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Algebraic L-Theory

IV. Polynomial Extension Rings

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Introduction

In Chapter XII of [1] Bass defines the notion of a *contracted functor*, as a functor

$$F: (\text{rings}) \rightarrow (\text{abelian groups})$$

such that the sequence

$$0 \rightarrow F(A) \xrightarrow{\begin{pmatrix} \bar{e}_+ \\ -\bar{e}_- \end{pmatrix}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_+ E_-)} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \rightarrow 0$$

is naturally split exact for any ring A (associative with 1), where

$$\bar{e}_\pm: A \rightarrow A[x^{\pm 1}] \quad \bar{E}_\pm: A[x^{\pm 1}] \rightarrow A[x, x^{-1}]$$

are inclusions in polynomial extensions of A , and

$$\begin{aligned} B: F(A[x, x^{-1}]) &\rightarrow LF(A) \\ &= \text{coker}((\bar{E}_+ \bar{E}_-): F(A[x]) \oplus F(A[x^{-1}]) \rightarrow F(A[x, x^{-1}])) \end{aligned}$$

is the natural projection. Theorem 7.4 of Chapter XII of [1], the “Fundamental Theorem” of algebraic K -theory, states that

$$K_1: (\text{rings}) \rightarrow (\text{abelian groups})$$

is a contracted functor such that

$$LK_1(A) = K_0(A)$$

up to natural isomorphism. Here, we obtain analogous results for the groups of algebraic L -theory considered in the previous instalments of this series ([5], [6], [7] – we shall refer to these as Parts I, II, III respectively). In Part I we defined L -theoretic functors

$$U_n, V_n: (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

for $n \pmod{4}$, using quadratic forms on $\begin{cases} \text{f.g. projective} \\ \text{f.g. free} \end{cases} A$ -modules for the $\begin{cases} U\text{-} \\ V\text{-} \end{cases}$ groups.

(The definitions are reviewed in §3 below, allowing this part to be read independently of the previous parts). It was shown in Part II that

$$V_n(A[x, x^{-1}]) = V_n(A) \oplus U_{n-1}(A)$$

if the involution $\bar{} : A \rightarrow A; a \mapsto \bar{a}$ is extended to $A[x, x^{-1}]$ by $\bar{x} = x^{-1}$. The main result of this part of the paper (Theorem 4.1) is a split exact sequence

$$0 \rightarrow V_n(A) \xrightarrow{\begin{pmatrix} \bar{E}+ \\ -\bar{E}- \end{pmatrix}} V_n(A[x]) \oplus V_n(A[x^{-1}]) \xrightarrow{(E+E-)} V_n(A[x, x^{-1}]) \xrightarrow{B} U_n(A) \rightarrow 0$$

for each $n \pmod{4}$, with the involution on A extended to $A[x^{\pm 1}]$, $A[x, x^{-1}]$ by $\bar{x} = x$. The proof depends on L -theoretic analogues (Lemmas 4.2, 4.3) of the Higman linearization trick (quoted in Lemma 2.2) and of a result from [2] (quoted in Lemma 2.3) on the automorphisms of $A[x, x^{-1}]$ -modules which are linear in x . A similar result has been obtained independently by Karoubi ([4]), using an L -theoretic analogue of the localization sequence of Chapter IX of [1].

Adopting the terminology of [1], we can say that each

$$V_n : (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

is a contracted functor, with

$$LV_n(A) = U_n(A)$$

up to natural isomorphism. Corollary 4.4 generalizes this “Fundamental Theorem” of algebraic L -theory to describe the intermediate L -groups $V_n^Q(A[x, x^{-1}])$, as defined in Part III, for suitable subgroups $Q \subseteq \tilde{K}_1(A[x, x^{-1}])$. Corollary 4.5 identifies the “lower L -theories” of Part II with the functors

$$L^m U_n : (\text{rings with involution}) \rightarrow (\text{abelian groups}) \quad (m > 0)$$

derived from U_n . (There is an obvious analogy here with the “lower K -theories” of Chapter XII of [1],

$$K_{-m} = L^m K_0 : (\text{rings}) \rightarrow (\text{abelian groups}).)$$

Corollary 4.6 describes the L -groups of polynomial extensions in several variables.

The work presented here was stimulated by a course of lectures on algebraic K -theory given by Hyman Bass at Cambridge University in the Lent Term of 1973.

§1. Contracted Functors

Let (rings) be the category of associative rings with 1, and 1-preserving ring morphisms. Let x be an invertible indeterminate over such a ring A commuting with every element of A , and define $A[x, x^{-1}]$, the ring of finite polynomials $\sum_{j=-\infty}^{\infty} x^j a_j$ in x, x^{-1} with coefficients $a_j \in A$. Let $A[x^{\pm 1}]$ be the subring of $A[x, x^{-1}]$ of poly-

nomials involving only non-negative powers of $x^{\pm 1}$. Let

$$\bar{e}_{\pm}: A \rightarrow A[x^{\pm 1}], \quad \bar{E}_{\pm}: A[x^{\pm 1}] \rightarrow A[x, x^{-1}], \quad \bar{e} = \bar{E}_{\pm} \bar{e}_{\pm}: A \rightarrow A[x, x^{-1}]$$

be the inclusions, and define left inverses

$$\varepsilon_{\pm}: A[x^{\pm 1}] \rightarrow A, \quad \varepsilon: A[x, x^{-1}] \rightarrow A$$

for \bar{e}_{\pm}, \bar{e} by $x^{\pm 1} \mapsto 1$.

A functor

$$F: (\text{rings}) \rightarrow (\text{abelian groups})$$

is *contracted* if the sequence

$$0 \rightarrow F(A) \xrightarrow{\begin{pmatrix} \bar{e}_{+} \\ -\bar{e}_{-} \end{pmatrix}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_{+} E_{-})} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \rightarrow 0$$

is exact for each A , and there is given a natural right inverse

$$\bar{B}: LF(A) \rightarrow F(A[x, x^{-1}])$$

for the natural projection

$$\begin{aligned} B: F(A[x, x^{-1}]) &\rightarrow LF(A) \\ &= \text{coker}((\bar{E}_{+} \bar{E}_{-}): F(A[x]) \oplus F(A[x^{-1}]) \rightarrow F(A[x, x^{-1}])), \end{aligned}$$

that is $B\bar{B} = 1: LF(A) \rightarrow LF(A)$. (This is just Definition 7.1 of Chapter XII of [1]).

LEMMA 1.1. *Let*

$$F, G: (\text{rings}) \rightarrow (\text{abelian groups})$$

be functors, and suppose given

i) *a natural left inverse*

$$E_{+}: F(A[x, x^{-1}]) \rightarrow F(A[x])$$

for

$$\bar{E}_{+}: F(A[x]) \rightarrow F(A[x, x^{-1}])$$

such that the square

$$\begin{array}{ccc} F(A[x^{-1}]) & \xrightarrow{E_{-}} & F(A[x, x^{-1}]) \\ \varepsilon_{-} \downarrow & & \downarrow E_{+} \\ F(A) & \xrightarrow{\bar{e}_{+}} & F(A[x]) \end{array}$$

commutes,

ii) *natural morphisms*

$$\bar{\eta}_{+}: G(A) \rightarrow L_{+}F(A) = \text{coker}(\bar{E}_{+}: F(A[x]) \rightarrow F(A[x, x^{-1}]))$$

$$\eta_{+}: L_{+}F(A) \rightarrow G(A)$$

such that $\eta_+ \bar{\eta}_+ = 1$, and such that the square

$$\begin{array}{ccc} L_+ F(A) & \xrightarrow{\eta_+} & G(A) \\ \Delta_+ \downarrow & & \downarrow \bar{\eta}_- \\ F(A[x, x^{-1}]) & \xrightarrow{\delta_-} & L_- F(A) \end{array}$$

commutes, where

$$\Delta_+ : L_+ F(A) \rightarrow F(A[x, x^{-1}])$$

is the right inverse for the natural projection

$$\delta_+ : F(A[x, x^{-1}]) \rightarrow L_+ F(A)$$

induced by

$$1 - \bar{E}_+ E_+ : F(A[x, x^{-1}]) \rightarrow F(A[x, x^{-1}]),$$

and δ_- , $\bar{\eta}_-$ are defined as δ_+ , $\bar{\eta}_+$ but with x^{-1} replacing x .

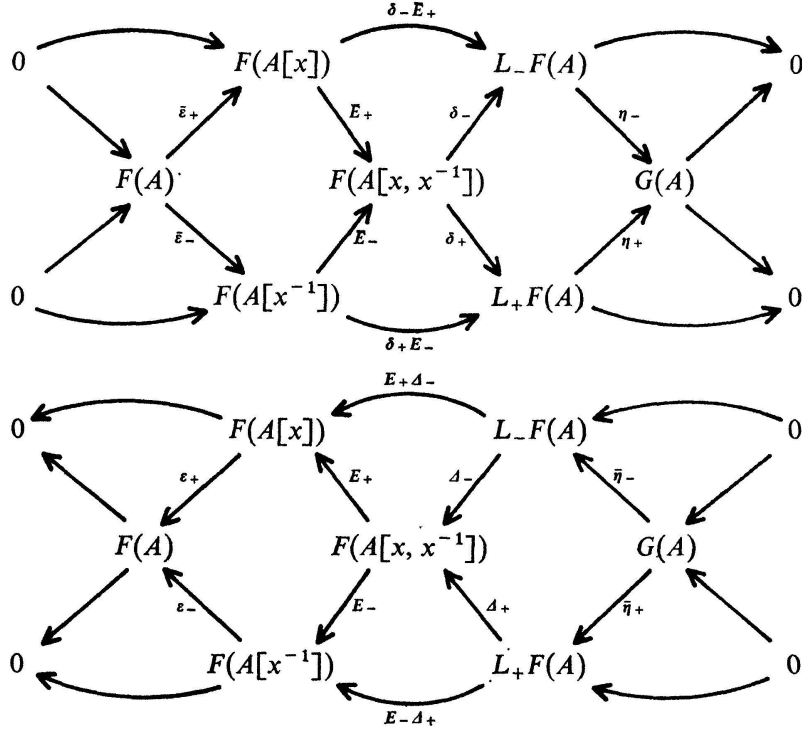
Then F is a contracted functor, and

$$B = \eta_+ \delta_+ : F(A[x, x^{-1}]) \rightarrow G(A)$$

induces a natural isomorphism

$$LF(A) = \text{coker}((\bar{E}_+ \bar{E}_-) : F(A[x]) \oplus F(A[x^{-1}]) \rightarrow F(A[x, x^{-1}])) \rightarrow G(A).$$

Proof. The diagrams



are commutative exact braids, where E_- , Δ_- , η_- are defined as E_+ , Δ_+ , η_+ but with x^{-1} replacing x . It follows that

$$0 \rightarrow F(A) \xrightarrow{\begin{pmatrix} \bar{\varepsilon}_+ \\ -\bar{\varepsilon}_- \end{pmatrix}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_+ E_-)} F(A[x, x^{-1}]) \xrightarrow{B} G(A) \rightarrow 0$$

is an exact sequence, with

$$\bar{B} = \Delta_{\pm} \bar{\eta}_{\pm} : G(A) \rightarrow F(A[x, x^{-1}])$$

a natural right inverse for

$$B = \eta_{\pm} \delta_{\pm} : F(A[x, x^{-1}]) \rightarrow G(A).$$

Thus F is a contracted functor, with

$$LF(A) = G(A)$$

up to natural isomorphism. \square

(The conditions of Lemma 1.1 are necessary, as well as sufficient, for a functor to be contracted. If

$$F : (\text{rings}) \rightarrow (\text{abelian groups})$$

is a contracted functor, then

$$F(A[x, x^{-1}]) = \bar{\varepsilon}F(A) \oplus \bar{E}_+ N_+ F(A) \oplus \bar{E}_- N_- F(A) \oplus \bar{B}LF(A)$$

where

$$N_{\pm} F(A) = \ker(\varepsilon_{\pm} : F(A[x^{\pm 1}]) \rightarrow F(A)),$$

and the morphisms

$$\begin{aligned} E_+ : F(A[x, x^{-1}]) &\rightarrow F(A[x]) = \bar{\varepsilon}_+ F(A) \oplus N_+ F(A); \\ \bar{\varepsilon}(r) \oplus \bar{E}_+(s_+) \oplus \bar{E}_-(s_-) \oplus \bar{B}(t) &\mapsto \bar{\varepsilon}_+(r) \oplus s_+ \\ \bar{\eta}_+ : LF(A) &\rightarrow L_+ F(A) = \bar{E}_- N_- F(A) \oplus \bar{B}LF(A); t \mapsto 0 \oplus \bar{B}(t) \\ \eta_+ : L_+ F(A) &\rightarrow LF(A); \bar{E}_-(s_-) \oplus \bar{B}(t) \mapsto t \end{aligned}$$

satisfy the conditions of Lemma 1.1, with $G = LF$.)

