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Functions of Bounded mean Oscillation and Quasiconformal Mappings

H. M. REIMANN

1. Introduction

A locally integrable real valued function u is said to be of bounded mean oscillation (BMO) in \mathbf{R}^n , if

$$\frac{1}{|Q|} \int_Q \left| u(x) - \frac{1}{|Q|} \int_Q u(x) dx \right| dx \leq K$$

for every cube $Q \subset \mathbf{R}^n$ and some constant K . The notations

$$u_Q = \frac{1}{|Q|} \int_Q u(x) dx \quad \text{with} \quad |Q| = \int_Q dx$$

will be used. On the space of BMO-functions modulo constants a norm can be defined by

$$\|u\|_* = \sup_{Q \subset \mathbf{R}^n} \int_Q |u(x) - u_Q| dx. \quad (1.1)$$

With this norm \mathbf{BMO}/\mathbf{R} is a Banach space. The space of BMO-functions was introduced by John and Nirenberg [8]. We will make use of their fundamental lemma:

LEMMA 1. *Assume that $u \in \mathbf{BMO}$. Then, if $\mu(\sigma) = |\{x \in Q : |u(x) - u_Q| > \sigma\}|$ is the measure of the set of points in the cube Q where $|u(x) - u_Q| > \sigma$, we have*

$$\mu(\sigma) \leq a e^{-b\sigma/\|u\|_*} |Q|, \quad (1.2)$$

where a and b are constants depending on n only.

BMO-functions have been used in many different contexts, first in a paper of John on rotation and strain [7] and at the same time by Moser [9] in his work on the continuity of solutions of elliptic differential equations. Later on applications arose in connection with singular integral operators (Stein [12]) and as spaces of interpolation (Stampacchia [11], Stein and Zygmund [13]). Most recently, Fefferman and Stein [3] characterized the space of BMO-functions as the dual of the Hardy space H^1 .

It seems that BMO-functions also have their place in the theory of quasiconformal mappings. We propose to show that the logarithm of the Jacobian determinant of a

quasiconformal mapping $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is in BMO. We then proceed to study transformations of BMO-functions by quasiconformal mappings. It turns out that a quasiconformal mapping of \mathbf{R}^n onto itself induces a continuous bijective isomorphism $\varphi: u \rightarrow u \circ f^{-1}$ of BMO/R. Moreover this situation is in a certain way typical for quasiconformal mappings: If $\varphi: u \rightarrow u \circ f^{-1}$ is a continuous bijective isomorphism of BMO/R which is induced by a homeomorphism f of \mathbf{R}^n satisfying certain regularity conditions, then f is quasiconformal.

2. The Jacobian of a Quasiconformal Mapping

For our considerations we adopt the so called analytic definition of quasiconformality. A function $f: G \rightarrow \mathbf{R}^n$ defined in a domain $G \subset \mathbf{R}^n$ is said to be absolutely continuous on lines (ACL), if it is continuous and if for each interval $I = \{x \in \mathbf{R}^n : a_i \leq x_i \leq b_i\} \subset G$ f is absolutely continuous on almost all line segments in I , parallel to the coordinate axes. The partial derivatives of an ACL-function f exist a.e. and the Jacobian matrix of f at x will be denoted by $F(x)$, its determinant by $J_f(x)$. By definition a K -quasiconformal mapping is a homeomorphism $f: G \rightarrow \mathbf{R}^n$ such that $f \in \text{ACL}$, f is totally differentiable a.e. and

$$\sup_{\xi \in \mathbf{R}^n, |\xi| = 1} |F(x) \cdot \xi|^n \leq K J_f(x) \quad \text{a.e.} \quad (2.1)$$

According to a theorem of Väisälä [14], in this definition the regularity conditions $f \in \text{ACL}$ and f differentiable a.e. can be replaced by the single hypothesis, that f has generalized derivatives which are locally L^n -integrable.

THEOREM 1. *If f is a quasiconformal mapping of \mathbf{R}^n onto itself with Jacobian determinant J_f , then $\log J_f \in \text{BMO}$.*

The proof of Theorem 1 is based on a converse to the lemma of John and Nirenberg (Lemma 3) and on the following result due to Gehring [5]:

LEMMA 2. *Assume that f is a K -quasiconformal mapping of G onto $G' \subset \mathbf{R}^n$ and that Q is a cube in the domain G with*

$$\text{dia } Q' < \text{dist}(Q', \partial G') \quad (2.2)$$

$(Q' = fQ)$. Then there exist constants c and p , $p > n$, which depend on K and n only, such that

$$\left(\int_Q J_f^{p/n} dx \right)^{n/p} \leq c \int_Q J_f dx. \quad (2.3)$$

We set

$$L_f(x) = \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|}.$$

Since f is K -quasiconformal, inequality (2.1) and the total differentiability imply $L_f^n \leq KJ_f$ a.e. and $J_f \leq L_f^n$ a.e. Hence according to Lemma 4 in [3] there exists a constant c_0 (depending on K and n) such that for every cube $Q \subset G$ with $\text{dia } Q' < \text{dist}(Q', \partial G')$

$$\int_Q J_f dx \leq c_0 \left(\int_Q J_f^{1/n} dx \right)^n, \quad (2.4)$$

If Q is such a cube (satisfying (2.3)), then (2.4) holds for any cube contained in Q and Lemma 3 in [5] shows that for some constants c and p , $p > n$,

$$\left(\int_Q J_f^{p/n} dx \right)^{n/p} \leq c \int_Q J_f dx.$$

For later reference let us note a simple consequence of this lemma (cf. [5] Theorem 2)

