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**Autor:** Werra, D. de
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# Partitions of Graphs into Coverings and Hypergraphs into Transversals

D. DE WERRA


## Abstract

A covering of a multigraph $G$ is a subset of edges which meet all vertices of $G$. Partitions of the edges of $G$ into coverings $C_1, C_2, \ldots, C_k$ are considered.

In particular we examine how close the cardinalities of these coverings may be. A result concerning matchings is extended to the decomposition into coverings. Finally these considerations are generalized to the decompositions of the vertices of a hypergraph into transversals (a transversal is a set of vertices meeting all edges of the hypergraph).


## Introduction

In this note a multigraph $G=(X, U)$ consists of a finite non-empty set $X$ of vertices and a set $U$ of edges.

A *covering* $C$ in $G$ is a subset of edges such that each vertex of $G$ is adjacent to at least one edge of $C$. Given a multigraph $G$ we will consider partitions of the edges of $G$ into coverings $C_1, C_2, \ldots, C_k$. (Such a partition exists only if each vertex $x$ has degree at least $k$, i.e. if any $x$ is adjacent to at least $k$ edges). The cardinality of $C_i$ will be denoted by $c_i$.

We will first examine the following question: given a multigraph $G$ when does a given finite sequence $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k \geqslant 0$ represent the cardinalities of a partition of $U$ into coverings?

A similar problem concerning partitions into matchings (i.e. subsets of nonadjacent edges) has been solved in [1] and [2].

In §2 the problem is formulated in terms of hypergraphs; we now have partitions of the vertices into transversals and we examine in particular how close the cardinalities of transversals in a partition can be.

All notions not defined here can be found in [3].

## §1. Partitions into Coverings

Let us call *covering index* $i(G)$ of $G$ the largest $k$ for which there exists a partition of the edges of $G$ into $k$ coverings $C_1, C_2, \ldots, C_k$.

To each such partition we associate a sequence $c_1, c_2, \ldots, c_k$ where $c_i$ is the cardinality of $C_i$ and where the indices are chosen in such a way that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_k$.

We may now formulate a theorem which is quite similar to the matching case.

THEOREM 1. *If the sequence $c_1, c_2, \ldots, c_k$ corresponds to a partition of the edges of G into coverings, then any sequence $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_k$ with*

$$\bar{c}_1 \leqslant \bar{c}_2 \leqslant \cdots \leqslant \bar{c}_k$$

$$\sum_{i=1}^{r} \bar{c}_i \geqslant \sum_{i=1}^{r} c_i \qquad r=1, \ldots, k$$

$$\sum_{i=1}^{k} \bar{c}_i = \sum_{i=1}^{k} c_i$$

*corresponds also to a partition of the edges of G into coverings.*

*Proof.* We only have to prove that any couple of coverings $C_i$, $C_j$ with $c_i - c_j = K \geqslant 2$ may be replaced by two disjoint coverings $\bar{C}_i$, $\bar{C}_j$ with $c_i - c_j = K-2$ and $\bar{C}_i \cup \bar{C}_j = C_i \cup C_j$; then by repeated transformations of this type we will obtain any sequence $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_k$ satisfying the above condition.

Let $G_{ij}$ be the graph formed by the edges of $C_i \cup C_j$; in $G_{ij}$ we construct an alternating chain (i.e. its edges belong alternately to $C_i$ and $C_j$) and extend it as far as possible (it may happen that this chain goes through the same vertex several times). We remove it from $G_{ij}$ and we construct another alternating chain which is as long as possible in the remaining graph. We remove it and continue until we obtain an alternating chain $Q$ starting and ending with edges in $C_i$ (such a chain must exist since $c_i - c_j = K \geqslant 2$). We interchange the edges of $Q \cap C_i$ and $Q \cap C_j$ and obtain two subsets $\bar{C}_i$, $\bar{C}_j$ with $\bar{c}_i - \bar{c}_j = K-2$. $\bar{C}_i$ and $\bar{C}_j$ are still coverings: at each endpoint of $Q$ there were (before the interchange) more edges of $C_i$ than of $C_j$ (i.e. at least 2 edges of $C_i$), so after the interchange $\bar{C}_i$ and $\bar{C}_j$ have at least one edge at each endpoint of $Q$ as well as at any other vertex of $G$.

