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The Theory of Foliations of Codimension Greater than One

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In this paper I will develop a construction which translates the problems of existence and classification of foliations, up to concordance, into homotopy theory. The main result is in close analogy to the theory developed by Haefliger for foliations of open manifolds [4, 5]. The technique of proof is different, however, since the Gromov-Phillips theorem concerning maps transversal to a foliation is not available for closed manifolds.

Many interesting foliations can be constructed as corollaries of the main result. For example, every plane field of codimension greater than one is homotopic to a completely integrable C^0 plane field (Theorem 2), and a sphere S^n has a C^∞ foliation in codimension k, when $1<k\leqslant n/2$, if and only if it has a k-plane field (Corollary 2). The corresponding result for spheres when $k=1$ is known by the work of Reeb [10], for $n=3$, Lawson [6], for n of the form 2^k+3, and Durfee [2] and Tamura [11], for general n.

Section 1 gives the definition of Haefliger structures from the point of view needed later, and states the main results.

Section 2 defines the notions of a plane field transverse to, and in general position with respect to, a smooth triangulation and gives an outline which puts the proof together.

Section 3 concerns the main step of the construction: given a nice foliation on all but one of the hyperfaces of a simplex, to construct a nice foliation on the simplex.

Section 4 fills in a hole in the foliation constructed in Section 3. This is the only part of the construction which does not work for codimension 1.

Sections 5 and 6 have to do with technical aspects of plane fields transverse to a triangulation.

For an excellent recent survey giving background on foliations, see Lawson [7].

I want to express many thanks to André Haefliger for asking me all the right questions relating to this work and for his interest and encouragement. I also want to thank Blaine Lawson for discussing and helping me understand this idea when it was still in an incipient form.

1. We will think of Haefliger structures as foliated micro-bundles. More precisely, a C^r-Haefliger structure $\mathscr{H}$ of codimension k on a manifold M^n is given by

(1) a differentiable $\mathbb{R}^k$ bundle, $\nu_{\mathscr{H}}$, called the normal bundle of $\mathscr{H}$, together with a section Z, the "zero-section";

(2) a foliation $\mathscr{F}_{\mathscr{H}}$ of codimension k defined in a neighborhood U of Z in $\nu_{\mathscr{H}}$.

$\mathscr{F}_{\mathscr{H}}$ is required to be transverse to the fibers of $\nu_{\mathscr{H}}$. Two such foliations $\mathscr{F}_{\mathscr{H}}$ and $\mathscr{F}'_{\mathscr{H}}$ are considered equivalent if they agree on some smaller neighborhood of Z. Haefliger structures are the same thing as Γ_k^r-structures, where Γ_k^r is the groupoid of germs of C^r diffeomorphisms of $\mathbb{R}^k$, cf. Haefliger [3 or 5]. Γ_k^r-structures were first introduced by Haefliger; they arose in his study of codimension one analytic foliations.

Given a codimension k, C^r foliation $\mathscr{F}$ of M^n, a Haefliger structure $\mathscr{H}(\mathscr{F})$ is associated in a canonical way as follows. There is a vector bundle $= \nu_{\mathscr{F}} = \nu(\mathscr{H}(\mathscr{F}))$ over M^n consisting of vectors normal to the leaves of $\mathscr{F}$. The exponential map exp maps $\nu_{\mathscr{F}}$ into M^n; in a neighborhood U of the zero-section Z, exp restricted to a fiber of ν is transverse to $\mathscr{F}$ so a foliation of U, $\mathscr{F}_{\mathscr{H}(\mathscr{F})} = \mathscr{F}$ is induced.

The converse is not true. Given a Haefliger structure $\mathscr{H}$, it is of the form $\mathscr{H}(\mathscr{G})$ if and only if Z is transverse to $\mathscr{F}_{\mathscr{H}}$, in which case $\mathscr{G}$ is the foliation induced on Z by $\mathscr{F}_{\mathscr{H}}$. In such a case we will say the Haefliger structure $\mathscr{H}$ *is* a foliation, namely $\mathscr{G}$.

Notice that whenever a Haefliger structure $\mathscr{H}$ is a foliation, that fact gives a piece of information about bundles: that is, it gives a bundle monomorphism $i_{\mathscr{H}}: \nu_{\mathscr{H}} \to T(M^n)$.

Two Haefliger structures $\mathscr{H}_1$ and $\mathscr{H}_2$ are *homotopic* if there is a Haefliger structure $\bar{\mathscr{H}}$ on $M \times I$ such that $\bar{\mathscr{H}}$ restricts to give $\mathscr{H}_1$ and $\mathscr{H}_2$ on $M \times 0$ and $M \times 1$. Haefliger structures can easily be defined over complexes instead of manifolds.

Haefliger showed [4 or 5] that there is a classifying space $B\Gamma_k^r$ for codimension k, C^r, Haefliger structures, so that homotopy classes of codimension k, C^r, Haefliger structures on a space X correspond 1-1 with homotopy classes $[X, B\Gamma_k^r]$ of maps of X to $B\Gamma_k^r$ $[0 \leqslant r \leqslant \omega]$.

THEOREM 1. *Let M^n be a manifold, $\mathscr{H}$ a C^r Haefliger structure* $[1 \leqslant r \leqslant \infty]$ *of codimension* > 1 *over M^n, and i a bundle monomorphism*

$$i: \nu_{\mathscr{H}} \to T(M^n).$$

Then, there is a C^r foliation $\mathscr{F}$ of M^n homotopic to $\mathscr{H}$ with $i_{\mathscr{F}} \simeq i$. (Here we identify $\nu_{\mathscr{H}}$ with $\nu_{\mathscr{F}}$).

THEOREM 1, RELATIVE VERSION. *If $\mathscr{H}$ is already a foliation in a neighborhood of a closed subset K of M, and if $i = i_{\mathscr{H}}$ in this neighborhood, then $\mathscr{F}$ can be taken to agree with $\mathscr{H}$ in a somewhat smaller neighborhood of K, and the homotopies of $\mathscr{H}$ to $\mathscr{F}$ and i to $i_{\mathscr{F}}$ can be taken as homotopies* rel K.

Theorem 1 for codimension one foliations in its absolute version is unknown, and in its relative version is false.[1] The counterexamples come from the Reeb stability theorem [10].

Theorem 1 says nothing about analytic foliations. Haefliger proved [3] that simply

[1]) See note added in proof on p. 231.

connected closed manifolds do not have codimension one analytic foliations, so even the absolute version of Theorem 1 for analytic codimension one foliations is false. Little seems to be known about higher codimension analytic foliations or Haefliger structures.

COROLLARY 1. *If M^n has a k-frame field φ, $(k>1)$ (i.e., k linearly independent vector fields), then φ is homotopic to a frame field spanning the normal plane field of a C^∞ foliation $\mathscr{F}$ of M^n.*

Proof. There is a trivial Haefliger structure $\mathscr{H}$ with $v_{\mathscr{H}} \approx \mathbb{R}^k \times M^n$ where the leaves of $\mathscr{F}_{\mathscr{H}}$ are $x \times M^n$, $x \in \mathbb{R}^k$. φ determines a bundle monomorphism $i: v_{\mathscr{H}} \to T(M^n)$.

COROLLARY 2. *A sphere S^n has a foliation in codimension k, provided $k<n/2$, iff S^n has a k-plane field.*

Proof. For $k=1$ this is already known, by Reeb [10], Lawson [6], and Durfee [2] or Tamura [11].