COROLLARY. *Assume that f is a K -quasiconformal mapping of G onto $G' \subset \mathbf{R}^n$ and that Q is a cube in the domain G with $\text{dia } Q' < \text{dist}(Q', \partial G')$. Then*

$$\frac{|A'|}{|Q'|} \leq c \left(\frac{|A|}{|Q|} \right)^{(p-n)/p} \quad (2.5)$$

for every measurable set $A \subset Q$ with image $A' = fA$. (As above $|A|$ stands for the n -dimensional measure of the set A .)

If A is a measurable set, $A \subset Q$, then

$$\begin{aligned} \frac{|A'|}{|A|} &= \int_A J_f dx \leq \left(\int_A J_f^{p/n} dx \right)^{n/p} \\ &\leq \left(\frac{|A|}{|Q|} \right)^{-n/p} \left(\int_Q J_f^{p/n} dx \right)^{n/p} \leq c \left(\frac{|A|}{|Q|} \right)^{-n/p} \frac{|Q'|}{|Q|} \end{aligned}$$

by Lemma 2 and Hölder's inequality.

Let us remark that for the case of plane quasiconformal mappings results similar to Lemma 2 have previously been established by Bojarski [1] and by Gehring and Reich [6].

LEMMA 3. $f = \log u \in \text{BMO}$ if and only if for all cubes $Q \subset \mathbf{R}^n$

$$\left(\int_Q u^a dx \right)^{1/a} \leq k \left(\int_Q u^{-b} dx \right)^{-1/b} \quad (2.6)$$

for some positive constants a, b and k .

The fact, that inequality (2.6) is a consequence of Lemma 1 has already been pointed out by John and Nirenberg. (The result has been stated in this form by Moser [9].) Let us therefore assume that (2.6) holds for $u = e^f$. It is well known that

$$M_t = M_t(u) = \begin{cases} \left(\int_Q u^t dx \right)^{1/t} & t \neq 0 \\ \exp \int_Q \log u dx & t = 0 \end{cases}$$

is a monotone increasing function of t . Our assumption therefore implies $M_s \leq KM_0$ and $M_0 \leq KM_{-s}$ for $s = \min(a, b)$. If we set

$$Q_1 = \left\{ x \in Q : \log u(x) \geq \int_Q \log u dx \right\}$$

and $Q_2 = Q \setminus Q_1$ we obtain the inequalities

$$|Q|^{-1} \int_{Q_1} u^s dx \leq \int_Q u^s dx = M_s^s \leq k^s M_0^s$$

and

$$|Q|^{-1} \int_{Q_2} u^{-s} dx \leq k^{-s} M_0^{-s}.$$

After adding these two inequalities and inserting the expressions

$$f = \log u \quad \text{and} \quad f_Q = \int_Q f dx = \log M_0$$

we have

$$\int_{Q_1} e^{s(f-f_Q)} dx + \int_{Q_2} e^{-s(f-f_Q)} dx = \int_Q e^{s|f-f_Q|} dx \leq (k^s + k^{-s}) |Q|.$$

Finally,

$$\exp \int_Q s|f-f_Q| dx \leq \int_Q e^{s|f-f_Q|} dx$$

upon applying Jensen's inequality. This shows that $f \in \text{BMO}$ with

$$\|f\|_* \leq s^{-1} \log (k^s + k^{-s}). \quad (2.7)$$

We shall also need a distortion lemma for quasiconformal mappings, to the effect that the image of a cube can still be compared with a cube. This kind of result is typical for the geometric theory of quasiconformal mappings. We choose a formulation, which is particularly suited for our purposes.

LEMMA 4. *Let $f: G \rightarrow \mathbf{R}^n$ be a K -quasiconformal mapping. There exists a constant k (which depends on K and n) such that to every cube $P' \subset G' = fG$ with*

$$\text{dist}(P', \partial G') > 2k \text{ dia } P'$$

there exists a cube $Q \subset G$ with $fQ = Q' \supset P'$, $\text{dia } Q' < \text{dist}(Q', \partial G')$ and

$$|Q'| \leq k^n n^{n/2} |P'|. \quad (2.8)$$

The proof is based on the geometric definition of quasiconformality, according to which a homeomorphism $f: G \rightarrow \mathbf{R}^n$ is K -quasiconformal if and only if

$$\text{mod } R' \leq K^{1/(n-1)} \text{ mod } R$$

for every ring $R \subset G$. We refer the reader to the literature (see e.g. [10], [2]) for the precise definitions and for a proof of the equivalence of the analytic and geometric definitions of quasiconformality.

