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Cancellation Properties of H-Spaces

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One defines the genus of a homotopy type X to be the set $G(X)$ of all homotopy types Y such that the p-localizations X_p and Y_p are homotopy equivalent for all primes p [11]. To have a good notion of p-localization, we will work throughout this paper in the homotopy category of nilpotent, connected, pointed CW-complexes [1, 7, 14]. Notice that all H-spaces are nilpotent spaces whereas H'-spaces are in general not nilpotent unless they are 1-connected. We call X *quasi-finite*, if $H_i(X; \mathbf{Z})$ is finitely generated for all i and zero for i big enough. It is not known whether a quasi-finite H-complexes is necessarily of the homotopy type of a finite complex.

It was observed by many authors [5, 8, 9, 10, 11, 13, 15] that H-spaces, cancellation phenomena and the genus of a space are closely related. The oldest example expressing this relation is the H-manifold E_{7w} of [6] for which one has

$$G(E_{7w}) = G(Sp(2)) \qquad \text{(see [11])}$$

and

$$E_{7w} \times S^3 \simeq Sp(2) \times S^3 \qquad \text{(see [6])}.$$

More complicated examples of the same nature, involving the Lie group G_2, were discussed in [8].

Our main results (Proposition 1 and 2) give conditions under which one can cancel factors in a product or summands in a wedge respectively. When applied to H-spaces, this yields a converse of a theorem of Zabrodsky [15], giving a very simple characterization of the genus of an H-complex (Theorem A below).

For a quasi-finite H-complex X one has by Hopf's theorem $H^*(X; \mathbf{Q}) \cong H^*(S^{n_1} \times \times \cdots \times S^{n_k}; \mathbf{Q})$; the array $(n_1, \ldots, n_k) = \tau(X)$ is called the *type* of X. We will prove

THEOREM A. *Let X and Y be quasi-finite H-complexes. Then the following are equivalent*

A1: $G(X) = G(Y)$

A2: $X \times (\prod_{\tau(X)} S^{n_i}) \simeq Y \times (\prod_{\tau(Y)} S^{m_j})$

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Dually, if X is a quasi-finite nilpotent H'-complex then $\pi_*(X)\otimes\mathbf{Q}\cong\pi_*(S^{m_1}\vee\vee\cdots\vee S^{m_k})\otimes\mathbf{Q}$ and we call $(m_1,\ldots,m_k)=\tau'(X)$ the *cotype* of X; notice that if X is not 1-connected, then X is of the form $S^1\vee Y$, and the nilpotency of X implies $X\simeq S^1$. Then one has

THEOREM B. *Let X and Y be quasi-finite nilpotent H'-complexes. Then the following are equivalent*

B1: $G(X)=G(Y)$

B2: $X\vee(\bigvee_{\tau'(X)} S^{m_i})\simeq Y\vee(\bigvee_{\tau'(Y)} S^{m_j})$

It was proved in [15] that in the 1-connected case A1 implies A2 and B1 implies B2 (under the conditions stated). Since these proofs easily extend to the nilpotent case, we will not repeat the arguments here. In order to get the converse, we proceed very much in the way one would argue in an "abelian" situation [2, 4]. We show that if the homotopy endomorphism set $[X, X]$ has a suitable "local" structure, then X-factors or X-summands may be cancelled. More precisely

DEFINITION 1 [2]. The set $\operatorname{End}X=[X,X]$ is called H-local (or π-local) if for all $f, g\in\operatorname{End}X$ which are not homotopy equivalences, $H_*f+H_*g\neq Id$ in $\operatorname{End}(H_*(X;\mathbf{Z}))$ (or $\pi_*f+\pi_*g\neq Id$ in $\operatorname{End}(\pi_*X)$ respectively).

For example it is obvious that S_p^n and $K(\mathbf{Z}_p, n)$ are both π-local and H-local for all $n\geqslant 1$ and all primes p. ($\mathbf{Z}_p$ denotes the integers localized at (p)).

LEMMA 1. *Let A be a retract of $X\times Y$ and suppose that End A is π-local. Then A is a retract of X or Y.*

Proof. There are maps $f:A\to X\times Y$ and $g:X\times Y\to A$ such that $g\circ f\simeq Id_A$. Let $f_X=pr_X\circ f$, $f_Y=pr_Y\circ f$ and $g_X=g\mid X$, $g_Y=g\mid Y$. Then $\pi_*(g_X\circ f_X)+\pi_*(g_Y\circ f_Y)=Id$. Hence, since End A is π-local, $g_X\circ f_X$ or $g_Y\circ f_Y$ must be a homotopy equivalence and therefore A is a retract of X or Y.

