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Algebraic L-Theory

IV. Polynomial Extension Rings

by A. A. RANICKI, Trinity College, Cambridge

Introduction

In Chapter XII of [1] Bass defines the notion of a contracted functor, as a functor $F:(rings) \rightarrow (abelian groups)$

such that the sequence

$$0 \to F(A) \xrightarrow{\left(-\frac{\bar{\epsilon}}{\epsilon}^{+}\right)} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E+E-)} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \to 0$$

is naturally split exact for any ring A (associative with 1), where

$$\bar{\varepsilon}_+: A \to A[x^{\pm 1}]$$
 $\bar{E}_+: A[x^{\pm 1}] \to A[x, x^{-1}]$

are inclusions in polynomial extensions of A, and

$$B: F(A[x, x^{-1}]) \to LF(A)$$

$$= \operatorname{coker}((\bar{E}_{+}\bar{E}_{-}): F(A[x]) \oplus F(A[x^{-1}]) \to F(A[x, x^{-1}]))$$

is the natural projection. Theorem 7.4 of Chapter XII of [1], the "Fundamental Theorem" of algebraic K-theory, states that

$$K_1: (rings) \rightarrow (abelian groups)$$

is a contracted functor such that

$$LK_1(A) = K_0(A)$$

up to natural isomorphism. Here, we obtain analogous results for the groups of algebraic L-theory considered in the previous instalments of this series ([5], [6], [7] – we shall refer to these as Parts I, II, III respectively). In Part I we defined L-theoretic functors

$$U_n, V_n$$
: (rings with involution) \rightarrow (abelian groups)

for $n \pmod{4}$, using quadratic forms on $\begin{cases} \text{f.g. projective} \\ \text{f.g. free} \end{cases}$ A-modules for the $\begin{cases} U_{\text{-groups}} \\ V_{\text{-groups}} \end{cases}$

(The definitions are reviewed in §3 below, allowing this part to be read independently of the previous parts). It was shown in Part II that

$$V_n(A[x, x^{-1}]) = V_n(A) \oplus U_{n-1}(A)$$

if the involution $\bar{a}: A \to A$; $a \mapsto \bar{a}$ is extended to $A[x, x^{-1}]$ by $\bar{x} = x^{-1}$. The main result of this part of the paper (Theorem 4.1) is a split exact sequence

$$0 \to V_n(A) \xrightarrow{\left(-\frac{\bar{\varepsilon}_+}{\bar{\varepsilon}_-}\right)} V_n(A[x]) \oplus V_n(A[x^{-1}]) \xrightarrow{(E_+E_-)} V_n(A[x,x^{-1}]) \xrightarrow{B} U_n(A) \to 0$$

for each $n \pmod{4}$, with the involution on A extended to $A[x^{\pm 1}]$, $A[x, x^{-1}]$ by $\bar{x} = x$. The proof depends on L-theoretic analogues (Lemmas 4.2, 4.3) of the Higman linearization trick (quoted in Lemma 2.2) and of a result from [2] (quoted in Lemma 2.3) on the automorphisms of $A[x, x^{-1}]$ -modules which are linear in x. A similar result has been obtained independently by Karoubi ([4]), using an L-theoretic analogue of the localization sequence of Chapter IX of [1].

Adopting the terminology of $\lceil 1 \rceil$, we can say that each

 V_n : (rings with involution) \rightarrow (abelian groups)

is a contracted functor, with

$$LV_n(A) = U_n(A)$$

up to natural isomorphism. Corollary 4.4 generalizes this "Fundamental Theorem" of algebraic L-theory to describe the intermediate L-groups $V_n^Q(A[x, x^{-1}])$, as defined in Part III, for suitable subgroups $Q \subseteq \tilde{K}_1(A[x, x^{-1}])$. Corollary 4.5 identifies the "lower L-theories" of Part II with the functors

$$L^m U_n$$
: (rings with involution) \rightarrow (abelian groups) $(m>0)$

derived from U_n . (There is an obvious analogy here with the "lower K-theories" of Chapter XII of $\lceil 1 \rceil$,

$$K_{-m} = L^m K_0: (rings) \rightarrow (abelian groups).$$

Corollary 4.6 describes the L-groups of polynomial extensions in several variables. The work presented here was stimulated by a course of lectures on algebraic K-theory given by Hyman Bass at Cambridge University in the Lent Term of 1973.

§1. Contracted Functors

Let (rings) be the category of associative rings with 1, and 1-preserving ring morphisms. Let x be an invertible indeterminate over such a ring A commuting with every element of A, and define $A[x, x^{-1}]$, the ring of finite polynomials $\sum_{j=-\infty}^{\infty} x^j a_j$ in x, x^{-1} with coefficients $a_j \in A$. Let $A[x^{\pm 1}]$ be the subring of $A[x, x^{-1}]$ of poly-

nomials involving only non-negative powers of $x^{\pm 1}$. Let

$$\bar{\varepsilon}_{\pm}: A \to A[x^{\pm 1}], \quad \bar{E}_{\pm}: A[x^{\pm 1}] \to A[x, x^{-1}], \quad \bar{\varepsilon} = \bar{E}_{\pm}\bar{\varepsilon}_{\pm}: A \to A[x, x^{-1}]$$

be the inclusions, and define left inverses

$$\varepsilon_+:A[x^{\pm 1}]\to A, \quad \varepsilon:A[x,x^{-1}]\to A$$

for $\bar{\varepsilon}_{\pm}$, $\bar{\varepsilon}$ by $x^{\pm 1} \mapsto 1$.

A functor

$$F: (rings) \rightarrow (abelian groups)$$

is contracted if the sequence

$$0 \to F(A) \xrightarrow{\left(-\frac{\bar{\delta}}{\bar{\epsilon}}\right)} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_+E_-)} F(A[x,x^{-1}]) \xrightarrow{B} LF(A) \to 0$$

is exact for each A, and there is given a natural right inverse

$$\bar{B}: LF(A) \to F(A[x, x^{-1}])$$

for the natural projection

$$B: F(A[x, x^{-1}]) \to LF(A)$$

$$= \operatorname{coker}((\bar{E}_+\bar{E}_-): F(A[x]) \oplus F(A[x^{-1}]) \to F(A[x, x^{-1}])),$$

that is $B\bar{B}=1:LF(A)\to LF(A)$. (This is just Definition 7.1 of Chapter XII of [1]).

LEMMA 1.1. Let

$$F, G: (rings) \rightarrow (abelian groups)$$

be functors, and suppose given

i) a natural left inverse

$$E_+: F(A[x, x^{-1}]) \to F(A[x])$$

for

$$\bar{E}_+: F(A[x]) \to F(A[x, x^{-1}])$$

such that the square

$$F(A[x^{-1}]) \xrightarrow{E_{-}} F(A[x, x^{-1}])$$

$$\downarrow^{E_{+}}$$

$$F(A) \xrightarrow{\bar{E}_{+}} F(A[x])$$

commutes,

ii) natural morphisms

$$\bar{\eta}_+: G(A) \to L_+F(A) = \operatorname{coker}(\bar{E}_+: F(A[x]) \to F(A[x, x^{-1}]))$$

 $\eta_+: L_+F(A) \to G(A)$

such that $\eta_+\bar{\eta}_+=1$, and such that the square

$$L_{+}F(A) \xrightarrow{\eta^{+}} G(A)$$

$$\downarrow^{\Delta_{+}} \downarrow^{\bar{\eta}_{-}} F(A[x, x^{-1}]) \xrightarrow{\delta_{-}} L_{-}F(A)$$

commutes, where

$$\Delta_+: L_+F(A) \to F(A[x, x^{-1}])$$

is the right inverse for the natural projection

$$\delta_{+}: F(A[x, x^{-1}]) \to L_{+}F(A)$$

induced by

$$1 - \bar{E}_+ E_+ : F(A[x, x^{-1}]) \to F(A[x, x^{-1}]),$$

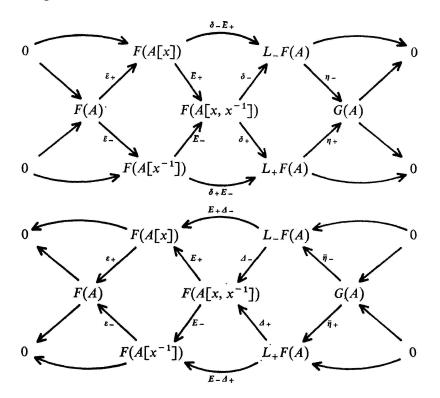
and δ_- , $\bar{\eta}_-$ are defined as δ_+ , $\bar{\eta}_+$ but with x^{-1} replacing x. Then F is a contracted functor, and

$$B = \eta_+ \delta_+ : F(A[x, x^{-1}]) \to G(A)$$

induces a natural isomorphism

$$LF(A) = \operatorname{coker}((\bar{E}_+\bar{E}_-): F(A[x]) \oplus F(A[x^{-1}]) \to F(A[x, x^{-1}])) \to G(A).$$

Proof. The diagrams



are commutative exact braids, where E_- , Δ_- , η_- are defined as E_+ , Δ_+ , η_+ but with x^{-1} replacing x. It follows that

$$0 \to F(A) \xrightarrow{\left(-\frac{\bar{\varepsilon}_{+}}{\bar{\varepsilon}_{-}}\right)} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_{+}E_{-})} F(A[x, x^{-1}]) \xrightarrow{B} G(A) \to 0$$

is an exact sequence, with

$$\bar{B} = \Delta_{\pm} \bar{\eta}_{\pm} : G(A) \to F(A[x, x^{-1}])$$

a natural right inverse for

$$B = \eta_{\pm} \delta_{\pm} : F(A[x, x^{-1}]) \to G(A).$$

Thus F is a contracted functor, with

$$LF(A) = G(A)$$

up to natural isomorphism.

(The conditions of Lemma 1.1 are necessary, as well as sufficient, for a functor to be contracted. If

$$F: (rings) \rightarrow (abelian groups)$$

is a contracted functor, then

$$F(A[x, x^{-1}]) = \bar{\varepsilon}F(A) \oplus \bar{E}_+ N_+ F(A) \oplus \bar{E}_- N_- F(A) \oplus \bar{B}LF(A)$$

where

$$N_{\pm}F(A) = \ker \left(\varepsilon_{\pm} : F(A[x^{\pm 1}]) \to F(A)\right),$$

and the morphisms

$$E_{+}:F(A[x,x^{-1}]) \to F(A[x]) = \bar{\varepsilon}_{+}F(A) \oplus N_{+}F(A);$$

$$\bar{\varepsilon}(r) \oplus \bar{E}_{+}(s_{+}) \oplus \bar{E}_{-}(s_{-}) \oplus \bar{B}(t) \mapsto \bar{\varepsilon}_{+}(r) \oplus s_{+}$$

$$\bar{\eta}_{+}:LF(A) \to L_{+}F(A) = \bar{E}_{-}N_{-}F(A) \oplus \bar{B}LF(A); t \mapsto 0 \oplus \bar{B}(t)$$

$$\eta_{+}:L_{+}F(A) \to LF(A); \bar{E}_{-}(s_{-}) \oplus \bar{B}(t) \mapsto t$$

satisfy the conditions of Lemma 1.1, with G=LF.)