Using preliminary translations, we can assume that the given cube P' is centered at 0 and that $f(0) = 0$. Consider the spherical ring $R' \subset G'$ with complementary components

$$C'_0 = \{z: |z| \leq r' = 2^{-1} \text{ dia } P'\} \quad \text{and} \quad C'_1 = \{z: |z| \geq s' = k2^{-1} \text{ dia } P'\}.$$

The constant $k > 1$ will be determined later on. The modulus of the ring R' is given by

$$\text{mod } R' = \log \frac{s'}{r'} = \log k. \quad (2.9)$$

We set $s = \inf_{z \in C'_1} |f^{-1}(z)|$ and $r = \sup_{z \in C'_0} |f^{-1}(z)|$ and observe that $|f^{-1}(z)| \leq r$ for all $z \in P'$.

The inner complementary component C_0 of $R = f^{-1}R'$ contains 0 and a point x_0 with $|x_0| = r$, the outer component C_1 a continuum connecting ∞ with a point x_1 , $|x_1| = s$. According to a theorem of Teichmüller and its space analogue (see [4], [2], [10])

$$\text{mod } R \leq \log \left(\lambda^2 \left(\frac{s}{r} + 1 \right) \right) \quad (2.10)$$

for some constant λ which depends on n only. Since K -quasiconformal mappings satisfy

$$\text{mod } R' \leq K_0 \text{ mod } R$$

with $K_0 = K^{1/(n-1)}$, we then have by (2.9) and (2.10)

$$\log k \leq K_0 \log \left(\lambda^2 \left(\frac{s}{r} + 1 \right) \right)$$

which is equivalent to

$$k^{-K_0} \leq \lambda^2 \left(\frac{s}{r} + 1 \right).$$

This shows that $s/r \geq n^{1/2}$ if we choose $k = (\lambda^2(1+n^{1/2}))^{K_0}$. In this situation any cube $Q \subset G$ centered at the origin with side length $2r$ (and diameter $2r n^{1/2}$) satisfies the requirements of the lemma: The construction shows that $\text{dia } Q' < k \text{ dia } P'$ and

$$|Q'| \leq k^n n^{n/2} |P'|$$

since $|z| \leq k 2^{-1} \text{ dia } P'$ for all $z \in Q'$. If we further assume that $\text{dist}(P', \partial G') > 2k \text{ dia } P' = 4s'$, then it is clear that $\text{dia } Q' \leq 2s' \leq \text{dist}(Q', \partial G')$.

LEMMA 5. *The Jacobian determinant $J = J_f$ of a K -quasiconformal mapping $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies*

$$\left(\int_Q J \, dx \right) \leq c_1 \left(\int_Q J^{-b} \, dx \right)^{-1/b} \quad (2.11)$$

for all cubes $Q \subset \mathbf{R}^n$. The constants b and c_1 depend on n and K only.

The inverse f^{-1} of a K -quasiconformal mapping is K^{n-1} -quasiconformal and its determinant J^{-1} satisfies Gehring's inequality (see (2.3)):

$$\left(\int_{P'} J^{-p/n} \, dy \right)^{n/p} \leq c \int_{P'} J^{-1} \, dy \quad (2.12)$$

for every cube $P' \subset \mathbf{R}^n$. To any cube $Q \subset \mathbf{R}^n$ let us choose in accordance with Lemma 4 a cube P' with $P = f^{-1} P' \supset Q$ and

$$|P| \leq k^n n^{n/2} |Q| = k' |Q|. \quad (2.13)$$

A transformation of variables for the integrals in inequality (2.12) then shows that

$$\left(|P'|^{-1} \int_P J^{-p/n} J \, dx \right)^{n/p} \leq c |P| |P'|^{-1}.$$

Because $|P'| = \int_P J dx$ this can be rewritten in the form

$$\left(\int_P J^{(n-p)/n} dx \right)^{n/p} \leq c |P| \left(\int_P J dx \right)^{(n-p)/p}$$

and together with inequality (2.13) this leads to

$$\left(\int_Q J^{(n-p)/n} dx \right)^{n/(n-p)} \geq (ck')^{p/(n-p)} \int_Q J dx.$$

If the two Lemmata 3 and 5 are combined, a bound for $\|\log J\|_*$ can be given by

$$\|\log J\|_* \leq \frac{n}{p-n} \log ((ck')^{p/n} + (ck')^{-p/n})$$

provided that $p \leq 2n$ and by

$$\|\log J\|_* \leq \log ((ck')^{p/(p-n)} + (ck')^{p/(n-p)})$$

otherwise.