Now if we are interested in knowing how close the cardinalities of coverings in a partition can be, we have the following immediate consequence of the theorem.

COROLLARY. *For any $k \leqslant i(G)$, there exists a partition of the edges of G into coverings $C_1, C_2, \ldots, C_k$ with cardinalities $c_1, c_2, \ldots, c_k$ satisfying: $|c_i - c_j| \leqslant 1$ $i, j = 1, \ldots, k$.*

*Remark.* The proof of Theorem 1 may be adapted to the case of $p$-bounded colorations [4] for which a similar result holds.

## §2. Transversals in Hypergraphs

A *hypergraph* $H = (X, U)$ consists of a finite set $X$ of vertices and a family $U$ of nonempty edges $E_j$ $(j=1, \ldots, m)$ satisfying $\bigcup_{j=1}^{m} U_j = X$.

A *transversal* is a subset $T$ of vertices such that $T \cap E_j \neq \emptyset$ for $j=1, \ldots, m$.

$H(p)$ will denote any hypergraph in which any vertex belongs to at most $p$ edges.

Let $T_1, T_2, \ldots, T_k$ be a partition of the vertices of a hypergraph $H(p)$ into transversals and let $t_1, t_2, \ldots, t_k$ be their cardinalities. If $p=1$, all edges are disjoint; in this case it is easy to obtain from $T_1, T_2, \ldots, T_k$ a partition $\bar{T}_1, \bar{T}_2, \ldots, \bar{T}_k$ with $|\bar{t}_i-\bar{t}_j|\leqslant 1$ $i,j=1,\ldots,k$.

Hence we will assume that $p\geqslant 2$ in the remainder of the note.

LEMMA. *Any two transversals $T_i, T_j$ of $H(p)$ with $t_j>(p-1)\,t_i+1$ may be replaced by two transversals $\bar{T}_i, \bar{T}_j$ with $\bar{t}_i\leqslant\bar{t}_j\leqslant(p-1)\,\bar{t}_i+1$.*

*Proof.* Consider the subhypergraph $H_{ij}=\langle T_i\cup T_j\rangle$ spanned by $T_i\cup T_j$ (its edges are $(T_i\cup T_j)\cap E_r$ for $r=1,\ldots,m$).

We will associate to $H_{ij}$ a graph $G_{ij}$ whose vertices are those of $T_i\cup T_j$; its edges which will be called *heavy edges* are obtained as follows:

initially there are no heavy edges. We examine consecutively all edges $E$ of $H_{ij}$ (note that for each $E$, $T_i\cap E\neq\emptyset$ and $T_j\cap E\neq\emptyset$)

a) if in edge $E$ no pair of vertices $x, y$ with $x\in T_i\cap E$ and $y\in T_j\cap E$ is joined by a heavy edge, then we pick up one such pair $(x, y)$ and it becomes a heavy edge.

b) if in edge $E$ there is already a pair $x, y$ with $x\in T_i\cap E$ and $y\in T_j\cap E$ which is a heavy edge, we simply examine the next edge of $H_{ij}$.

By construction, $G_{ij}$ is bipartite; besides no vertex in $G_{ij}$ has a degree greater than $p$ (since no vertex belongs to more than $p$ edges of $H_{ij}$).

Assume now that $t_j=t_i+M>(p-1)\,t_i+1$. $G_{ij}$ has at most $t_i\cdot p$ edges and $2t_i+ +M\geqslant t_i\cdot p+2$ vertices, hence it cannot be connected.

