2 \to IG/IG^2 \to 0. \tag{4.5}$$

From (4.3) one deduces that $\ker(1_{\mathbf{Z}}\otimes_G\kappa)=H_2(G, \mathbf{Z})$. Using the isomorphisms $IF/IF^2\cong F_{ab}$ and $IG/IG^2\cong G_{ab}$, the sequence (4.5) can be written as

$$0\to H_2(G, \mathbf{Z})\to R/[F, R] \xrightarrow{1_{\mathbf{Z}}\otimes_G\kappa} F_{ab} \xrightarrow{\pi_*} G_{ab}\to 0. \tag{4.6}$$

*Remark.* If $\{r_i\}_{i\in I}$ generates $R$ as a normal subgroup of $F$, then $\{r_i[R, R]\}_{i\in I}$ generates $R_{ab}$ as a $G$-module, and $\{r_i[R, F]\}_{i\in I}$ generates $R/[F, R]$ as an abelian group.

Suppose next that $G$ is finitely presentable. Let

$$\mathscr{P}=\langle z_1,\ldots, z_M : r_1,\ldots, r_N\rangle$$

be a finite presentation of $G$. For a finitely generated abelian group $A$, denote the minimal number of generators by $s(A)$. Since, in (4.6), $\ker\pi_* < F_{ab}$ is free abelian, it follows that

$$\begin{aligned} N\geqslant s(R/[F, R]) &= s(H_2(G, \mathbf{Z}))+\mathrm{rk}(\ker\pi_*)\\ &= s(H_2(G, \mathbf{Z})+M-\mathrm{rk}(G_{ab}). \end{aligned}$$

Therefore

$$\operatorname{def}\mathscr{P}=M-N\leqslant \mathrm{rk}(G_{ab})-s(H_2(G, \mathbf{Z}))$$

and

$$\operatorname{def}G\leqslant \mathrm{rk}(G_{ab})-s(H_2(G, \mathbf{Z})). \tag{4.7}$$

(The above deduction is taken from [20, p. 295].)

The inequality (4.7) shows that the deficiency of a finitely presentable group is bounded by $\operatorname{rk}(G_{ab})$, and is thus finite. So there always exists a presentation $\mathscr{P}$ of $G$ whose deficiency equals $\operatorname{def} G$.

Suppose, finally, that $G$ lies in $\mathbf{M}(\mathbf{Z})$. Let $\mathscr{P}=\langle z_1,\dots, z_M : r_1,\dots, r_N\rangle$ be a presentation of $G$ whose deficiency equals $\operatorname{rk}(G_{ab})$. Let (4.4) denote the associated beginning of a $G$-free resolution of $\mathbf{Z}$. Consider

$$\mathbf{1}_{\mathbf{Z}}\otimes_G\partial_2=(\mathbf{1}_{\mathbf{Z}}\otimes_G\kappa)\circ(\mathbf{1}_{\mathbf{Z}}\otimes_G\beta).$$

The map $\mathbf{1}_{\mathbf{Z}}\otimes_G\kappa$ is injective by (4.6) and (4.7), whereas $\mathbf{1}_{\mathbf{Z}}\otimes_G\beta:\coprod(\mathbf{Z})_i\twoheadrightarrow R/[F, R]$ maps a free abelian group of rank $N$ onto an abelian group of rank

$$M-\operatorname{rk}(G_{ab})=M-\operatorname{def} G=N,$$

and thus $\mathbf{1}_{\mathbf{Z}}\otimes_G\beta$ is isomorphic. So $\mathbf{1}_{\mathbf{Z}}\otimes_G\partial_2$ is injective.

We summarize part of the discussion in

PROPOSITION 4.1. *Suppose $G$ has a presentation*

$$\mathscr{P}=\langle z_1,\dots, z_M : r_1,\dots, r_N\rangle$$

*whose deficiency $M-N$ equals* $\operatorname{rk}(G_{ab})$. *Then*

$$\begin{tikzcd}
 & R_{ab} \arrow[dr, "\kappa"] & & IF \arrow[dr, "\mu"] & & \\
\coprod_{1=i}^{N}(\mathbf{Z}G)_i \arrow[ur, "\beta"] \arrow[rr, "\partial_2"'] & & \mathbf{Z}G\otimes_F IF \arrow[ur, "\nu"] \arrow[rr, "\partial_1"'] & & \mathbf{Z}G \arrow[r, two heads, "\varepsilon"'] & \mathbf{Z}
\end{tikzcd}$$