§2. K-Theory of Polynomial Extensions

Let $\mathbf{P}(A)$ be the category of finitely generated (f.g.) projective left A -modules. Write $|\mathbf{P}(A)|$ for the class of objects, and $\text{Hom}_A(P, Q)$ for the additive group of

morphisms $g: P \rightarrow Q \in \mathbf{P}(A)$. A ring morphism

$$f: A \rightarrow A'$$

induces a functor

$$f: \mathbf{P}(A) \rightarrow \mathbf{P}(A'); \begin{cases} P \in |\mathbf{P}(A)| \mapsto fP = A' \otimes_A P \in |\mathbf{P}(A')| \\ g \in \text{Hom}_A(P, Q) \mapsto fg = 1 \otimes g \in \text{Hom}_{A'}(fP, fQ). \end{cases}$$

Given $P \in |\mathbf{P}(A)|$, let

$$P[x^{\pm 1}] = \bar{e}_{\pm} P \in |\mathbf{P}(A[x^{\pm 1}])|, \quad P_x = \bar{e} P \in |\mathbf{P}(A[x, x^{-1}])|.$$

Defining complementary A -submodules

$$P^+ = \sum_{j=0}^{\infty} x^j P, \quad P^- = \sum_{j=-\infty}^{-1} x^j P$$

of P_x (where $x^j P = x^j \otimes P$) we shall identify

$$P^+ = P[x], \quad xP^- = P[x^{-1}]$$

in the obvious way.

Let $\mathbf{N}(A)$ be the category with objects pairs

$$(P \in |\mathbf{P}(A)|, v \in \text{Hom}_A(P, P) \text{ nilpotent})$$

and morphisms

$$f: (P, v) \rightarrow (P', v') \in \mathbf{N}(A)$$

isomorphisms $f \in \text{Hom}_A(P, P')$ such that

$$v'f = fv \in \text{Hom}_A(P, P').$$

As usual, there are defined functors

$$K_i: (\text{rings}) \rightarrow (\text{abelian groups}); \quad A \mapsto K_i(\mathbf{P}(A))$$

for $i=0,1$. Theorem 7.4 of Chapter XII of [1], the “Fundamental Theorem” of algebraic K -theory, may be stated and proved as follows:

THEOREM 2.1 *The functor K_1 is contracted, with*

$$L_+ K_1(A) = K_0 \mathbf{N}(A), \quad LK_1(A) = K_0(A)$$

up to natural isomorphism.

Proof. Given an automorphism

$$f: G_x \rightarrow G_x \in \mathbf{P}(A[x, x^{-1}]) \quad (G \in |\mathbf{P}(A)|)$$

let $F = f(G) \subseteq G_x$, and define

$$(P, v) = (G^- / x^{-N} F^-, x^{-1}) \in |\mathbf{N}(A)|$$

for $N \geq 0$ so large that $x^{-N} F^- \subseteq G^-$. Then

$$\begin{aligned} E_+ : K_1(A[x, x^{-1}]) &\rightarrow K_1(A[x]); \\ \tau(f: G_x \rightarrow G_x) &\mapsto \bar{e}_+ \tau(\varepsilon f: G \rightarrow G) \oplus \tau((1-v)^{-1}(1-xv): P^+ \rightarrow P^+) \end{aligned}$$

is a well-defined morphism.

LEMMA 2.2 *Every element of $K_1(A[x])$ can be represented by an automorphism*

$$f = f_0 + x f_1: G^+ \rightarrow G^+ \in \mathbf{P}(A[x])$$

with $f_0, f_1 \in \text{Hom}_A(G, G)$.

Proof. Given an automorphism

$$f = f_0 + x f_1 + x^2 f_2 + \cdots + x^r f_r \in \text{Hom}_{A[x]}(G^+, G^+) \quad (f_j \in \text{Hom}_A(G, G), 0 \leq j \leq r)$$

we can apply the usual Higman linearization trick (first used in the proof of Theorem 15 of [3]), the identity

$$\begin{aligned} \begin{pmatrix} 1 & -x^{r-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x f_r & 1 \end{pmatrix} \\ = \begin{pmatrix} f_0 + x f_1 + \cdots + x^{r-1} f_{r-1} & -x^{r-1} \\ x f_r & 1 \end{pmatrix}: G^+ \oplus G^+ \rightarrow G^+ \oplus G^+ \end{aligned}$$

$(r-1)$ times, to obtain a representative automorphism for $\tau(f) \in K_1(A[x])$ which is linear in x (with $r=1$). \square

Given an automorphism

$$f = f_0 + x f_1 \in \text{Hom}_{A[x]}(G^+, G^+)$$

let $\gamma = (f_0 + f_1)^{-1} f_1 \in \text{Hom}_A(G, G)$. Then

$$f = (f_0 + f_1) (1 + (x-1) \gamma): G^+ \rightarrow G^+$$

and (up to isomorphism)

$$(G^- / x^{-1} f(G^-), x^{-1}) = (G^- / x^{-1} (1 + (x-1) \gamma) G^-, x^{-1}) = (G, -\gamma(1-\gamma)^{-1}) \in |\mathbf{N}(A)|.$$

It follows that

$$\begin{aligned} E_+ \bar{E}_+ \tau(f) &= \tau(f_0 + f_1: G^+ \rightarrow G^+) \oplus \tau((1 + \gamma(1 - \gamma)^{-1})^{-1} \\ &\quad \times (1 + x\gamma(1 - \gamma)^{-1}): G^+ \rightarrow G^+) \\ &= \tau(f_0 + f_1: G^+ \rightarrow G^+) \oplus \tau(1 + (x - 1)\gamma: G^+ \rightarrow G^+) \\ &= \tau(f) \in K_1(A[x]). \end{aligned}$$

Thus the composite

$$K_1(A[x]) \xrightarrow{E_+} K_1(A[x, x^{-1}]) \xrightarrow{E_+} K_1(A[x])$$

is the identity. Similarly, it can be shown that the square

$$\begin{array}{ccc} K_1(A[x^{-1}]) & \xrightarrow{E_-} & K_1(A[x, x^{-1}]) \\ \varepsilon_- \downarrow & & \downarrow E_+ \\ K_1(A) & \xrightarrow{\bar{\varepsilon}_+} & K_1(A[x]) \end{array}$$

commutes.

Higman's trick also shows that every element of $K_1(A[x, x^{-1}])$ may be expressed as

$$\tau = \tau(f_0 + xf_1: P_x \rightarrow P_x) \oplus \tau(x^N: Q_x \rightarrow Q_x) \in K_1(A[x, x^{-1}])$$

for some $P, Q \in |\mathbf{P}(A)|$, $f_0, f_1 \in \text{Hom}_A(P, P)$, $N \in \mathbb{Z}$.

LEMMA 2.3. *If $\gamma \in \text{Hom}_A(P, P)$ is such that*

$$1 + (x - 1)\gamma \in \text{Hom}_{A[x, x^{-1}]}(P_x, P_x)$$

is an isomorphism then there exist integers $r, s \geq 0$ such that

$$\gamma^r(1 - \gamma)^s = 0 \in \text{Hom}_A(P, P),$$

and $R = \ker \gamma^r$, $S = \ker(1 - \gamma)^s$ are complementary submodules of P , such that

$$\gamma = \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix}: P = R \oplus S \rightarrow P = R \oplus S$$

with $\gamma_R \in \text{Hom}_A(R, R)$, $1 - \gamma_S \in \text{Hom}_A(S, S)$ nilpotent.

Proof. See Corollary 2.4 of [2] and pp. 232–34 of [8]. \square

If $f_0, f_1 \in \text{Hom}_A(P, P)$ are such that

$$f = f_0 + xf_1 \in \text{Hom}_{A[x, x^{-1}]}(P_x, P_x)$$

is an isomorphism, then

$$\varepsilon f = f_0 + f_1 \in \text{Hom}_A(P, P)$$

is an isomorphism, and $\gamma = (f_0 + f_1)^{-1} f_1 \in \text{Hom}_A(P, P)$ satisfies the hypothesis of Lemma 2.3. Hence

$$\begin{aligned} \tau(f) &= \bar{\varepsilon} \tau(f_0 + f_1 : P \rightarrow P) \oplus \tau(1 + (x-1) \gamma : P_x \rightarrow P_x) \\ &= \bar{\varepsilon} \tau(f_0 + f_1 : P \rightarrow P) \\ &\quad \oplus \bar{E}_+ \tau(1 + (x-1) \gamma_R : R[x] \rightarrow R[x]) \\ &\quad \oplus \bar{E}_- \tau(1 + (x^{-1}-1) (1 - \gamma_S) : S[x^{-1}] \rightarrow S[x^{-1}]) \\ &\quad \oplus \tau(x : S_x \rightarrow S_x) \in K_1(A[x, x^{-1}]) \end{aligned}$$

It is now easy to verify that

$$K_1(A[x]) \begin{matrix} \xrightarrow{E_+} \\ \xleftarrow{E_+} \end{matrix} K_1(A[x, x^{-1}]) \begin{matrix} \xrightarrow{\delta_+} \\ \xleftarrow{\delta_+} \end{matrix} K_0\mathbf{N}(A)$$

is a direct sum system, with

$$\begin{aligned} \delta_+ : K_0\mathbf{N}(A) &\rightarrow K_1(A[x, x^{-1}]); [P, v] \mapsto \tau((1-v)^{-1}(x-v) : P_x \rightarrow P_x) \\ \delta_+ : K_1(A[x, x^{-1}]) &\rightarrow K_0\mathbf{N}(A); \tau(f : G_x \rightarrow G_x) \mapsto [G^+ / x^N F^+, x] - [F^+ / x^N F^+, x] \end{aligned}$$

where $F = f(G) \subseteq G_x$ (as before) and $N \geq 0$ is so large that $x^N F^+ \subseteq G^+$, (so that, in particular,

$$\delta_+ \tau(f_0 + x f_1 : P_x \rightarrow P_x) = [S, -\gamma_S^{-1}(1 - \gamma_S)] \in K_0\mathbf{N}(A).$$

Identifying

$$L_+ K_1(A) = K_0\mathbf{N}(A)$$

in this way, note that the morphisms

$$\begin{aligned} \eta_+ : K_0\mathbf{N}(A) &\rightarrow K_0(A); [P, v] \mapsto [P] \\ \bar{\eta}_+ : K_0(A) &\rightarrow K_0\mathbf{N}(A); [P] \mapsto [P, 0] \end{aligned}$$

are such that the conditions of Lemma 1.1 are satisfied. Hence

$$K_1 : (\text{rings}) \rightarrow (\text{abelian groups})$$

is a contracted functor, with

$$LK_1(A) = K_0(A)$$

up to natural isomorphism. This completes the proof of Theorem 2.1. \square

§3. Review of the Definitions of the L-Groups

Let (rings with involution) be the category of rings A (as in §1) with involution $\bar{} : A \rightarrow A; a \mapsto \bar{a}$ such that

$$\bar{\bar{1}} = 1, \overline{a+b} = \bar{a} + \bar{b}, \overline{ab} = \bar{b} \cdot \bar{a}, a = \bar{a} \quad \text{for all } a, b \in A.$$

As in Part I it will be assumed that f.g. free A -modules have a well-defined dimension.