§ 2. K-Theory of Polynomial Extensions

Let P(A) be the category of finitely generated (f.g.) projective left A-modules. Write |P(A)| for the class of objects, and $Hom_A(P,Q)$ for the additive group of

morphisms $g: P \to Q \in \mathbf{P}(A)$. A ring morphism

$$f: A \to A'$$

induces a functor

$$f: \mathbf{P}(A) \to \mathbf{P}(A'); \begin{cases} P \in |\mathbf{P}(A)| \mapsto fP = A' \otimes_A P \in |\mathbf{P}(A')| \\ g \in \mathrm{Hom}_A(P, Q) \mapsto fg = 1 \otimes g \in \mathrm{Hom}_{A'}(fP, fQ). \end{cases}$$

Given $P \in |\mathbf{P}(A)|$, let

$$P[x^{\pm 1}] = \bar{\varepsilon}_{\pm} P \in |\mathbf{P}(A[x^{\pm 1}])|, P_x = \bar{\varepsilon} P \in |\mathbf{P}(A[x, x^{-1}])|.$$

Defining complementary A-submodules

$$P^{+} = \sum_{j=0}^{\infty} x^{j} P$$
, $P^{-} = \sum_{j=-\infty}^{-1} x^{j} P$

of P_x (where $x^jP = x^j \otimes P$) we shall identify

$$P^+ = P[x], \quad xP^- = P[x^{-1}]$$

in the obvious way.

Let N(A) be the category with objects pairs

$$(P \in |\mathbf{P}(A)|, v \in \mathrm{Hom}_A(P, P) \text{ nilpotent})$$

and morphisms

$$f:(P, v) \rightarrow (P', v') \in \mathbf{N}(A)$$

isomorphisms $f \in \text{Hom}_{A}(P, P')$ such that

$$v'f = fv \in \operatorname{Hom}_{A}(P, P').$$

As usual, there are defined functors

$$K_i$$
: (rings) \rightarrow (abelian groups); $A \mapsto K_i(\mathbf{P}(A))$

for i=0,1. Theorem 7.4 of Chapter XII of [1], the "Fundamental Theorem" of algebraic K-theory, may be stated and proved as follows:

THEOREM 2.1 The functor K_1 is contracted, with

$$L_+K_1(A) = K_0N(A), LK_1(A) = K_0(A)$$

up to natural isomorphism.

Proof. Given an automorphism

$$f: G_x \to G_x \in \mathbf{P}(A[x, x^{-1}]) \quad (G \in |\mathbf{P}(A)|)$$

let $F=f(G)\subseteq G_x$, and define

$$(P, v) = (G^{-}/x^{-N}F^{-}, x^{-1}) \in |\mathbf{N}(A)|$$

for $N \ge 0$ so large that $x^{-N}F^- \subseteq G^-$. Then

$$E_{+}:K_{1}(A[x, x^{-1}]) \to K_{1}(A[x]);$$

$$\tau(f:G_{x} \to G_{x}) \mapsto \bar{\varepsilon}_{+}\tau(\varepsilon f:G \to G) \oplus \tau((1-v)^{-1}(1-xv):P^{+} \to P^{+})$$

is a well-defined morphism.

LEMMA 2.2 Every element of $K_1(A[x])$ can be represented by an automorphism

$$f = f_0 + x f_1 : G^+ \to G^+ \in \mathbf{P}(A[x])$$

with $f_0, f_1 \in \text{Hom}_A(G, G)$.

Proof. Given an automorphism

$$f = f_0 + xf_1 + x^2f_2 + \dots + x^rf_r \in \text{Hom}_{A[x]}(G^+, G^+) \quad (f_j \in \text{Hom}_A(G, G), 0 \le j \le r)$$

we can apply the usual Higman linearization trick (first used in the proof of Theorem 15 of [3]), the identity

$$\begin{pmatrix} 1 & -x^{r-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ xf_r & 1 \end{pmatrix}$$

$$= \begin{pmatrix} f_0 + xf_1 + \dots + x^{r-1}f_{r-1} & -x^{r-1} \\ xf_r & 1 \end{pmatrix} : G^+ \oplus G^+ \to G^+ \oplus G^+$$

(r-1) times, to obtain a representative automorphism for $\tau(f) \in K_1(A[x])$ which is linear in x (with r=1). \square

Given an automorphism

$$f = f_0 + x f_1 \in \text{Hom}_{A[x]}(G^+, G^+)$$

let $\gamma = (f_0 + f_1)^{-1} f_1 \in \text{Hom}_A(G, G)$. Then

$$f = (f_0 + f_1) (1 + (x - 1) \gamma) : G^+ \to G^+$$

and (up to isomorphism)

$$(G^{-}/x^{-1}f(G^{-}), x^{-1}) = (G^{-}/x^{-1}(1+(x-1)\gamma)G^{-}, x^{-1}) = (G, -\gamma(1-\gamma)^{-1}) \in |\mathbf{N}(A)|.$$

It follows that

$$E_{+}\bar{E}_{+}\tau(f) = \tau(f_{0} + f_{1}: G^{+} \to G^{+}) \oplus \tau((1 + \gamma(1 - \gamma)^{-1})^{-1} \times (1 + x\gamma(1 - \gamma)^{-1}): G^{+} \to G^{+})$$

$$= \tau(f_{0} + f_{1}: G^{+} \to G^{+}) \oplus \tau(1 + (x - 1) \gamma: G^{+} \to G^{+})$$

$$= \tau(f) \in K_{1}(A[x]).$$

Thus the composite

$$K_1(A[x]) \xrightarrow{E_+} K_1(A[x, x^{-1}]) \xrightarrow{E_+} K_1(A[x])$$

is the identity. Similarly, it can be shown that the square

$$K_{1}\left(A\left[x^{-1}\right]\right) \xrightarrow{E_{-}} K_{1}\left(A\left[x, x^{-1}\right]\right)$$

$$\downarrow^{E_{+}}$$

$$K_{1}\left(A\right) \xrightarrow{\bar{E}_{+}} K_{1}\left(A\left[x\right]\right)$$

commutes.

Higman's trick also shows that every element of $K_1(A[x, x^{-1}])$ may be expressed as

$$\tau = \tau (f_0 + x f_1 : P_x \to P_x) \oplus \tau (x^N : Q_x \to Q_x) \in K_1 (A \lceil x, x^{-1} \rceil)$$

for some $P, Q \in |\mathbf{P}(A)|, f_0, f_1 \in \mathrm{Hom}_A(P, P), N \in \mathbf{Z}$.

LEMMA 2.3. If $\gamma \in \text{Hom}_A(P, P)$ is such that

$$1 + (x-1) \gamma \in \text{Hom}_{A[x, x^{-1}]} (P_x, P_x)$$

is an isomorphism then there exist integers $r, s \ge 0$ such that

$$\gamma^r(1-\gamma)^s = 0 \in \operatorname{Hom}_A(P, P),$$

and $R = \ker \gamma^r$, $S = \ker (1 - \gamma)^s$ are complementary submodules of P, such that

$$\gamma = \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} : P = R \oplus S \to P = R \oplus S$$

with $\gamma_R \in \text{Hom}_A(R, R)$, $1 - \gamma_S \in \text{Hom}_A(S, S)$ nilpotent. Proof. See Corollary 2.4 of [2] and pp. 232-34 of [8]. \square If $f_0, f_1 \in \text{Hom}_A(P, P)$ are such that

$$f = f_0 + x f_1 \in \text{Hom}_{A[x, x^{-1}]} (P_x, P_x)$$

is an isomorphism, then

$$\varepsilon f = f_0 + f_1 \in \operatorname{Hom}_A(P, P)$$

is an isomorphism, and $\gamma = (f_0 + f_1)^{-1} f_1 \in \text{Hom}_A(P, P)$ satisfies the hypothesis of Lemma 2.3. Hence

$$\tau(f) = \bar{\epsilon}\tau(f_0 + f_1: P \to P) \oplus \tau(1 + (x - 1) \gamma: P_x \to P_x)$$

$$= \bar{\epsilon}\tau(f_0 + f_1: P \to P)$$

$$\oplus \bar{E}_+\tau(1 + (x - 1) \gamma_R: R[x] \to R[x])$$

$$\oplus \bar{E}_-\tau(1 + (x^{-1} - 1) (1 - \gamma_S): S[x^{-1}] \to S[x^{-1}])$$

$$\oplus \tau(x: S_x \to S_x) \in K_1(A[x, x^{-1}])$$

It is now easy to verify that

$$K_1(A[x]) \stackrel{E_+}{\rightleftharpoons} K_1(A[x, x^{-1}]) \stackrel{\delta_+}{\rightleftharpoons} K_0 \mathbf{N}(A)$$

is a direct sum system, with

$$\Delta_{+}: K_{0}\mathbf{N}(A) \to K_{1}(A[x, x^{-1}]); [P, v] \mapsto \tau((1-v)^{-1}(x-v): P_{x} \to P_{x})$$

$$\delta_{+}: K_{1}(A[x, x^{-1}]) \to K_{0}\mathbf{N}(A); \tau(f: G_{x} \to G_{x}) \mapsto [G^{+}/x^{N}F^{+}, x] - [F^{+}/x^{N}F^{+}, x]$$

where $F=f(G)\subseteq G_x$ (as before) and $N\geqslant 0$ is so large that $x^NF^+\subseteq G^+$, (so that, in particular,

$$\delta_{+}\tau(f_{0}+xf_{1}:P_{x}\to P_{x})=[S,-\gamma_{S}^{-1}(1-\gamma_{S})]\in K_{0}\mathbf{N}(A)).$$

Identifying

$$L_+K_1(A) = K_0\mathbf{N}(A)$$

in this way, note that the morphisms

$$\eta_+: K_0\mathbf{N}(A) \to K_0(A); [P, v] \mapsto [P]$$
 $\bar{\eta}_+: K_0(A) \to K_0\mathbf{N}(A); [P] \mapsto [P, 0]$

are such that the conditions of Lemma 1.1 are satisfied. Hence

$$K_1: (rings) \rightarrow (abelian groups)$$

is a contracted functor, with

$$LK_1(A) = K_0(A)$$

up to natural isomorphism. This completes the proof of Theorem 2.1.

§3. Review of the Definitions of the L-Groups

Let (rings with involution) be the category of rings A (as in §1) with involution $\overline{}: A \to A$; $a \mapsto \bar{a}$ such that

$$\overline{1}=1$$
, $\overline{a+b}=\overline{a}+\overline{b}$, $\overline{ab}=\overline{b}\cdot\overline{a}$, $a=a$ for all $a, b\in A$.

