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Scattering Theory for Elliptic Operators of Arbitrary Order

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Abstract

We prove the main theorems of scattering theory for selfadjoint elliptic partial differential operators of arbitrary order. Under various hypotheses we show that the wave operators exist and are complete, that the intertwining relations hold, and that the invariance principle holds. One of our main hypotheses is that each lower order coefficient $q(x)$ satisfies

$$(1+|x|)^\alpha \int_{|x-y|<a} |q(y)|\, dy \in L^p(E^n)$$

for some $\alpha \geqslant 0$, $a>0$ and for $p \leqslant \infty$ such that

$$\alpha > 1 - \frac{2n}{(n+1)p}$$

1. Introduction

In this paper we study scattering theory for operators of the form

$$P(D)+Q(x, D) \tag{1.1}$$

in E^n, where $P(D)$ is an elliptic operator of order $2m$ with constant real coefficients, and $Q(x, D)$ is a symmetric operator of order $\leqslant 2m-2$ with variable coefficients. The prototype of (1.1) is the operator

$$-\Delta + q(x), \tag{1.2}$$

which has been studied extensively by many authors. Our results are new even in this case.

One of our results (Theorem 2.7) shows that for $P(D)$ rotationally invariant the main conclusions of scattering theory hold for (1.1) (and consequently for (1.2)) if each coefficient $q(x)$ of $Q(x, D)$ satisfies

$$\sup_x \int_{|x-y|<\delta} |q(y)|\, |x-y|^{2-n}\, dy \to 0 \quad \text{as} \quad \delta \to 0, \tag{1.3}$$

$$\int_{|x-y|<1} |q(y)|\, dy \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.4}$$

and there are numbers $a>0$, $\alpha\geqslant 0$ and $p\leqslant\infty$ such that

$$\alpha>1-\frac{2n}{(n+1)\,p} \tag{1.5}$$

and

$$(1+|x|)^{\alpha}\int\limits_{|x-y|<a}|q(y)|\,dy\in L^{p}(E^{n}). \tag{1.6}$$

In particular the wave operators exist and are complete; the intertwining relations and the invariance principle hold (see Section 2). Another result (Theorem 2.6) comes to the same conclusion if (1.6) is replaced by

$$\sup_{x}\int|q(y)|\,(1+|x-y|)^{-\beta}\,dy<+\infty \tag{1.7}$$

holds for some $\beta<\frac{1}{2}(n-1)$. Still others (Theorems 2.3 and 2.4) give hypotheses depending on the operator $P(D)$ and hold even for $P(D)$ not rotationally invariant.

Kuroda [7] proved these results (and the discreteness of the continuum eigenvalues) under the hypothesis

$$(1+|x|)^{1+\varepsilon}\,q(x)\in L^{\infty} \tag{1.8}$$

for some $\varepsilon>0$. He does not require that $P(D)$ be rotationally invariant. Note that (1.8) implies (1.3)–(1.6). Beals [17] and Birman [18] were the first to study scattering theory for higher order operators. Although his results are weaker, Beals was able to treat operators of the form

$$P(x,D)+q(x), \tag{1.9}$$

where $P(x,D)$ is a uniformly strongly elliptic operator with smooth coefficients. One of his assumptions is

$$\int\limits_{|x-y|<1}|q(y)|\,|x-y|^{2m-n}\,dy\leqslant C(1+|x|)^{-a} \tag{1.10}$$

for some $a>n$. This implies our hypotheses (cf. Theorem 2.7). Birman obtains similar results for $q(x)$ bounded and

$$(1+|x|)^{\alpha}\,q(x)\in L^{p} \tag{1.11}$$

with $p=2$ and $\alpha>\frac{1}{2}n$. Again this implies (1.3)–(1.6).

The operator (1.2) has received considerable attention in recent years. We quote only some of the more recent results. Most of them obtained eigenfunctions evpansions as well. Ikebe [9] assumed $q(x)=0(|x|^{-2-\varepsilon})$ as $|x|\to\infty$. Rejto [11] allowed the estimate $q(x)=0(|x|^{-\varepsilon-4/3})$, while Kato [13] showed that (1.8) suffices. Kuroda [6] assumed (1.3), (1.4) and (1.11) with

$$\alpha>1-p^{-1}. \tag{1.12}$$

Alsholm and Schmidt [10] assumed

$$\int |q(y)|\,|x-y|^{(1-n)/2}\,dy \tag{1.13}$$

bounded and tending to 0 as $|x|\to\infty$,

$$\sup_x \int_{|x-y|<1} |q(y)|^2\,|x-y|^{-\delta}\,dy<\infty \tag{1.14}$$

for some $\delta>n-4$, and (1.11) holds for $p=2$ and some $\alpha>1-\frac{1}{2}n$. For $n=3$, Kato and Kuroda [1] showed that $q\in L^{3/2}$ is sufficient, while Simon [13], using a result of Kato [3], extended this to those q satisfying