In general, the hypothesis implies that S^n has a k-frame field, as D. Husemoller pointed out to me: Consider the clutching map for $T(S^n)$,

$$S^{n-1} \xrightarrow{f} O(n).$$

A k-plane field corresponds to liftings of f to $O(k) \times O(n-k)$, and k-frame fields correspond to liftings to $O(n-k)$:

$$\begin{tikzcd}
& O(n-k) \arrow[d] \\
& O(k) \times O(n-k) \arrow[d] \\
S^{n-1} \arrow[r, "f"] \arrow[ruu, dashed, "k\text{-frame field}"] \arrow[ru, dashed, "k\text{-plane field}"'] & O(n)
\end{tikzcd}$$

There is a k-plane field iff $[f]$ in $\pi_{n-1}(O(n))$ lies in the image of $\pi_{n-1}(O(k) \times O(n-k)) = \pi_{n-1}(O(k)) \times \pi_{n-1}(O(n-k))$ and there is a k-frame field iff $[f]$ lies in the image of $\pi_{n-1}(O(n-k))$. But since $n-k \geqslant k$, the image of $\pi_{n-1}(O(k))$, and therefore the image of $\pi_{n-1}(O(k)) \times \pi_{n-1}(O(n-k))$ is contained in the image of $\pi_{n-1}(O(n-k))$.

Corollary 2 now follows from Corollary 1.

COROLLARY 3. *Every C^∞ 2-plane field τ^2 on a manifold M^n, $n>3$, is homo-*

topic to a completely integrable C^∞ plane field. This is also true rel K, *if K is a compact set such that τ^2 is already completely integrable in a neighborhood of K.*

Proof. This follows from the theorem, announced in [12] and due to Mather [8, 9] when $k=1$, that $\pi_{(k+1)}(B\bar{\Gamma}_k^\infty)\approx 0$.

Remarks. A proof of Corollary 3 by itself was the starting point leading to Theorem 1.

John Wood [14] proved that every *transversely oriented* plane field τ^2 on a manifold M^3 is homotopic to a foliation. I now have a proof of this for a general plane field τ^2 on a 3-manifold. However, this is false in the relative version: $(S^2\times I)\# M^3$, where $M^3\neq S^3$, does not have a foliation with the boundary components leaves, by the Reeb stability theorem [10].

THEOREM 2. *Every plane field of codimension* >1, *is homotopic to a* C^0, *completely integrable plane field, giving a foliation with* C^∞ *leaves.*

This theorem follows from some still unpublished work of mine concerning Lipschitz foliations, using the method of proof of Theorem 1 (which is not stated to include C^0 foliations). The proof of Theorem 2 will not be given here.

Theorem 2 should be compared with Bott's vanishing theorem [1] which says that all polynomials in the real Pontrjagin classes of total dimensions greater than $2k$ vanish for a plane field ν^k homotopic to the normal plane field of a C^2 foliation.

In Haefliger's classifying theorem for foliations on open manifolds U^n [4 or 5], such foliations are classified, up to integrable homotopy, by homotopy classes of pairs $(\mathscr{H}, i: \nu_{\mathscr{H}}\to T(U^n))$. This is false on compact manifolds M^n, since if two foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ of M^n are integrably homotopic, then they are conjugate by a diffeomorphism isotopic to the identity. A statement about concordance classes can be made, however. Two bundle monomorphisms $i_0, i_1: \nu\to T(M^n)$ are *concordant* if there is a bundle monomorphism $I: \nu\times I\to T(M^n\times I)$ such that $I_t=i_t$ for $t=0, 1$. Two codimension k foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ are concordant if there is a codimension k foliation of $M^n\times I$, transverse to $M^n\times 0$ and $M^n\times 1$ and inducing there $\mathscr{F}_0$ and $\mathscr{F}_1$.

COROLLARY 4. *Concordance classes of foliations correspond* 1-1 *with homotopy classes of Haefliger structures $\mathscr{H}$ together with concordance classes of maps $i: \nu_{\mathscr{H}}\to T(M^n)$.*

Proof. Immediate from the relative version of Theorem 1.

Going further, one can build a space of concordances of codimension k foliations of M^n, $C_{\mathscr{F}^k}(M^n)$, by a semi-simplicial construction: namely, the 0-simplices of $C_{\mathscr{F}^k}(M^n)$ are foliations of M^n; the 1-simplices are foliations of $M^n\times I$ transverse to $M^n\times 0$ and $M^n\times 1$; and so on. The j-simplices are foliations of $M^n\times\Delta^j$ transverse to each $M^n\times\Delta^l$, where Δ^l is a face of Δ^j. The attaching maps for the simplices are obvious.

Similarly, one builds a space of concordances of k-plane fields on M^n, $C_{V^k}(M^n)$. The 0-simplices are k-plane fields on M^n; in general, the j-simplices are k-plane fields on $M^n \times \Delta^j$, tangent to each $M^n \times \Delta^l$, where Δ^l is a face of Δ^j.

There are maps

$$C_{\mathscr{F}^k}(M^n) \xrightarrow{v} C_{V^k}(M^n)$$

and

$$C_{\mathscr{F}^k}(M^n) \xrightarrow{\mathscr{H}} (B\Gamma_k)^{M^n},$$

where $B\Gamma_k$ is the classifying space for codimension k Haefliger structures, and X^{M^n} designates the space of maps of M^n to X. In fact, we have a commutative diamond:

$$\begin{tikzcd}
& C_{\mathscr{F}^k}(M^n) \arrow[dl, "v"'] \arrow[dr, "\mathscr{H}"] & \\
C_{V^k}(M^n) \arrow[dr] & & (B\Gamma_k)^{M^n} \arrow[dl] \\
& (BO_k)^{M^n} &
\end{tikzcd}$$

COROLLARY 5. *$C_{\mathscr{F}^k}(M^n)$ is the homotopy fiber product of $C_{V^k}(M^n)$ and $(B\Gamma_k)^{M^n}$ over $(BO_k)^{M^n}$.*

Proof. Given a pair of simplicial complexes (K, L), with a homotopy-commutative diagram:

$$\begin{tikzcd}
& & & C_{\mathscr{F}^k}(M^n) \arrow[ddl] \arrow[dddr] & \\
L \arrow[urrr] & & & & \\
\cap & & C_{V^k}(M^n) \arrow[ddr] & & \\
K \arrow[uuurrr, dashed] \arrow[urr] \arrow[rrrr] & & & & (B\Gamma_k)^{M^n} \arrow[dl] \\
& & & (BO_k)^{M^n} &
\end{tikzcd}$$

then, the dotted arrow can be fitted in skeleton by skeleton using Theorem 1. This is the defining property of the homotopy fiber product.

2. The Construction

I will now outline the construction specified by Theorem 1, postponing some of the steps and technicalities to later sections.

A foliation will be constructed from a Haefliger structure by a simplicial process. In order for this process to work nicely, simplex by simplex, it is necessary to have a plane field or a foliation, transverse to a smooth triangulation. A plane field τ^{n-k} is *transverse* to a triangulation α of a manifold M^n if τ^{n-k} is transverse to each simplex α_i^l of α, when $l \geqslant k$, and the subspace $\tau^{n-k} \oplus T(\alpha_i^l)$ of $T(M^n)$ along a_i^l has dimension $(n-k)+l$, when $l \leqslant k$. A foliation $\mathscr{F}$ is *transverse* to a triangulation α if its tangent plane field $T\mathscr{F}$ is transverse to α. (See Fig. 1a.)