As in Part I it will be assumed that f.g. free A-modules have a well-defined dimension. Given a ring with involution A define a duality involution

*:
$$\mathbf{P}(A) \to \mathbf{P}(A)$$

$$\begin{cases}
P \in |\mathbf{P}(A)| \mapsto P^* = \operatorname{Hom}_A(P, A), & \text{left} \quad A \text{-action by} \\
A \times P^* \to P^*; (a, p^*) \mapsto (p \mapsto p^*(p) \cdot \bar{a}) \\
f \in \operatorname{Hom}_A(P, Q) \mapsto (f^* : Q^* \to P^*; q^* \mapsto (p \mapsto q^*(f(p)))),
\end{cases}$$

using the natural isomorphisms

$$P \rightarrow P^{**}; p \mapsto (p^* \mapsto \overline{p^*(p)}) \quad (P \in |\mathbf{P}(A)|)$$

to identify

$$**=1:P(A) \to P(A).$$

An ε -hermitian product (over A) is a morphism

$$\theta: Q \to Q^* \in \mathbf{P}(A)$$

such that

$$\theta^* = \varepsilon \theta \in \operatorname{Hom}_A(Q, Q^*),$$

where $\varepsilon = \pm 1$. $A \pm form$ (over A) is a pair

$$(Q \in |\mathbf{P}(A)|, \varphi \in \mathrm{Hom}_A(Q, Q^*)),$$

and

$$\theta = \varphi \pm \varphi * \in \operatorname{Hom}_{A}(Q, Q^{*})$$

is the associated \pm hermitian product. An isomorphism of \pm forms

$$(f,\chi):(Q,\varphi)\to(Q',\varphi')$$

is an isomorphism $f \in \text{Hom}_A(Q, Q')$ together with a morphism $\chi \in \text{Hom}_A(Q, Q^*)$ such that

$$f * \varphi' f - \varphi = \chi \mp \chi * \in \text{Hom}_A(Q, Q^*).$$

Such an isomorphism preserves the associated \pm hermitian products, in that

$$f*(\varphi'\pm\varphi'*)f=(\varphi\pm\varphi*)\in \operatorname{Hom}_A(Q,Q*).$$

A \pm form (Q, φ) is non-singular if the associated \pm hermitian product $(\varphi \pm \varphi^*) \in \text{Hom}_A(Q, Q^*)$ is an isomorphism. The hamiltonian \pm form on $P \in |\mathbf{P}(A)|$,

$$H\pm(P)=(P\oplus P^*,\begin{pmatrix}0&1\\0&0\end{pmatrix})$$

is non-singular. A sublagrangian of a non-singular \pm form (Q, φ) is a direct summand L of Q such that

$$i^* \varphi i = \lambda \mp \lambda^* \in \text{Hom}_{A}(L, L^*)$$

for some $\lambda \in \operatorname{Hom}_A(L, L^*)$, denoting by $j \in \operatorname{Hom}_A(L, Q)$ the inclusion. It was shown in Theorem 1.1 of Part I that if L is a sublagrangian of (Q, φ) there is defined a non-singular \pm form $(L^1/L, \hat{\varphi})$ on a direct complement L^1/L to L in the *annihilator* of L in (Q, φ) ,

$$L^{\perp} = \ker(i^*(\varphi \pm \varphi^*): Q \rightarrow L^*),$$

and that there is defined an isomorphism of \pm forms

$$(f,\chi):(Q,\varphi)\to H\pm(L)\oplus(L^1/L,\hat{\varphi})$$

with f the identity on $L^{\perp} = L \oplus L^{\perp}/L$. A lagrangian is a sublagrangian L such that $L^{\perp} = L$,

in which case there is defined an isomorphism of \pm forms

$$(f, \chi): (Q, \varphi) \rightarrow H \pm (L).$$

A \pm formation (over A), $(Q, \varphi; F, G)$, is a triple consisting of

- i) a non-singular \pm form over A, (Q, φ) ,
- ii) a lagrangian F of (Q, φ) ,
- iii) a sublagrangian G of (Q, φ) .

An isomorphism of \pm formations

$$(f, \chi): (Q, \varphi; F, G) \rightarrow (Q', \varphi'; F', G')$$

is an isomorphism of ± forms

$$(f,\chi):(Q,\varphi)\to(Q',\varphi')$$

such that f(F) = F', f(G) = G'. A stable isomorphism of \pm formations

$$[f,\chi]:(Q,\varphi;F,G)\rightarrow(Q',\varphi';F',G')$$

is an isomorphism of \pm formations

$$(f,\chi):(Q,\varphi;F,G)\oplus(H\pm(P);P,P^*)\to(Q',\varphi';F',G')\oplus(H\pm(P');P',P'^*)$$

defined for some $P, P' \in |\mathbf{P}(A)|$.

Let $T \subseteq \widetilde{K}_0(A) = \operatorname{coker}(K_0(\mathbf{Z}) \to K_0(A))$ be a subgroup invariant under the duality involution

*:
$$\tilde{K}_0(A) \to \tilde{K}_0(A)$$
; $\lceil P \rceil \mapsto \lceil P^* \rceil$ (that is, *(T)=T).

For $n \pmod{4}$ define the abelian monoid $X_n^T(A)$ of $\begin{cases} \text{isomorphism} \\ \text{stable isomorphism} \end{cases}$

classes of
$$\begin{cases} \pm \text{ forms } (Q, \varphi) \\ \pm \text{ formations } (Q, \varphi; F, G) \end{cases}$$
 over A such that the projective class $\begin{cases} [Q] \\ [G] - [F^*] \end{cases}$ lies in $T \subseteq \tilde{K}_0(A)$, under the direct sum \oplus , with $\pm = (-)^i$ if $n = \begin{cases} 2i \\ 2i+1 \end{cases}$.

The monoid morphisms

$$\partial^{T}: X_{n}^{T}(A) \rightarrow X_{n-1}^{T}(A); \begin{cases} (Q, \varphi) \mapsto (H_{\mp}(Q); Q, \Gamma_{(Q, \varphi)}) \\ (Q, \varphi; F, G) \mapsto (G^{\perp}/G, \hat{\varphi}) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

are such that $(\partial^T)^2 = 0$, where

$$\Gamma_{(Q,\varphi)} = \{ (x, (\varphi \pm \varphi^*) x) \mid x \in Q \} \subseteq Q \oplus Q^*.$$

Define an equivalence relation \sim on $\ker(\partial^T: X_n^T(A) \to X_{n-1}^T(A))$ by $z_1 \sim z_2$ if there exist $b_1, b_2 \in X_{n+1}^T(A)$ such that $z_1 \oplus \partial^T b_1 = z_2 \oplus \partial^T b_2 \in X_n^T(A)$. It was shown in Theorem 2.1 of Part III that the quotient monoids

$$U_n^T(A) = \ker(\partial^T : X_n^T(A) \to X_{n-1}^T(A)) / \operatorname{im}(\partial^T : X_{n+1}^T(A) \to X_n^T(A))$$

of equivalence classes are abelian groups, generalizing the definitions in Part I of

$$U_n(A) = U_n^{R_0(A)}(A), \quad V_n(A) = U_n^{\{0\}}(A).$$

Theorem 2.3 of Part III established an exact sequence

$$\cdots \rightarrow H^{n+1}(T'/T) \rightarrow U_n^T(A) \rightarrow U_n^{T'}(A) \rightarrow H^n(T'/T) \rightarrow U_{n-1}^T(A) \rightarrow \cdots$$

for *-invariant subgroups $T \subseteq T' \subseteq \widetilde{K}_0(A)$, where

$$H^{n}(G) = \{g \in G \mid g^{*} = (-)^{n} g\} / \{h + (-)^{n} h^{*} \mid h \in G\}$$

are the Tate cohomology groups (abelian, of exponent 2).

There are analogous definitions and results for L-groups associated with subgroups $R \subseteq \widetilde{K}_1(A) = \operatorname{coker}(K_1(\mathbf{Z}) \to K_1(A))$ invariant under the duality involution

$$*: \widetilde{K}_1(A) \to \widetilde{K}_1(A); \tau(f: P \to Q) \mapsto \tau(f^*: Q^* \to P^*)$$

denoting by $P \approx 1$ a f.g. free A-module P with a prescribed base, and by $P \approx 1$ the dual based A-module.

A based \pm form (Q, φ) is a \pm form (Q, φ) on a based A-module Q. The torsion of a based \pm form (Q, φ) is

$$\tau(Q,\varphi) = \begin{cases} \tau(\varphi \pm \varphi^* \colon Q \to Q^*) \in \widetilde{K}_1(A) & \text{if } (Q,\varphi) \text{ is non-singular} \\ 0 \in \widetilde{K}_1(A) & \text{otherwise.} \end{cases}$$

An R-isomorphism of based ± forms

$$(f,\chi):(Q,\varphi)\rightarrow(Q',\varphi')$$

is an isomorphism of the underlying forms

$$(f,\chi):(Q,\varphi)\to(Q',\varphi')$$

such that

$$\tau(f:Q\to Q')\in R\subseteq \widetilde{K}_1(A).$$

A based \pm formation $(Q, \varphi; F, G)$ is a \pm formation $(Q, \varphi; F, G)$ with bases for F, G and G^{\perp}/G . The torsion $\tau(Q, \varphi; F, G) \in \tilde{K}_1(A)$ of a based \pm formation is the torsion of the isomorphism

$$f: F \oplus F^* \to G \oplus G^* \oplus G^{\perp}/G$$

in the isomorphism of \pm forms

$$(f,\chi): H\pm(F) \rightarrow H\pm(G)\oplus(G^{\perp}/G,\hat{\varphi})$$

given by Theorem 1.1 of Part I. An R-isomorphism of based \pm formations

$$(f,\chi):(Q,\varphi;\mathcal{F},\mathcal{G})\to(Q',\varphi';\mathcal{F}',\mathcal{G}')$$

is an isomorphism of the underlying \pm formations such that the restrictions

$$F \to F', G \to G', G^{\perp}/G \to G'^{\perp}/G'$$

of f have torsions in $R \subseteq \tilde{K}_1(A)$. A stable R-isomorphism of based \pm formations

$$[f,\chi]:(Q,\varphi;F,G)\rightarrow (Q',\varphi';E',G')$$

is an R-isomorphism

$$(f,\chi):(Q,\varphi; F,G)\oplus (H\pm (P); P,P^*)\rightarrow (Q',\varphi'; F',G')\oplus (H\pm (P'); P',P'^*)$$

defined for some based A-modules P, P'.