$$\int\int |q(x)\,q(y)|\,|x-y|^{-2}\,d^3x\,d^3y<\infty. \tag{1.15}$$

(He calls this the Rollnik class.) Greifenegger, Jorgens, Weidmann and Winkler [12] assumed

$$\sup_x (1+|x|)^{\tau} \int_{|x-y|<1} |q(y)|^2\,|x-y|^{-\delta}\,dy<\infty$$

for some $\tau>2$ and $\delta>n-4$. It is simple to find potentials which are covered by the present paper but which are not included in any of the results quoted. Moreover, we recapture most of these results.

We follow the method introduced by Kato and Kuroda [1–7], which first proves an abstract theorem in Hilbert space and then applies this to the partial differential operator. The abstract theorems that they proved did not quite suit our needs, although the version given in [7, I] comes fairly close. In Section 3 we present a theorem which differs from that of [7, I] in several crucial respects, but retains most of the basic features. In Section 4 we show how this theorem can be applied to obtain our main results, which are stated in the next section. We would like to thank S. T. Kuroda for making a manuscript of [7] available to us.

In [19] and at a seminar at the Hebrew University given in May, 1973, Agmon described his results on general eigenfunction expansions for symmetric operators of the form

$$P(D)+\sum_{|\mu|\leqslant 2m} b_\mu^0(x)\,D^\mu+\sum_{|\mu|<2m} b_\mu^1(x)\,D^\mu .$$

He assumes that the b_μ^0 satisfy (1.8) and that b_μ^1 has compact support and is a compact operator from $H^{2m-|\mu|,2}$ to L^2. After these lectures, we realized that the results of the present paper hold even when $Q(x, D)$ is of order $2m$ as long as its highest order coefficients are bounded and $P(D)+Q(x, D)$ is bounded from below. All of the proofs apply unchanged.

M. Thompson (Comm. Pure Appl. Math. *25* (1972) 499–532) obtains eigenfunction expansions for higher order elliptic operators. Because his expansions are done in the classical sense, he is required to make assumptions on the operator which are more restrictive than those of Agmon.

2. The Main Results

Let $x=(x_1,\ldots, x_n)$ represent a generic point in E^n, and set $|x|^2=x_1^2+\cdots+x_n^2$. $D_j=\partial/i\partial x_j$, $1\leqslant j\leqslant n$. For any multi-index $\mu=(\mu_1,\cdots,\mu_n)$ of nonnegative integers, we set $|\mu|=\mu_1+\cdots+\mu_n$ and $D^\mu=D_1^{\mu_1}\ldots D_n^{\mu_n}$. We also write $x^\mu=x_1^{\mu_1}\ldots x_n^{\mu_n}$, and consider a polynomial

$$P(\xi)=\sum_{|\mu|\leqslant 2m} a_\mu \xi^\mu \tag{2.1}$$

of degree $2m$ with real coefficients, where $\xi=(\xi_1,\ldots,\xi_n)$. The partial differential operator corresponding to $P(\xi)$ is

$$P(D)=\sum_{|\mu|\leqslant 2m} a_\mu D^\mu . \tag{2.2}$$

Let C_0^∞ denote the set of complex valued functions in $C_0^\infty(E^n)$ which have compact supports. Consider $P(D)$ as an operator on $L^2=L^2(E^n)$ with domain C_0^∞. It is known that it has a closure P_0 which is self-adjoint (cf., e.g., [14, p. 62]). Our first assumption is

(I) There are positive constants a and K such that

$$P(\xi)\geqslant a|\xi|^{2m}-K, \qquad \xi\in E^n. \tag{2.3}$$

Next we consider the variable coefficient operator

$$Q(x, D)=\sum_{j,k=1}^{N} \bar{P}_j(D)\,q_{jk}(x)\,P_k(D), \tag{2.4}$$

where the $P_k(D)$ are constant coefficient operators of orders $m_k<m$ and $\bar{P}(\xi)$ denotes the polynomial whose coefficients are the complex conjugates of those of $P(\xi)$. We assume that the coefficients satisfy $\overline{q_{jk}(x)}=q_{kj}(x)$. Our hypotheses concerning the coefficients will be expressed in terms of the quantities

$$N_{\alpha,\delta}(h)=\sup_x \int_{|x-y|<\delta} |h(y)|^2\,\omega_\alpha(x-y)\,dy,$$

where

$$\begin{aligned}\omega_\alpha(x)&=|x|^{\alpha-n} && \text{for} \quad \alpha<n\\ &=1-\log|x| && \text{for} \quad \alpha=n\\ &=1 && \text{for} \quad \alpha>n.\end{aligned}$$