*is the beginning of a $G$-free resolution of $\mathbf{Z}$ for which $\mathbf{1}_{\mathbf{Z}}\otimes_G\partial_2$ is injective.*

*Every* $\mathbf{M}(\mathbf{Z})$*-group is therefore an* $\mathbf{E}(\mathbf{Z})$*-group.*

COROLLARY 4.2. *Every* $\mathbf{M}$*-group is an* $\mathbf{E}$*-group.*

### 4.3. *An Isomorphism Criterion*

We recall some notions from the *free differential calculus* [6]. Let $F$ be a free group, free on $\{x_i\}_{i\in I}$. The corresponding augmentation ideal $IF$ is an $F$-free (left) module, free on $\{1-x_i\}_{i\in I}$ (see e.g. [10, p. 196, Theorem 5.5]). So every element of the form $1-f$ $(f\in F)$ can uniquely be written as

$$1-f=\sum D_{x_i}(f)\cdot(1-x_i) \tag{4.8}$$

This representation defines for every $i\in I$ a function

$$D_{x_i}: F\to \mathbf{Z}F.$$

From (4.8) one derives in a straightforward manner the following properties:

(i) $D_{x_i}(x_k)=\begin{cases}1 & \text{if } i=k,\\ 0 & \text{otherwise}\end{cases}$

(ii) $D_{x_i}(f\cdot \bar{f})=D_{x_i}(f)+f\cdot D_{x_i}(\bar{f})$

(iii) $D_{x_i}(f^{-1})=-f^{-1}\cdot D_{x_i}(f)$

Property (ii) says the function $D_{x_i}$ is a *derivation* which, in view of (i), we shall call the *partial derivative* with respect to $x_i$. Properties (i), (ii) and (iii) show how to calculate explicitly the partial derivative $D_{x_i}$ of a given element $f\in F$.

Consider now a group $G$ in $\mathbf{M}$. As proved by W. Magnus [13, p. 310, Hilfssatz 1], $G$ has a presentation of the form

$$G=\langle x_1,\dots, x_t, y_1,\dots, y_d : x_1C_1(x_m, y_n),\dots, x_tC_t(x_m, y_n)\rangle_\pi, \tag{4.9}$$

where every $C_i(x_m, y_n)$ has zero exponent sum on all generators. (The index $\pi$ appearing in (4.9) indicates the name of the projection from the free group onto $G$.) Let $F$ be free on $y_1,\dots, y_d$, and let $F^x * F^y$ denote the free group

$$\langle x_1,\dots, x_t, y_1,\dots, y_d : \rangle$$

occurring in the presentation (4.9). The augmentation ideal of $\mathbf{Z}(F^x*F^y)$ is naturally isomorphic with

$$(\mathbf{Z}(F^x*F^y)\otimes_{F^x}IF^x)\oplus(\mathbf{Z}(F^x*F^y)\otimes_{F^y}IF^y)$$

(see e.g. [10, p. 196, Theorem 5.5], or [10, p. 220, Lemma 14.1]). Therefore the exact sequence (4.4) can in our particular case be written as

$$P_2\xrightarrow{\partial_2}(\mathbf{Z}G\otimes_{F^x}IF^x)\oplus(\mathbf{Z}G\otimes_{F^y}IF^y)\xrightarrow{\partial_1}\mathbf{Z}G\overset{\varepsilon}{\twoheadrightarrow}\mathbf{Z}. \tag{4.10}$$

Denote the projection from $(\mathbf{Z}G\otimes_{F^x}IF^x)\oplus(\mathbf{Z}G\otimes_{F^y}IF^y)$ onto its first summand $\mathbf{Z}G\otimes_{F^x}IF^x$ by $p_x$. The announced criterion reads then as follows:

PROPOSITION 4.3. *Let* $\mathbf{k}$ *be an s-tuple out of* $\mathbf{K}$, *let* $F$ *be free on* $y_1,\dots, y_d$, *let*

$$G=\langle x_1,\dots, x_t, y_1,\dots, y_d : x_1C_1(x_m, y_n),\dots, x_tC_t(x_m, y_n)\rangle_\pi,$$

*where* $C_i(x_m, y_n)$ *has zero exponent sum on all the generators, and let* $\varphi: F\to G$ *be given by* $y_n\mapsto y_n^\pi$ $(1\leqslant n\leqslant d)$.