A more refined notion is also needed: a plane field τ^{n-k} is in *general position* with respect to a triangulation α, if, for each n-simplex α_i^n of α, and for each point $x \in \alpha_i^n$, the linear projection defined by τ_x^{n-k}

$$L_x : \alpha_i^n \to \alpha_i^n / \tau_x^{n-k} \subset \mathbb{R}^k$$

takes each k-face of α_i^n to a non-degenerate k-simplex in $\mathbb{R}^k$. (See Fig. 1b.) This condi-

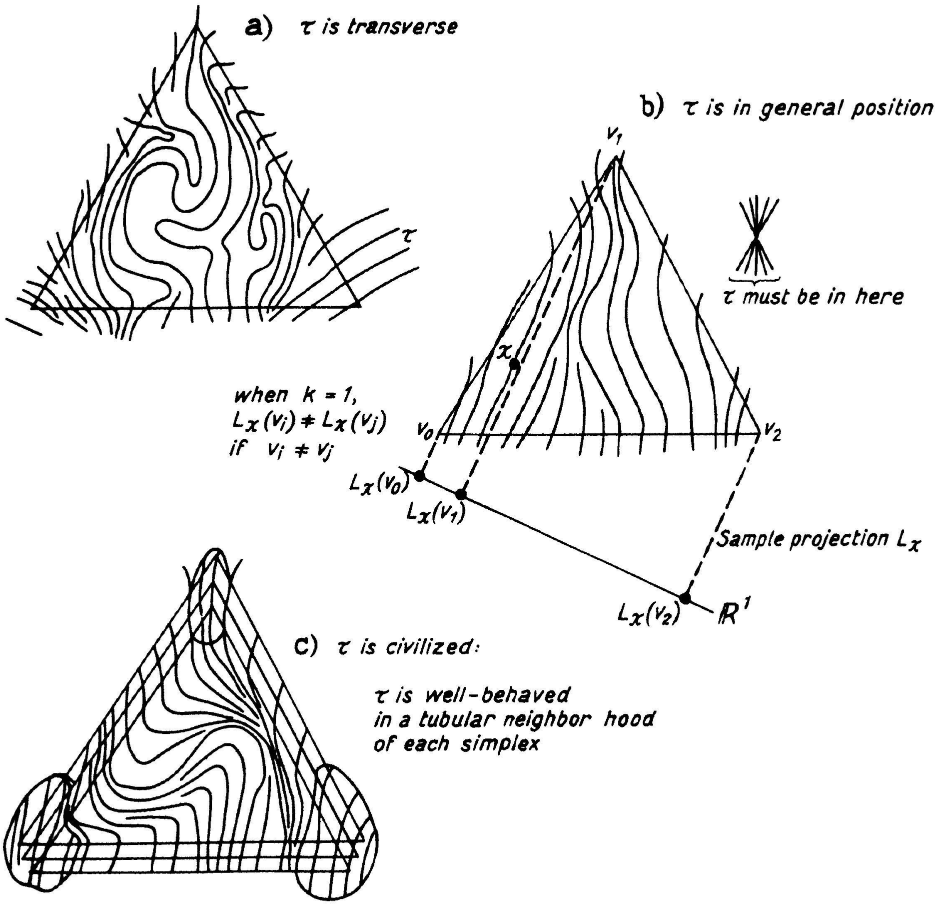


Fig. 1

tion guarantees that τ^{n-k} is transverse to α, and it also puts a restriction on the amount τ can wobble in any simplex. It guarantees that many qualitative features of the projection L_x, such as the subset of α_i^n projecting to the boundary of the image in $\mathbb{R}^k$, remain unchanged as x varies through α_i^n.

In Section 5, it will be shown that for any C^0 plane field τ and any smooth triangulation α, there is a subdivision α' of α which can be slightly jiggled to obtain a triangulation α'' in general position with τ. Some people (including me) can accept this intuitively without proof.

The setting for the construction is this. We begin with a Haefliger structure $\mathscr{H}$ over M^n, as well as a bundle monomorphism $i: v_{\mathscr{H}} \to T(M^n)$. This means that the tangent plane field $\tau^n = T(\mathscr{F}_{\mathscr{H}})$ of the associated foliation of $v_{\mathscr{H}}$ is homotopic to a plane field τ_1^n transverse both to the zero-section Z of $v_{\mathscr{H}}$, and to the fibers of $(v_{\mathscr{H}})$. Such a τ_1^n defines a projection of the tangent space of the fibers to $T(Z) \approx T(M^n)$; and the specification that this projection equals i then completely determines τ_1^n, along Z. The space of all plane fields transverse to the fibers is contractible: this gives us a homotopy τ_t^n, which we assume constant in a neighborhood of $t=0$, and constant in a neighborhood of $v_{\mathscr{H}} \mid K$ (K is where $\mathscr{H}$ is already a foliation). Form the plane field $\bar{\tau}^{n+1} = \tau_t \oplus \langle \partial/\partial t \rangle$ on $v_{\mathscr{H}} \times [0, 1]$. Then $\bar{\tau}^{n+1}$ is completely integrable in a neighborhood of $(v_{\mathscr{H}} \times 0) \cup (v_{\mathscr{H}} \mid K \times [0, 1])$.

Now choose a fine product triangulation α of $v_{\mathscr{H}} \times I$ which can be jiggled to give a triangulation β, in general position with respect to $\bar{\tau}$. $Z \times 1$ should be a subcomplex of α', and the jiggling should take $Z \times 1$ to itself, so $Z \times 1$ is a subcomplex of α''. This is made possible by the fact that $\bar{\tau}$ is transverse to $Z \times 1$. Also, each simplex of α'' intersecting $K \times [0, 1]$ should be contained in the neighborhood where $\bar{\tau}$ is completely integrable.

Let $G = v_{\mathscr{H}} \mid K \times [0, 1] \cup v_{\mathscr{H}} \times 0$. We will construct a homotopy of $\bar{\tau}$ rel G among plane fields transverse to α'', to a completely integrable plane field. Then the resulting homotopy, and foliation, will be transverse to each n-simplex of $Z \times 1$. But since $Z \times 1$ is a differentiable submanifold, this implies they are transverse to Z, and the construction will be finished.

First, we perturb $\bar{\tau}$ in a neighborhood of the k-skeleton until it is integrable in a smaller neighborhood and still remains in general position. This is possible roughly because $\bar{\tau}$ intersects a k-simplex in a 0-dimensional plane field: which is integrable. A careful construction is contained in Section 6.

From now on $\bar{\tau}$ will not be changed in some small neighborhood of the k-skeleton.

Next, as in Section 6, $\bar{\tau}$ is perturbed to be civilized: it will remain civilized from now on. (See Fig. 1c.)

The rest of the construction has to do with collapsing. The point is that the triangulation β collapses to G, since it is a product triangulation. Collapsing means removing, one by one, the interior of a simplex $\beta_{j_i}^{l_i}$ together with the interior of one hyperface $\beta_{m_i}^{l_i-1}$, with the restriction that $\beta_{m_i}^{l_i-1}$ must not be a face of any other simplex still remaining. The sequence $\{l_i\}$ of dimensions is more or less random. We reverse the collapsing procedure, and "inflate" $v_{\mathscr{H}} \times [0, 1]$ from G. A foliation $\mathscr{F}$, and a civilized homotopy of $\bar{\tau}$ to $T\mathscr{F}$, is constructed inductively as simplices of dimension greater than k are inflated. Then only one step, inflating a foliation with a simplex is needed to complete the construction. This step will be stated precisely in the next section, and it will be the subject of the next two sections.

There is one more technicality of the construction, namely, the extension of a civilized homotopy of a plane field τ on a simplex β_i^l to a civilized homotopy in a neighborhood of β_i^l. This is done in Section 6.

3. Inflation

Now it is time to analyze the main inductive step referred to in Section 2: "inflating" a foliation on a simplex.

Refer to Section 6 for the definition of "civilized". It depends on the choice of a codimension k plane field τ in general position, as well as the choice of tubular neighborhoods of simplices.

Let β^{k+l+1} be a $k+l+1$-simplex $[l \geqslant 0]$; let β_0^{k+l} be a face of it. Let λ be the union of the other $(k+l)$-faces of β^{k+l+1}.

INFLATION LEMMA. *Civilization leads to inflation. More precisely,*

(1) *Let $\mathscr{F}$ be a civilized foliation of λ. Then there is a civilized foliation $\mathscr{F}'$ of β^{k+l+1} extending $\mathscr{F}$.*

(2) *If there is a civilized homotopy $\{\tau_t\}$ of $\tau \cap \lambda$ to $T\mathscr{F}$, then $\mathscr{F}'$ is constructed so that there is a civilized homotopy $\{\tau_t'\}$ of $\tau \cap \beta^{k+l+1}$ to $T\mathscr{F}'$, extending $\{\tau_t\}$.*

Proof. In the case $l=0$ any civilized line field on β^{k+1} homotopic to τ defines such a foliation $\mathscr{F}'$. The general proof also works, if we consider the sphere S^{-1} to be the empty set.