For $n \pmod{4}$ define the abelian monoid $Y_n^R(A)$ of $\begin{cases} R\text{-isomorphism} \\ \text{stable } R\text{-isomorphism} \end{cases}$ classes of based $\begin{cases} \pm \text{ forms} \\ \pm \text{ formations} \end{cases}$ over A with torsion in $R \subseteq \tilde{K}_1(A)$, under the direct sum \oplus ,

with $\pm = (-)^i$ if $n = \begin{cases} 2i \\ 2i+1 \end{cases}$. The monoid morphisms

$$\partial^{R}: Y_{n}^{R}(A) \to Y_{n-1}^{R}(A); \begin{cases} (Q, \varphi) \mapsto (H_{\mp}(Q); Q, \Gamma_{(Q, \varphi)}) \\ (Q, \varphi; F, G) \mapsto (G^{\perp}/G, \widehat{\varphi}) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

are such that $(\partial^R)^2 = 0$, and the quotient monoids

$$V_n^R(A) = \ker\left(\partial^R: Y_n^R(A) \to Y_{n-1}^R(A)\right) / \overline{\operatorname{im}\left(\partial^R: Y_{n+1}^R(A) \to Y_n^R(A)\right)}$$

are abelian groups (by Theorem 3.1 of Part III) generalizing the definitions in Part I of

$$V_n(A) = V_n^{R_1(A)}(A) (= U_n^{\{0\}}(A)), \quad W_n(A) = V_n^{\{0\}}(A).$$

Theorem 3.3 in Part III established an exact sequence

$$\cdots \rightarrow H^{n+1}(R'/R) \rightarrow V_n^R(A) \rightarrow V_n^{R'}(A) \rightarrow H^n(R'/R) \rightarrow V_{n-1}^R(A) \rightarrow \cdots$$

for *-invariant subgroups $R \subseteq R' \subseteq \tilde{K}_1(A)$.

A morphism of rings with involution

$$f: A \to A'$$

such that $f(T) \subseteq T'$ (for some *-invariant subgroups $T \subseteq \widetilde{K}_0(A)$, $T' \subseteq \widetilde{K}_0(A')$) induces abelian group morphisms

$$f: U_n^T(A) \to U_n^{T'}(A'); \begin{cases} (Q, \varphi) \mapsto (fQ, f\varphi) \\ (Q, \varphi; F, G) \mapsto (fQ, f\varphi; fF, fG) \end{cases} \quad n = \begin{cases} 2i \\ 2i+1. \end{cases}$$

Similarly, if $f(R) \subseteq R'$ (for *-invariant subgroups $R \subseteq \tilde{K}_1(A)$, $R' \subseteq \tilde{K}_1(A')$) there are induced morphisms

$$f: V_n^R(A) \rightarrow V_n^{R'}(A') \pmod{4}$$
.

§4. L-Theory of Polynomial Extensions

Given a ring with involution A and an indeterminate x over A commuting with

every element of A extend the involution on A to the involution

$$\overline{}: A[x, x^{-1}] \to A[x, x^{-1}]; \qquad \sum_{j=-\infty}^{\infty} x^j a_j \mapsto \sum_{j=-\infty}^{\infty} x^j \bar{a}_j$$

on $A[x, x^{-1}]$. This restricts to involutions on the subrings A[x], $A[x^{-1}]$ of $A[x, x^{-1}]$. F. g, free A[x]-modules have well-defined dimension, as do those over $A[x^{-1}]$, $A[x, x^{-1}]$. Thus the rings with involution $A[x^{\pm 1}]$, $A[x, x^{-1}]$ satisfy the conditions imposed on A in §3.

Call a functor

 $F: (rings with involution) \rightarrow (abelian groups)$

contracted if the sequence

$$0 \to F(A) \xrightarrow{\left(-\frac{\bar{\varepsilon}}{\bar{\varepsilon}}^{-}\right)} F(A[x]) \oplus F(A[x^{-1}]) \xrightarrow{(E_{+}E_{-})} F(A[x, x^{-1}]) \xrightarrow{B} LF(A) \to 0$$

is exact for every ring with involution A and there is given a natural right inverse

$$\bar{B}: LF(A) \to F(A[x, x^{-1}])$$

for the natural projection

$$B: F(A[x, x^{-1}]) \to LF(A)$$

$$= \operatorname{coker}((\bar{E}_{+}\bar{E}_{-}): F(A[x] \oplus F(A[x^{-1}]) \to F(A[x, x^{-1}])).$$

The obvious analogue to Lemma 1.1 holds for functors

(rings with involution) \rightarrow (abelian groups)

as does the following analogue of Theorem 2.1 for the L-theoretic functors of §3:

THEOREM 4.1. Each of the functors

 V_n : (rings with involution) \rightarrow (abelian groups) $(n \pmod{4})$

is contracted, with

$$LV_n(A) = U_n(A), \quad L_{\pm}V_n(A) = U_n^{R_0(A)}(A[x^{\pm 1}])$$

up to natural isomorphism, where $\tilde{K}_0(A) \equiv \bar{\varepsilon}_{\mp} \tilde{K}_0(A) \subseteq \tilde{K}_0(A[x^{\mp 1}])$. \square

The proof of Theorem 4.1 in the case n=2i will be similar to the proof of Theorem 2.1. The case n=2i+1 will follow by an application of the results of Part II on the *L*-theory of Laurent extensions (that is, of the ring $A[x, x^{-1}]$ with involution by $\bar{x} = x^{-1}$).

Recall from Part II that a modular A-base of an $A[x, x^{-1}]$ -module Q is an A-submodule Q_0 of Q such that every element q of Q has a unique expression as

$$q = \sum_{j=-\infty}^{\infty} x^{j} q_{j} \quad (q_{j} \in Q_{o}, \{j \mid q_{j} \neq 0\} \quad \text{finite}),$$

so that $Q = A[x, x^{-1}] \otimes_A Q_0$ up to $A[x, x^{-1}]$ -module isomorphism. For example the A-modules generated by the bases of free $A[x, x^{-1}]$ -modules are modular A-bases. Define a morphism

$$\delta_{+}: V_{2i}(A[x, x^{-1}]) \to U_{2i}^{R_{0}(A)}(A[x^{-1}]);$$

$$(Q, \varphi) \mapsto (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})$$

by choosing a modular A-base Q_0 for Q (which is a f.g. free $A[x, x^{-1}]$ -module) and an integer $N \ge 0$ so large that

$$(\varphi \pm \varphi^*)(x^N Q_0^+) \subseteq x^{-N} Q_0^{*+} \qquad (\pm = (-)^i),$$

defining

$$P = x^{N}Q_{0}^{-} \cap (\varphi \pm \varphi^{*})^{-1} (x^{-N}Q_{0}^{*+}) \in |\mathbf{P}(A)|,$$

with $[\varphi]_j \in \operatorname{Hom}_A(P, P^*)$ given by

$$[\varphi]_j(y)(y')=a_j\in A \quad (y,y'\in P,j\in \mathbb{Z})$$

it

$$\varphi(y)(y') = \sum_{j=-\infty}^{\infty} x^j a_j \in A[x, x^{-1}] \quad (a_j \in A),$$

and writing $P[x^{-1}]$ for $\bar{\varepsilon}_- P = A[x^{-1}] \otimes_A P \in |\mathbf{P}(A[x^{-1}])|$.

The A-module isomorphism

$$[\varphi \pm \varphi^*]_{-1}: Q \to Q^*$$

may be expressed as

$$[\varphi \pm \varphi^*]_{-1} = \begin{pmatrix} [\varphi]_{-1} \pm ([\varphi]_{-1})^* & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \pm 1 & 0 \end{pmatrix} : P \oplus L \oplus L^* \to P^* \oplus L^* \oplus L$$

where $L = (\varphi \pm \varphi^*)^{-1} (x^{-N} Q_0^{*-})$, $L^* = x^N Q_0^+ \subseteq Q$, so that $(P, [\varphi]_{-1})$ is a non-singular \pm form over A.

For any $y, y' \in P$

$$[\varphi \pm \varphi^*]_{-2}(y)(y') = [\varphi \pm \varphi^*]_{-1}(xy)(y')$$

= $[\varphi \pm \varphi^*]_{-1}(xy - x^N y_{N-1})(y') \in A$,

where $y_{N-1} \in Q_0$ is such that

$$y-x^{N-1}y_{N-1} \in x^{N-1}Q_0^- \cap (\varphi \pm \varphi^*)^{-1}(x^{-N-1}Q_0^*) = x^{-1}P.$$

Thus

$$(P, ([\varphi \pm \varphi^*]_{-1})^{-1} ([\varphi \pm \varphi^*]_{-2})) = ((\varphi \pm \varphi^*)^{-1} (x^{-N}Q_0^{*+})/x^NQ_0^+, x) \in |\mathbf{N}(A)|,$$

and $(P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2})$ is a non-singular \pm form over $A[x^{-1}]$.

Suppose that Q'_0 is a different modular A-base of Q. Let $M \ge 0$ be so large that

$$Q_0' \subseteq \sum_{j=-M}^M x^j Q_0, \quad Q_0 \subseteq \sum_{j=-M}^M x^j Q_0'.$$

Then N' = N + M is so large that

$$(\varphi \pm \varphi^*)(x^{N'}Q_0^{\prime +}) \subseteq x^{-N'}Q_0^{\prime *+},$$

and

$$P' = x^{N'} Q_0'^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N'} Q_0'^{*+}) \quad \text{(definition)}$$

= $x^N (x^M Q_0'^- \cap Q_0^+) \oplus P \oplus x^{-N} (\varphi \pm \varphi^*)^{-1} (Q_0^{*-} \cap x^{-M} Q_0'^{*+}).$

Now

$$L = (x^N (x^M Q_0'^- \cap Q_0^+)) [x^{-1}] \subseteq P'[x^{-1}]$$

is a sublagrangian of $(P'[x^{-1}], [\varphi]_{-1} - x^{-1} [\varphi]_{-2})$ with $L^{\perp}/L = P[x^{-1}]$, so that

$$(P'[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) = (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) \oplus H_{\pm}(L)$$

$$= (P[x^{-1}], [\varphi]_{-1} - x^{-1}[\varphi]_{-2}) \in U_{2i}^{R_0(A)}(A[x^{-1}]).$$

Thus the choice of N and Q_0 is immaterial to the definition of δ_+ .

Finally, suppose that

$$(Q, \varphi) = \bar{E}_+(Q_0^+, \varphi_0) \in V_{2i}(A[x, x^{-1}])$$

for some $(Q_0^+, \varphi_0) \in V_{2i}(A[x])$. Then we can choose N=0, and

$$\delta_+(Q,\varphi)=0\in U_{2i}^{R_0(A)}(A[x^{-1}]).$$

Hence the morphism

$$\delta_+: V_{2i}(A[x, x^{-1}]) \to U_{2i}^{R_0(A)}(A[x^{-1}])$$

is well-defined, and such that the composite

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{R_0(A)}(A[x^{-1}])$$

is zero. Before going on to show that this sequence is in fact split exact, we need an L-theoretic analogue of Lemma 2.2 (the Higman linearization trick):

LEMMA 4.2. Every element of $U_{2i}^{R_0(A)}\left(A[x]\right)$ (resp. $V_{2i}(A[x,x^{-1}])$) can be represented by a linear \pm form, $(Q^+, \varphi_0 + x\varphi_1)$ over A[x] (resp. $(Q_x, \varphi_0 + x\varphi_1)$ over $A[x, x^{-1}]$) where $\varphi_0, \varphi_1 \in \operatorname{Hom}_A(Q, Q^*)$.