Given a ring with involution A define a *duality* involution

$$*: \mathbf{P}(A) \rightarrow \mathbf{P}(A) \begin{cases} P \in |\mathbf{P}(A)| \mapsto P^* = \text{Hom}_A(P, A), \text{ left } A\text{-action by} \\ A \times P^* \rightarrow P^*; (a, p^*) \mapsto (p \mapsto p^*(p) \cdot \bar{a}) \\ f \in \text{Hom}_A(P, Q) \mapsto (f^*: Q^* \rightarrow P^*; q^* \mapsto (p \mapsto q^*(f(p)))) \end{cases}$$

using the natural isomorphisms

$$P \rightarrow P^{**}; p \mapsto (p^* \mapsto \overline{p^*(p)}) \quad (P \in |\mathbf{P}(A)|)$$

to identify

$$** = 1 : \mathbf{P}(A) \rightarrow \mathbf{P}(A).$$

An ε -hermitian product (over A) is a morphism

$$\theta : Q \rightarrow Q^* \in \mathbf{P}(A)$$

such that

$$\theta^* = \varepsilon \theta \in \text{Hom}_A(Q, Q^*),$$

where $\varepsilon = \pm 1$. A \pm form (over A) is a pair

$$(Q \in |\mathbf{P}(A)|, \varphi \in \text{Hom}_A(Q, Q^*)),$$

and

$$\theta = \varphi \pm \varphi^* \in \text{Hom}_A(Q, Q^*)$$

is the associated \pm hermitian product. An *isomorphism* of \pm forms

$$(f, \chi) : (Q, \varphi) \rightarrow (Q', \varphi')$$

is an isomorphism $f \in \text{Hom}_A(Q, Q')$ together with a morphism $\chi \in \text{Hom}_A(Q, Q^*)$ such that

$$f^* \varphi' f - \varphi = \chi \mp \chi^* \in \text{Hom}_A(Q, Q^*).$$

Such an isomorphism preserves the associated \pm hermitian products, in that

$$f^*(\varphi' \pm \varphi'^*)f = (\varphi \pm \varphi^*) \in \text{Hom}_A(Q, Q^*).$$

A \pm form (Q, φ) is *non-singular* if the associated \pm hermitian product $(\varphi \pm \varphi^*) \in \text{Hom}_A(Q, Q^*)$ is an isomorphism. The *hamiltonian* \pm form on $P \in |\mathbf{P}(A)|$,

$$H\pm(P) = (P \oplus P^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$$

is non-singular. A *sublagrangian* of a non-singular \pm form (Q, φ) is a direct summand L of Q such that

$$j^*\varphi j = \lambda \mp \lambda^* \in \text{Hom}_A(L, L^*)$$

for some $\lambda \in \text{Hom}_A(L, L^*)$, denoting by $j \in \text{Hom}_A(L, Q)$ the inclusion. It was shown in Theorem 1.1 of Part I that if L is a sublagrangian of (Q, φ) there is defined a non-singular \pm form $(L^\perp/L, \hat{\varphi})$ on a direct complement L^\perp/L to L in the *annihilator* of L in (Q, φ) ,

$$L^\perp = \ker(j^*(\varphi \pm \varphi^*): Q \rightarrow L^*),$$

and that there is defined an isomorphism of \pm forms

$$(f, \chi): (Q, \varphi) \rightarrow H\pm(L) \oplus (L^\perp/L, \hat{\varphi})$$

with f the identity on $L^\perp = L \oplus L^\perp/L$. A *lagrangian* is a sublagrangian L such that

$$L^\perp = L,$$

in which case there is defined an isomorphism of \pm forms

$$(f, \chi): (Q, \varphi) \rightarrow H\pm(L).$$

A \pm *formation* (over A), $(Q, \varphi; F, G)$, is a triple consisting of

- i) a non-singular \pm form over A , (Q, φ) ,
- ii) a lagrangian F of (Q, φ) ,
- iii) a sublagrangian G of (Q, φ) .

An *isomorphism* of \pm formations

$$(f, \chi): (Q, \varphi; F, G) \rightarrow (Q', \varphi'; F', G')$$

is an isomorphism of \pm forms

$$(f, \chi): (Q, \varphi) \rightarrow (Q', \varphi')$$

such that $f(F)=F'$, $f(G)=G'$. A *stable isomorphism* of \pm formations

$$[f, \chi]: (Q, \varphi; F, G) \rightarrow (Q', \varphi'; F', G')$$

is an isomorphism of \pm formations

$$(f, \chi): (Q, \varphi; F, G) \oplus (H \pm (P); P, P^*) \rightarrow (Q', \varphi'; F', G') \oplus (H \pm (P'); P', P'^*)$$

defined for some $P, P' \in |\mathbf{P}(A)|$.

Let $T \subseteq \tilde{K}_0(A) = \text{coker}(K_0(\mathbf{Z}) \rightarrow K_0(A))$ be a subgroup invariant under the duality involution

$$*: \tilde{K}_0(A) \rightarrow \tilde{K}_0(A); [P] \mapsto [P^*] \quad (\text{that is, } *(T) = T).$$

For $n \pmod{4}$ define the abelian monoid $X_n^T(A)$ of $\begin{cases} \text{isomorphism} \\ \text{stable isomorphism} \end{cases}$

classes of $\begin{cases} \pm \text{ forms } (Q, \varphi) \\ \pm \text{ formations } (Q, \varphi; F, G) \end{cases}$ over A such that the projective class $\begin{cases} [Q] \\ [G] - [F^*] \end{cases}$ lies in $T \subseteq \tilde{K}_0(A)$, under the direct sum \oplus , with $\pm = (-)^i$ if $n = \begin{cases} 2i \\ 2i+1 \end{cases}$.

The monoid morphisms

$$\partial^T: X_n^T(A) \rightarrow X_{n-1}^T(A); \begin{cases} (Q, \varphi) \mapsto (H_{\mp}(Q); Q, \Gamma_{(Q, \varphi)}) \\ (Q, \varphi; F, G) \mapsto (G^\perp/G, \hat{\varphi}) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

are such that $(\partial^T)^2 = 0$, where

$$\Gamma_{(Q, \varphi)} = \{(x, (\varphi \pm \varphi^*)x) \mid x \in Q\} \subseteq Q \oplus Q^*.$$

Define an equivalence relation \sim on $\ker(\partial^T: X_n^T(A) \rightarrow X_{n-1}^T(A))$ by $z_1 \sim z_2$ if there exist $b_1, b_2 \in X_{n+1}^T(A)$ such that $z_1 \oplus \partial^T b_1 = z_2 \oplus \partial^T b_2 \in X_n^T(A)$. It was shown in Theorem 2.1 of Part III that the quotient monoids

$$U_n^T(A) = \ker(\partial^T: X_n^T(A) \rightarrow X_{n-1}^T(A)) / \overline{\text{im}(\partial^T: X_{n+1}^T(A) \rightarrow X_n^T(A))}$$

of equivalence classes are abelian groups, generalizing the definitions in Part I of

$$U_n(A) = U_n^{K_0(A)}(A), \quad V_n(A) = U_n^{\{0\}}(A).$$

Theorem 2.3 of Part III established an exact sequence

$$\cdots \rightarrow H^{n+1}(T'/T) \rightarrow U_n^T(A) \rightarrow U_n^{T'}(A) \rightarrow H^n(T'/T) \rightarrow U_{n-1}^T(A) \rightarrow \cdots$$

for $*$ -invariant subgroups $T \subseteq T' \subseteq \tilde{K}_0(A)$, where

$$H^n(G) = \{g \in G \mid g^* = (-)^n g\} / \{h + (-)^n h^* \mid h \in G\}$$

are the Tate cohomology groups (abelian, of exponent 2).

There are analogous definitions and results for L -groups associated with subgroups $R \subseteq \tilde{K}_1(A) = \text{coker}(K_1(\mathbb{Z}) \rightarrow K_1(A))$ invariant under the duality involution

$$*: \tilde{K}_1(A) \rightarrow \tilde{K}_1(A); \tau(f: \underline{P} \rightarrow \underline{Q}) \mapsto \tau(f^*: \underline{Q}^* \rightarrow \underline{P}^*)$$

denoting by \underline{P} a f.g. free A -module P with a prescribed base, and by \underline{P}^* the dual based A -module.

A *based* \pm form (\underline{Q}, φ) is a \pm form (Q, φ) on a based A -module \underline{Q} . The *torsion* of a based \pm form (\underline{Q}, φ) is

$$\tau(\underline{Q}, \varphi) = \begin{cases} \tau(\varphi \pm \varphi^*: \underline{Q} \rightarrow \underline{Q}^*) \in \tilde{K}_1(A) & \text{if } (Q, \varphi) \text{ is non-singular} \\ 0 \in \tilde{K}_1(A) & \text{otherwise.} \end{cases}$$

An R -isomorphism of based \pm forms

$$(f, \chi): (\underline{Q}, \varphi) \rightarrow (\underline{Q}', \varphi')$$

is an isomorphism of the underlying forms

$$(f, \chi): (Q, \varphi) \rightarrow (Q', \varphi')$$

such that

$$\tau(f: \underline{Q} \rightarrow \underline{Q}') \in R \subseteq \tilde{K}_1(A).$$

A *based* \pm formation $(Q, \varphi; \underline{F}, \underline{G})$ is a \pm formation $(Q, \varphi; F, G)$ with bases for F, G and G^\perp/G . The *torsion* $\tau(Q, \varphi; \underline{F}, \underline{G}) \in \tilde{K}_1(A)$ of a based \pm formation is the torsion of the isomorphism

$$f: \underline{F} \oplus \underline{F}^* \rightarrow \underline{G} \oplus \underline{G}^* \oplus \underline{G^\perp/G}$$

in the isomorphism of \pm forms

$$(f, \chi): H \pm (F) \rightarrow H \pm (G) \oplus (G^\perp/G, \hat{\varphi})$$

given by Theorem 1.1 of Part I. An R -isomorphism of based \pm formations

$$(f, \chi): (Q, \varphi; \underline{F}, \underline{G}) \rightarrow (Q', \varphi'; \underline{F}', \underline{G}')$$

is an isomorphism of the underlying \pm formations such that the restrictions

$$\underline{F} \rightarrow \underline{F}', \underline{G} \rightarrow \underline{G}', \underline{G^\perp/G} \rightarrow \underline{G'^\perp/G'}$$

of f have torsions in $R \subseteq \tilde{K}_1(A)$. A *stable* R -isomorphism of based \pm formations

$$[f, \chi]: (Q, \varphi; F, G) \rightarrow (Q', \varphi'; F', G')$$

is an R -isomorphism

$$(f, \chi): (Q, \varphi; \underline{F}, \underline{G}) \oplus (H \pm (P); \underline{P}, \underline{P}^*) \rightarrow (Q', \varphi'; \underline{F}', \underline{G}') \oplus (H \pm (P'); \underline{P}', \underline{P}'^*)$$

defined for some based A -modules $\underline{P}, \underline{P}'$.