In general, first we need to round the corners of $\partial\beta^{k+l+1}$ so we can see what's going on. In Section 6, we chose nice tubular neighborhoods $N(\beta_j^m)$ of the simplices β_j^m of each m. Then let $f: \partial\beta^{k+l+1} \to \beta^{k+l+1}$ be a homeomorphism close to inclusion such that $f(x)$ is in the normal fiber through x, for $x \in N(\beta_i^m) - \bigcup_{p<m} \bigcup_j N(\beta_j^m)$, and the image of f, ∂b^{k+l+1} is a smooth convex hypersurface.

For each $x \in \beta^{k+l+1}$ we have the linear projection $L_x: \beta^{k+l+1} \to \beta^{k+l+1}/\tau_x \subset \mathbb{R}^k$ (the inclusion in $\mathbb{R}^k$ is defined only up to affine transformations). The image of L_x is convex, and by the assumption that τ is in general position, its boundary has a natural, linear triangulation. (Since no $k+1$ vertices β_i^0 project to the same hyperplane, each point in ∂ (Image L_x) is a *unique* convex combination of the $L_x(\beta_i^0)$ lying in ∂ (Image L_x)). Then L_x^{-1} (∂ Image L_x) is a $k-1$ sphere Σ^{k-1} which is a subcomplex of β^{k+l+1}. Σ^{k-1} does not depend on x.

$L_x(b^{k+l+1})$ is almost as big as $L_x(\beta^{k+l+1})$ by assumption on f τ is transverse to ∂b^{k+l+1} everywhere except within $U_x L_x^{-1}$ (∂ Image $L_x(^{k+l+1})$) which we may assume is contained in $N(\Sigma)$. [If τ is not transverse at y, this means $T_y(\partial b^{k+l+1})$ projects degenerately by L_y; since this hyperplane lies on one side of ∂b^{k+l+1}, $L_y(y) \subset \partial L_y \times (b^{k+l+1})$].

It follows that every civilized plane field σ is transverse to ∂b^{k+l+1} except in $N(\Sigma^{k-1})$: it is transverse outside of $N(k-1\text{-skeleton})$ because f restricted to β^{k+j}, $[j \geq 0]$, preserves $\sigma \cap \beta^{k+j}$; yet σ agrees with τ in $N(k\text{-skeleton})$.

There is exactly one vertex β_0^0 of β^{k+l+1} which is not a vertex of β_0^{k+l}. Let x be an arbitrary point of β^{k+l+1}. There is one case when it is very easy to inflate a foliation on β^{k+l+1}: when $L_x(\beta_0^0)$ lies in the interior of $L_x(\beta^{k+l+1})$ so that $L_x(\beta^{k+l+1}) = L_x(\beta_0^{k+l}) = L_x(\lambda)$. (See Fig. 2a.) To handle this case, first chop out a neighborhood U of Σ, contained in $N(\Sigma)$ and containing all points where τ is not transverse to ∂b, chopping along the leaves of the foliation given by τ in $N(\Sigma)$. These leaves are all disks D^{l+1}, by the construction of N. Let $b' = b - U$: b' has corners, so differentiably $b' = D^k \times D^{l+1}$, with a projection to D^k coming from L_x. $f(\beta_0^{k+l})$ intersects each factor $D^{l+1} \times y$ in a disk D^l (at least if f was chosen nicely). Now chop out a tubular neigh-

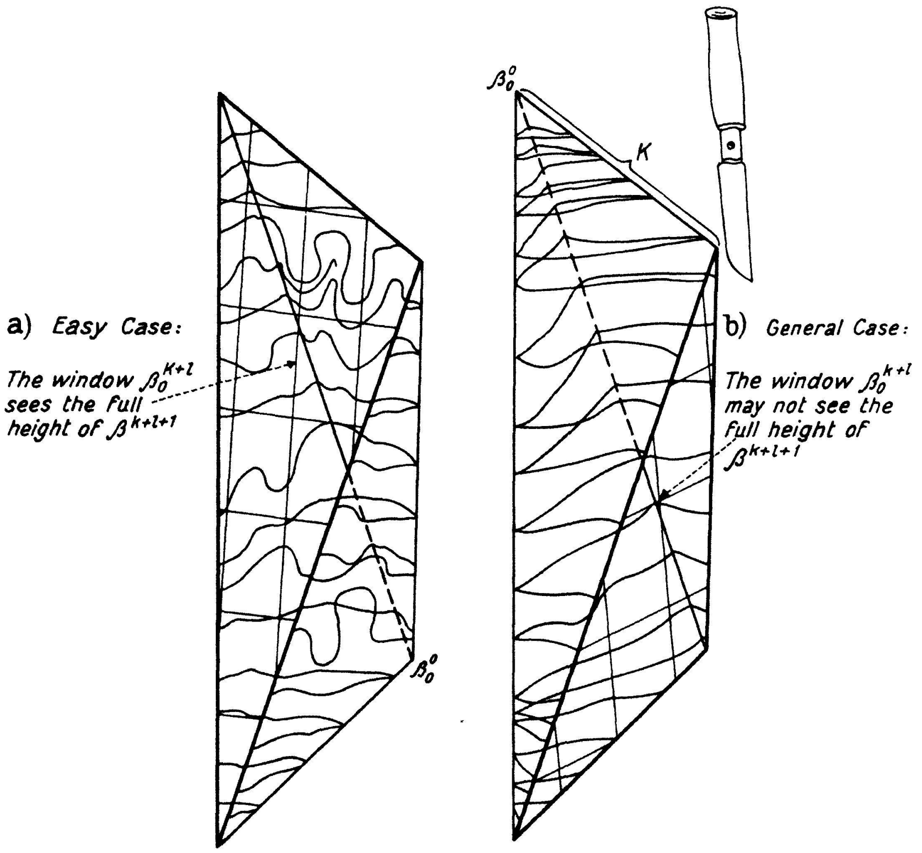


Fig. 2
A simplex β^{k+l+1} before inflation. In (b), one must cut along κ for the construction

borhood of $\partial f(\beta_0^{k+l}) = \partial f(\lambda)$ to obtain b'' with the structure $D^k \times D^l \times D^1$ where $D^k \times D^l \times 1 \subset f(\lambda)$ and $D^k \times D^l \times -1 \subset f(\beta_0^{k+l})$. Straighten out the D^1-factors so they are tangent to normal fibers in $N(\beta_j^m) - \bigcup_{p \leq m} \bigcup_l \beta_l^p$, and so that the projection $p: D^k \times D^l \times D^1 \to D^k \times D^l \times 1$ takes each $N(\beta_j^m)$ to itself for $\beta_j^m \subset \lambda$. Then we set $\mathscr{F}' = p^* \mathscr{F}$ and $\tau_t' = p^* \tau_t$ in b'; elsewhere $\mathscr{F}'$ and τ' are determined, using the tubular neighborhoods by the condition of being civilized.