Dually one has

LEMMA 2. *Let A be a retract of $X\vee Y$ and suppose that End A is H-local. Then A is a retract of X or Y.*

The proof is completely analogous to that of Lemma 1, with π_* replaced by H_*.

DEFINITION 2. X is called irreducible, if X has no non-trivial retracts.

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[2]) H_* denotes reduced homology.

Examples of irreducible spaces are S^n, S^n_p, $K(\mathbf{Z}, n)$, $K(\mathbf{Z}_p, n)$, CP^n, HP^n, BS^3, $SU(r)$ etc.

DEFINITION 3. X is called completely reducible (resp. completely coreducible) if X is a finite product (resp. finite wedge) of irreducible spaces.

Notice that if X is an H-complex and Y a retract of X with retraction map $\varphi: X \to Y$, then $X \simeq \mathrm{fib}(\varphi) \times Y$ and Y as well as the "fiber" $\mathrm{fib}(\varphi)$ are H-spaces. Hence, if in addition X is quasi-finite, one arrives after finitely many steps at a decomposition $X \simeq X_1 \times \cdots \times X_n$ with X_i irreducible for $1 \leqslant i \leqslant n$. By essentially the same argument we see that for a quasi-finite H-complex X the p-localization X_p is completely reducible. Similarly, if Y is a quasi-finite H'-complex then Y and Y_p are both completely coreducible.

DEFINITION 4. X is called balanced if for all $f, g \in \mathrm{End}\,X$ with $f \circ g$ a homotopy equivalence both f and g are homotopy equivalences.

If X is nilpotent and of finite type (i.e. $\pi_i X$ or equivalently $H_i X$ is finitely generated for all i) then X is balanced. This follows easily by checking induced maps in homology. Similarly, if X is nilpotent and of finite type, then X_p is balanced for p a prime of 0.

PROPOSITION 1. *Let Y be a completely reducible space and A a balanced space with* $\mathrm{End}\,A$ *a π-local set. Then $X \times A \simeq Y \times A$ implies $X \simeq Y$.*

Proof. For a map $g: U \times V \to S \times T$ we write $g(U, T): U \to T$ for $pr_T \circ g \circ in_U$. Let $f: X \times A \to Y \times A$ be a homotopy equivalence. We will distinguish two cases

1) $f(A, A)$ is a homotopy equivalence. Then one has in homotopy
$f_*(A, A) \circ f_*^{-1}(Y, A) + f_*(X, A) \circ f_*^{-1}(Y, X) = 0$
$f_*(A, Y) \circ f_*^{-1}(Y, A) + f_*(X, Y) \circ f_*^{-1}(Y, X) = Id$

Hence

$$(f_*(X, Y) - f_*(A, Y)\,(f_*(A, A))^{-1} f_*(X, A)) \circ f_*^{-1}(Y, X) = Id.$$

Similarly

$$f_*^{-1}(Y, X) \circ (f_*(X, Y) - f_*(A, Y)\,(f_*(A, A))^{-1} f_*(X, A)) = Id.$$

It follows that $f^{-1}(Y, X): Y \to X$ is a homotopy equivalence.

2) $f(A, A)$ is not a homotopy equivalence. Then, since A is a balanced space, $f^{-1}(A, A) \circ f(A, A)$ is not a homotopy equivalence. But $f_*^{-1}(A, A) \circ f_*(A, A) + f_*^{-1}(Y, A) \circ f_*(A, Y) = Id$ and $\mathrm{End}\,A$ is π-local. Therefore $f^{-1}(Y, A) \circ f(A, Y)$ is a homotopy equivalence and we conclude that A is a retract of Y. Now Y is comcompletely reducible: $Y \simeq Y_1 \times \cdots \times Y_n$ with Y_i irreducible $1 \leqslant i \leqslant n$. It follows by

Lemma 1 and induction that A is a retract of one of the Y_i's. More precisely, for some i_0, $\varphi = pr_{i_0} \circ f(A, Y): A \to Y_{i_0}$ has a homotopy left inverse. Since Y_{i_0} is irreducible, we conclude that φ is actually a homotopy equivalence. We consider now the automorphism

$$\theta: Y \times A \to Y \times A, \theta(y_1, \ldots, y_{i_0}, \ldots, y_n, a) = (y_1, \ldots, \varphi a, \ldots, y_n, \varphi^{-1} y_{i_0})$$

and form the new homotopy equivalence $\bar{f} = \theta \circ f$. It is immediate that $\bar{f}(A, A) = \varphi^{-1} \circ \varphi = Id_A$. Hence, by replacing f by $\bar{f}$, we are back in case 1.