Proof. Given $(Q^+, \varphi) \in U_{2i}^{R_0(A)}(A[x])$, let

$$\varphi = \sum_{j=0}^{N} x^{j} \varphi_{j} \operatorname{Hom}_{A[x]}(Q^{+}, Q^{*+}) \quad (\varphi_{j} \in \operatorname{Hom}_{A}(Q, Q^{*})),$$

and suppose N > 1. Now

$$\begin{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ \pm x^{N-1} \varphi_N^* & 0 & 1
\end{pmatrix}, \begin{pmatrix} 0 & -x^{N-1} \varphi_N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0
\end{pmatrix} \\
\vdots (Q^+, \varphi) \oplus H_{\pm}(Q^+) \to \begin{pmatrix} Q^+ \oplus Q^+ \oplus Q^{*+}, \begin{pmatrix} \varphi - x^N \varphi_N & -x^{N-1} \varphi_N & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

is an isomorphism of \pm forms over A[x], so that

$$(Q'^+, \varphi') = (Q^+, \varphi) \in U_{2i}^{R_0(A)}(A[x])$$

with $Q' = Q \oplus Q \oplus Q^*$ such that

$$\varphi' = \sum_{j=0}^{N-1} x^j \varphi_j' \in \text{Hom}_{A[x]}(Q'^+, Q'^{*+}) \qquad (\varphi_j' \in \text{Hom}_A(Q', Q'^*)).$$

Iterating this procedure (N-1) times we obtain a representative for

$$(Q^+, \varphi) \in U_{2i}^{R_0(A)} (A[x]) \text{ with } N=1.$$

The same method works for elements $(Q_x, \varphi) \in V_{2i}(A[x, x^{-1}])$ provided we can assume that

$$(\varphi \pm \varphi^*) (Q^+) \subseteq Q^{*+}.$$

Choosing $N \ge 0$ so large that

$$(\varphi \pm \varphi^*) (x^N Q^+) \subseteq x^{-N} Q^{*+},$$

note that

$$(x^{N}, 0): (Q_{x}, \varphi' = x^{2N}\varphi) \rightarrow (Q_{x}, \varphi)$$

as an isomorphism of \pm forms over $A[x, x^{-1}]$, so that

$$(Q_x, \varphi') = (Q_x, \varphi) \in V_{2i}(A[x, x^{-1}]),$$

and that

$$(\varphi' \pm \varphi'^*) (Q^+) \subseteq Q^{*+}. \quad \square$$

The morphism

$$\Delta_{+}: U_{2i}^{\mathcal{R}_{0}(A)}(A[x^{-1}]) \to V_{2i}(A[x, x^{-1}]);$$

$$(Q[x^{-1}], \varphi) \mapsto (Q_{x}, x\varphi) \oplus \bar{\epsilon} \varepsilon_{-}(Q[x^{-1}], -\varphi) \oplus H_{\pm}(-Q_{x})$$

is clearly well-defined, with $-Q \in |\mathbf{P}(A)|$ such that $Q \oplus -Q$ is f.g. free. The composite

$$U_{2i}^{\mathcal{R}_0(A)}(A\left[x^{-1}\right]) \xrightarrow{\Delta_+} V_{2i}(A\left[x, x^{-1}\right]) \xrightarrow{\delta_+} U_{2i}^{\mathcal{R}_0(A)}(A\left[x^{-1}\right])$$

is the identity: by Lemma 4.2 it is sufficient to consider $\delta_{+}\Delta_{+}(Q[x^{-1}], \varphi)$ with

$$\varphi = \varphi_0 + x^{-1} \varphi_{-1} \in \text{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q^*[x^{-1}]) (\varphi_0, \varphi_{-1} \in \text{Hom}_A(Q, Q^*)),$$

and

$$\delta_{+}\Delta_{+}(Q[x^{-1}], \varphi_{0} + x^{-1}\varphi_{-1})$$

$$= \delta_{+}((Q_{x}, x\varphi_{0} + \varphi_{-1}) \oplus (Q_{x}, -(\varphi_{0} + \varphi_{-1})) \oplus H_{\pm}(-Q_{x}))$$

$$= ((Q^{-} \cap (x(\varphi_{0} \pm \varphi_{0}^{*}) + (\varphi_{-1} \pm \varphi_{-1}^{*}))^{-1}(Q^{*+}))[x^{-1}],$$

$$[x\varphi_{0} + \varphi_{-1}]_{-1} - x^{-1}[x\varphi_{0} + \varphi_{-1}]_{-2})$$

$$= ((1 + x^{-1}\gamma)^{-1}(x^{-1}Q), [x\varphi_{0} + \varphi_{-1}]_{-1} - x^{-1}[x\varphi_{0} + \varphi_{-1}]_{-2})$$

where $\gamma = (\varphi_0 \pm \varphi_0^*)^{-1} (\varphi_{-1} \pm \varphi_{-1}^*) \in \text{Hom}_A(Q, Q)$ is nilpotent. Now

$$(1+x^{-1}\gamma)^{-1} = \sum_{j=0}^{\infty} (-)^{j} x^{-j} \gamma^{j} \in \operatorname{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q[x^{-1}]),$$

so that

$$[x\varphi_{0} + \varphi_{-1}]_{j} (1 + y^{-1}\gamma)^{-1} (x^{-1}y) (1 + x^{-1}\gamma)^{-1} (x^{-1}y')$$

$$= \begin{cases} \varphi_{0}(y) (y') \\ (\varphi_{-1} - \varphi_{0}\gamma - \gamma^{*}\varphi_{0}) (y) (y') \end{cases} \text{ if } j = \begin{cases} -1 \\ -2 \end{cases} (y, y' \in Q),$$

and

$$\varphi_{-1} - \varphi_0 \gamma - \gamma^* \varphi_0 = -\varphi_{-1} + \chi \mp \chi^* \in \operatorname{Hom}_A(Q, Q^*),$$

where $\chi = \varphi_{-1} - \gamma^* \varphi_0 \in \text{Hom}_A(Q, Q^*)$. Thus

$$\delta_{+}\Delta_{+}(Q[x^{-1}], \varphi_{0}+x^{-1}\varphi_{-1})=(Q[x^{-1}], \varphi_{0}+x^{-1}(\varphi_{-1}-(\chi\mp\chi^{*})))$$

$$=(Q[x^{-1}], \varphi_{0}+x^{-1}\varphi_{-1})\in U_{2i}^{\mathcal{R}_{0}(A)}(A[x^{-1}])$$

and

$$\delta_{+}\Delta_{+}=1:U_{2i}^{\tilde{K}_{0}(A)}(A[x^{-1}])\to U_{2i}^{\tilde{K}_{0}(A)}(A[x^{-1}]).$$

It is therefore sufficient to prove that $V_{2i}(A[x, x^{-1}])$ is generated by the images of $\bar{E}_+: V_{2i}(A[x]) \to V_{2i}(A[x, x^{-1}])$, $\Delta_+: U_{2i}^{\bar{R}_0(A)}(A[x^{-1}]) \to V_{2i}(A[x, x^{-1}])$ for the exactness of

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{R_0(A)}(A[x^{-1}]).$$

We shall do this using the following L-theoretic analogue of Lemma 2.3:

LEMMA 4.3. Let (Q_x, φ) be a non-singular \pm form over $A[x, x^{-1}]$ such that $\varphi = \mu + (x-1) v \in \operatorname{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$ $(\mu, v \in \operatorname{Hom}_A(Q, Q^*))$.

Then (Q_x, φ) is isomorphic to the sum

$$(R_x, \mu_R + (x-1) \nu_R) \oplus (S_x, \mu_S + (x-1) \nu_S)$$

of non-singular \pm forms over $A[x, x^{-1}]$ such that

$$(R[x], \mu_R + (x-1) \nu_R)$$

is a non-singular \pm form over A[x], and

$$(S[x^{-1}], x^{-1}(\mu_S + (x-1)\nu_S))$$

is a non-singular \pm form over $A[x^{-1}]$.

Proof. The invertibility of

$$\varphi \pm \varphi^* = (\mu \pm \mu^*) + (x-1)(\nu \pm \nu^*) \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$$

implies that

$$\varepsilon(\varphi \pm \varphi^*) = \mu \pm \mu^* \in \operatorname{Hom}_A(Q, Q^*)$$
$$(\mu \pm \mu^*)^{-1} (\varphi \pm \varphi^*) = 1 + (x - 1) \gamma \in \operatorname{Hom}_{A[x, x^{-1}]}(Q_x, Q_x)$$

are isomorphisms, where

$$\gamma = (\mu \pm \mu^*)^{-1} (\nu \pm \nu^*) \in \text{Hom}_A(Q, Q).$$

Hence, by Lemma 2.3,

$$\gamma = \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} : Q = R \oplus S \rightarrow Q = R \oplus S$$

with $\gamma_R \in \text{Hom}_A(R, R)$, $1 - \gamma_S \in \text{Hom}_A(S, S)$ nilpotent.

Adding on some \mp hermitian products of type $\chi \mp \chi^* \in \text{Hom}_A(Q, Q^*)$ to μ and ν if necessary, it may be assumed that $\mu(R)(S) = 0$, $\nu(R)(S) = 0$. Let

$$\mu = \begin{pmatrix} \mu_R & \mu_{RS} \\ 0 & \mu_S \end{pmatrix} : R \oplus S \to R^* \oplus S^*, \quad \nu = \begin{pmatrix} \nu_R & \nu_{RS} \\ 0 & \nu_S \end{pmatrix} : R \oplus S \to R^* \oplus S^*$$

so that

$$\begin{pmatrix} \mu_R \pm \mu_R^* & \mu_{RS} \\ \pm \mu_{RS}^* & \mu_S \pm \mu_S^* \end{pmatrix} \quad \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} = \begin{pmatrix} \nu_R \pm \nu_R^* & \nu_{RS} \\ \pm \nu_{RS}^* & \nu_S \pm \nu_S^* \end{pmatrix} : R \oplus S \to R^* \oplus S^*.$$

Working as in the calculation of $\delta_+ \Delta_+$ above,

$$\begin{split} \delta_{+}\left(Q_{x},\varphi\right) &= \left(\left(Q^{-} \cap \left(\varphi \pm \varphi^{*}\right)^{-1} \left(Q^{*+}\right)\right) \left[x^{-1}\right], \left[\varphi\right]_{-1} - x^{-1} \left[\varphi\right]_{-2}\right) \\ &= \left(\left(1 + (x - 1) \gamma_{S}\right)^{-1} \left(S\right) \left[x^{-1}\right], \left[\mu_{S} + (x - 1) \nu_{S}\right]_{-1} - x^{-1} \left[\mu_{S} + (x - 1) \nu_{S}\right]_{-2}\right) \\ &= \left(S \left[x^{-1}\right], x^{-1} \left(\mu_{S} + (x - 1) \nu_{S}\right)\right) \in U_{2i}^{R_{0}(A)}\left(A \left[x^{-1}\right]\right). \end{split}$$