Refer to Fig. 3 for the general case. The complication is that in the product structure of $b' = D^k \times D^{l+1}$, a factor $y \times D^{l+1}$ does not intersect β_0^{k+l} unless y is in the image of $L_x(\beta_0^{k+l})$. The hyperface β_0^{k+l} is like a window in β^{k+l+1} which does not extend the

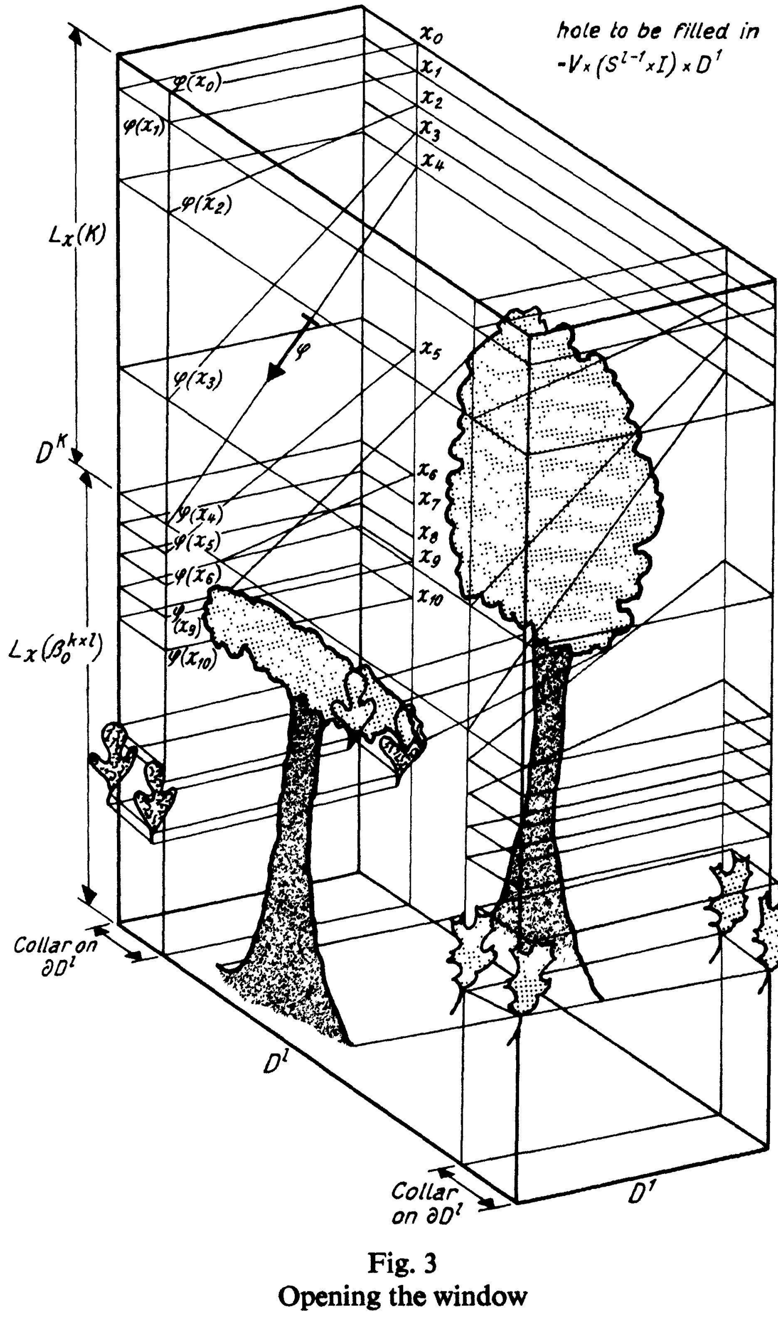


Fig. 3
Opening the window

full height (when $k=1$). The first step is to cut open λ and extend the window in this cut. To do this, we observe that the region $L_x(\beta^{k+l+1})-L_x(\beta_0^{k+l})$ has a natural triangulation, namely, the join of v_0 with the natural triangulation of the part of $\partial(L_x(\beta_0^{k+l}))$ visible from v_0. This region lifts simplicially in a unique way to β^{k+l+1}, defining there a subcomplex κ, homeomorphic with a k-disk and meeting β_0^{k+l} in a $(k-1)$-disk. Choose a tubular neighborhood $N'(\kappa)\subset N(\kappa)\cap\partial\beta^{k+l+1}$ and form the union $N'(\kappa)\cup\beta_0^{k+l}$, obtaining a new "window" γ. Unfortunately $\mathscr{F}'$ is determined in part of γ (in $N'(\kappa)$) since this is part of λ; in fact, $\mathscr{F}$ is a trivial foliation there, induced from a projection to $\mathbb{R}^k$ along the normal fibers of $N(\kappa)$. Let $\lambda'=\lambda-N'(\kappa)$. Chop out from b a neighborhood of Σ along leaves of $\mathscr{F}$, and chop out a neighborhood of $\partial\lambda'$, as before, to obtain b' with a product structure

$$b'=D^k\times D^l\times D^1$$

where $D^k\times D^l\times 1\subset f(\lambda')$ and $D^k\times D^l\times -1\subset f(\gamma)$. Again, straighten out the D^1-factors to agree with the tubular neighborhood structure. $\mathscr{F}$ is defined in a neighborhood of $f(\lambda)\cup f(N'(\kappa))\cup\partial(D^k\times D^l)\times D^1$. $\mathscr{F}$ is trivial in $f(N'(\kappa))$ and $(\partial D^k)\times D^l\times D^1$, since these parts come from the k-skeleton of β^{k+l+1}. (See Fig. 3.) In $\partial(D^k\times D^l)\times D^1$, $\mathscr{F}$ is induced from projection to $\partial(D^k\times D^l)\times 1$. Let $S^{l-1}\times I$ be a collar neighborhood of D^l. We will fill in $\mathscr{F}'$ except for a hole. If V is a small neighborhood of $L_x(\kappa)$ in $L_x(\beta^{k+l+1})$, the hole to be filled in later is $V\times(S^l\times I)\times D^1$. To fill in $\mathscr{F}'$ except for the hole, let φ_t be an isotopy of $\mathbb{R}^k$ supported in interior $(L_x\beta^{k+l+1})\cap V$ and such that φ_1 carries $L_x(\beta_0^{k+l})$ to almost all of $L_x(\beta^{k+l+1})$. Then define a foliation $\mathscr{F}''$ in $D^k\times D^l\times D^1$, where $D^k\approx L_x(\beta^{k+l+1})$, by pushing $\mathscr{F}$ along by φ_t: *i.e.*, there is a projection

$$p':D^k\times D^l\times D^1\to D^k\times D^l\times\{1\}$$

defined by

$$p'(x,y,s)=(\varphi_t(x),y,1)$$

where we reparameterize $D^1=[-1,1]$ by the parameter $t\in[0,1]$, $t(s)=1/2-s/2$. Then we fit $\mathscr{F}''$ inside b'-(collar neighborhood), stretching it out a little around its boundary so it becomes civilized. Since φ_1 squashes $L_x(\kappa)$ to a small neighborhood of $\partial L_x(\beta^{k+l+1})$, where $\mathscr{F}$ is trivial, we obtain a foliation $\mathscr{F}'$ consistent with $\mathscr{F}$ and defined everywhere except in the hole.

$\mathscr{F}'$ around the boundary of the hole, $V\times(S^{l-1})\times I)\times D^1$ does not depend on the S^{l-1} factor at all: it is induced from projection to factors $V\times t$, and the leaves of this foliation, going once around S^1, trace out the isotopy φ_t.

4. Filling the Hole

There is a hole in Section 3 of the form $V \times (S^{k-1} \times I) \times D^1$, where $V \subset \mathbb{R}^k$, with a foliation $\mathscr{F}'$ around its boundary is induced from projection to $V \times I \times D' = V \times D^2$. The foliation around $V \times \partial D^2 = V \times S$ determined by the compactly supported isotopy $\{\varphi_t\}$ of V.

To make the construction more elementary, we will choose more explicitly a $\{\varphi_t\}$ meeting the specifications of Section 3. This is exactly the place where the construction fails for codimension $k=1$.

Toward this purpose, let $D^{k-1} \times S$ be imbedded in $\mathbb{R}^k$, and let $\partial/\partial\theta$ be the unit length vector field tangent to the S^1-factors (where S^1 has length 2π). Let f be a C^∞ function on D^{k-1} which is 0 in a neighborhood of ∂D^{k-1}, 1 in the middle of D^{k-1}, and bounded between 0 and 1. Let $X = f(\partial/\partial\theta)$, and let ψ_t be the flow of X. We can take φ_t to be conjugate to ψ_t. (See Fig. 4.)

$\{\psi_t\}_{0 \leqslant t \leqslant 1}$ defines a codimension k foliation $\mathscr{I}$ of $R^k \times S^1$ which is trivial (induced from projection to $\mathbb{R}^k$) outside of $(D^{k-1} \times S^1) \times S^1$. If we extend this foliation to a codimension k foliation $\mathscr{I}'$ of $\mathbb{R}^k \times D^2$ which is trivial outside of $(D^{k-1} \times S^1) \times D^2$, this will suffice to fill in the hole (after conjugating everything so ψ_t goes to φ_t).