For $n \pmod{4}$ define the abelian monoid $Y_n^R(A)$ of $\begin{cases} R\text{-isomorphism} \\ \text{stable } R\text{-isomorphism} \end{cases}$ classes of based $\begin{cases} \pm \text{ forms} \\ \pm \text{ formations} \end{cases}$ over A with torsion in $R \subseteq \tilde{K}_1(A)$, under the direct sum \oplus , with $\pm = (-)^i$ if $n = \begin{cases} 2i \\ 2i+1 \end{cases}$. The monoid morphisms

$$\partial^R: Y_n^R(A) \rightarrow Y_{n-1}^R(A); \begin{cases} (Q, \varphi) \mapsto (H_{\mp}(Q); \underline{Q}, \Gamma_{(Q, \varphi)}) \\ (Q, \varphi; \underline{F}, \underline{G}) \mapsto (\underline{G}^\perp/\underline{G}, \hat{\varphi}) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

are such that $(\partial^R)^2 = 0$, and the quotient monoids

$$V_n^R(A) = \ker(\partial^R: Y_n^R(A) \rightarrow Y_{n-1}^R(A)) / \overline{\text{im}(\partial^R: Y_{n+1}^R(A) \rightarrow Y_n^R(A))}$$

are abelian groups (by Theorem 3.1 of Part III) generalizing the definitions in Part I of

$$V_n(A) = V_n^{K_1(A)}(A) (= U_n^{\{0\}}(A)), \quad W_n(A) = V_n^{\{0\}}(A).$$

Theorem 3.3 in Part III established an exact sequence

$$\cdots \rightarrow H^{n+1}(R'/R) \rightarrow V_n^R(A) \rightarrow V_n^{R'}(A) \rightarrow H^n(R'/R) \rightarrow V_{n-1}^R(A) \rightarrow \cdots$$

for $*$ -invariant subgroups $R \subseteq R' \subseteq \tilde{K}_1(A)$.

A morphism of rings with involution

$$f: A \rightarrow A'$$

such that $f(T) \subseteq T'$ (for some $*$ -invariant subgroups $T \subseteq \tilde{K}_0(A)$, $T' \subseteq \tilde{K}_0(A')$) induces abelian group morphisms

$$f: U_n^T(A) \rightarrow U_n^{T'}(A'); \begin{cases} (Q, \varphi) \mapsto (fQ, f\varphi) \\ (Q, \varphi; F, G) \mapsto (fQ, f\varphi; fF, fG) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1 \end{cases}.$$

Similarly, if $f(R) \subseteq R'$ (for $*$ -invariant subgroups $R \subseteq \tilde{K}_1(A)$, $R' \subseteq \tilde{K}_1(A')$) there are induced morphisms

$$f: V_n^R(A) \rightarrow V_n^{R'}(A') \quad (n \pmod{4}).$$

§4. L-Theory of Polynomial Extensions

Given a ring with involution A and an indeterminate x over A commuting with

every element of A extend the involution on A to the involution

$$\bar{\cdot} : A[x, x^{-1}] \rightarrow A[x, x^{-1}]; \quad \sum_{j=-\infty}^{\infty} x^j a_j \mapsto \sum_{j=-\infty}^{\infty} x^j \bar{a}_j$$

on $A[x, x^{-1}]$. This restricts to involutions on the subrings $A[x]$, $A[x^{-1}]$ of $A[x, x^{-1}]$. F. g, free $A[x]$ -modules have well-defined dimension, as do those over $A[x^{-1}]$, $A[x, x^{-1}]$. Thus the rings with involution $A[x^{\pm 1}]$, $A[x, x^{-1}]$ satisfy the conditions imposed on A in §3.

Call a functor

$$F: (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

contracted if the sequence

$$0 \rightarrow F(A) \xrightarrow{\begin{pmatrix} \bar{E}_+ \\ -\bar{E}_- \end{pmatrix}} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_+ E_-)} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \rightarrow 0$$

is exact for every ring with involution A and there is given a natural right inverse

$$\bar{B}: LF(A) \rightarrow F(A[x, x^{-1}])$$

for the natural projection

$$\begin{aligned} B: F(A[x, x^{-1}]) &\rightarrow LF(A) \\ &= \text{coker}((\bar{E}_+ \bar{E}_-): F(A[x]) \oplus F(A[x^{-1}]) \rightarrow F(A[x, x^{-1}])). \end{aligned}$$

The obvious analogue to Lemma 1.1 holds for functors

$$(\text{rings with involution}) \rightarrow (\text{abelian groups})$$

as does the following analogue of Theorem 2.1 for the L -theoretic functors of §3:

THEOREM 4.1. *Each of the functors*

$$V_n: (\text{rings with involution}) \rightarrow (\text{abelian groups}) \quad (n \pmod{4})$$

is contracted, with

$$LV_n(A) = U_n(A), \quad L_{\pm} V_n(A) = U_n^{K_0(A)}(A[x^{\mp 1}])$$

up to natural isomorphism, where $\tilde{K}_0(A) \equiv \bar{E}_{\mp} \tilde{K}_0(A) \subseteq \tilde{K}_0(A[x^{\mp 1}])$. \square

The proof of Theorem 4.1 in the case $n=2i$ will be similar to the proof of Theorem 2.1. The case $n=2i+1$ will follow by an application of the results of Part II on the L -theory of Laurent extensions (that is, of the ring $A[x, x^{-1}]$ with involution by $\bar{x} = x^{-1}$).

Recall from Part II that a *modular A-base* of an $A[x, x^{-1}]$ -module Q is an A -submodule Q_0 of Q such that every element q of Q has a unique expression as

$$q = \sum_{j=-\infty}^{\infty} x^j q_j \quad (q_j \in Q_0, \{j \mid q_j \neq 0\} \text{ finite}),$$

so that $Q = A[x, x^{-1}] \otimes_A Q_0$ up to $A[x, x^{-1}]$ -module isomorphism. For example the A -modules generated by the bases of free $A[x, x^{-1}]$ -modules are modular A -bases.

Define a morphism

$$\begin{aligned} \delta_+ : V_{2i}(A[x, x^{-1}]) &\rightarrow U_{2i}^{K_0(A)}(A[x^{-1}]); \\ (Q, \varphi) &\mapsto (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) \end{aligned}$$

by choosing a modular A -base Q_0 for Q (which is a f.g. free $A[x, x^{-1}]$ -module) and an integer $N \geq 0$ so large that

$$(\varphi \pm \varphi^*)(x^N Q_0^+) \subseteq x^{-N} Q_0^{*+} \quad (\pm = (-)^i),$$

defining

$$P = x^N Q_0^- \cap (\varphi \pm \varphi^*)^{-1}(x^{-N} Q_0^{*+}) \in |\mathbf{P}(A)|,$$

with $[\varphi]_j \in \text{Hom}_A(P, P^*)$ given by

$$[\varphi]_j(y)(y') = a_j \in A \quad (y, y' \in P, j \in \mathbf{Z})$$

if

$$\varphi(y)(y') = \sum_{j=-\infty}^{\infty} x^j a_j \in A[x, x^{-1}] \quad (a_j \in A),$$

and writing $P[x^{-1}]$ for $\bar{\varepsilon}_- P = A[x^{-1}] \otimes_A P \in |\mathbf{P}(A[x^{-1}])|$.

The A -module isomorphism

$$[\varphi \pm \varphi^*]_{-1} : Q \rightarrow Q^*$$

may be expressed as

$$[\varphi \pm \varphi^*]_{-1} = \begin{pmatrix} [\varphi]_{-1} \pm ([\varphi]_{-1})^* & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \pm 1 & 0 \end{pmatrix} : P \oplus L \oplus L^* \rightarrow P^* \oplus L^* \oplus L$$

where $L = (\varphi \pm \varphi^*)^{-1}(x^{-N} Q_0^{*-})$, $L^* = x^N Q_0^+ \subseteq Q$, so that $(P, [\varphi]_{-1})$ is a non-singular \pm form over A .

For any $y, y' \in P$

$$\begin{aligned} [\varphi \pm \varphi^*]_{-2}(y)(y') &= [\varphi \pm \varphi^*]_{-1}(xy)(y') \\ &= [\varphi \pm \varphi^*]_{-1}(xy - x^N y_{N-1})(y') \in A, \end{aligned}$$

where $y_{N-1} \in Q_0$ is such that

$$y - x^{N-1} y_{N-1} \in x^{N-1} Q_0^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N-1} Q_0^*) = x^{-1} P.$$

Thus

$$(P, ([\varphi \pm \varphi^*]_{-1})^{-1} ([\varphi \pm \varphi^*]_{-2})) = ((\varphi \pm \varphi^*)^{-1} (x^{-N} Q_0^{*+}) / x^N Q_0^+, x) \in |\mathbf{N}(A)|,$$

and $(P[x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2})$ is a non-singular \pm form over $A[x^{-1}]$.

Suppose that Q'_0 is a different modular A -base of Q . Let $M \geq 0$ be so large that

$$Q'_0 \subseteq \sum_{j=-M}^M x^j Q_0, \quad Q_0 \subseteq \sum_{j=-M}^M x^j Q'_0.$$

Then $N' = N + M$ is so large that

$$(\varphi \pm \varphi^*) (x^{N'} Q_0'^+) \subseteq x^{-N'} Q_0'^{**+},$$

and

$$\begin{aligned} P' &= x^{N'} Q_0'^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N'} Q_0'^{**+}) \quad (\text{definition}) \\ &= x^N (x^M Q_0'^- \cap Q_0^+) \oplus P \oplus x^{-N} (\varphi \pm \varphi^*)^{-1} (Q_0^{*-} \cap x^{-M} Q_0'^{**+}). \end{aligned}$$

Now

$$L = (x^N (x^M Q_0'^- \cap Q_0^+)) [x^{-1}] \subseteq P' [x^{-1}]$$

is a sublagrangian of $(P' [x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2})$ with $L^\perp / L = P [x^{-1}]$, so that

$$\begin{aligned} (P' [x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2}) &= (P [x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2}) \oplus H_\pm(L) \\ &= (P [x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2}) \in U_{2i}^{K_0(A)}(A[x^{-1}]). \end{aligned}$$

Thus the choice of N and Q_0 is immaterial to the definition of δ_+ .

Finally, suppose that

$$(Q, \varphi) = \bar{E}_+(Q_0^+, \varphi_0) \in V_{2i}(A[x, x^{-1}])$$

for some $(Q_0^+, \varphi_0) \in V_{2i}(A[x])$. Then we can choose $N=0$, and

$$\delta_+(Q, \varphi) = 0 \in U_{2i}^{K_0(A)}(A[x^{-1}]).$$

Hence the morphism

$$\delta_+ : V_{2i}(A[x, x^{-1}]) \rightarrow U_{2i}^{K_0(A)}(A[x^{-1}])$$

is well-defined, and such that the composite

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{K_0(A)}(A[x^{-1}])$$

is zero. Before going on to show that this sequence is in fact split exact, we need an L -theoretic analogue of Lemma 2.2 (the Higman linearization trick):

LEMMA 4.2. *Every element of $U_{2i}^{K_0(A)}(A[x])$ (resp. $V_{2i}(A[x, x^{-1}])$) can be represented by a linear \pm form, $(Q^+, \varphi_0 + x\varphi_1)$ over $A[x]$ (resp. $(Q_x, \varphi_0 + x\varphi_1)$ over $A[x, x^{-1}]$) where $\varphi_0, \varphi_1 \in \text{Hom}_A(Q, Q^*)$.*

Proof. Given $(Q^+, \varphi) \in U_{2i}^{K_0(A)}(A[x])$, let

$$\varphi = \sum_{j=0}^N x^j \varphi_j \in \text{Hom}_{A[x]}(Q^+, Q^{*+}) \quad (\varphi_j \in \text{Hom}_A(Q, Q^*)),$$

and suppose $N > 1$. Now

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ \pm x^{N-1} \varphi_N^* & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -x^{N-1} \varphi_N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) : (Q^+, \varphi) \oplus H_{\pm}(Q^+) \rightarrow \left(Q^+ \oplus Q^+ \oplus Q^{*+}, \begin{pmatrix} \varphi - x^N \varphi_N & -x^{N-1} \varphi_N & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

is an isomorphism of \pm forms over $A[x]$, so that

$$(Q'^+, \varphi') = (Q^+, \varphi) \in U_{2i}^{K_0(A)}(A[x])$$

with $Q' = Q \oplus Q \oplus Q^*$ such that

$$\varphi' = \sum_{j=0}^{N-1} x^j \varphi'_j \in \text{Hom}_{A[x]}(Q'^+, Q'^{*+}) \quad (\varphi'_j \in \text{Hom}_A(Q', Q'^*)).$$