We write $N_\alpha(h)=N_{\alpha,1}(h)$, and we shall say that $h\in N_\alpha$ if $N_\alpha(h)<\infty$. Our second assumption is

(II) For each j and k, $q_{jk}(x)=g_{jk}(x)\,h_{jk}(x)$, where $g_{jk}\in N_{2m-2m_j}$ and $h_{jk}\in N_{2m-2m_k}$. Also

$$\int_{|x-y|<1} |h_{jk}(y)|^2\,dy\to 0 \quad \text{as} \quad |x|\to\infty. \tag{2.5}$$

Moreover, for those j satisfying $2m\leqslant 2m_j+n$, we assume

$$N_{2m-2m_j,\delta}(g_{jk})\to 0 \quad \text{as} \quad \delta\to 0, \qquad \text{each } k. \tag{2.6}$$

We shall make use of

LEMMA 2.1. *If* (I) *and* (II) *hold, then there is a unique self-adjoint operator H on L^2 such that $C_0^\infty\subset D(|H|^{1/2})$ and*

$$(Hu, v)=(P(D)\,u+Q(x, D)\,u, v), \qquad u, v\in C_0^\infty.$$

Next, let Γ be the set of those real λ for which there is a $\xi\in E^n$ satisfying $P(\xi)=\lambda$, $\operatorname{grad} P(\xi)=0$. It can be shown that Γ consists of a finite number of points (cf. Agmon [19]). We shall prove

LEMMA 2.2. *If λ is not in Γ, then the limit*

$$K_\lambda(x)=\lim_{\varepsilon\to 0}\int_{|P(\xi)-\lambda|<1} \frac{\varepsilon\, e^{ix\xi}\,d\xi}{(P(\xi)-\lambda)^2+\varepsilon^2} \tag{2.7}$$

exists for each $x\in E^n$ (here $x\xi=x_1\xi_1+\cdots+x_n\xi_n$). The convergence is uniform on bounded sets and $K_\lambda(x)$ is infinitely differentiable.

The proof of this lemma will be given in Section 5. Our final hypothesis will be given in terms of the function (2.7).

(III) For each j, k, s, t and each $\lambda \notin \Gamma$

$$\sup_x \int (|g_{jk}(y)|^2 + |h_{st}(y)|^2)\, |P_j(D)\, P_t(D)\, K_\lambda(x-y)|\, dy < \infty, \tag{2.8}$$

and there are positive constants α, C such that

$$\begin{aligned} \sup_x \int (|g_{jk}(y)|^2 + |h_{st}(y)|^2)\, P_j(D)\, P_t(D) \\ \times [K_\lambda(x-y) - K_{\lambda'}(x-y)]\, dy \leqslant C|\lambda-\lambda'|^\alpha. \end{aligned} \tag{2.9}$$

One of our main results is

THEOREM 2.3. *Under hypotheses* (I)–(III) *the operator* H *given in Lemma* 2.1 *has the following properties:*

1) *There is a closed set* $e \subset E^1$ *of (Lebesque) measure* 0 *such that* H *restricted to* $E[\sigma(P_0) - (\Gamma \cup e)]\, L^2$ *is absolutely continuous (here* $E(I)$ *denotes the spectral measure associated with* H*).*

2) *The absolutely continuous part of* H *is unitarily equivalent to* P_0.

3) *The wave operators*

$$W = s - \lim_{t \to \pm\infty} e^{itH}\, e^{-itP_0} \tag{2.10}$$

exist and are complete.

4) *The invariance principle holds.*

5) *The intertwining relations*

$$HW_\pm = W_\pm P_0 \tag{2.11}$$

hold.

6) *The operator* $S = W_+^* W_-$ *is unitary on* L^2 *and commutes with* P_0.

Perhaps a few words of explanation are in order. The limit in (2.10) is the strong limit (taken when the operator is applied to each $u \in L^2$). Completeness of the wave operators means that their ranges coincide. This is required for S to be unitary. The invariance principle means that

$$W_\pm = s - \lim_{t \to \pm\infty} e^{it\varphi(H)}\, e^{-it\varphi(P_0)} \tag{2.12}$$

holds for any function $\varphi(\lambda)$ satisfying

$$\int_0^\infty \Big|\int_I e^{-it\varphi(\lambda)-i\xi\lambda}\, d\lambda\Big|^2 d\xi \to 0 \quad \text{as} \quad t\to+\infty \tag{2.13}$$

for any compact interval I contained in $\sigma(P_0)-e$. Note that (2.13) holds trivially for $\varphi(\lambda)=\lambda$. Further explanations of the conclusions of Theorem 2.3 may be found in Section 3 and in Kato [4, Chapter X].