Remark 1. By definition, a K -quasiconformal mapping $f = (f_1, \dots, f_n)$ of \mathbf{R}^n onto itself satisfies for $i = 1, \dots, n$

$$K^{-n+1} J_f \leq |\operatorname{grad} f_i|^n \leq K J_f \quad \text{a.e. in } \mathbf{R}^n.$$

Hence there exist $g_i \in L^\infty(\mathbf{R}^n)$, $\|g_i\|_\infty \leq (n-1) \log K$, such that $n \log |\operatorname{grad} f_i| = \log J_f + g_i$ ($i = 1, \dots, n$). But functions in $L^\infty(\mathbf{R}^n)$ are also in BMO and therefore $\log |\operatorname{grad} f_i| \in \text{BMO}$ ($i = 1, \dots, n$).

Remark 2. Local variants of Theorem 1 can be obtained. If $f: G \rightarrow G'$ is a quasiconformal mapping and if $Q \subset G$ is a cube such that both $\operatorname{dist}(Q, \partial G)$ and $\operatorname{dist}(Q', \partial G')$ are big enough, then $\log J_f$ considered as a function in Q is in BMO.

3. The Invariance of the Space BMO

THEOREM 2. *If f is a K -quasiconformal mapping of \mathbf{R}^n onto itself, then $\varphi: u \rightarrow u' = u \circ f^{-1}$ is a bijective isomorphism of BMO and*

$$\|u'\|_* \leq C \|u\|_* \tag{3.1}$$

for all $u \in \text{BMO}$, where C is a constant depending on K and n only.

We note that since the inverse of a quasiconformal mapping is also a quasiconformal mapping, all that has to be shown is inequality (3.1). It then follows directly from the definition that φ is an isomorphism of BMO.

For the proof of Theorem 2 we assume that $u \in \text{BMO}$ and set $u' = u \circ f^{-1}$. To a given cube P' we determine Q as in Lemma 4 such that $P' \subset Q'$ and $|Q'| \leq k^n n^{n/2} |P'|$. The

set $A_\sigma = \{x \in Q : |u(x) - u_Q| > \sigma\}$ is mapped onto the set $A'_\sigma = \{z \in Q' : |u'(z) - u_Q| > \sigma\}$ and by the corollary to Lemma 2 one knows that

$$\frac{|A'_\sigma|}{|Q'|} \leq c \left(\frac{|A_\sigma|}{|Q|} \right)^{(p-n)/p}.$$

Because of Lemma 1

$$|A_\sigma| \leq a e^{-b\sigma/\|u\|_*} |Q|$$

hence

$$\frac{|A'_\sigma|}{|Q'|} \leq c a^{(p-n)/p} \exp\left(-\frac{b\sigma(p-n)}{\|u\|_* p}\right).$$

An integration of this inequality with respect to σ shows that

$$\int_{Q'} |u'(z) - u_Q| dz = |Q'|^{-1} \int_0^\infty |A'_\sigma| d\sigma \leq c a^{(p-n)/p} b^{-1} p (p-n)^{-1} \|u\|_*$$

and in combination with inequality (2.8) of Lemma 4

$$\int_{P'} |u'(z) - u_Q| dz \leq k^n n^{n/2} \int_{Q'} |u'(z) - u_Q| dz \leq \text{const} \|u\|_*$$

One is left with the task of replacing u_Q by

$$u'_{P'} = \int_{P'} u'(z) dz,$$

but

$$|u'_{P'} - u_Q| = \left| \int_{P'} (u'(z) - u_Q) dz \right|,$$

so

$$\int_{P'} |u'(z) - u'_{P'}| dz \leq |u'_{P'} - u_Q| + \int_{P'} |u'(z) - u_Q| dz \leq 2 \int_{P'} |u'(z) - u_Q| dz.$$

This shows that $\|u'\|_* \leq C \|u\|_*$ with $C = 2k^n n^{n/2} c a^{(p-n)/p} b^{-1} p (p-n)^{-1}$.

THEOREM 3. *Assume that f is a (orientation preserving) homeomorphism of \mathbf{R}^n onto itself, that $f \in \text{ACL}$ and that f is totally differentiable a.e. If the induced mapping $\varphi : u \rightarrow u' = u \circ f^{-1}$ is a bijective isomorphism of BMO and if*

$$\|u'\|_* \leq C \|u\|_* \quad \text{for all } u \in \text{BMO} \tag{3.1}$$

then f is a quasiconformal mapping.

Let us make precise that the hypothesis of φ being a bijective isomorphism of BMO is meant to include the assumption that f and its inverse are absolutely continuous with respect to n -dimensional measure. If this were not the case, the isomorphism φ could not be defined properly: if the zero set N were mapped onto a set N' of positive measure, then both $u' = 0$ and $u'' = \chi_{N'}$, the characteristic function of N' , were in BMO and both would satisfy

$$u' = u \circ f^{-1} \quad u'' = u \circ f^{-1}$$

with $u = 0$ a.e. in \mathbf{R}^n , $u \in \text{BMO}$.

Our first aim is to construct suitable functions $u \in \text{BMO}$.