If we replace, in this proof, induced maps in homotopy by induced maps in homology and using the nilpotency of the spaces involved, we will get the following dual proposition.

PROPOSITION 2. *Let Y be a completely coreducible space and A a balanced space with* End$A$ *an H-local set. Then $X \vee A \simeq Y \vee A$ implies $X \simeq Y$.*

Remark. In the proofs of Lemma 1 and 2 as well as Proposition 1 we made no use of the nilpotency of the spaces involved. Therefore these results remain true if the spaces are only assumed to be pointed connected CW-complexes. However for Proposition 2 and the following applications we have to restrict to nilpotent spaces.

COROLLARY 1. *Let Y be a finite product of quasi-finite H-complexes and spheres. Then $G(X \times S^n) = G(Y \times S^n)$ implies $G(X) = G(Y)$.*

Proof. Suppose $G(X \times S^n) = G(Y \times S^n)$. Then $X_p \times S^n_p \simeq Y_p \times S^n_p$ with Y_p completely reducible, EndS^n_p π-local and S^n_p balanced. Hence we conclude by Proposition 1 that $X_p \simeq Y_p$ for all primes p.

COROLLARY 2. A2 *implies* A1 *in Theorem A.*

Namely one has only to observe that A2 implies $\tau(X) = \tau(Y)$, by checking rational cohomology. Then the result follows by iterated application of Corollary 1.

Similarly one has in the dual situation

COROLLARY 3. *Let Y be a quasi-finite H'-complex. If $G(X \vee S^n) = G(Y \vee S^n)$ for some $n > 1$, then $G(X) = G(Y)$.*

COROLLARY 4. B2 *implies* B1 *in Theorem* B.

Notice that Theorem B is trivial in case Y is not 1-connected, because then $Y \simeq S^1$.

We can also use Proposition 1 to find new H-complexes. For instance one has:

COROLLARY 5. *Let Y be a quasi-finite H-complex. Then $X \times S^{n_1} \times \cdots \times S^{n_k} \simeq Y \times S^{n_1} \times \cdots \times S^{n_k}$ implies that X is a quasi-finite H-complex.*

Proof. Corollary 1 implies $G(X)=G(Y)$ or $X\in G(Y)$. Notice that X is of finite type. Since Y is a quasi-finite H-complex, the same is true for X (cf [7]).

It is known [3] that for a finite 1-connected H-complex X there can be only finitely many finite H-complexes $Y\in G(X)$. The following examples should illustrate that the set $G(X)$ can still be quite big.

EXAMPLE 1 (compare also [10] and [16]). Let p denote a prime and consider the principal fibration $SU(p)\to SU(p+1)\to S^{2p+1}$ classified by $\gamma=\alpha+\beta\in\pi_{2p}SU(p)$, where β is of order p and α of order prime to p. Define X_n as the total space of the principal $SU(p)$-fibration over S^{2p+1} classified by $\alpha+n\beta$. Then it is easily seen that $X_n\simeq X_m$ if and only if $n\equiv\pm m(p)$ and, if $n\not\equiv 0(p)$ then $X_n\in G(SU(p+1))$. Hence we conclude

$$|G(SU(p+1))|\geqslant\frac{p-1}{2}$$

and, by Theorem A, $X_n\times S^3\times S^5\times\cdots\times S^{2p+1}\simeq SU(p+1)\times S^3\times S^5\times\cdots\times S^{2p+1}$ if $n\not\equiv 0(p)$.

EXAMPLE 2 (compare also [12]). Let p be a prime and $Y_m=S^3\bigcup_{m\gamma}e^{2p+1}$ where $\gamma\in\pi_{2p}S^3$ is of order p. Then Y_m is a H'-space since γ is a primitive element. Furthermore it is easy to see that $Y_m\simeq Y_n$ if and only if $m\equiv\pm n(p)$. Hence

$$|G(Y_1)|\geqslant\frac{p-1}{2}$$

and, by Theorem B, $Y_1\vee S^3\vee S^{2p+1}\simeq Y_m\vee S^3\vee S^{2p+1}$ if $m\not\equiv 0(p)$.

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