*Then the mappings* $\varphi: F/F_{(\mathbf{k},j)}\xrightarrow{\simeq} G/G_{(\mathbf{k},j)}$ *are isomorphic for all* $j$ $(2\leqslant j<\omega)$ *if and only if*

$$\varphi: F/F_{(\mathbf{k},2)}\xrightarrow{\simeq} G/G_{(\mathbf{k},2)} \tag{4.11}$$

*is isomorphic, and* $\varphi$ *in* (4.11) *is isomorphic if and only if*

$$\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G(p_x\circ\partial_2)$$

*is surjective.*

*Proof.* It is readily checked that $\varphi:F/F_2\to G/G_2$ is an isomorphism. So Proposition 3.2 applies saying that

$$\varphi:F/F_{\mathbf{k}_1}\rightarrowtail G/G_{\mathbf{k}_1} \tag{4.12}$$

is injective for every $\mathbf{k}_1\in\mathbf{K}$. One also easily checks that the map $\mathbf{1}_{\mathbf{Z}}\otimes_G(p_x\circ\partial_2)$ is given by $1_{\mathbf{Z}}\otimes 1_m\mapsto 1_{\mathbf{Z}}\otimes(1-x_m)$ $(1\leqslant m\leqslant t)$ and thus is onto (even isomorphic).

The Frattini subgroup of a nilpotent group contains the derived group. In other words, every set $S\subset H$, generating the (arbitrary) group $H$ modulo its derived group $H_2$, generates $H$ modulo any term $H_j$ $(2\leqslant j<\omega)$ of its lower central series. Thus $\varphi:F/F_{(\mathbf{k},j)}\to G/G_{(\mathbf{k},j)}$ is onto for a given $j$ $(2\leqslant j<\omega)$ if and only if $\varphi:F/F_{(\mathbf{k},2)}\to\to G/G_{(\mathbf{k},2)}$ is onto. Together with (4.12), this remark proves the first part of Proposition 4.3. It also shows that the map

$$\varphi:F/F_{(k_1,\dots,k_i,\dots,k_s,2)}\to G/G_{(k_1,\dots,k_i,\dots,k_s,2)}$$

is surjective if and only if all the maps

$$\varphi:(F_{(k_1,\dots,k_i)})_{ab}\to(G_{(k_1,\dots,k_i)})_{ab}$$

are surjective $(i=1,2,\dots,s)$. Since

$$\mathbf{1}_{\mathbf{Z}(G/G_{(k_1,\dots,k_i)})}\otimes_G(p_x\circ\partial_2)$$

is surjective when $\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G(p_x\circ\partial_2)$ is surjective, we can apply induction on the length of $\mathbf{k}$. Taking into account (4.12), only the following claim (*) remains to be verified:

(*) Suppose $\varphi:F/F_{\mathbf{k}}\overset{\sim}{\to}G/G_{\mathbf{k}}$ is isomorphic. Then $\varphi:(F_{\mathbf{k}})_{ab}\to(G_{\mathbf{k}})_{ab}$ is surjective if and only if $\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G(p_x\circ\partial_2)$ is surjective.

In order to prove (*), we analyse the natural transformation $\varphi(-)$ from

$$H_1(F,-):\mathcal{M}od_G\to\mathcal{A}b$$

to

$$H_1(G,-):\mathcal{M}od_G\to\mathcal{A}b$$

(cf. the proof of Lemma 3.1).

Let $IF\rightarrowtail\mathbf{Z}F\twoheadrightarrow\mathbf{Z}$ be the $F$-free augmentation resolution of $\mathbf{Z}$, and let

$$P_2\xrightarrow{\partial_2}(\mathbf{Z}G\otimes_{F^x}IF^x)\oplus(\mathbf{Z}G\otimes_{F^y}IF^y)\xrightarrow{\partial_1}\mathbf{Z}G\overset{\varepsilon}{\twoheadrightarrow}\mathbf{Z}$$

be the beginning of a $G$-free resolution of $\mathbf{Z}$, as described in the previous section (see also (4.10)). Choose $\tau_0=\varphi:\mathbf{Z}F\to\mathbf{Z}G$ and

$$\tau_1=\varphi_*:\begin{array}{l} IF\to(\mathbf{Z}G\otimes_{F^x}IF^x)\oplus(\mathbf{Z}G\otimes_{F^y}IF^y)\\ 1-y_n\mapsto\quad\; 0\quad\;\; \oplus(1_{\mathbf{Z}G}\otimes(1-y_n)).\end{array}$$