LEMMA 6. (John-Nirenberg).

$$u(x) = \log^+ \frac{1}{|x|} = \begin{cases} \log \frac{1}{|x|} & |x| \leq 1 \\ 0 & |x| \geq 1 \end{cases}$$

is in BMO.

For a proof see [9].

LEMMA 7. Assume that g is a continuous function defined on \mathbf{R} with

$$k_g = \sup_{x \in \mathbf{R}} |g(x)| + \sup_{x, y \in \mathbf{R}} |g(x) - g(y)| \left(1 + \log^+ \frac{1}{|x-y|} \right) < \infty. \quad (3.2)$$

If $u \in \text{BMO}(\mathbf{R}^n)$ and if

$$k_u = \sup_{|Q| \geq 1} \left| \int_Q u(x) dx \right| < \infty,$$

then $v(x, y) = u(x)g(y) \in \text{BMO}(\mathbf{R}^{n+1})$.

Let $Q_r \subset \mathbf{R}^n$ denote the cube with side length r , centred at the origin. If $u \in \text{BMO}$, then for $r \leq 1$

$$|u_{Q_1} - u_{Q_r}| \leq (2^n + 1) \left(1 - \frac{\log r}{\log 2} \right) \|u\|_* \quad (3.3)$$

This can be seen as follows: Set $u_s = u_{Q_{2^{-s}}}$, $s = 0, 1, \dots$. Then

$$\begin{aligned} |u_s - u_{s-1}| &= 2^{ns} \int_{Q_{2^{-s}}} |u_s - u_{s-1}| dx \\ &\leq 2^{ns} \int_{Q_{2^{-s+1}}} |u(x) - u_{s-1}| dx + \|u\|_* \leq (2^n + 1) \|u\|_*, \end{aligned}$$

$$|u_s - u_{Q_1}| \leq \sum_{k=1}^s |u_k - u_{k-1}| \leq s(2^n + 1) \|u\|_*.$$

For $r = 2^{-s}$ this is equivalent with

$$|u_{Q_r} - u_{Q_1}| \leq (2^n + 1) \frac{-\log r}{\log 2} \|u\|_*.$$

For arbitrary r , $0 < r \leq 1$, inequality (3.3) can now easily be derived.

A cube $Q \subset \mathbf{R}^{n+1}$ with sides parallel to the coordinate axes can be represented as a direct product $Q = P \times S$ of cubes $P \subset \mathbf{R}^n$, $S \subset \mathbf{R}$. Set $a_Q = u_P g_0$, where g_0 is the value of g at the center of S . Then

$$\begin{aligned} & \oint_Q |v(x, y) - a_Q| dx dy \\ & \leq \int_S |g(y)| dy \int_P |u(x) - u_P| dx + |u_P| \int_S |g(y) - g_0| dy. \end{aligned}$$

If $|Q| \geq 1$ this gives immediately

$$\oint_Q |v(x, y) - a_Q| dx dy \leq k_g (\|u\|_* + k_u).$$

If $|Q| < 1$, we make use of (3.3) and (3.2) to conclude that

$$|u_P| \leq (2^n + 1) \left(1 - \frac{\log |P|}{n \log 2} \right) + k_u \|u\|_*$$

and

$$\int_S |g(y) - g_0| dy \leq k_g (1 - n^{-1} \log |P|)^{-1}.$$

Therefore

$$\oint_Q |v(x, y) - a_Q| dx dy \leq 2k_g (k_u + 2^{n+1} \|u\|_*)$$

for any cube with sides parallel to the coordinate axes. If $Q \subset \mathbf{R}^{n+1}$ is an arbitrary cube, there exists a cube $Q_0 \supset Q$ with sides parallel to the coordinate axes and with

$$|Q_0| \leq (n+1)^{(n+1)/2} |Q|.$$

So the estimate

$$\int_Q |v - v_Q| \, dx \, dy \leq 2 \int_Q |v - a_{Q_0}| \, dx \, dy \leq 2(n+1)^{(n+1)/2} \int_{Q_0} |v - a_{Q_0}| \, dx \, dy$$

for the mean oscillation over Q shows that

$$\|v\|_* \leq 4(n+1)^{(n+1)/2} k_g (k_u + 2^{n+1} \|u\|_*).$$

As an application set $u(x) = \log^+ 1/|x|$, $x \in \mathbb{R}$ and let $g(y)$ be the piecewise linear, continuous odd function defined for $y \in \mathbb{R}$ by

$$g(y) = \begin{cases} 1 - |y - 1| & 0 \leq y \leq 2, \\ 0 & 2 \leq y, \\ -g(-y) & y \leq 0. \end{cases}$$