Iterating this procedure $(N-1)$ times we obtain a representative for

$$(Q^+, \varphi) \in U_{2i}^{K_0(A)}(A[x]) \text{ with } N=1.$$

The same method works for elements $(Q_x, \varphi) \in V_{2i}(A[x, x^{-1}])$ provided we can assume that

$$(\varphi \pm \varphi^*)(Q^+) \subseteq Q^{*+}.$$

Choosing $N \geq 0$ so large that

$$(\varphi \pm \varphi^*)(x^N Q^+) \subseteq x^{-N} Q^{*+},$$

note that

$$(x^N, 0) : (Q_x, \varphi' = x^{2N} \varphi) \rightarrow (Q_x, \varphi)$$

as an isomorphism of \pm forms over $A[x, x^{-1}]$, so that

$$(Q_x, \varphi') = (Q_x, \varphi) \in V_{2i}(A[x, x^{-1}]),$$

and that

$$(\varphi' \pm \varphi'^*) (Q^+) \subseteq Q^{*+}. \quad \square$$

The morphism

$$\begin{aligned} \Delta_+ : U_{2i}^{K_0(A)}(A[x^{-1}]) &\rightarrow V_{2i}(A[x, x^{-1}]); \\ (Q[x^{-1}], \varphi) &\mapsto (Q_x, x\varphi) \oplus \bar{\varepsilon}\varepsilon_-(Q[x^{-1}], -\varphi) \oplus H_{\pm}(-Q_x) \end{aligned}$$

is clearly well-defined, with $-Q \in |\mathbf{P}(A)|$ such that $Q \oplus -Q$ is f.g. free.

The composite

$$U_{2i}^{K_0(A)}(A[x^{-1}]) \xrightarrow{\Delta_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{K_0(A)}(A[x^{-1}])$$

is the identity: by Lemma 4.2 it is sufficient to consider $\delta_+ \Delta_+(Q[x^{-1}], \varphi)$ with

$$\varphi = \varphi_0 + x^{-1}\varphi_{-1} \in \text{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q^*[x^{-1}]) \quad (\varphi_0, \varphi_{-1} \in \text{Hom}_A(Q, Q^*)),$$

and

$$\begin{aligned} &\delta_+ \Delta_+(Q[x^{-1}], \varphi_0 + x^{-1}\varphi_{-1}) \\ &= \delta_+ ((Q_x, x\varphi_0 + \varphi_{-1}) \oplus (Q_x, -(\varphi_0 + \varphi_{-1})) \oplus H_{\pm}(-Q_x)) \\ &= ((Q^- \cap (x(\varphi_0 \pm \varphi_0^*) + (\varphi_{-1} \pm \varphi_{-1}^*)))^{-1} (Q^{*+})) [x^{-1}], \\ &[x\varphi_0 + \varphi_{-1}]_{-1} - x^{-1}[x\varphi_0 + \varphi_{-1}]_{-2} \\ &= ((1 + x^{-1}\gamma)^{-1} (x^{-1}Q), [x\varphi_0 + \varphi_{-1}]_{-1} - x^{-1}[x\varphi_0 + \varphi_{-1}]_{-2}) \end{aligned}$$

where $\gamma = (\varphi_0 \pm \varphi_0^*)^{-1} (\varphi_{-1} \pm \varphi_{-1}^*) \in \text{Hom}_A(Q, Q)$ is nilpotent. Now

$$(1 + x^{-1}\gamma)^{-1} = \sum_{j=0}^{\infty} (-)^j x^{-j} \gamma^j \in \text{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q[x^{-1}]),$$

so that

$$\begin{aligned} [x\varphi_0 + \varphi_{-1}]_j (1 + y^{-1}\gamma)^{-1} (x^{-1}y) (1 + x^{-1}\gamma)^{-1} (x^{-1}y') \\ = \begin{cases} \varphi_0(y) (y') \\ (\varphi_{-1} - \varphi_0\gamma - \gamma^*\varphi_0) (y) (y') \end{cases} \quad \text{if } j = \begin{cases} -1 \\ -2 \end{cases} \quad (y, y' \in Q), \end{aligned}$$

and

$$\varphi_{-1} - \varphi_0\gamma - \gamma^*\varphi_0 = -\varphi_{-1} + \chi \mp \chi^* \in \text{Hom}_A(Q, Q^*),$$

where $\chi = \varphi_{-1} - \gamma^*\varphi_0 \in \text{Hom}_A(Q, Q^*)$. Thus

$$\begin{aligned} \delta_+ \Delta_+(Q[x^{-1}], \varphi_0 + x^{-1}\varphi_{-1}) &= (Q[x^{-1}], \varphi_0 + x^{-1}(\varphi_{-1} - (\chi \mp \chi^*))) \\ &= (Q[x^{-1}], \varphi_0 + x^{-1}\varphi_{-1}) \in U_{2i}^{K_0(A)}(A[x^{-1}]) \end{aligned}$$

and

$$\delta_+ \Delta_+ = 1 : U_{2i}^{K_0(A)}(A[x^{-1}]) \rightarrow U_{2i}^{K_0(A)}(A[x^{-1}]).$$

It is therefore sufficient to prove that $V_{2i}(A[x, x^{-1}])$ is generated by the images of $\bar{E}_+ : V_{2i}(A[x]) \rightarrow V_{2i}(A[x, x^{-1}])$, $\Delta_+ : U_{2i}^{K_0(A)}(A[x^{-1}]) \rightarrow V_{2i}(A[x, x^{-1}])$ for the exactness of

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{K_0(A)}(A[x^{-1}]).$$

We shall do this using the following L-theoretic analogue of Lemma 2.3:

LEMMA 4.3. *Let (Q_x, φ) be a non-singular \pm form over $A[x, x^{-1}]$ such that*

$$\varphi = \mu + (x-1) \nu \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*) \quad (\mu, \nu \in \text{Hom}_A(Q, Q^*)).$$

Then (Q_x, φ) is isomorphic to the sum

$$(R_x, \mu_R + (x-1) \nu_R) \oplus (S_x, \mu_S + (x-1) \nu_S)$$

of non-singular \pm forms over $A[x, x^{-1}]$ such that

$$(R[x], \mu_R + (x-1) \nu_R)$$

is a non-singular \pm form over $A[x]$, and

$$(S[x^{-1}], x^{-1}(\mu_S + (x-1) \nu_S))$$

is a non-singular \pm form over $A[x^{-1}]$.

Proof. The invertibility of

$$\varphi \pm \varphi^* = (\mu \pm \mu^*) + (x-1)(\nu \pm \nu^*) \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$$

implies that

$$\varepsilon(\varphi \pm \varphi^*) = \mu \pm \mu^* \in \text{Hom}_A(Q, Q^*)$$

$$(\mu \pm \mu^*)^{-1}(\varphi \pm \varphi^*) = 1 + (x-1)\gamma \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x)$$

are isomorphisms, where

$$\gamma = (\mu \pm \mu^*)^{-1}(\nu \pm \nu^*) \in \text{Hom}_A(Q, Q).$$

Hence, by Lemma 2.3,

$$\gamma = \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} : Q = R \oplus S \rightarrow Q = R \oplus S$$

with $\gamma_R \in \text{Hom}_A(R, R)$, $1 - \gamma_S \in \text{Hom}_A(S, S)$ nilpotent.

Adding on some \mp hermitian products of type $\chi \mp \chi^* \in \text{Hom}_A(Q, Q^*)$ to μ and ν if necessary, it may be assumed that $\mu(R)(S)=0$, $\nu(R)(S)=0$. Let

$$\mu = \begin{pmatrix} \mu_R & \mu_{RS} \\ 0 & \mu_S \end{pmatrix} : R \oplus S \rightarrow R^* \oplus S^*, \quad \nu = \begin{pmatrix} \nu_R & \nu_{RS} \\ 0 & \nu_S \end{pmatrix} : R \oplus S \rightarrow R^* \oplus S^*$$

so that

$$\begin{pmatrix} \mu_R \pm \mu_R^* & \mu_{RS} \\ \pm \mu_{RS}^* & \mu_S \pm \mu_S^* \end{pmatrix} \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} = \begin{pmatrix} \nu_R \pm \nu_R^* & \nu_{RS} \\ \pm \nu_{RS}^* & \nu_S \pm \nu_S^* \end{pmatrix} : R \oplus S \rightarrow R^* \oplus S^*.$$

Working as in the calculation of $\delta_+ A_+$ above,

$$\begin{aligned} \delta_+(Q_x, \varphi) &= ((Q^- \cap (\varphi \pm \varphi^*)^{-1} (Q^{*+})) [x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2}) \\ &= ((1 + (x-1) \gamma_S)^{-1} (S) [x^{-1}], [\mu_S + (x-1) \nu_S]_{-1} - x^{-1} [\mu_S + (x-1) \nu_S]_{-2}) \\ &= (S [x^{-1}], x^{-1} (\mu_S + (x-1) \nu_S)) \in U_{2i}^{K_0(A)}(A [x^{-1}]). \end{aligned}$$

Thus $\varepsilon_- \delta_+(Q_x, \varphi) = (S, \mu_S)$ is a non-singular \pm form over A , and hence so is (S, ν_S) , because

$$(\nu_S \pm \nu_S^*) = (\mu_S \pm \mu_S^*) \gamma_S \in \text{Hom}_A(S, S^*)$$

and $\gamma_S \in \text{Hom}_A(S, S)$ is an isomorphism (being unipotent). Let

$$\begin{aligned} g &= \pm (\nu_S \pm \nu_S^*)^{-1} \nu_{RS}^* \in \text{Hom}_A(R, S) \\ \mu' &= \begin{pmatrix} \mu'_R = \mu_R - g^* \mu_S g & 0 \\ 0 & \mu_S \end{pmatrix} : R \oplus S \rightarrow R^* \oplus S^* \\ \nu' &= \begin{pmatrix} \nu'_R = \nu_R - g^* \nu_S g & 0 \\ 0 & \nu_S \end{pmatrix} : R \oplus S \rightarrow R^* \oplus S^*. \end{aligned}$$

Now

$$\begin{aligned} (f, \chi) &= \left(\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ (\mu_S + (x-1) \nu_S) g & 0 \end{pmatrix} \right) \\ &: (Q_x, \varphi) = (R_x \oplus S_x, \mu + (x-1) \nu) \rightarrow (Q_x, \varphi') = (R_x \oplus S_x, \mu' + (x-1) \nu') \end{aligned}$$

is an isomorphism of \pm forms over $A[x, x^{-1}]$. It follows that

$$f^*(\varphi' \pm \varphi'^*) f = (\varphi \pm \varphi^*) \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$$

and as f is defined over A

$$\begin{aligned} f^*(\mu' \pm \mu'^*) f &= (\mu \pm \mu^*) \in \text{Hom}_A(Q, Q^*) \\ f^*(\nu' \pm \nu'^*) f &= (\nu \pm \nu^*) \in \text{Hom}_A(Q, Q^*). \end{aligned}$$

Defining

$$\gamma' = (\mu' \pm \mu'^*)^{-1} (v' \pm v'^*) = \begin{pmatrix} \gamma'_R = (\mu'_R \pm \mu'^*_R)^{-1} (v_R \pm v'_R) & 0 \\ 0 & \gamma_S \end{pmatrix} : R \oplus S \rightarrow R \oplus S,$$

we have that

$$\gamma' = f\gamma f^{-1} = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} = \begin{pmatrix} \gamma_R & 0 \\ g\gamma_R - \gamma_S g & \gamma_S \end{pmatrix} : R \oplus S \rightarrow R \oplus S.$$