Thus $\varepsilon_-\delta_+(Q_x, \varphi) = (S, \mu_S)$ is a non-singular \pm form over A, and hence so is (S, ν_S) , because

$$(v_S \pm v_S^*) = (\mu_S \pm \mu_S^*) \gamma_S \in \text{Hom}_A(S, S^*)$$

and $\gamma_S \in \text{Hom}_A(S, S)$ is an isomorphism (being unipotent). Let

$$g = \pm (v_{S} \pm v_{S}^{*})^{-1} v_{RS}^{*} \in \text{Hom}_{A}(R, S)$$

$$\mu' = \begin{pmatrix} \mu'_{R} = \mu_{R} - g^{*} \mu_{S} g & 0 \\ 0 & \mu_{S} \end{pmatrix} : R \oplus S \to R^{*} \oplus S^{*}$$

$$v' = \begin{pmatrix} v'_{R} = v_{R} - g^{*} v_{S} g & 0 \\ 0 & v_{S} \end{pmatrix} : R \oplus S \to R^{*} \oplus S^{*}.$$

Now

$$(f,\chi) = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ (\mu_S + (x-1) \nu_S) g & 0 \end{pmatrix}$$

: $(Q_x, \varphi) = (R_x \oplus S_x, \mu + (x-1) \nu) \rightarrow (Q_x, \varphi') = (R_x \oplus S_x, \mu' + (x-1) \nu')$

is an isomorphism of \pm forms over $A[x, x^{-1}]$. It follows that

$$f^*(\varphi' \pm \varphi'^*) f = (\varphi \pm \varphi^*) \in \text{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*)$$

and as f is defined over A

$$f * (\mu' \pm \mu'^*) f = (\mu \pm \mu^*) \in \text{Hom}_A(Q, Q^*)$$
$$f * (\nu' \pm \nu'^*) f = (\nu \pm \nu^*) \in \text{Hom}_A(Q, Q^*).$$

Defining

$$\gamma' = (\mu' \pm \mu'^*)^{-1} (\nu' \pm \nu'^*) = \begin{pmatrix} \gamma'_R = (\mu'_R \pm \mu'^*_R)^{-1} (\nu_R \pm \nu^*_R) & 0 \\ 0 & \gamma_S \end{pmatrix} : R \oplus S \to R \oplus S,$$

we have that

$$\gamma' = f \gamma f^{-1} = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} = \begin{pmatrix} \gamma_R & 0 \\ g \gamma_R - \gamma_S g & \gamma_S \end{pmatrix} : R \oplus S \to R \oplus S.$$

Hence

$$\gamma_R' = \gamma_R \in \operatorname{Hom}_A(R, R)$$

is nilpotent, and $(R[x], \mu'_R + (x-1)\nu'_R)$ is a non-singular \pm form over A[x]. This completes the proof of Lemma 4.3. \square

Given $(Q_x, \varphi) \in V_{2i}(A[x, x^{-1}])$ it may be assumed, by Lemma 4.2, that $\varphi = \mu + (x-1) \ v \in \operatorname{Hom}_{A[x, x^{-1}]}(Q_x, Q_x^*) \ (\mu, v \in \operatorname{Hom}_A(Q, Q^*))$. Applying the decomposition of Lemma 4.3,

$$(Q_{x}, \varphi) = (R_{x}, \mu_{R} + (x-1) \nu_{R}) \oplus (S_{x}, \mu_{S} + (x-1) \nu_{S})$$

$$= \{ (R_{x}, \mu_{R} + (x-1) \nu_{R}) \oplus (S_{x}, \mu_{S}) \} \oplus \{ (S_{x}, \mu_{S} + (x-1) \nu_{S}) \} \oplus \{ (S_{x}, \mu_{S} + (x-1) \nu_{S}) \} \oplus \{ (R[x], \mu_{R} + (x-1) \nu_{R}) \} \oplus \{ (S[x], \mu_{S}) \} \oplus \{ (R[x], \mu_{R} + (x-1) \nu_{R}) \} \oplus \{ (S[x], \mu_{S}) \} \oplus \{ (R[x], \mu_{R} + (x-1) \nu_{R}) \} \oplus \{ (R[x], \mu_{S} + (x-1) \nu_{S}) \} \oplus \{ ($$

As pointed out above, this suffices to prove the exactness of

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta_+} U_{2i}^{R_0(A)}(A[x^{-1}]).$$

Define next a morphism

$$E_{+}: V_{2i}(A[x, x^{-1}]) \to V_{2i}(A[x]);$$

$$(Q_{x}, \varphi) \mapsto ((\varphi \pm \varphi^{*})^{-1} (x^{N_{1}+1}Q^{*-}) \cap x^{-N_{1}}Q^{*+}) [x], [\varphi]_{0} - x([\varphi]_{1})$$

$$\oplus ((x^{N}Q^{-} \cap (\varphi \pm \varphi^{*})^{-1} (x^{-N}Q^{*+})) [x], [\varphi]_{-1} - [\varphi]_{-2})$$

for $N, N_1 \ge 0$ so large that

$$(\varphi \pm \varphi^*)(Q) \subseteq \sum_{j=-2N}^{2N_1+1} x^j Q^*$$

with $Q \in |\mathbf{P}(A)|$ f.g. free. The verification that E_+ is well-defined is by analogy with that for δ_+ . Moreover, if

$$(Q_x, \varphi) = (R_x, \mu_R + (x-1) \nu_R) \oplus (S_x, \mu_S + (x-1) \nu_S)$$

(as in Lemma 4.3), then

$$E_{+}(Q_{x}, \varphi) = (R[x], \mu_{R} + (x-1)\nu_{R}) \oplus (S[x], \mu_{S}) \in V_{2i}(A[x]),$$

so that the composites

$$U_{2i}^{R_0(A)}(A[x^{-1}]) \xrightarrow{A_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x])$$

$$V_{2i}(A[x]) \xrightarrow{E_+} V_{2i}(A[x, x^{-1}]) \xrightarrow{E_+} V_{2i}(A[x])$$

are 0, 1 respectively. Thus

$$V_{2i}(A[x]) \underset{E_{+}}{\rightleftharpoons} V_{2i}(A[x, x^{-1}]) \underset{A_{+}}{\rightleftharpoons} U_{2i}^{R_{0}(A)}(A[x^{-1}])$$

defines a direct sum system, and we can identify

$$L_+V_{2i}(A) = U_{2i}^{R_0(A)}(A[x^{-1}]).$$

Similarly, replacing x with x^{-1} , there is defined a direct sum system

$$V_{2i}(A[x^{-1}]) \underset{E_{-}}{\rightleftharpoons} V_{2i}(A[x, x^{-1}]) \underset{A_{-}}{\rightleftharpoons} U_{2i}^{R_{0}(A)}(A[x]),$$

allowing the identification

$$L_{-}V_{2i}(A) = U_{2i}^{R_{0}(A)}(A[x]).$$

The proof of Lemma 4.2 shows that every element $(Q[x^{-1}], \varphi) \in V_{2i}(A[x^{-1}])$ has a representative with

$$\varphi = \varphi_0 + x^{-1} \varphi_{-1} \in \text{Hom}_{A[x^{-1}]}(Q[x^{-1}], Q^*[x^{-1}]) \quad (\varphi_0, \varphi_{-1} \in \text{Hom}_A(Q, Q^*)).$$

The composite

$$V_{2i}(A \lceil x^{-1} \rceil) \xrightarrow{E_{-}} V_{2i}(A \lceil x, x^{-1} \rceil) \xrightarrow{E_{+}} V_{2i}(A \lceil x \rceil)$$

sends such a representative to

$$E_{+}\bar{E}_{-}(Q[x^{-1}], \varphi) = (((\varphi \pm \varphi^{*})^{-1} (xQ^{*-}) \cap Q^{+}) [x], [\varphi]_{0} - [\varphi]_{1})$$

$$\oplus ((xQ^{-} \cap (\varphi \pm \varphi^{*})^{-1} (x^{-1}Q^{*})) [x], [\varphi]_{-1} - [\varphi]_{-2})$$

$$= (Q[x], \varphi_{0}) \oplus ((\varphi \pm \varphi^{*})^{-1} (Q^{*} \oplus x^{-1}Q^{*}) [x], [\varphi]_{-1}$$

$$- [\varphi]_{-2}) \in V_{2i}(A[x, x^{-1}]).$$

The A-module isomorphism

$$Q \oplus Q \to (\varphi \pm \varphi^*)^{-1} (Q^* \oplus x^{-1} Q^*);$$

$$(y, y') \mapsto (\varphi \pm \varphi^*)^{-1} ((\varphi_0 \pm \varphi_0^*) y, x^{-1} (((\varphi_0 \pm \varphi_0^*) + \varphi_{-1} \pm \varphi_{-1}^*)) y + (\varphi_0 \pm \varphi_0^*) y'))$$

defines an isomorphism of \pm forms over A

$$(Q \oplus Q, \begin{pmatrix} \varphi_0 + \varphi_{-1} & 0 \\ 0 & -\varphi_0 \end{pmatrix}) \rightarrow ((\varphi \pm \varphi^*)^{-1} (Q^* \oplus x^{-1} Q^*), [\varphi]_{-1} - [\varphi]_{-2}).$$

Therefore

$$E_{+}\bar{E}_{-}(Q[x^{-1}], \varphi_{0} + x^{-1}\varphi_{-1}) = (Q[x], \varphi_{0} + \varphi_{-1}) \oplus (Q[x] \oplus Q[x], \varphi_{0} \oplus -\varphi_{0})$$

$$= (Q[x], \varphi_{0} + \varphi_{-1})$$

$$= \bar{\varepsilon}_{+}\varepsilon_{-}(Q[x^{-1}], \varphi_{0} + x^{-1}\varphi_{-1}) \in V_{2i}(A[x]),$$

and the square

$$V_{2i}(A[x^{-1}]) \xrightarrow{E_{-}} V_{2i}(A[x, x^{-1}])$$

$$\downarrow^{E_{-}} \qquad \downarrow^{E_{+}}$$

$$V_{2i}(A) \xrightarrow{\bar{e}_{+}} V_{2i}(A[x])$$

commutes. Similarly, we can verify that the square

$$U_{2i}^{\tilde{R}_{0}(A)}(A[x^{-1}]) \xrightarrow{\eta_{+}} U_{2i}(A)$$

$$\downarrow^{\bar{\eta}_{-}}$$

$$V_{2i}(A[x, x^{-1}]) \xrightarrow{\delta} U_{2i}^{\tilde{R}_{0}(A)}(A[x])$$

commutes, where

$$\eta_{\pm}: U_{2i}^{\mathcal{R}_{0}(A)}(A[x^{\mp 1}]) \to U_{2i}(A), \quad \bar{\eta}_{\pm}: U_{2i}(A) \to U_{2i}^{\mathcal{R}_{0}(A)}(A[x^{\mp 1}])$$

are the morphisms induced by

$$\eta_{\pm}: A\left[x^{\mp 1}\right] \to A; \sum_{j=0}^{\infty} x^{\mp j} a_j \mapsto a_0, \quad \bar{\varepsilon}_{\mp}: A \to A\left[x^{\mp 1}\right]$$

respectively (so that $\eta_{\pm}\bar{\eta}_{\pm}=1$). For

$$\begin{split} \delta_{-} \Delta_{+} \left(Q \left[x^{-1} \right], \varphi = \varphi_{0} + x^{-1} \varphi_{-1} \right) \\ &= \delta_{-} \left(\left(Q_{x}, x \varphi \right) \oplus \left(Q_{x}, - (\varphi_{0} + \varphi_{-1}) \right) \oplus H_{\pm} \left(- Q_{x} \right) \right) \\ &= \left(\left(x^{-1} Q^{+} \cap (\varphi \pm \varphi^{*})^{-1} \left(Q^{*-} \right) \right) \left[x \right], \left[x \varphi \right]_{-1} - x \left[x \varphi \right]_{0} \right) \\ &= \left(\left(x^{-1} Q \right) \left[x \right], \left[x \varphi \right]_{-1} \right) = \left(Q \left[x \right], \varphi_{0} \right) \\ &= \bar{\eta}_{-} \eta_{+} \left(Q \left[x^{-1} \right], \varphi \right) \in U_{2i}^{R_{0}(A)} \left(A \left[x \right] \right). \end{split}$$