Since $|g(x) - g(y)| \leq \min\{2, |x - y|\}$, g satisfies the assumptions of Lemma 7 (with $k_g \leq 3$). From the Lemmata 6 and 7 we conclude that

$$v(x_1, x_2) = g(x_2) \log^+ \frac{1}{|x_1|}$$

is in BMO. For dimensions $n > 2$ let us define $v \in \text{BMO}$ by

$$v(x) = \log^+ 1/|x_1| \, h(x_2) \dots h(x_{n-1}) \, g(x_n)$$

with

$$h(x) = \begin{cases} 1 & |x| \leq 1, \\ 2 - |x| & 1 \leq |x| \leq 2, \\ 0 & |x| \geq 2, \end{cases}$$

v then has compact support and $v(x) = g(x_n) \log^+ 1/|x_1|$ for $|x_i| \leq 1$, $i = 1, \dots, n$. Finally, for $r > 0$ let v_r be defined by

$$v_r(x) = v\left(\frac{x}{r}\right).$$

Certainly $\|v_r\|_* = \|v\|_*$, since the space of BMO-functions is invariant under dilations.

With $a = (a_1, \dots, a_n) \in \mathbb{R}_+^n$ (i.e. $a_i > 0$, $i = 1, \dots, n$) we associate the sets $U_{a,r} = \{x : |x_i| \leq a_i r, i = 1, \dots, n\}$ and the functions

$$\varphi_{a,r} = \begin{cases} |U_{a,r}|^{-1} v_r(x) & x \in U_{a,r} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi_{a,r} = |\varphi_{a,r}|.$$

LEMMA 8. If $h \in L^1(\mathbf{R}^n)$, then there exists a sequence r_j converging to 0 such that a.e. in \mathbf{R}^n

$$\lim_{j \rightarrow \infty} \varphi_{a, r_j} * h(t) = \lim_{j \rightarrow \infty} \int_{\mathbf{R}^n} \varphi_{a, r_j}(x) h(t-x) dx = 0$$

and

$$\lim_{j \rightarrow \infty} \psi_{a, r_j} * h(t) = c_a h(t)$$

with

$$c_a = \int_{\mathbf{R}^n} \psi_{a, r} dx = \int_{U_{a, 1}} |v(x)| dx.$$

c_a is a continuous function of $a \in \mathbf{R}_+^n$. For $a = (\alpha_1, \dots, \alpha_{n-1}, 1)$, $\alpha_i \leq 1$, c_a can easily be calculated:

$$c_a = \frac{1}{2}(1 - \log \alpha_1). \quad (3.4)$$

For the proof of Lemma 8 consider the differences

$$d_1(t) = \varphi_{a, r} * h(t) - 0 = \int_{\mathbf{R}^n} \varphi_{a, r}(x) (h(t-x) - h(t)) dx$$

and

$$d_2(t) = \psi_{a, r} * h(t) - c_a h(t) = \int_{\mathbf{R}^n} \psi_{a, r}(x, (h(t-x) - h(t))) dx.$$

They satisfy

$$\lim_{r \rightarrow 0} \int_{\mathbf{R}^n} |d_k(t)| dt \leq \lim_{r \rightarrow 0} \int_{\mathbf{R}^n} \psi_{a, r}(x) \int_{\mathbf{R}^n} |h(t-x) - h(t)| dt dx = 0.$$

$k = 1, 2$ because

$$\lim_{x \rightarrow 0} \int_{\mathbf{R}^n} |h(t-x) - h(t)| dt = 0$$

and $\psi_{a, r}$ has its support in $U_{a, r}$. For some sequence r_j with $\lim_{j \rightarrow \infty} r_j = 0$ the differences d_1 and d_2 will therefore converge pointwise a.e.

For any rotation ϱ of \mathbf{R}^n set $\varphi_{\varrho, a, r}(x) = \varphi_{a, r}(\varrho^{-1}x)$, $U_{\varrho, a, r} = \varrho^{-1}(U_{a, r})$ and $v_{\varrho, r}(x) = v_r(\varrho^{-1}x) = v(\varrho^{-1}x/r)$. Observe that $\|v_{\varrho, r}\|_* = \|v\|_*$, since BMO is invariant under rotations. We think of ϱ as being given by an element in $O(n)$, the group of orthogonal $n \times n$ -matrices. As an immediate consequence of Lemma 8 let us note:

COROLLARY. *Let $\{\varrho_i\}$ and $\{a_m\}$ be countable dense subsets of $O(n)$ and \mathbf{R}_+^n respectively. If $h \in L^1(\mathbf{R}^n)$, then for any pair ϱ_i, a_m there exists a sequence $\{r_j\}$ with $\lim_{j \rightarrow \infty} r_j = 0$ such that*

$$\lim_{j \rightarrow \infty} \varphi_{\varrho_i, a_m, r_j} * h(t) = 0 \quad (3.5)$$

and

$$\lim_{j \rightarrow \infty} \psi_{\varrho_i, a_m, r_j} * h(t) = c_{a_m} h(t) \quad (3.6)$$

for all $t \in \mathbf{R}^n \setminus N$, where N is a set of measure zero which is independent of the pair ϱ_i, a_m .