Hence

$$\gamma'_R = \gamma_R \in \text{Hom}_A(R, R)$$

is nilpotent, and $(R[x], \mu'_R + (x-1)v'_R)$ is a non-singular \pm form over $A[x]$. This completes the proof of Lemma 4.3. \square

Given $(Q_x, \varphi) \in V_{2i}(A[x, x^{-1}])$ it may be assumed, by Lemma 4.2, that $\varphi = \mu + (x-1)v \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$ ($\mu, v \in \text{Hom}_A(Q, Q^*)$). Applying the decomposition of Lemma 4.3,

$$\begin{aligned} (Q_x, \varphi) &= (R_x, \mu_R + (x-1)v_R) \oplus (S_x, \mu_S + (x-1)v_S) \\ &= \{(R_x, \mu_R + (x-1)v_R) \oplus (S_x, \mu_S)\} \oplus \{(S_x, \mu_S + (x-1)v_S) \\ &\quad \oplus (S_x, -\mu_S) \oplus H_{\pm}(-S_x)\} \\ &= \bar{E}_+((R[x], \mu_R + (x-1)v_R) \oplus (S[x], \mu_S)) \\ &\quad \oplus \Delta_+(S[x^{-1}], x^{-1}(\mu_S + (x-1)v_S)) \in V_{2i}(A[x, x^{-1}]). \end{aligned}$$

As pointed out above, this suffices to prove the exactness of

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{K_0(A)}(A[x^{-1}]).$$

Define next a morphism

$$\begin{aligned} E_+ : V_{2i}(A[x, x^{-1}]) &\rightarrow V_{2i}(A[x]); \\ (Q_x, \varphi) &\mapsto ((\varphi \pm \varphi^*)^{-1} (x^{N_1+1} Q^{*-}) \cap x^{-N_1} Q^{*+})[x], [\varphi]_0 - x([\varphi]_1) \\ &\quad \oplus ((x^N Q^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N} Q^{*+})) [x], [\varphi]_{-1} - [\varphi]_{-2}) \end{aligned}$$

for $N, N_1 \geq 0$ so large that

$$(\varphi \pm \varphi^*)(Q) \subseteq \sum_{j=-2N}^{2N_1+1} x^j Q^*$$

with $Q \in |\mathbf{P}(A)|$ f.g. free. The verification that E_+ is well-defined is by analogy with that for δ_+ . Moreover, if

$$(Q_x, \varphi) = (R_x, \mu_R + (x-1)v_R) \oplus (S_x, \mu_S + (x-1)v_S)$$

(as in Lemma 4.3), then

$$E_+(Q_x, \varphi) = (R[x], \mu_R + (x-1)\nu_R) \oplus (S[x], \mu_S) \in V_{2i}(A[x]),$$

so that the composites

$$\begin{aligned} U_{2i}^{K_0(A)}(A[x^{-1}]) &\xrightarrow{A_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x]) \\ V_{2i}(A[x]) &\xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x]) \end{aligned}$$

are 0, 1 respectively. Thus

$$V_{2i}(A[x]) \begin{smallmatrix} \xrightarrow{E_+} \\ \xleftarrow{E_+} \end{smallmatrix} V_{2i}(A[x, x^{-1}]) \begin{smallmatrix} \xrightarrow{\delta_+} \\ \xleftarrow{A_+} \end{smallmatrix} U_{2i}^{K_0(A)}(A[x^{-1}])$$

defines a direct sum system, and we can identify

$$L_+ V_{2i}(A) = U_{2i}^{K_0(A)}(A[x^{-1}]).$$

Similarly, replacing x with x^{-1} , there is defined a direct sum system

$$V_{2i}(A[x^{-1}]) \begin{smallmatrix} \xrightarrow{E_-} \\ \xleftarrow{E_-} \end{smallmatrix} V_{2i}(A[x, x^{-1}]) \begin{smallmatrix} \xrightarrow{\delta_-} \\ \xleftarrow{A_-} \end{smallmatrix} U_{2i}^{K_0(A)}(A[x]),$$

allowing the identification

$$L_- V_{2i}(A) = U_{2i}^{K_0(A)}(A[x]).$$

The proof of Lemma 4.2 shows that every element $(Q[x^{-1}], \varphi) \in V_{2i}(A[x^{-1}])$ has a representative with

$$\varphi = \varphi_0 + x^{-1}\varphi_{-1} \in \text{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q^*[x^{-1}]) \quad (\varphi_0, \varphi_{-1} \in \text{Hom}_A(Q, Q^*)).$$

The composite

$$V_{2i}(A[x^{-1}]) \xrightarrow{E_-} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x])$$

sends such a representative to

$$\begin{aligned} E_+ \bar{E}_-(Q[x^{-1}], \varphi) &= (((\varphi \pm \varphi^*)^{-1} (xQ^{*-}) \cap Q^+) [x], [\varphi]_0 - [\varphi]_1) \\ &\quad \oplus ((xQ^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-1}Q^*)) [x], [\varphi]_{-1} - [\varphi]_{-2}) \\ &= (Q[x], \varphi_0) \oplus ((\varphi \pm \varphi^*)^{-1} (Q^* \oplus x^{-1}Q^*) [x], [\varphi]_{-1} \\ &\quad - [\varphi]_{-2}) \in V_{2i}(A[x, x^{-1}]). \end{aligned}$$

The A -module isomorphism

$$\begin{aligned} Q \oplus Q &\rightarrow (\varphi \pm \varphi^*)^{-1} (Q^* \oplus x^{-1}Q^*); \\ (y, y') &\mapsto (\varphi \pm \varphi^*)^{-1} ((\varphi_0 \pm \varphi_0^*) y, x^{-1} (((\varphi_0 \pm \varphi_0^*) + \varphi_{-1} \pm \varphi_{-1}^*) y + (\varphi_0 \pm \varphi_0^*) y')) \end{aligned}$$

defines an isomorphism of \pm forms over A

$$(Q \oplus Q, \begin{pmatrix} \varphi_0 + \varphi_{-1} & 0 \\ 0 & -\varphi_0 \end{pmatrix}) \rightarrow ((\varphi \pm \varphi^*)^{-1} (Q^* \oplus x^{-1} Q^*), [\varphi]_{-1} - [\varphi]_{-2}).$$

Therefore

$$\begin{aligned} E_+ \bar{E}_- (Q[x^{-1}], \varphi_0 + x^{-1} \varphi_{-1}) &= (Q[x], \varphi_0 + \varphi_{-1}) \oplus (Q[x] \oplus Q[x], \varphi_0 \oplus -\varphi_0) \\ &= (Q[x], \varphi_0 + \varphi_{-1}) \\ &= \bar{\varepsilon}_+ \varepsilon_- (Q[x^{-1}], \varphi_0 + x^{-1} \varphi_{-1}) \in V_{2i}(A[x]), \end{aligned}$$

and the square

$$\begin{array}{ccc} V_{2i}(A[x^{-1}]) & \xrightarrow{E_-} & V_{2i}(A[x, x^{-1}]) \\ \varepsilon_- \downarrow & & \downarrow E_+ \\ V_{2i}(A) & \xrightarrow{\bar{\varepsilon}_+} & V_{2i}(A[x]) \end{array}$$

commutes. Similarly, we can verify that the square

$$\begin{array}{ccc} U_{2i}^{K_0(A)}(A[x^{-1}]) & \xrightarrow{\eta_+} & U_{2i}(A) \\ \Delta_+ \downarrow & & \downarrow \bar{\eta}_- \\ V_{2i}(A[x, x^{-1}]) & \xrightarrow{\delta_-} & U_{2i}^{K_0(A)}(A[x]) \end{array}$$

commutes, where

$$\eta_{\pm} : U_{2i}^{K_0(A)}(A[x^{\mp 1}]) \rightarrow U_{2i}(A), \quad \bar{\eta}_{\pm} : U_{2i}(A) \rightarrow U_{2i}^{K_0(A)}(A[x^{\mp 1}])$$

are the morphisms induced by

$$\eta_{\pm} : A[x^{\mp 1}] \rightarrow A; \sum_{j=0}^{\infty} x^{\mp j} a_j \mapsto a_0, \quad \bar{\varepsilon}_{\mp} : A \rightarrow A[x^{\mp 1}]$$

respectively (so that $\eta_{\pm} \bar{\eta}_{\pm} = 1$). For

$$\begin{aligned} \delta_- \Delta_+ (Q[x^{-1}], \varphi = \varphi_0 + x^{-1} \varphi_{-1}) &= \delta_- ((Q_x, x\varphi) \oplus (Q_x, -(\varphi_0 + \varphi_{-1})) \oplus H_{\pm}(-Q_x)) \\ &= ((x^{-1} Q^+ \cap (\varphi \pm \varphi^*)^{-1} (Q^{*-})) [x], [x\varphi]_{-1} - x[x\varphi]_0) \\ &= ((x^{-1} Q) [x], [x\varphi]_{-1}) = (Q[x], \varphi_0) \\ &= \bar{\eta}_- \eta_+ (Q[x^{-1}], \varphi) \in U_{2i}^{K_0(A)}(A[x]). \end{aligned}$$

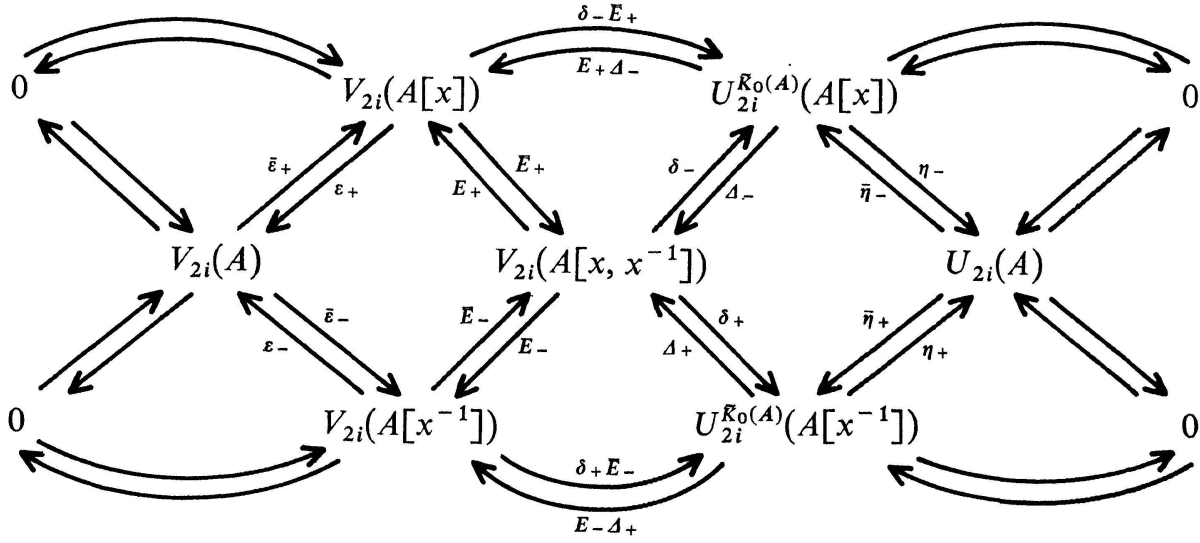
The conditions of Lemma 1.1 are now satisfied, and so

$$V_{2i} : (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

is a contracted functor, with

$$L_{\pm} V_{2i}(A) = U_{2i}^{K_0(A)}(A[x^{\mp 1}]), \quad LV_{2i}(A) = U_{2i}(A)$$