The diagram of $F$-modules

$$\begin{array}{ccccccc} 0 & \to & IF & \to & \mathbf{Z}F & \to & \mathbf{Z}\\ \downarrow & & \downarrow{\scriptstyle\tau_1} & & \downarrow{\scriptstyle\tau_0} & & \downarrow{\scriptstyle=}\\ P_2 & \to & (\mathbf{Z}G\otimes_{F^x}IF^x)\oplus(\mathbf{Z}G\otimes_{F^y}IF^y) & \to & \mathbf{Z}G & \to & \mathbf{Z}\end{array}\tag{4.13}$$

is then commutative and may be used to compute $\varphi(A):H_1(F,A)\to H_1(G,A)$ for any $G$-module $A$.

Suppose now that $\varphi:F/F_{\mathbf{k}}\cong G/G_{\mathbf{k}}$ is an isomorphism. Then, modulo $G_{\mathbf{k}}$, every generator $x_m^\pi$ $(1\leqslant m\leqslant t)$ is congruent to the image $w_m^\varphi$ of some element $w_m$ of $F$. As $1-w_m$ has a representation

$$1-w_m=\sum(D_{y_n}(w_m))\cdot(1-y_n),$$

there results a $G$-one-cycle, to wit

$$1_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes(1-x_m)\oplus\sum(-D_{y_n}(w_m))^\varphi\otimes(1-y_n).\tag{4.14}$$

It follows that every element $\alpha_x\in(\mathbf{Z}(G/G_{\mathbf{k}})\otimes_{F^x}IF^x)$ occurs as the first component of some $G$-one-cycle $\alpha_x\oplus\alpha_y$.

On the other hand, it is clear from (4.13) that the images of the $F$-one-cycles under $\tau_1$ are exactly the $G$-one-cycles of the form $0+\alpha_y$. The natural transformation

$$\varphi(\mathbf{Z}(G/G_{\mathbf{k}})):H_1(F,\mathbf{Z}(G/G_{\mathbf{k}}))\to H_1(G,\mathbf{Z}(G/G_{\mathbf{k}}))$$

is therefore surjective if and only if

$$\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G(p_x\circ\partial_2)$$

is surjective. But $\varphi:(F_{\mathbf{k}})_{ab}\to(G_{\mathbf{k}})_{ab}$ is naturally isomorphic with the composition

$$H_1(F,\mathbf{Z}(F/F_{\mathbf{k}}))\underset{\varphi_1}{\cong}H_1(F,\mathbf{Z}(G/G_{\mathbf{k}}))\xrightarrow[\varphi(\mathbf{Z}(G/G_{\mathbf{k}}))]{}H_1(G,\mathbf{Z}(G/G_{\mathbf{k}})),$$

in which $\varphi_1$ is isomorphic by hypothesis. So $(*)$ is established, proving Proposition 4.3.

*Remarks.* The composition $p_x\circ\partial_2:P_2\to\mathbf{Z}G\otimes_{F^x}IF^x$ induces for every $\mathbf{k}\in\mathbf{K}$ an embedding

$$\eta_{\mathbf{k}}=\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G(p_x\circ\partial_2).$$

For $\eta_{\mathbf{k}}$ is a map between $G/G_{\mathbf{k}}$-free modules, for which $\mathbf{1}_{\mathbf{Z}}\otimes_{G/G_{\mathbf{k}}}\eta_{\mathbf{k}}$ is injective, and $G/G_{\mathbf{k}}$, being poly-(torsion-free nilpotent) lies in $\mathbf{D}(\mathbf{Z})$. The condition

"$\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G(p_x\circ\partial_2)$ is surjective" appearing in the statement of Proposition 4.3, is therefore equivalent to the condition

"$\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})} \otimes_G (p_x \circ \partial_2)$ is isomorphic".