The proof of Theorem 3 consists in showing that $\sup_{\xi \in \mathbf{R}^n, |\xi|=1} |F(x) \cdot \xi|^n \leq K \det F(x)$ a.e. in \mathbf{R}^n . ($F(x)$ is the Jacobian matrix of f at x .) This clearly is a local property. Based on our hypothesis and on the corollary to Lemma 8 we can assume that at $x=0$ the following conditions are satisfied:

- i) f is (totally) differentiable
- ii) $J_f = \det F \neq 0$, in view of the remark following Theorem 3
- iii) (3.5) and (3.6) hold for $h(x) = \begin{cases} J_f(x) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$

F can be written in the form $F = \varrho D \sigma$, where $\varrho, \sigma \in O(n)$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{pmatrix}$ is a diagonal matrix with $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n > 0$ (this is true for any $n \times n$ -matrix M with $\det M \neq 0$). Let us exclude the case $\det \sigma = -1$ by possibly interchanging the order of the coordinates. If we compose f with the rotation σ^{-1} , then the resulting mapping $g = \sigma^{-1} \circ f$ still satisfies the assumptions of Theorem 3 and the three additional conditions above. Note that $J_g = J_f$ and that the Jacobian matrix of g at 0 is $G = \varrho D$. The same is true if we consider the mapping cf , $c > 0$, instead of f (with $J_{cf} = c^n J_f$). Therefore we can assume without loss of generality, that $F = \varrho D$ with $D = \begin{pmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{pmatrix}$ and

$\lambda_1 \geq \lambda_2 \dots \geq \lambda_n > 0$. We then have to show that $\lambda_1^n \leq K \lambda_1 \dots \lambda_n$.

With this in mind let us choose ϱ_i and $a_m = (\alpha_{m1}, \dots, \alpha_{mn})$ in such a way that

$$|\lambda_k \alpha_{mk} - 1| < \varepsilon \quad k = 1, \dots, n \quad (3.7)$$

$$|c_{a_m} - \frac{1}{2}(1 - \log \alpha_{m1})| < \varepsilon \quad (3.8)$$

(cf. (3.4)) and such that for r small enough, say $r < \delta_1$, $U'_r = f U_{\varrho_i, a_m, r}$ contains the cube $S = \{z : |z_i| \leq r(1 - \varepsilon)\}$ and is contained in the cube $Q = \{z : |z_i| \leq r(1 + \varepsilon)\}$. We remind that $U_{\varrho, a, r}$ was defined by $U_{\varrho, a, r} = \varrho^{-1} \{x : |x_i| \leq r \alpha_i, i = 1, \dots, n\}$.

By the main hypothesis (3.1) $w_r = v_{\varrho, r} \circ f^{-1}$ is in BMO and $\|w_r\|_* \leq C \|v\|_*$. So with

$$\begin{aligned} w_Q &= \int_Q w_r(z) dz \\ \int_{U'_r} |w_r(z) - w_Q| dz &\leq |S|^{-1} \int_Q |w_r(z) - w_Q| dz \\ &\leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n \int_Q |w_r(z) - w_Q| dz \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C \|v\|_*. \end{aligned}$$

In this inequality we can replace w_Q by the mean value

$$\tilde{v}_r = \int_{U'_r} w_r(z) dz$$

if we instead write

$$\int_{U'_r} |w_r(z) - \tilde{v}_r| dz \leq 2 \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C \|v\|_*.$$

Due to the absolute continuity of f with respect to n -dimensional Lebesgue measure

$$\begin{aligned} \tilde{v}_r &= \frac{|U_{\varrho_i, a_m, r}|}{|U'_r|} \int_{U_{\varrho_i, a_m, r}} v_{\varrho_i, r}(x) J_f(x) dx \\ &= \frac{|U_{\varrho_i, a_m, r}|}{|U'_r|} \varphi_{\varrho_i, a_m, r} * h(0) \end{aligned}$$

so by the corollary to Lemma 6

$$\lim_{j \rightarrow \infty} \tilde{v}_{r_j} = 0.$$

Hence there exists $\delta_2 > 0$ ($\delta_2 \leq \delta_1$) such that for $r = r_j < \delta_2$

$$\int_{U'_r} |w_r(z)| dz \leq \varepsilon + 2 \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n C \|v\|_*. \quad (3.9)$$

On the other side

$$\int_{U'_r} |w_r(z)| dz = \frac{|U_{\varrho_i, a_m, r}|}{|U'_r|} \psi_{\varrho_i, a_m, r} * h(0)$$

so by (3.6)

$$\lim_{j \rightarrow \infty} \int_{U'_{r_j}} |w_{r_j}(z)| dz = (J_f(0))^{-1} c_{a_m} J_f(0) = c_{a_m}. \quad (3.10)$$

Combining the two results (3.9) and (3.10) with (3.8) we conclude that for j big enough

$$\frac{1}{2}(1 - \log \alpha_{m1}) - \varepsilon \leq \int_{U'_{r_j}} |w_{r_j}(z)| dz \leq \varepsilon + 2 \left(\frac{1+\varepsilon}{1-\varepsilon} \right)^n C \|v\|_*.$$

Together with (3.7) this implies

$$1 + \log \lambda_1 \leq 4 C \|v\|_*$$

since $\varepsilon > 0$ was arbitrary. The inequality $\lambda_1^n \leq K \lambda_1 \dots \lambda_n$ therefore holds with $K = e^{(n-1)(4C\|v\|_* - 1)}$.