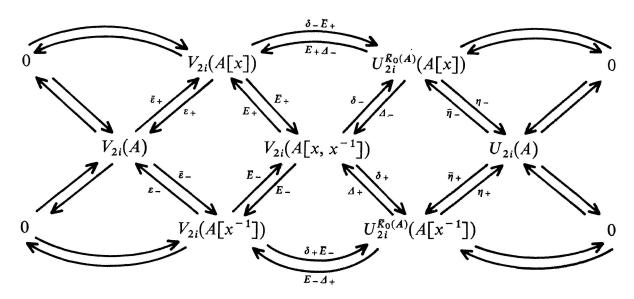
The conditions of Lemma 1.1 are now satisfied, and so

 V_{2i} : (rings with involution) \rightarrow (abelian groups)

is a contracted functor, with

$$L_{\pm}V_{2i}(A) = U_{2i}^{R_0(A)}(A[x^{\mp 1}]), \quad LV_{2i}(A) = U_{2i}(A)$$

(up to natural isomorphisms), and the diagram

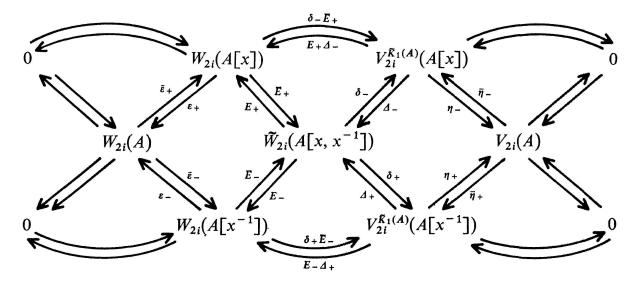


incorporates two commutative exact braids.

Let $S_0 \subseteq \widetilde{K}_1(A[x, x^{-1}])$ be the infinite cyclic subgroup generated by $\overline{B}([A])$ = $\tau(x: A_x \to A_x)$, and define

$$\widetilde{W}_n(A[x, x^{-1}]) = V_n^{S_0}(A[x, x^{-1}]) \quad (n \pmod{4}).$$

Working as for $V_{2i}(A[x, x^{-1}])$, it is possible to define morphisms to fit into a diagram



(with $E_+\bar{E}_+=1$ etc.) incorporating two commutative exact braids. For example,

$$\begin{split} \delta_{+} \colon & \widetilde{W}_{2i}(A\left[x,\,x^{-1}\right]) \to V_{2i}^{R_{1}(A)}(A\left[x^{-1}\right]); \, (Q_{x},\,\varphi) \mapsto (P_{2i}^{x-1},\,[\varphi]_{-1} - x^{-1}\,[\varphi]_{-2}) \\ & E_{+} \colon & \widetilde{W}_{2i}(A\left[x,\,x^{-1}\right]) \to W_{2i}(A\left[x\right]); \\ & (Q_{x},\,\varphi) \mapsto (P_{1}\left[x\right],\,[\varphi]_{0} - x\,[\varphi]_{1}) \oplus (P_{2i}^{x},\,[\varphi]_{-1} - [\varphi]_{-2}) \end{split}$$

for any A-base P of $P = x^N Q^- \cap (\varphi \pm \varphi^*)^{-1} (x^{-N} Q^{*+})$ (which is free for sufficiently large $N \ge 0$, as $\tau(Q_x, \varphi) \in S_0$ and $[P] = B\tau(Q_x, \varphi) = 0 \in \tilde{K}_0(A)$) with

$$\underset{\sim}{P_1} = (\varphi \pm \varphi^*)^{-1} (x^N Q^*) \oplus (\varphi \pm \varphi^*)^{-1} (P^*)$$

the corresponding A-base of $P_1 = (\phi \pm \phi^*)^{-1} (x^{N+1}Q^{*-}) \cap x^{-N}Q^+$, for N so large that

$$(\varphi \pm \varphi^*)(Q) \subseteq \sum_{j=-2N}^{2N+1} x^j Q^*.$$

Also, let

$$\Delta_{+}: V_{2i}^{R_{1}(A)}(A[x^{-1}]) \to \widetilde{W}_{2i}(A[x, x^{-1}]); (Q[x^{-1}], \varphi) \mapsto (Q_{x}, x\varphi) \oplus (Q_{x}, x\varphi) \oplus (Q_{x}, x\varphi)$$

where $Q = (\varepsilon_- (\varphi \pm \varphi^*))^{-1} (Q^*)$.

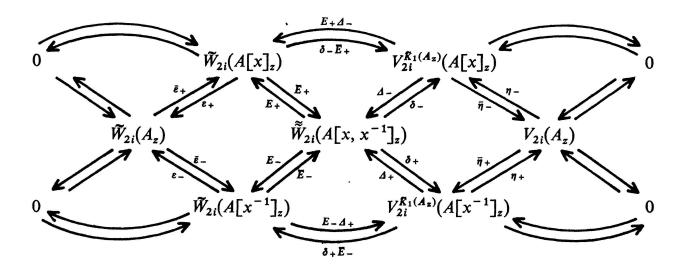
Given an invertible indeterminate z over A commuting with every element of A define A_z as $A[z, z^{-1}]$ but with involution by $\bar{z} = z^{-1}$. Similarly, define $A[x^{\pm 1}]_z$, $A[x, x^{-1}]_z$, and identify

$$A[x^{\pm 1}]_z = A_z[x^{\pm 1}], \quad A[x, x^{-1}]_z = A_z[x, x^{-1}].$$

Let $S_0' \subseteq \tilde{K}_1(A_z)$ be the infinite cyclic subgroup generated by $\tau(z:A_z \to A_z)$ and define

$$\begin{split} \widetilde{W}_{n}(A_{z}) &= V_{n}^{S'o}(A_{z}) \\ \widetilde{W}_{n}(A[x^{\pm 1}]_{z}) &= V_{n}^{\bar{e}_{\pm}(x)S'o}(A[x^{\pm 1}]_{z}) \\ &\stackrel{\approx}{W}_{n}(A[x, x^{-1}]_{z}) &= V_{n}^{\bar{e}(z)S_{0} \oplus \bar{e}(x)S'o}(A[x, x^{-1}]_{z}) \end{split}$$

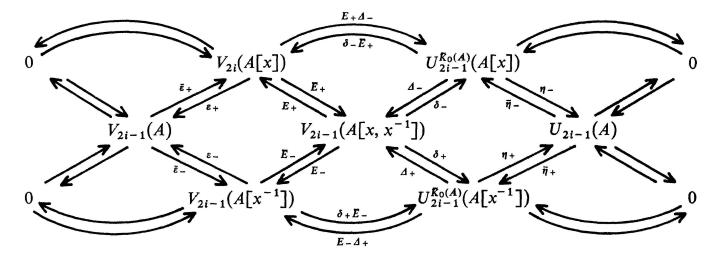
for $n \pmod{4}$. By analogy with $\widetilde{W}_{2i}(A[x,x^{-1}])$, $\widetilde{W}_{2i}(A[x,x^{-1}]_z)$ fits into a diagram incorporating two commutative exact braids (where $A_z = A[z,z^{-1}]$, with $\bar{z} = z^{-1}$).



We can now apply the decompositions

$$\begin{split} \widetilde{W}_{2i}(A_z) &= \bar{\varepsilon}(z) \ W_{2i}(A) \oplus \bar{B}(z) \ V_{2i-1}(A) \\ \widetilde{W}_{2i}(A[x]_z) &= \bar{\varepsilon}(z) \ W_{2i}(A[x]) \oplus \bar{B}(z) \ V_{2i-1}(A[x]) \\ \overset{\approx}{W}_{2i}(A[x, x^{-1}]_z) &= \bar{\varepsilon}(z) \ \widetilde{W}_{2i}(A[x, x^{-1}]) \oplus \bar{B}(z) \ V_{2i-1}(A[x, x^{-1}]) \\ V_{2i}^{\tilde{K}_1(A_z)}(A[x]_z) &= \bar{\varepsilon}(z) \ V_{2i}^{\tilde{K}_1(A)}(A) \oplus \bar{B}(z) \ U_{2i-1}^{\tilde{K}_0(A)}(A) \\ V_{2i}(A_z) &= \bar{\varepsilon}(z) \ V_{2i}(A) \oplus \bar{B}(z) \ U_{2i-1}(A) \end{split}$$

given by Theorem 1.1 of Part II (and extended to the intermediate *L*-groups in Part III). The above diagram splits naturally (via $\bar{\epsilon}(z)$, $\bar{B}(z)$) into two similar ones: the diagram for $\tilde{W}_{2i}(A[x, x^{-1}])$ and the diagram



where

$$E_{+}: V_{2i-1}(A[x, x^{-1}]) \xrightarrow{\bar{B}(z)} \overset{\approx}{W}_{2i}(A[x, x^{-1}]_{z}) \xrightarrow{E_{+}} \tilde{W}_{2i}(A[x]_{z}) \xrightarrow{B(z)} V_{2i-1}(A[x])$$

$$\delta_{+}: V_{2i-1}(A[x, x^{-1}]) \xrightarrow{\bar{B}(z)} \overset{\approx}{W}_{2i}(A[x, x^{-1}]_{z})$$

$$\xrightarrow{\delta_{+}} V_{2i}^{\bar{R}i(A_{z})}(A[x^{-1}]_{z}) \xrightarrow{B(z)} U_{2i-1}^{\bar{R}_{0}(A)}(A[x^{-1}])$$

$$\Delta_{+}: U_{2i-1}^{\bar{R}_{0}(A)}(A[x^{-1}]) \xrightarrow{\bar{B}(z)} V_{2i}^{\bar{R}_{1}(A_{z})}(A[x^{-1}]_{z})$$

$$\xrightarrow{\Delta_{+}} \overset{\approx}{W}_{2i}(A[x, x^{-1}]_{z}) \xrightarrow{B(z)} V_{2i-1}(A[x, x^{-1}])$$

(and similarly for E_- , δ_- , Δ_-). Thus the conditions of Lemma 1.1 are also satisfied in the odd-dimensional case, and

 V_{2i-1} : (rings with involution) \rightarrow (abelian groups)

is a contracted functor, with identifications

$$L_{\pm}V_{2i-1}(A) = U_{2i-1}^{R_0(A)}(A[x^{\mp 1}]), \quad LV_{2i-1}(A) = U_{2i-1}(A).$$

This completes the proof of Theorem 4.1
The groups

$$\operatorname{Nil}_{\pm}(A) = \ker \left(\varepsilon_{\pm} : K_1(A[x^{\pm 1}]) \to K_1(A) \right)$$

are such that

$$K_{1}(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm} K_{1}(A) \oplus \operatorname{Nil}_{\pm}(A)$$

$$K_{1}(A[x, x^{-1}]) = \bar{\varepsilon} K_{1}(A) \oplus \bar{E}_{+} \operatorname{Nil}_{+}(A) \oplus \bar{E}_{-} \operatorname{Nil}_{-}(A) \oplus \bar{B} K_{0}(A),$$

fitting into direct sum systems

$$\operatorname{Nil}_{\pm}(A) \underset{E_{\pm} A_{\pm}}{\overset{\delta_{\pm} E_{\pm}}{\rightleftharpoons}} K_{0} \mathbf{N}(A) \underset{\bar{\eta}_{\pm}}{\overset{\eta_{\pm}}{\rightleftharpoons}} K_{0}(A)$$

(by Theorem 2.1).