(up to natural isomorphisms), and the diagram

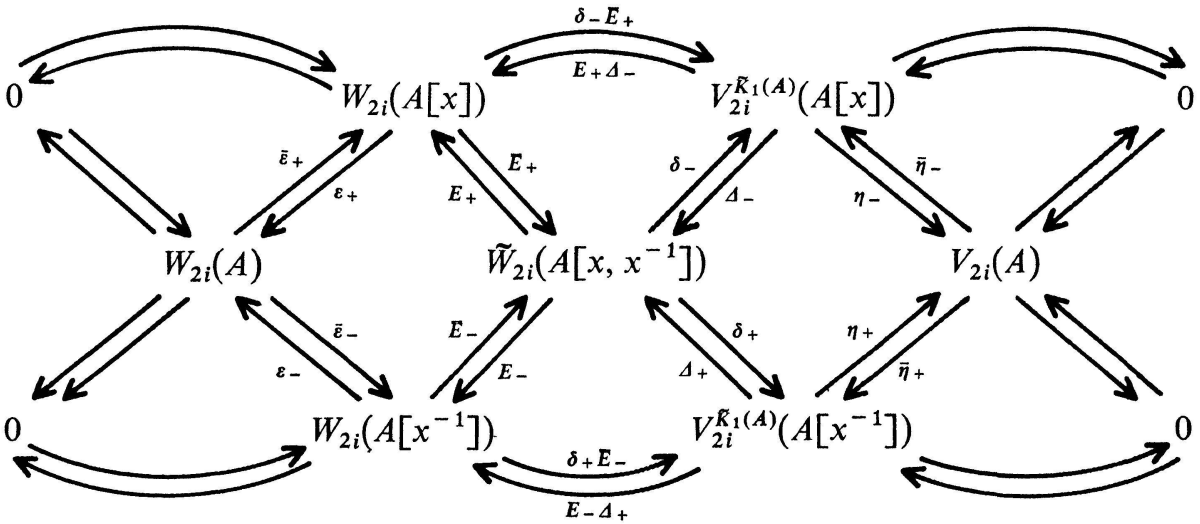


incorporates two commutative exact braids.

Let $S_0 \subseteq \tilde{K}_1(A[x, x^{-1}])$ be the infinite cyclic subgroup generated by $\bar{B}([A]) = \tau(x: A_x \rightarrow A_x)$, and define

$$\tilde{W}_n(A[x, x^{-1}]) = V_n^{S_0}(A[x, x^{-1}]) \quad (n \pmod{4}).$$

Working as for $V_{2i}(A[x, x^{-1}])$, it is possible to define morphisms to fit into a diagram



(with $E_+ \bar{E}_+ = 1$ etc.) incorporating two commutative exact braids. For example,

$$\begin{aligned} \delta_+ : \tilde{W}_{2i}(A[x, x^{-1}]) &\rightarrow V_{2i}^{K_1(A)}(A[x^{-1}]); (\mathcal{Q}_x, \varphi) \mapsto (\mathcal{P}[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) \\ E_+ : \tilde{W}_{2i}(A[x, x^{-1}]) &\rightarrow W_{2i}(A[x]); \\ &(\mathcal{Q}_x, \varphi) \mapsto (\mathcal{P}_1[x], [\varphi]_0 - x[\varphi]_1) \oplus (\mathcal{P}[x], [\varphi]_{-1} - [\varphi]_{-2}) \end{aligned}$$

for any A -base P of $P = x^N Q^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N} Q^{*+})$ (which is free for sufficiently large $N \geq 0$, as $\tau(\tilde{Q}_x, \varphi) \in S_0$ and $[P] = B\tau(\tilde{Q}_x, \varphi) = 0 \in \tilde{K}_0(A)$) with

$$P_1 = (\varphi \pm \varphi^*)^{-1} (x^N \tilde{Q}^*) \oplus (\varphi \pm \varphi^*)^{-1} (P^*)$$

the corresponding A -base of $P_1 = (\varphi \pm \varphi^*)^{-1} (x^{N+1} Q^{*-}) \cap x^{-N} Q^+$, for N so large that

$$(\varphi \pm \varphi^*) (Q) \subseteq \sum_{j=-2N}^{2N+1} x^j Q^*.$$

Also, let

$$\Delta_+ : V_{2i}^{K_1(A)}(A[x^{-1}]) \rightarrow \tilde{W}_{2i}(A[x, x^{-1}]); (Q[x^{-1}], \varphi) \mapsto (Q_x, x\varphi) \oplus (Q_x, -\bar{\varepsilon}\varepsilon_- \varphi)$$

where $Q = (\varepsilon_- (\varphi \pm \varphi^*))^{-1} (Q^*)$.

Given an invertible indeterminate z over A commuting with every element of A define A_z as $A[z, z^{-1}]$ but with involution by $\bar{z} = z^{-1}$. Similarly, define $A[x^{\pm 1}]_z$, $A[x, x^{-1}]_z$, and identify

$$A[x^{\pm 1}]_z = A_z[x^{\pm 1}], \quad A[x, x^{-1}]_z = A_z[x, x^{-1}].$$

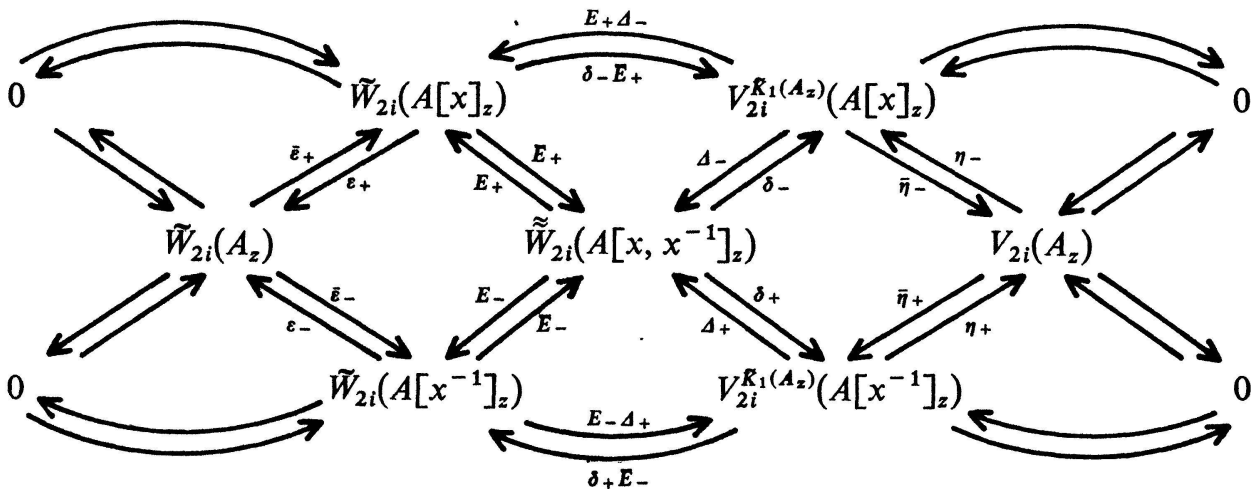
Let $S'_0 \subseteq \tilde{K}_1(A_z)$ be the infinite cyclic subgroup generated by $\tau(z : A_z \rightarrow A_z)$ and define

$$\tilde{W}_n(A_z) = V_n^{S'_0}(A_z)$$

$$\tilde{W}_n(A[x^{\pm 1}]_z) = V_n^{\bar{\varepsilon}_{\pm}(x)S'_0}(A[x^{\pm 1}]_z)$$

$$\tilde{W}_n(A[x, x^{-1}]_z) = V_n^{\bar{\varepsilon}(z)S_0 \oplus \bar{\varepsilon}(x)S'_0}(A[x, x^{-1}]_z)$$

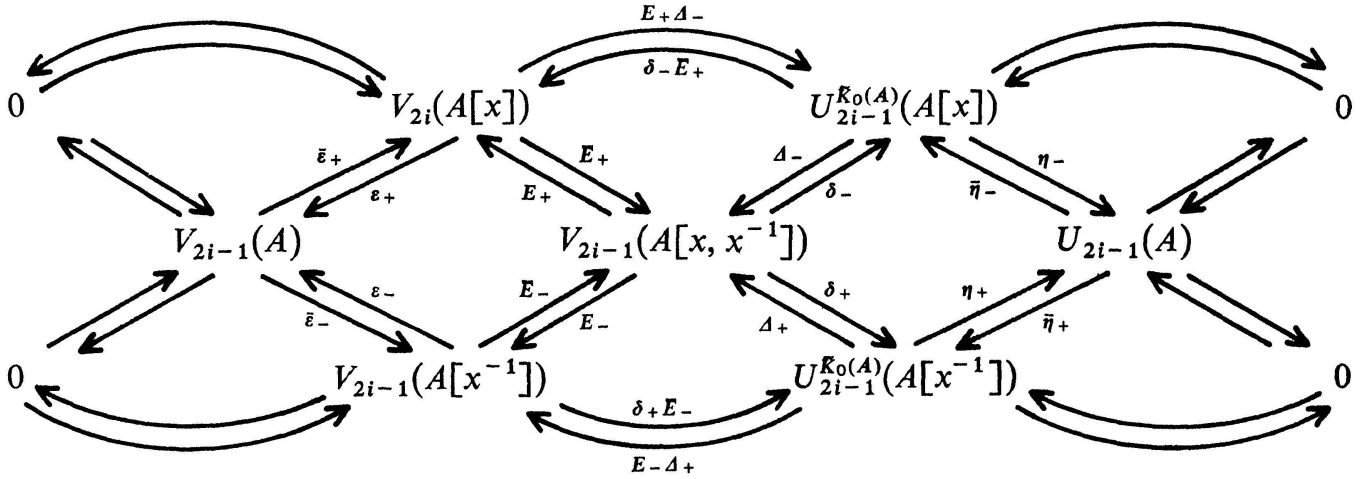
for $n \pmod{4}$. By analogy with $\tilde{W}_{2i}(A[x, x^{-1}])$, $\tilde{W}_{2i}(A[x, x^{-1}]_z)$ fits into a diagram incorporating two commutative exact braids (where $A_z = A[z, z^{-1}]$, with $\bar{z} = z^{-1}$).