Any *knot group* $G$, i.e. the fundamental group $\pi_1(S^3 \backslash \ell)$ of the complement of a (tame) knot $\ell$ in a 3-sphere, is an **M**-group, whose abelianization $G_{ab}$ is free cyclic (see e.g. [5, p. 83, (2.5)] and [5, p. 112, (1.2)]), and thus has a presentation of the form

$$G = \langle x_1, \cdots, x_t, y : x_1 C_1(x_m, y), \cdots, x_t C_t(x_m, y) \rangle_\pi.$$

Let $C = (c)$ denote a free cyclic group. Consider the map $\varphi: C \to G$ given by sending $c$ onto $y^\pi$. Then

$$\varphi: C/C'' \to G/G''$$

is an isomorphism if and only $G'$ is perfect, i.e. if $G' = G''$. It can be verified that the determinant of $\mathbf{1}_{\mathbf{Z}(G/G')} \otimes_G (p_x \circ \partial_2)$ is, up to a unit of $\mathbf{Z}(G/G')$, the Alexander polynomial $\Delta(t)$. So Proposition 4.3 generalizes the well-known criterion that the derived group $G'$ of a knot group $G$ is perfect if and only if the Alexander polynomial $\Delta(t)$ equals 1 (see e.g. [16, p. 46, Theorem 4.9.1]).

## 5. The Proof of Theorem C

In this section we shall prove Theorem C. Moreover, we shall discuss a similar result due to G. Baumslag [2].

5.1. THEOREM C. *Let $V$ and $W$ be non-trivial elements of the free group $\hat{F}$ on $y_1$ and $y_2$. Let $G$ denote the group*

$$\langle x, y_1, y_2, y_3 : x[V, x]([W, y_3])^\delta \rangle_\pi \quad (\delta = \pm 1),$$

*and let $\varphi: F \to G$ be the map from the free group on $y_1, y_2$ and $y_3$ into $G$, given by sending $y_n$ onto $y_n^\pi$ $(n = 1, 2, 3)$. Then the following statements hold:*

(i) *$\varphi: F/F_j \xrightarrow{\sim} G/G_j$ is isomorphic for every $j$ $(2 \leqslant j < \omega)$.*

(ii) *$\varphi: F/F_{(2,j)} \xrightarrow{\sim} G/G_{(2,j)}$ is isomorphic for every $j$ $(2 \leqslant j < \omega)$.*

(iii) *If $V$ belongs to $\hat{F}_{\mathbf{k}}$ then $\varphi: F/F_{(\mathbf{k},j)} \xrightarrow{\sim} G/G_{(\mathbf{k},j)}$ is isomorphic for every $j$ $(2 \leqslant j < \omega)$*

(iv) *If $W$ is not in $\hat{F}_2$ then $G$ is an extension of a free group by a free cyclic group. $G$ is, moreover, residually nilpotent.*

(v) *If $V$ belongs to $\hat{F}_2$ then $G$ is not free.*

*Remark.* If $a$ and $b$ are two elements of a group, we denote by $[a, b]$ the element $aba^{-1}b^{-1}$.

5.2. *Proof.* Plainly $\varphi: F/F_2 \xrightarrow{\sim} G/G_2$ is an isomorphism. The deficiency $\operatorname{def} G$ is equal to the rank of $G/G_2$ and $G/G_2$ is torsion-free. So $G$ is an **M**-group, and by

Corollary 4.2 it is an **E**-group. We then deduce from Theorem B that $\varphi$ induces, for every $\mathbf{k}\in\mathbf{K}$, an embedding

$$\varphi: F/F_{\mathbf{k}} \rightarrowtail G/G_{\mathbf{k}}.$$

In particular, $\varphi: F/F_j \to G/G_j$ is injective for every $j$ $(2\leqslant j<\omega)$. So claim (i) is established provided we can show that $\varphi: F/F_j \to G/G_j$ is onto for every $j$. This follows from the surjectivity of $\varphi: F/F_2 \to G/G_2$ and from the fact that the Frattini subgroup of a nilpotent group contains the derived group.