Since ψ_t preserves the circles $x \times S^1$ in $D^{k-1} \times S^1$, the codimension k foliation determined by ψ_t on $(D^{k-1} \times S^1) \times S^1$ is actually the union of a $(k-1)$-parameter family of codimension one foliations of the torus $S^1 \times S^1$. We will extend each foliation in this family to a codimension one foliation of $S^1 \times D^2$, in a way so that it depends differentiably on the parameters. We will meet the boundary conditions (that the foliations are trivial for parameters near ∂D^{k-1}), so the union of this family of codimension one foliations will fill in the hole.

Let $S^1 \times D^2$ be parameterized (φ, r, θ) where (r, θ) are polar coordinates for D^2; and let x be a parameter for D^{k-1}. $\mathscr{I}_x$, $\mathscr{I}'_x$, ω_x and ω'_x will designate the codimenson one foliations and integrable one forms on the slices $x \times S^1 \times 1 \times S^1$ and $x \times S^1 \times D^2$. Then for x very near ∂D^{k-1}, ω_x can be taken as $d\varphi$; in general, $\omega_x = d\varphi - f(x)\, d\theta$ is a closed form, defining the linear foliation $\mathscr{I}_x$ of $x \times S^1 \times S^1$.

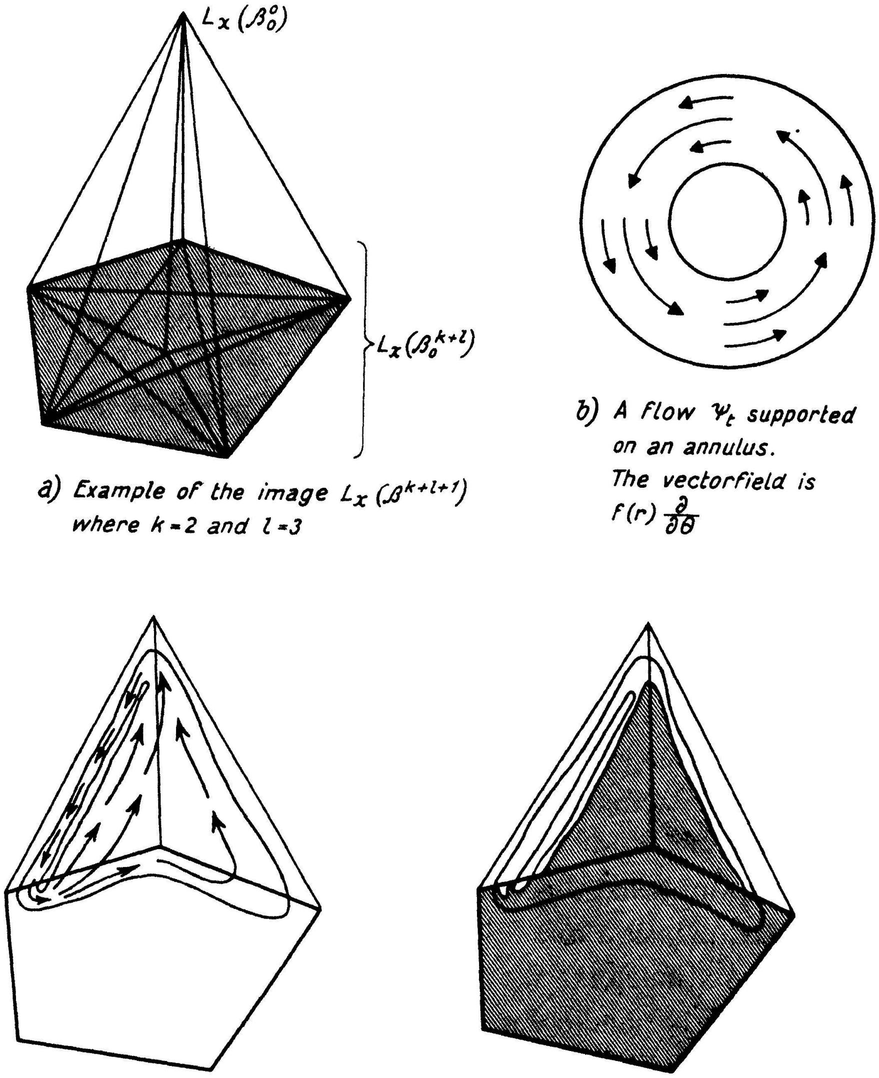
Let $g(x)$ be a C^∞ function which is 0 in a neighborhood of ∂D^{k-1} and 1 in support(f).

Let $\{\lambda_0, \lambda_{1/2}, \lambda_1\}$ be a partition of unity for the unit interval $[0, 1]$ with $\lambda_t = 1$ in a neighborhood of t.

Then we define

$$\omega'_x = (1 - g(x)\, d\varphi + g(x)[\lambda_1(r) \cdot \omega_x + \lambda_{1/2}(r)\, dr + \lambda_0(r)\, d\varphi].$$

Here, ω_x is a closed form defined on $S^1 \times D^2$ except for $r<0$, by the same formula $\omega_x = d\varphi - f(x)\, d\theta$.



a) Example of the image $L_x(\beta^{k+l+1})$ where $k=2$ and $l=3$

b) A flow ψ_t supported on an annulus. The vectorfield is $f(r)\frac{\partial}{\partial\theta}$

c) The flow φ_t: it is conjugate to ψ_t

d) The image $\varphi_1(L_x\beta_0^{k+l})$

Fig. 4

Observe first that ω_x' agrees with ω_x near $x\times S^1\times S^1$. (When $g(x)\neq 1$, note that $\omega_x=d\varphi$.) Second, ω_x' is non-singular, since either $\omega_x'(\partial/\partial\varphi)$ or $\omega_x'(\partial/\partial r)$ is non-zero. Third, ω_x' is integrable, since near any point in $D^2\times S^1$ it can be written in terms of at most two variables in $D^2\times S^1$. (Locally, away from $r=0$, ω_x is dh for some function h.)

Note. The original, less elementary, construction was to use the theorem (see [12]) that for *any* compactly supported isotopy $\{\varphi_t\}$, and for *any* compactly supported isotopy $\{\psi_t\}$ such that $\psi_1\neq$id., φ_1 is a product of conjugates of ψ_1 in such a way that the isotopy $\{\varphi_t\}$ is up to homotopy relative end points the product of the conjugates of the isotopies $\{\psi_t\}$.

5. Jiggling Triangulations

We need the following lemma for Section 2:

JIGGLING LEMMA. *Let M^n be a manifold and K a compact subset of M^n. Let τ^{n-k} be a C^0 plane field on M^n, and let α be a triangulation of M^n. Then there is a subdivision α' of α and a jiggling α'' of α' which is in general position with respect to τ in a neighborhood of K.*

Actually, some further conditions are required. These will be discussed at the end of the section.

Proof. For simplicity, consider first the case $M^n = \mathbb{R}^n$.

Let the vertices meeting some bounded neighborhood of K be indexed v_i, $0 \leqslant i \leqslant M$. We will look at a sequence of crystalline subdivisions α^l of α. (cf. Whitney [13, pp. 358–360] for a similar subdivision which would serve here.) For each n-simplex $\langle v_{i_0}, \ldots, v_{i_n} \rangle$ where $i_0 < i_1 < \cdots < i_n$, there is a linear map f to the cube $\{(x_1 \ldots x_n)\colon 0 \leqslant x_i \leqslant 1\}$ defined by the condition

$$f(v_{i_j}) = (\underbrace{0, 0, \ldots, 0}_{j\,0\text{'s}}, \underbrace{1, 1, \ldots, 1}_{(n-j)\,1\text{'s}}).$$

Now subdivide the cube into l_n little cubes, and subdivide each little cube in the standard way into simplices. The standard way to subdivide the cube $\{(x_1, \ldots, x_n)\colon 0 \leqslant x_i \leqslant 1\}$ is into $n!\,n$-simplices β_σ where σ is a permutation of $\{1, \ldots, n\}$ and $\beta_\sigma = \{(x, \ldots, x_n)\colon 0 \leqslant x_{\sigma_1} \leqslant \cdots \leqslant x_{\sigma_n} \leqslant 1\}$. The standard subdivision of a little cube comes from $\{\beta_\sigma\}$ by a contraction followed by a translation, sending the big cube to the little cube. Our simplex $\langle v_{i_0}, \ldots, v_{i_n} \rangle$ inherits a crystalline subdivision from all this, and these crystalline subdivisions fit together to give a subdivision a^l of α. Note that when $n \geqslant 3$, α^l depends, in an essential way, on the ordering of the vertices.