So there must exist a connected component $G'_{ij}$ of $G_{ij}$ with $t'_i<t'_j=t'_i+L\leqslant(p-1)\times \times t'_i+1$ where $t'_i$ and $t'_j$ are the cardinalities of the subsets $T'_i$ and $T'_j$ of vertices of $G'_{ij}$ belonging to $T_i$ and $T_j$ respectively.

We now interchange the vertices of $T'_i$ and $T'_j$, thus $T_i$ and $T_j$ are replaced by subsets $\bar{T}_i, \bar{T}_j$. We have to show that $\bar{T}_i$ and $\bar{T}_j$ are transversals of $H_{ij}$ and consequently of $H(p)$.

Notice that each edge of $H$ contains exactly one heavy edge of $G_{ij}$ and possibly isolated vertices of $G_{ij}$ (it may occur that a heavy edge belongs to several edges of $H$).

So changing the colour of an isolated vertex of $G_{ij}$ will still give two transversals $\bar{T}_i, \bar{T}_j$. Furthermore by interchanging the colours of the vertices in a connected component of $G_{ij}$ we also obtain transversals: all edges containing a heavy edge of $G'_{ij}$ will still be met by $\bar{T}_i$ and $\bar{T}_j$ and the edges containing only nonadjacent vertices of $G'_{ij}$ must contain a heavy edge of another component of $G_{ij}$; hence they will also be met by $\bar{T}_i$ and $\bar{T}_j$.

Finally observe that

$$0<L\leqslant(p-2)\,t'_i+1\leqslant(p-2)\,t_i+1<M$$

So the cardinalities $\bar{t}_i$ and $\bar{t}_j$ satisfy

$$t_i < \bar{t}_i = \bar{t}_i + L < t_i + M = t_j$$
$$t_i = t_j - M < t_j - L = \bar{t}_j < t_j$$

which implies

$$\max(\bar{t}_i, \bar{t}_j) < t_j$$
$$\min(\bar{t}_i, \bar{t}_j) > t_i$$

Let us choose the indices so that $\bar{t}_j \geqslant \bar{t}_i$; if we still have $\bar{t}_j > (p-1)\,\bar{t}_i + 1$, we may repeat the interchange procedure; we will ultimately obtain transversals $\bar{T}_i$, $\bar{T}_j$ satisfying

$$\bar{t}_i \leqslant \bar{t}_j \leqslant (p-1)\,\bar{t}_i + 1.$$

We denote by $q_H$ the greatest number $k$ of transversals $T_1, T_2, \ldots, T_k$ in a partition of $H$.

THEOREM 2. *For any $k \leqslant q_H$, there exists a partition of the vertices of $H(p)$ into transversals $T_1, T_2, \ldots, T_k$ with cardinalities $t_1, t_2, \ldots, t_k$ satisfying:* $\max_i(t_i) \leqslant (p-1)\min_i(t_i) + 1$.

*Proof.* The theorem follows directly from the previous lemma: as long as we have in the partition two transversals $T_i$, $T_j$ satisfying $t_j > (p-1)\,t_i + 1$ we perform the interchange procedure described in the lemma. Finally we will obtain a partition with cardinalities $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k$ satisfying $(p-1)\,t_k + 1 \geqslant t_1$.

*Remark.* The partitioning problem of § 1 is in fact a problem of transversals in the dual hypergraph $H$ of $G$: each edge of $G$ is a vertex of $H$; to each vertex $x$ of $G$ we associate an edge $E_x$; it contains all vertices corresponding to edges of $G$ which are adjacent to $x$. Clearly no vertex of $H$ belongs to more than 2 edges. Coverings in $G$ correspond to transversals in $H$.

Since $p = 2$, interchanges may be performed whenever $|t_j - t_i| > 1$, this means that Theorem 1 holds.

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*Département de Mathématiques*
*EPFL,*
*26, Av. de Conr,*
*1007, Lausanne,*
*Switzerland*