(ii) and (iii). The mapping $\varphi: F\to G$ is of the form considered in Proposition 4.3. So look at

$$\mathbf{1}_{\mathbf{Z}(G/G_{\mathbf{k}})}\otimes_G (p_x\circ\partial_2).$$

It is onto if, and only if, the image of $(D_x(r))^{\pi}$ under the canonical projection $p_{\mathbf{k}}: \mathbf{Z}G \twoheadrightarrow \mathbf{Z}(G/G_{\mathbf{k}})$ is a unit in $\mathbf{Z}(G/G_{\mathbf{k}})$, i.e. an element of the multiplicative group of $\mathbf{Z}(G/G_{\mathbf{k}})$. The partial derivative $D_x(r)$ of the relator with respect to $x$ may be computed as follows:

$$\begin{aligned} D_x(x[V,x]\cdot[W,y_3]^{\delta}) &= 1+xV-x\cdot VxV^{-1}x^{-1}\\ &= 1+xV-x\cdot[V,x]. \end{aligned}$$

(Remember $[a,b]=aba^{-1}b^{-1}$.) Clearly $x^{\pi}$ belongs to $G_2$. So $((D_x(r))^{\pi})^{p_2}=(V^{\pi})^{p_2}$ is a unit of $\mathbf{Z}(G/G_2)$. This proves (ii).

On the other hand, if $V$ lies in $\hat{F}_{\mathbf{k}}$ the element $((D_x(r))^{\pi})^{p_{\mathbf{k}}}$ is equal to 1, which proves (iii).

(iv) By hypothesis, $W$ is in $\hat{F}\backslash\hat{F}_2$. Suppose $\sigma_{y_1}(W)$, the exponent sum on $y_1$, differs from zero. We claim that the normal subgroup

$$N(x^{\pi}, y_2^{\pi}, y_3^{\pi}) \underset{i}{\lhd} G,$$

i.e. the normal closure of $x^{\pi}, y_2^{\pi}, y_3^{\pi}$ in $G$, is free. To see this, present $N(x^{\pi}, y_2^{\pi}, y_3^{\pi})$ by the method of Reidemeister-Schreier (cf. [14, pp. 253–258, Case 2 or Case 3]). Choose the powers $\{(y_1^{\pi})^h\}_{h\in\mathbf{Z}}$ as a transversal of $N(x^{\pi}, y_2^{\pi}, y_3^{\pi})$ in $G$. Choose the symbols

$$\begin{aligned} x_h &\rightsquigarrow y_1^h x y_1^{-h} && (h\in\mathbf{Z})\\ y_{2,h} &\rightsquigarrow y_1^h y_2 y_1^{-h} && (h\in\mathbf{Z})\\ y_{3,h} &\rightsquigarrow y_1^h y_3 y_1^{-h} && (h\in\mathbf{Z}) \end{aligned}$$

as generators for $N(x^{\pi}, y_2^{\pi}, y_3^{\pi})$. Since the powers $\{y_1^h\}_{h\in\mathbf{Z}}$ form a Schreier system, $N$ can be presented by

$$\langle x_h, y_{2,h}, y_{3,h}\ (h\in\mathbf{Z}) : x_h\cdot V_h x_{h+\sigma_V} V_h^{-1} x_h^{-1}\cdot(y_{3,h} W_h y_{3,h+\sigma_W}^{-1} W_h^{-1})^{\delta}\rangle_{\pi},$$

where $V_h$ and $W_h$ ($h \in \mathbf{Z}$) denote words in $\{y_{2,h}\}_{h\in\mathbf{Z}}$, and where $\sigma_V$ is short for $\sigma_{y_1}(V)$, $\sigma_W$ is short for $\sigma_{y_1}(W) \neq 0$. From this presentation it is patent that $N(x^\pi, y_2^\pi, y_3^\pi)$ is free on $\{x_h^\pi\}_{h\in\mathbf{Z}}$, $\{y_{2,h}^\pi\}_{h\in\mathbf{Z}}$ and $|\sigma_{y_1}(W)|$ consecutive generators chosen out of the sequence

$$\ldots, y_{3,-2}^\pi, y_{3,-1}^\pi, y_{3,0}^\pi, y_{3,+1}^\pi, y_{3,+2}^\pi, y_{3,+3}^\pi, \ldots.$$

The quotient $G/N(x^\pi, y_2^\pi, y_3^\pi)$ is generated by the image of $y_1^\pi$ and is free cyclic. So $G$ is an extension of a free group by a free cyclic group. There remains the question why $G$ is residually nilpotent.

The normal subgroup $N(x^\pi, y_2^\pi, y_3^\pi)$, being free, is residually nilpotent and contains $G_2$. Therefore

$$\bigcap_{j<\omega} G_{(2,j)} = e.$$

By (ii) the groups $G/G_{(2,j)}$ are free nilpotent by abelian. Such groups are known to be residually nilpotent ([7, p. 52, Theorem 6.3 or Theorem 7.1], cf. [15, p. 76, 26.33]). Consequently, $G$ is residually (residually nilpotent), i.e. residually nilpotent.