There are two properties of the sequence of subdivisions α^l that we will need. The first property is that there are a finite number of model n-simplices μ_i in $\mathbb{R}^n$, such that for any l, each n-simplex in α^l can be obtained from a model simplex by a contraction with a factor of $1/l$ followed by a translation. The second property (a consequence of the first) is that there is a uniform bound P to the number of simplices in the link of a vertex in α^l. Let δ be a number such that for each i and for each choice of points ω_j in the δ_i ball about the jth vertex of μ_i, $\omega_0, \ldots, \omega_n$ do not lie on a common hyperplane: *i.e.*, they determine a nondegenerate n-simplex.

It follows that for each choice of points x'_m in the δ/l ball about the mth vertex x_m of α^l, the x'_m determine a triangulation α'^l of most of the region coveıed by α. Indeed, there is a map, linear on each simplex, $\alpha^l \to \mathbb{R}^n$ sending x_m to x'_m; it sends each n-simplex to a non-degenerate n-simplex with the same orientation, so it is a local homeomorphism within δ/l of the identity, except on the boundary of the region.

Let d be a metric on the Grassmannian $G(l, n)$ of l-planes in $\mathbb{R}^n$, and let μ be the function on $G(l, n) \times G(q, n)$ defined as

$$\mu(\tau, \beta) = \text{g.l.b.}\{d(\tau, \tau') : \tau' \text{ is not transverse to } \beta\}.$$

Here, when we say two subspaces τ' and β of a vector space are transverse, we mean

$$\dim(\tau' \cap \beta) = \max(l + q - n, 0).$$

For each model simplex μ_i, for each vertex ω_j of μ_i, for each q-face $\langle \omega_{j_0}, \ldots, \omega_{j_p} \rangle$ of μ_i, $[q\_k-1]$, not incident to ω_j, for each choice of $w'_{j_0}, \ldots, w'_{j_q}$ in the δ-balls about $\omega_{j_0}, \ldots, \omega_{j_q}$, and for each $(q+n-k)$ plane π in $\mathbb{R}^n$ containing $\langle \omega'_{j_0} \ldots \omega'_{j_q} \rangle$ consider the sector S_ε of $\mathbb{R}^n$ consisting of all points on $q+n-k$-planes π' through $\langle \omega'_{j_0}, \ldots, \omega'_{j_q} \rangle$ such that $d(\pi, \pi') < \varepsilon$. The measure of the intersection of S_ε with the δ-ball about ω_j decreases to zero with ε. Since all the above choices ranges over a compact set, we can choose ε so that this measure is less than $1/P$ times the measure of the ball of radius δ.

We define inductively a sequence $\{\varepsilon_q\}$ for $0 \leqslant q \leqslant k$, as follows. Let $\varepsilon_0 = \infty$, $\varepsilon_1 = \varepsilon$. When ε_q has been defined, choose $\varepsilon_{q+1} > 0$, $\varepsilon_{q+1} \leqslant \varepsilon_q/2$ to satisfy the condition that whenever τ_1, τ_2 are $(n-k)$-planes satisfying $d(\tau_1, \tau_2) \leqslant \varepsilon_{q+1}$, and when α is a q-plane satisfying $\mu(\tau_1, \alpha) \geqslant \varepsilon_q/3$, then $d(\tau_1 \oplus \alpha, \tau_2 \oplus \alpha) \leqslant \varepsilon$.

For each vertex x in α^l define N_x to be the δ/l neighborhood of the star neighborhood of x (so that the star neighborhood of any open simplex in a jiggling of α^0 is contained in some N_x). Let l be great enough that for every u, v in N_x, $d(\tau_u, \tau_v) \leqslant \varepsilon_k/3$.

Assume, by induction on p, that $x'_1, \ldots, x'_p$ have been chosen, with $x'_1 \in B_{\delta/l}(x_i)$, so that for every q-simplex $\beta = \langle x_{i_0}, \ldots, x_{i_q} \rangle$ $[0 \leqslant q \leqslant k]$ of α^l, where $0 \leqslant i_0 < \cdots < i_q \leqslant p$, the jiggled simplex $\beta' = \langle x'_{i_0}, \ldots, x'_{i_q} \rangle$ satisfies $\mu(\tau_x, \beta') \geqslant 2\varepsilon_q/3$, for $x \in N_{x_{i_0}}$. When $q=0$, this condition is automatically satisfied. Now if β is in the link of x_{p+1}, and if τ is any plane such that $d(\tau, \tau_x) \leqslant 2\varepsilon_{q+1}/3 \leqslant \varepsilon_q/3$ for some $x \in N_{x_{i_0}}$, then $\mu(\tau, \beta') \geqslant \mu(\tau_x, \beta') - d(\tau, \tau_x) \geqslant \varepsilon_{q/3}$. As τ varies over all such choices it varies within a radius of $\varepsilon_k/3 + 2\varepsilon_{q+1}/3 \leqslant \varepsilon_{q+1}$ from a fixed τ_x, so by the choice of ε_{q+1}, $(\tau \oplus \beta')$ varies by a distance at most ε. By the choice of ε there is some point x'_{p+1} in $B_{\delta/l}(x_{p+1}$ not lying on any plane through β' parallel to $\tau \oplus \beta'$, for any choice of β' in the link of x_{p+1}, and τ as above. Consider $\gamma' = \langle x'_{i_0}, \ldots, x'_{i_q}, x'_{p+1} \rangle$. γ' is transverse to each such τ. Since the choices of τ contain a $2\varepsilon_{q+1}/3$ neighborhood of τ_x for $x \in N_{x_{i_0}}$, it follows that $\mu(\tau_x, \gamma') \geqslant 2\varepsilon_{q+1}/3$ for $x \in N_{x_{i_0}}$. Thus the inductive hypothesis is true for every simplex containing x'_{p+1}, so the inductive argument is complete.

The triangulation α' required in the lemma we take as α^l, and α'' we take as the triangulation determined by the x'_j.

This completes the proof for $\mathbb{R}^n$.

To do this for arbitrary M^n, the best way, which Haefliger suggested to me, is to imbed M^n in $\mathbb{R}^N$ (for N large). Triangulate M^n (smoothly) and subdivide until the triangulation is the projection of PL-approximation to M^n along the normal fibers of a tubular neighborhood of M^n. Now everything is the same: do a fine crystalline subdivision of the PL-approximation, jiggle, and project back to M^n.

This completes the proof of the Jiggling Lemma.

There are two further conditions required for Section 2. The first is that α' be a product triangulation; this is no problem since the product triangulation coming from a crystalline triangulation of $v_{\mathscr{H}}$ and a crystalline triangulation of I satisfies the two properties (p. 227) used in the proof. The second property is that the submanifold $Z \times 1$ be invariant under the jiggling. This is also easy to satisfy: Imbeded $v_{\mathscr{H}} \times I$ in $\mathbb{R}^N$ so that $Z \times 1$ is the intersection with $\mathbb{R}^M$. Then take δ, ε and l in the proof so that they work for the induced plane field on $Z \times 1$ as well as the plane field on $v_{\mathscr{H}} \times I$, and order the vertices of α_l, for the inductive jiggling, so that the vertices lying on $Z \times 1$ come first. They they can be jiggled in such a way as to leave $Z \times 1$ invariant.