4. A Local Version

The object of this section is a local version of the Theorems 2 and 3. If G is a domain in \mathbf{R}^n we denote by $\text{BMO}(G)$ the subspace of BMO consisting of all functions $u \in \text{BMO}$ with support in G (the support of a locally integrable function $u \in L^1_{\text{loc}}(\mathbf{R}^n)$ is the complement of the largest open set $O \subset \mathbf{R}^n$ with $u(x) = 0$ a.e. on O).

LEMMA 9. *If $u \in L^1_{\text{loc}}(\mathbf{R}^n)$ and if $\text{supp } u \subset G$, then*

$$\|u\|_* \leq 4 \sup_P \int_P |u(x) - u_P| dx,$$

where the supremum is extended over all cubes P with $\text{dia } P \leq 4n^{1/2} \text{ dia } G$.

G is contained in a ball B with radius $\text{dia } G$. If $Q \cap G \neq \emptyset$ for some cube with $\text{dia } Q \geq 4n^{1/2} \text{ dia } G$, then there exists a cube $P \subset Q$ with side length $n^{-1/2} \text{ dia } P = 2 \text{ dia } B$ such that $P \cap B = Q \cap B \supset Q \cap G$. With $N = \{x \in P : u(x) = 0\}$, $|N| \geq |P| - |G| \geq (1 - 2^{-n})|P|$, we obtain

$$\int_P |u - u_P| dx = |P|^{-1} \int_N |u_P| dx + |P|^{-1} \int_{P \setminus N} |u - u_P| dx$$

which shows that

$$|u_P| \leq (1 - 2^{-n})^{-1} \int_P |u - u_P| dx.$$

Finally

$$\begin{aligned}
 \int_Q |u - u_Q| dx &\leq 2 \int_Q |u - u_P| dx \\
 &\leq 2|Q|^{-1} \int_{Q \setminus P} |u_P| dx + 2|Q|^{-1} \int_P |u - u_P| dx \\
 &\leq 2(1 - 2^{-n})^{-1} \int_P |u - u_P| dx
 \end{aligned}$$

and the proof is complete.

THEOREM 4. *Assume that $f: G \rightarrow \mathbf{R}^n$ is a (orientation preserving) homeomorphism, $f \in \text{ACL}$, and that f is differentiable a.e. Then f is a quasiconformal mapping if and only if every point $x \in G$ has a neighbourhood U such that $\varphi: u \rightarrow u' = u \circ f^{-1}$ is an isomorphism of $\text{BMO}(U)$ onto $\text{BMO}(fU)$ which satisfies*

$$\|u'\|_* \leq C_0 \|u\|_*$$

for all $u \in \text{BMO}(U)$ with a fixed constant C_0 independent of U .

The proof for the quasiconformality of a homeomorphism $f: G \rightarrow \mathbf{R}^n$ satisfying all the above hypotheses is contained in the proof of Theorem 3. In order to show that a quasiconformal mapping gives rise to local isomorphisms of $\text{BMO}(U)$ onto $\text{BMO}(U')$, $U' = fU$, the full strength of the Lemmata 2 and 4 has to be used.

For $x \in G$ we choose a neighbourhood U in such a way that

$$4n^{1/2}(1+2k) \text{dia } U' < \text{dist}(U', \partial G'),$$

where k is the constant of Lemma 4. If P' is a cube with $\text{dia } P' \leq 4n^{1/2} \text{dia } U'$ and with $P' \cap U' \neq \emptyset$, then

$$\text{dist}(P', \partial G') \geq \text{dist}(U', \partial G') - \text{dia } P' > \text{dia } P'(1+2k) - \text{dia } P' = 2k \text{dia } P'.$$

Hence by Lemma 4 there exists a cube $Q \subset G$ with $fQ = Q' \supset P'$, $|Q'| \leq k^n n^{n/2} |P'|$ and with

$$\text{dist}(Q', \partial G') > \text{dia } Q'. \tag{4.1}$$

For a given function $u \in \text{BMO}(U)$ with $u' = u \circ f^{-1}$ we can then proceed as in the proof of Theorem 2. Condition (4.1) ensures the validity of Lemma 2. It follows (see (2.11)) that

$$\int_{P'} |u'(z) - u'_{P'}| dz \leq C \|u\|_*$$

and this inequality holds for all P' with $\text{dia } P' \leq 4n^{1/2} \text{dia } U'$. Therefore by Lemma 9

$$\|u'\|_* \leq 4C \|u\|_* = C_0 \|u\|_*.$$

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