Claim (v) is proved by applying the algorithm due to J. H. C. Whitehead [25] whereby one can decide whether a one-relator group is free.

5.3. We conclude Section 5 with a word on the model for Theorem C, namely on Theorem 2.1 in Baumslag's second paper on parafree groups [2, p. 512].

THEOREM D (G. Baumslag [2]). *Let $m$ and $n$ be integers, both different from zero. Let $H$ denote the group*

$$\langle x, y_1, y_2 : x[y_1^m, x][y_1^n, y_2]\rangle_\pi,$$

*and let $\varphi: F \to H$ be the map from the free group on $y_1$ and $y_2$ into $H$ given by sending $y_h$ onto $y_h^\pi$ ($h=1, 2$). Then the following statements hold:*

(i) *$\varphi: F/F_j \simeq H/H_j$ is isomorphic for every $j$ ($2 \leq j < \omega$).*

(ii) *$\varphi: F/[F, F''] \simeq H/[H, H'']$ is isomorphic.*

(iv) *$H$ is an extension of a free group by a free cyclic group. It is, moreover, residually nilpotent.*

(v) *$H$ is not free.*

The statements (i), (iv) and (v) can be proved by arguments, analogous to those used in the proofs of the statements (i), (iv) and (v) of Theorem C. Claim (ii) does not follow solely from the methods of this paper but needs, in addition, a result of U. Stammbach [18]. In detail:

Consider the partial derivative $D_x(r)$ of the relator with respect to $x$. It reads:

$$D_x(x[y_1^m, x][y_1^n, y_2]) = 1 + xy_1^m - x[y_1^m, x].$$

Because $x^\pi$ is in $H_2$, the image $((D_x(r))^\pi)^{p_2}$ of $(D_x(r))^\pi$ in $\mathbf{Z}(H/H_2)$ equals $((y_1^m)^\pi)^{p_2}$ which is a unit of $\mathbf{Z}(H/H_2)$. By Proposition 4.3, the map

$$\varphi: F/F_{(2,j)} \cong H/H_{(2,j)}$$

is therefore isomorphic for every $j$ $(2 \leqslant j < \omega)$. Consider $j=2$. From the properties

(1) $\varphi: F/F_2 \cong H/H_2$

(2) $H_2(H, \mathbf{Z})=0$

(3) $\varphi: F/F_{(2,2)} \cong H/H_{(2,2)}$

a result of Stammbach [18, p. 166, Satz] allows to infer that

$$\varphi: F/[\ldots[[F'', \underbrace{F], F], \ldots F}_{j-1}] \to H/[\ldots[[H'', \underbrace{H], H], \ldots H}_{j-1}]$$

is isomorphic for every $j$ $(2 \leqslant j < \omega)$. This proves (ii) in Theorem D.

We add a remark. In $H/H_3$ the image $\bar{x}=(x^\pi)^{p_3}$ of $x^\pi$ is equal to $[\bar{y}_1^n, \bar{y}_2]^{-1} = [\bar{y}_1, \bar{y}_2]^{-n}$. The image $\overline{D_x(r)}$ of $(D_x(r))^\pi$ in $\mathbf{Z}(H/H_3)$ can therefore be written as

$$\overline{D_x(r)} = 1 + [\bar{y}_1, \bar{y}_2]^{-n} \cdot \bar{y}_1^m - [\bar{y}_1, \bar{y}_2]^{-n}.$$

As $H/H_3$ is free nilpotent it is indicable in the sense of G. Higman ([9, p. 241ff.], cf. [8, pp. 61–62, §4.5]). It follows that all units of $\mathbf{Z}(H/H_3)$ are of the form $\pm \bar{h}$ $(\bar{h} \in H/H_3)$. So $\overline{D_x(r)}$ is not a unit in $\mathbf{Z}(H/H_3)$. By Proposition 4.3 this implies that

$$\varphi: F/F_{(3,2)} \to H/H_{(3,2)}$$

is injective but not onto.

I do, however, not know whether $H/H_{(3,2)}$ is free (abelian by nilpotent-of-class-two), or not.

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