6. Civilization

In this section, we will first do carefully the construction of perturbing a plane field in general position to be integrable in a neighborhood of the k-skeleton. Then we will give the rules for a "civilized" plane field and show that civilized homotopies can always be extended.

So let τ^{n-k} be a plane field on M^n in general position with respect to a triangulation β. Assume inductively that τ^{n-k} has been perturbed in a neighborhood of the $(l-1)$-skeleton to be integrable in a smaller neighborhood, $0 \leqslant l \leqslant k$. Then for each l-simplex β_i^l, imbed $\beta_i^l \times D^{k-l}$ transverse to τ in a small neighborhood of β_i^l, with $\beta_i^l \times 0$ imbedded by the identity map. The foliation in a neighborhood of $\partial\beta_i^l$ defines a tubular neighborhood structure for (a neighborhood of $\partial\beta_i^l) \times D^{k-l}$, the leaves of the foliation being the normal fibers.

Extend this tubular neighborhood structure over $\beta_i^l \times D^{k-l}$, making the normal fibers tangent to τ along $\beta_i^l \times D^{k-l}$, and straighten out τ to be tangent to the normal fibers in a smaller neighborhood. The homotopy can be small enough that τ remain in general position with respect to β (since this is an open condition).

In the course of the inductive procedure, outlined in Section 2, for the construction specified by Theorem 1, it is necessary to extend a homotopy of a plane field $\bar{\tau}$ defined on a simplex to a homotopy in $v_{\mathscr{H}} \times [0, 1]$. There is a simple example (see Fig. 5) which shows this is not always possible, if we are working with plane fields transverse to the triangulation. There are similar examples where $\bar{\tau}$ is kept fixed in a neighborhood of the k-skeleton. Therefore, we must find a subset of the plane fields transverse to a triangulation, civilized plane fields, among which such extensions are always possible.

Let M^n be a manifold, β a smooth triangulation and τ^{n-k} a plane field in general position with respect to β, and integrable in a neighborhood of the k-skeleton. Then for each simplex β_i^l choose a small tubular neighborhood $N(\beta_i^l)$, diffeomorphic with $\beta_i^l \times D^{n-l}$, so that

(i) $N(\beta_i^l) \cap N(\beta_j^m) \subset N(\beta_q^p)$ where β_q^p is the largest common face of β_i^l and β_j^m.

(ii) If β_i^l is a face of β_j^m and if $x \in \beta_i^l$, $y \in \beta_j^m$ are such that $x \times D_i^{n-l}$ intersects $y \times D^{n-m}$, then $x \times D^{n-l} \subset y \times D^{n-m}$.

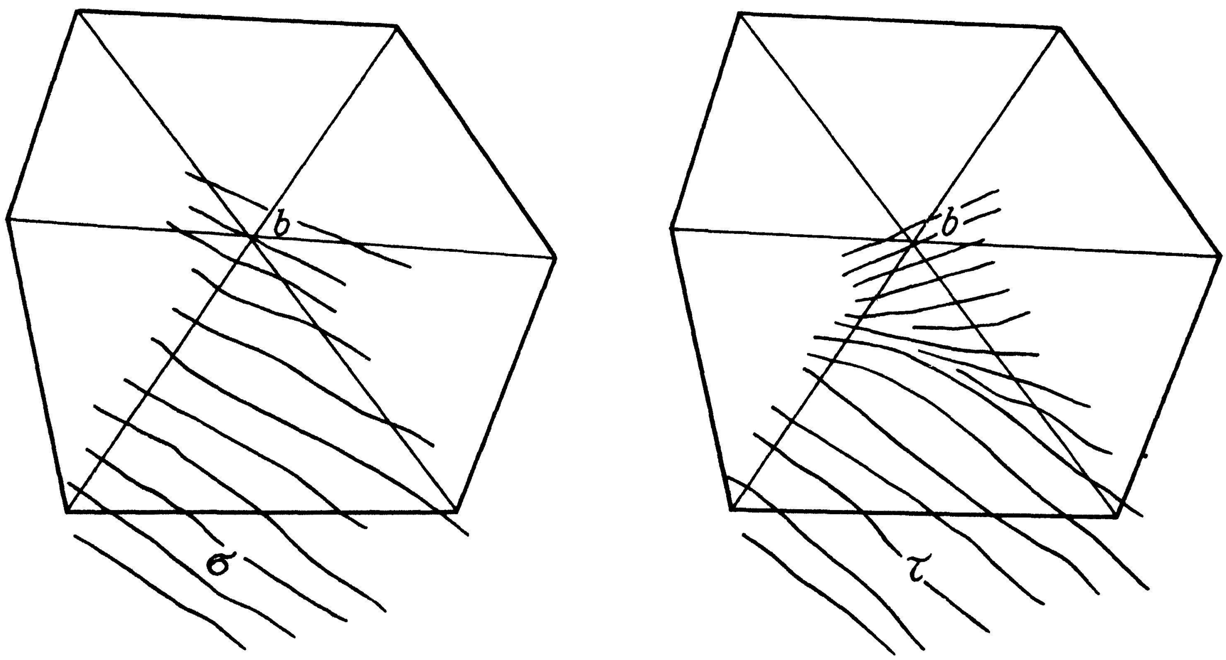


Fig. 5
There is a homotopy of σ to τ, transverse on the bottom simplex, but this homotopy can not be transverse in a neighborhood of b. Civilized plane fields are controlled in the normal directions to a simplex so this problem does not occur

(iii) If $l \leqslant k$ then τ is tangent to the normal fibers $x \times D^{n-l}$ of $N(\beta_i^l)$, and integrable there. If $l \geqslant k$, then the normal fibers of $N(\beta_i^l)$ are tangent to τ along β_i^l.

If τ is completely integrable in a neighborhood of a subcomplex K, then we add another condition.

(iv) If $\beta_i^l \subset K$, $l \geqslant k$, then τ is integrable in $N(\beta_i^l)$, and the normal fibers of $N(\beta_i^l)$ are tangent to τ.

Now τ can be homotoped, rel N (k-skeleton $\cup K$), so that it remains in general position with respect to β, and in $N(\beta_i^l)$ $[l \geqslant k]$, τ is induced from β_i^l by projection: that is, τ consists of all vectors which are taken to $\tau \cap \beta_i^l$ by the derivative of the projection $N(\beta_i^l) \to \beta_i^l$.

If a plane field σ is transverse to β, agrees with τ in N (k-skeleton), and if in $N(\beta_i^l)$ $[l \geqslant k]$ σ is induced from β_i^l by projection, then σ is *civilized.*

Thus, given a civilized codimension k plane field σ on a subcomplex L, it defines a unique, differentiable, civilized codimension k plane field τ^{n-k} in a neighborhood $T(L)$. Furthermore, τ^{n-k} is integrable iff $\sigma = \tau^{n-k} \cap L$ is integrable.

A *civilized homotopy* is a homotopy among civilized plane fields.

PROPOSITION. *Given a civilized plane field τ^{n-k} and M^n, a civilized homotopy of $\tau^{n-k} \cap L$, where L is a subcomplex, extends to a civilized homotopy of τ^{n-k}.*

Proof. Very easy. Induction on simplices $\beta_i^l \subset M^n - L$ of dimension $l > k$. Let $L^l = L \cup l$-skeleton of $M^n - l$. We are given to begin a civilized plane field on L^k. Assume that we have extended the homotopy to L^l. This defines a homotopy in $N(L^l)$. Now for each $l+1$-simplex β_i^{l+1} we have a homotopy in $N(\partial\beta_i^{l+1})$, which can be extended to a homotopy on β_i^{l+1}. Piecing together, we have a homotopy defined on L^{l+1}.

Remark. The extended homotopy can obviously be kept fixed in $N(L')$, where L' is a subcomplex of M^n, if the original homotopy is constant on $L' \cap L$.

Added in proof: I now have a proof of Theorem 1 in codimension 1. Thus if M^n is closed and $\chi(M^n) = 0$, M^n has a C^∞ codimension 1 foliation. The relative version is more complicated.

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