Given *-invariant subgroups $S_+ \subseteq Nil_+(A)$, define

$$\begin{split} N_{\pm}V_{n}^{S_{\pm}}(A) &= \ker \left(\varepsilon_{\pm} : V_{n}^{\bar{\varepsilon}_{\pm}\bar{K}_{1}(A) \oplus S_{\pm}} \left(A \left[x^{\pm 1} \right] \right) \to V_{n}(A) \right) \quad (n \, (\text{mod} \, 4)) \\ \text{writing} \quad \begin{cases} N_{\pm}V_{n}(A) \\ N_{\pm}W_{n}(A) \end{cases} \quad \text{for} \quad \begin{cases} N_{\pm}V_{n}^{\text{Nil}_{\pm}(A)}(A) \\ N_{\pm}V_{n}^{\{0\}}(A) \end{cases} \end{split}.$$

COROLLARY 4.4. Given *-invariant subgroups

$$R \subseteq \widetilde{K}_1(A)$$
, $S_{\pm} \subseteq \operatorname{Nil}_{\pm}(A)$, $\widetilde{T} \subseteq \widetilde{K}_0(A)$

there are direct sum decompositions

$$V_{n}^{\bar{\varepsilon}_{\pm}R \oplus S_{\pm}} (A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm} V_{n}^{R}(A) \oplus N_{\pm} V_{n}^{S_{\pm}} (A)$$

$$U_{n}^{\bar{\varepsilon}_{\pm}T} (A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm} U_{n}^{T}(A) \oplus N_{\pm} V_{n} (A)$$

$$V_{n}^{Q} (A[x, x^{-1}]) = \bar{\varepsilon} V_{n}^{R}(A) \oplus \bar{E}_{+} N_{+} V_{n}^{S_{+}} (A) \oplus \bar{E}_{-} N_{-} V_{n}^{S_{-}} (A) \oplus \bar{B} U_{n}^{T} (A)$$

for $n \pmod{4}$, where

$$Q = \bar{\varepsilon}R \oplus \bar{E}_{+}S_{+} \oplus \bar{E}_{-}S_{-} \oplus \bar{B}T \subseteq \tilde{K}_{1}(A[x, x^{-1}])$$

= $\bar{\varepsilon}\tilde{K}_{1}(A) \oplus \bar{E}_{+} \operatorname{Nil}_{+}(A) \oplus \bar{E}_{-} \operatorname{Nil}_{-}(A) \oplus \bar{B}K_{0}(A)$

with $T \subseteq K_0(A)$ the preimage of \tilde{T} under the natural projection $K_0(A) \to \tilde{K}_0(A)$.

Proof. The forgetful map

$$V_n(A[x^{\pm 1}]) \rightarrow U_n^{\bar{\epsilon}_{\pm}T}(A[x^{\pm 1}])$$

fits into the exact sequence of Theorem 2.3 of Part III, which splits, via $\bar{\epsilon}_{\pm}$, ϵ_{\pm} into two exact sequences

$$\rightarrow 0 \rightarrow N_{\pm}V_{n}(A) \rightarrow N_{\pm}V_{n}(A) \rightarrow 0 \rightarrow$$

$$\uparrow \downarrow \qquad \uparrow \downarrow \qquad \uparrow \downarrow \qquad \uparrow \downarrow$$

$$\rightarrow H^{n+1}(\bar{\varepsilon}_{\pm}\tilde{T}) \rightarrow V_{n}(A[x^{\pm 1}]) \rightarrow U_{n}^{\bar{\varepsilon}_{\pm}\tilde{T}}(A[x^{\pm 1}]) \rightarrow H^{n}(\bar{\varepsilon}_{\pm}\tilde{T}) \rightarrow$$

$$\bar{\varepsilon}_{\pm} \uparrow \downarrow \varepsilon_{\pm} \qquad \bar{\varepsilon}_{\pm} \uparrow \downarrow \varepsilon_{\pm} \qquad \bar{\varepsilon}_{\pm} \uparrow \downarrow \varepsilon_{\pm} \qquad \bar{\varepsilon}_{\pm} \uparrow \downarrow \varepsilon_{\pm}$$

$$\rightarrow H^{n+1}(\tilde{T}) \rightarrow V_{n}(A) \rightarrow U_{n}^{\tilde{T}}(A) \rightarrow H^{n}(\tilde{T}) \rightarrow .$$

Hence $N_{\pm}V_n(A) \subseteq V_n(A[x^{\pm 1}])$ is mapped isomorphically to ker $(\varepsilon_{\pm}: U_n^{\bar{\varepsilon}_{\pm}\bar{T}}(A[x^{\pm 1}]) \to U_n^{\bar{T}}(A))$ and so (up to isomorphism)

$$U_n^{\bar{\varepsilon}_{\pm}T}(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm}U_n^T(A) \oplus N_{\pm}V_n(A).$$

In particular,

$$U_n^{R_0(A)}(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm} U_n(A) \oplus N_{\pm} V_n(A),$$

$$V_n(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm} V_n(A) \oplus N_{\pm} V_n(A).$$

It now follows from Theorem 4.1 that

$$V_n(A[x,x^{-1}]) = \bar{\epsilon}V_n(A) \oplus \bar{E}_+ N_+ V_n(A) \oplus \bar{E}_- N_- V_n(A) \oplus \bar{B}U_n(A).$$

The expressions for $V_n^{\bar{\epsilon}_{\pm}R\oplus S_{\pm}}(A[x^{\pm 1}])$, $V_n^Q(A[x,x^{-1}])$ may be deduced from those for $V_n(A[x^{\pm 1}])$, $V_n(A[x,x^{-1}])$, working as for $U_n^{\bar{\epsilon}_{\pm}T}(A[x^{\pm 1}])$ above. (In particular, for R=0, $S_+=0$, $S_-=0$, $\tilde{T}=0$ we have

$$Q = S_0 \subseteq \tilde{K}_1 (A[x, x^{-1}])$$

and

$$W_n(A[x^{\pm 1}]) = \bar{\varepsilon}_{\pm} W_n(A) \oplus N_{\pm} W_n(A),$$

$$\tilde{W}_n(A[x, x^{-1}]) = \bar{\varepsilon} W_n(A) \oplus \bar{E}_{\pm} N_{\pm} W_n(A) \oplus \bar{E}_{-} N_{-} W_n(A) \oplus \bar{B} V_n(A).) \quad \Box$$

In §4 of Part II there were defined lower L-theories, functors

 $L_n^{(m)}$: (rings with involution) \rightarrow (abelian groups)

for m < 0, $n \pmod{4}$ by

$$L_n^{(m)}(A) = \ker \left(\varepsilon : L_{n+1}^{(m+1)}(A_z) \to L_{n+1}^{(m+1)}(A) \right)$$

with $L_n^{(0)}(A) = U_n(A)$. By convention, $L_n^{(1)}(A) = V_n(A)$.

COROLLARY 4.5. The lower L-theories $L_n^{(m)}$ coincide (up to natural isomorphism)

with the functors LV_n , L^2V_n , ... derived from V_n , with

$$L_n^{(m)}(A) = L^{1-m}V_n(A) \quad (m \le 0, n \pmod{4}).$$

Proof. By Theorem 4.1,

$$LV_n(A) = U_n(A) = L_n^{(0)}(A)$$
.

Assume inductively that

$$L_n^{(p)}(A) = L^{1-p}V_n(A) \quad (n \pmod{4})$$

for $0 \ge p > m$, for some $m \le -1$. Then

$$L_{n}^{(m)}(A) = \ker \left(\varepsilon : L_{n+1}^{(m+1)}(A_{z}) \to L_{n+1}^{(m+1)}(A)\right)$$

$$= \ker \left(\varepsilon : L^{-m}V_{n+1}(A_{z}) \to L^{-m}V_{n+1}(A)\right)$$

$$= L\left(\ker \left(\varepsilon : L^{-m-1}V_{n+1}(A_{z}) \to L^{-m-1}V_{n+1}(A)\right)\right)$$

$$= L\left(\ker \left(\varepsilon : L_{n+1}^{(m+2)}(A_{z}) \to L_{n+1}^{(m+2)}(A)\right)\right)$$

$$= LL_{n}^{(m+1)}(A)$$

$$= LL^{-m}V_{n}(A) = L^{1-m}V_{n}(A)$$

giving the induction step.

Given a functor

 $F: (rings with involution) \rightarrow (abelian groups)$

define

$$N_{\pm}F(A) = \ker(\varepsilon_{\pm}: F(A[x^{\pm 1}]) \rightarrow F(A)).$$

(By Corollary 4.4, the previous definitions of $N_{\pm}V_n(A)$, $N_{\pm}W_n(A)$ agree with this, up to natural isomorphism).

By analogy with the first part of Corollary 7.6 of Chapter XII of [1] we have

COROLLARY 4.6. Let $x_1, x_2, ..., x_p$ be independent commuting indeterminates over A, with $\bar{x}_1 = x_i$ $(1 \le j \le p)$. Then

$$L_n^{(m)}(A[x_1, x_2, ..., x_p]) = (1 \oplus N_+)^p L_n^{(m)}(A)$$

$$L_n^{(m)}(A[x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_p, x_p^{-1}]) = (1 \oplus N_+ \oplus N_- \oplus L)^p L_n^{(m)}(A)$$

up to natural isomorphism, for $m \le 1$, $n \pmod{4}$, $p \ge 1$. \square

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