We can now apply the decompositions

$$\begin{aligned}\tilde{W}_{2i}(A_z) &= \bar{\varepsilon}(z) W_{2i}(A) \oplus \bar{B}(z) V_{2i-1}(A) \\ \tilde{W}_{2i}(A[x]_z) &= \bar{\varepsilon}(z) W_{2i}(A[x]) \oplus \bar{B}(z) V_{2i-1}(A[x]) \\ \tilde{W}_{2i}(A[x, x^{-1}]_z) &= \bar{\varepsilon}(z) \tilde{W}_{2i}(A[x, x^{-1}]) \oplus \bar{B}(z) V_{2i-1}(A[x, x^{-1}]) \\ V_{2i}^{K_1(A_z)}(A[x]_z) &= \bar{\varepsilon}(z) V_{2i}^{K_1(A)}(A) \oplus \bar{B}(z) U_{2i}^{K_0(A)}(A) \\ V_{2i}(A_z) &= \bar{\varepsilon}(z) V_{2i}(A) \oplus \bar{B}(z) U_{2i-1}(A)\end{aligned}$$

given by Theorem 1.1 of Part II (and extended to the intermediate L -groups in Part III). The above diagram splits naturally (via $\bar{\varepsilon}(z), \bar{B}(z)$) into two similar ones: the diagram for $\tilde{W}_{2i}(A[x, x^{-1}])$ and the diagram



where

$$\begin{aligned}E_+ : V_{2i-1}(A[x, x^{-1}]) &\xrightarrow{\bar{B}(z)} \tilde{W}_{2i}(A[x, x^{-1}]_z) \xrightarrow{E_+} \tilde{W}_{2i}(A[x]_z) \xrightarrow{B(z)} V_{2i-1}(A[x]) \\ \delta_+ : V_{2i-1}(A[x, x^{-1}]) &\xrightarrow{\bar{B}(z)} \tilde{W}_{2i}(A[x, x^{-1}]_z) \\ &\xrightarrow{\delta_+} V_{2i}^{K_1(A_z)}(A[x^{-1}]_z) \xrightarrow{B(z)} U_{2i-1}^{K_0(A)}(A[x^{-1}]) \\ \Delta_+ : U_{2i-1}^{K_0(A)}(A[x^{-1}]) &\xrightarrow{\bar{B}(z)} V_{2i}^{K_1(A_z)}(A[x^{-1}]_z) \\ &\xrightarrow{\Delta_+} \tilde{W}_{2i}(A[x, x^{-1}]_z) \xrightarrow{B(z)} V_{2i-1}(A[x, x^{-1}])\end{aligned}$$

(and similarly for E_- , δ_- , Δ_-). Thus the conditions of Lemma 1.1 are also satisfied in the odd-dimensional case, and

$$V_{2i-1} : (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

is a contracted functor, with identifications

$$L_{\pm} V_{2i-1}(A) = U_{2i-1}^{K_0(A)}(A[x^{\mp 1}]), \quad LV_{2i-1}(A) = U_{2i-1}(A).$$

This completes the proof of Theorem 4.1 \square

The groups

$$\text{Nil}_{\pm}(A) = \ker(\varepsilon_{\pm} : K_1(A[x^{\pm 1}]) \rightarrow K_1(A))$$

are such that

$$\begin{aligned} K_1(A[x^{\pm 1}]) &= \bar{\varepsilon}_{\pm} K_1(A) \oplus \text{Nil}_{\pm}(A) \\ K_1(A[x, x^{-1}]) &= \bar{\varepsilon} K_1(A) \oplus \bar{E}_+ \text{Nil}_+(A) \oplus \bar{E}_- \text{Nil}_-(A) \oplus \bar{B} K_0(A), \end{aligned}$$

fitting into direct sum systems

$$\text{Nil}_{\pm}(A) \begin{matrix} \xrightarrow{\delta_{\pm} E_{\pm}} \\ \xleftarrow{E_{\pm} A_{\pm}} \end{matrix} K_0 \mathbf{N}(A) \begin{matrix} \xrightarrow{\eta_{\pm}} \\ \xleftarrow{\bar{\eta}_{\pm}} \end{matrix} K_0(A)$$

(by Theorem 2.1).

Given $*$ -invariant subgroups $S_{\pm} \subseteq \text{Nil}_{\pm}(A)$, define

$$N_{\pm} V_n^{S_{\pm}}(A) = \ker(\varepsilon_{\pm} : V_n^{\bar{\varepsilon}_{\pm} K_1(A) \oplus S_{\pm}}(A[x^{\pm 1}]) \rightarrow V_n(A)) \quad (n \pmod{4})$$

$$\text{writing } \begin{cases} N_{\pm} V_n(A) \\ N_{\pm} W_n(A) \end{cases} \text{ for } \begin{cases} N_{\pm} V_n^{\text{Nil}_{\pm}(A)}(A) \\ N_{\pm} V_n^{\{0\}}(A) \end{cases}.$$

COROLLARY 4.4. *Given $*$ -invariant subgroups*

$$R \subseteq \tilde{K}_1(A), \quad S_{\pm} \subseteq \text{Nil}_{\pm}(A), \quad \tilde{T} \subseteq \tilde{K}_0(A)$$

there are direct sum decompositions

$$\begin{aligned} V_n^{\bar{\varepsilon}_{\pm} R \oplus S_{\pm}}(A[x^{\pm 1}]) &= \bar{\varepsilon}_{\pm} V_n^R(A) \oplus N_{\pm} V_n^{S_{\pm}}(A) \\ U_n^{\bar{\varepsilon}_{\pm} \tilde{T}}(A[x^{\pm 1}]) &= \bar{\varepsilon}_{\pm} U_n^{\tilde{T}}(A) \oplus N_{\pm} V_n(A) \\ V_n^Q(A[x, x^{-1}]) &= \bar{\varepsilon} V_n^R(A) \oplus \bar{E}_+ N_+ V_n^{S_+}(A) \oplus \bar{E}_- N_- V_n^{S_-}(A) \oplus \bar{B} U_n^{\tilde{T}}(A) \end{aligned}$$

for $n \pmod{4}$, where

$$\begin{aligned} Q &= \bar{\varepsilon} R \oplus \bar{E}_+ S_+ \oplus \bar{E}_- S_- \oplus \bar{B} \tilde{T} \subseteq \tilde{K}_1(A[x, x^{-1}]) \\ &= \bar{\varepsilon} \tilde{K}_1(A) \oplus \bar{E}_+ \text{Nil}_+(A) \oplus \bar{E}_- \text{Nil}_-(A) \oplus \bar{B} K_0(A) \end{aligned}$$

with $T \subseteq K_0(A)$ the preimage of \tilde{T} under the natural projection $K_0(A) \rightarrow \tilde{K}_0(A)$.

Proof. The forgetful map

$$V_n(A[x^{\pm 1}]) \rightarrow U_n^{\bar{\varepsilon}_{\pm} \tilde{T}}(A[x^{\pm 1}])$$

fits into the exact sequence of Theorem 2.3 of Part III, which splits, via $\bar{\varepsilon}_\pm, \varepsilon_\pm$ into two exact sequences

$$\begin{array}{ccccccc} \rightarrow & 0 & \rightarrow & N_\pm V_n(A) & \rightarrow & N_\pm V_n(A) & \rightarrow 0 \rightarrow \\ & \updownarrow & & \updownarrow & & \updownarrow & \updownarrow \\ \rightarrow & H^{n+1}(\bar{\varepsilon}_\pm \tilde{T}) & \rightarrow & V_n(A[x^{\pm 1}]) & \rightarrow & U_n^{\bar{\varepsilon}_\pm \tilde{T}}(A[x^{\pm 1}]) & \rightarrow H^n(\bar{\varepsilon}_\pm \tilde{T}) \rightarrow \\ & \bar{\varepsilon}_\pm \updownarrow \varepsilon_\pm & & \bar{\varepsilon}_\pm \updownarrow \varepsilon_\pm & & \bar{\varepsilon}_\pm \updownarrow \varepsilon_\pm & \bar{\varepsilon}_\pm \updownarrow \varepsilon_\pm \\ \rightarrow & H^{n+1}(\tilde{T}) & \rightarrow & V_n(A) & \rightarrow & U_n^{\tilde{T}}(A) & \rightarrow H^n(\tilde{T}) \rightarrow . \end{array}$$

Hence $N_\pm V_n(A) \subseteq V_n(A[x^{\pm 1}])$ is mapped isomorphically to $\ker(\varepsilon_\pm : U_n^{\bar{\varepsilon}_\pm \tilde{T}}(A[x^{\pm 1}]) \rightarrow U_n^{\tilde{T}}(A))$ and so (up to isomorphism)

$$U_n^{\bar{\varepsilon}_\pm \tilde{T}}(A[x^{\pm 1}]) = \bar{\varepsilon}_\pm U_n^{\tilde{T}}(A) \oplus N_\pm V_n(A).$$

In particular,

$$\begin{aligned} U_n^{K_0(A)}(A[x^{\pm 1}]) &= \bar{\varepsilon}_\pm U_n(A) \oplus N_\pm V_n(A), \\ V_n(A[x^{\pm 1}]) &= \bar{\varepsilon}_\pm V_n(A) \oplus N_\pm V_n(A). \end{aligned}$$

It now follows from Theorem 4.1 that

$$V_n(A[x, x^{-1}]) = \bar{\varepsilon} V_n(A) \oplus \bar{E}_+ N_+ V_n(A) \oplus \bar{E}_- N_- V_n(A) \oplus \bar{B} U_n(A).$$

The expressions for $V_n^{\bar{\varepsilon}_\pm R \oplus S_\pm}(A[x^{\pm 1}])$, $V_n^Q(A[x, x^{-1}])$ may be deduced from those for $V_n(A[x^{\pm 1}])$, $V_n(A[x, x^{-1}])$, working as for $U_n^{\bar{\varepsilon}_\pm \tilde{T}}(A[x^{\pm 1}])$ above. (In particular, for $R=0$, $S_+=0$, $S_-=0$, $\tilde{T}=0$ we have

$$Q = S_0 \subseteq \tilde{K}_1(A[x, x^{-1}])$$

and

$$\begin{aligned} W_n(A[x^{\pm 1}]) &= \bar{\varepsilon}_\pm W_n(A) \oplus N_\pm W_n(A), \\ \tilde{W}_n(A[x, x^{-1}]) &= \bar{\varepsilon} W_n(A) \oplus \bar{E}_+ N_+ W_n(A) \oplus \bar{E}_- N_- W_n(A) \oplus \bar{B} V_n(A). \quad \square \end{aligned}$$

In §4 of Part II there were defined lower L -theories, functors

$$L_n^{(m)} : (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

for $m < 0$, $n \pmod{4}$ by

$$L_n^{(m)}(A) = \ker(\varepsilon : L_{n+1}^{(m+1)}(A_z) \rightarrow L_{n+1}^{(m+1)}(A))$$

with $L_n^{(0)}(A) = U_n(A)$. By convention, $L_n^{(1)}(A) = V_n(A)$.

COROLLARY 4.5. *The lower L -theories $L_n^{(m)}$ coincide (up to natural isomorphism)*

with the functors LV_n, L^2V_n, \dots derived from V_n , with

$$L_n^{(m)}(A) = L^{1-m}V_n(A) \quad (m \leq 0, n \pmod{4}).$$

Proof. By Theorem 4.1,

$$LV_n(A) = U_n(A) = L_n^{(0)}(A).$$

Assume inductively that

$$L_n^{(p)}(A) = L^{1-p}V_n(A) \quad (n \pmod{4})$$

for $0 \geq p > m$, for some $m \leq -1$. Then

$$\begin{aligned} L_n^{(m)}(A) &= \ker(\varepsilon: L_{n+1}^{(m+1)}(A_z) \rightarrow L_{n+1}^{(m+1)}(A)) \\ &= \ker(\varepsilon: L^{-m}V_{n+1}(A_z) \rightarrow L^{-m}V_{n+1}(A)) \\ &= L(\ker(\varepsilon: L^{-m-1}V_{n+1}(A_z) \rightarrow L^{-m-1}V_{n+1}(A))) \\ &= L(\ker(\varepsilon: L_{n+1}^{(m+2)}(A_z) \rightarrow L_{n+1}^{(m+2)}(A))) \\ &= LL_n^{(m+1)}(A) \\ &= LL^{-m}V_n(A) = L^{1-m}V_n(A) \end{aligned}$$

giving the induction step. \square

Given a functor

$$F: (\text{rings with involution}) \rightarrow (\text{abelian groups})$$

define

$$N_{\pm}F(A) = \ker(\varepsilon_{\pm}: F(A[x^{\pm 1}]) \rightarrow F(A)).$$

(By Corollary 4.4, the previous definitions of $N_{\pm}V_n(A)$, $N_{\pm}W_n(A)$ agree with this, up to natural isomorphism).

By analogy with the first part of Corollary 7.6 of Chapter XII of [1] we have

COROLLARY 4.6. *Let x_1, x_2, \dots, x_p be independent commuting indeterminates over A , with $\bar{x}_j = x_j$ ($1 \leq j \leq p$). Then*

$$\begin{aligned} L_n^{(m)}(A[x_1, x_2, \dots, x_p]) &= (1 \oplus N_+)^p L_n^{(m)}(A) \\ L_n^{(m)}(A[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_p, x_p^{-1}]) &= (1 \oplus N_+ \oplus N_- \oplus L)^p L_n^{(m)}(A) \end{aligned}$$

up to natural isomorphism, for $m \leq 1, n \pmod{4}, p \geq 1$. \square

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