A slight variation of Theorem 2.3 is given by

THEOREM 2.4. *The conclusions of Theorem* 2.3 *hold if hypothesis* (III) *is replaced by*

(III′) For each j, k, s, t and each $\lambda \notin \Gamma$

$$\iint |g_{jk}(x)|^2 (|g_{st}(y)|^2+|h_{st}(y)|^2) \tag{2.14}$$
$$|P_j(D)\,P_t(D)\,K_\lambda(x-y)|^2\, dx\, dy<\infty$$

and

$$\iint |g_{jk}(x)^2 (|g_{st}(y)|^2+|h_{st}(y)|^2 \tag{2.15}$$
$$|P_j(D)\,P_t(D)\,[K_\lambda(x-y)-K_{\lambda'}(x-y)]|\, dx\, dy\leqslant C|\lambda-\lambda'|^\alpha$$

for some positive constants α, C.

Next we restrict our attention to operators which are rotationally invariant. Assume that

$$P(\xi)=p(|\xi|^2), \tag{2.16}$$

where $p(t)$ is a polynomial of degree m in one variable. In this case we can compute the functions (2.7) explicitly. In fact we have

THEOREM 2.5. *If* (2.16) *holds and* $\lambda\notin\Gamma$, *then*

$$K_\lambda(x)=\pi^2(2\pi/|x|)^\gamma \sum_{p(t_k{}^2)=\lambda} t_k^\gamma J_\gamma(t_k|x|)/p'(t_k^2), \tag{2.17}$$

where $\gamma=\frac{1}{2}n-1$ *and the* t_k *are the positive roots of* $p(t^2)=\lambda$ *(here* p' *denotes the derivative of* p*). Moreover, one has*

$$|D^\mu K_\lambda(x)|\leqslant C_{\lambda,\mu}(1+|x|)^{(1-n)/2} \tag{2.18}$$

for each μ *and*

$$|D^{\mu}[K_{\lambda}(x)-K_{\lambda'}(x)]| \leqslant C_{\lambda,\mu,\alpha}|\lambda-\lambda'|^{\alpha} \\ (1+|x|)^{\alpha+(1-n)/2} \tag{2.19}$$

for each $\alpha>0$ *and* μ.

Using the estimates (2.18) and (2.19), we have

THEOREM 2.6. *Suppose* $P(\xi)$ *is rotationally invariant and that* (I) *and* (II) *hold. Assume also that for each* j *and* k *there is a* $\beta<\frac{1}{2}(n-1)$ *such that*

$$\sup_{x} \int (|g_{jk}(y)|^2+(h_{jk}(y)|^2)(1+|x-y|)^{-\beta}\, dy<\infty. \tag{2.20}$$

Then all of the conclusions of Theorem 2.3 *hold.*

THEOREM 2.7. *Suppose* $P(\xi)$ *is rotationally invariant and that* (I) *and* (II) *hold. Assume also that for each* j *and* k *there are numbers* $\delta>0$, $\alpha\geqslant 0$ *and* $p\leqslant\infty$ *such that*

$$\alpha>1-\frac{2n}{(n+1)\,p}$$

and

$$(1+|x|)^{\alpha} \int_{|x-y|<\delta} (|g_{jk}(y)|^2+|h_{jk}(y)|^2)\, dy \in L^p.$$

Then all of the conclusions of Theorem 2.3 *hold.*

3. An Abstract Theorem

We now give a theorem in Hilbert space which we use in proving our results in Section 2. It is a variation of the Kato-Kuroda approach (cf. [1–7]), and we found it most useful for the applications at hand. We believe that it is of interest in its own right. Our results are similar to Theorems 3.11–3.13 of [7, I], but there are important differences. In particular, we were unable to use the results of Kuroda to obtain the theorems of Section 2.

First we discuss notations and conventions. For a Hilbert space $\mathscr{H}$, the norm and scalar product will be denoted by $\| \ \|_{\mathscr{H}}$ and $(\ ,\ )_{\mathscr{H}}$, respectively. $B(\mathscr{H},\mathscr{K})$ will denote the set of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. We shall write $B(\mathscr{H})$ for $B(\mathscr{H},\mathscr{H})$. All operators will be densely defined. The doman and range of an operator will be denoted by $D(A)$ and $R(A)$, respectively. Its resolvent and spectrum sets will be denoted by $\varrho(A)$ and $\sigma(A)$, respectively. The closure of a closable operator will be denoted by $[A]^a$. All subsets of the real line will be Borel sets. For each such set I, χ_I will denote its characteristic function. The Hilbert space of all strongly measurable $\mathscr{H}$-valued functions with square integrable norms over I will be denoted by $L^2(I, \mathscr{H})$. $I \subset \subset J$ will mean that the closure $\bar{I}$ of I is a compact subset of J.

A bilinear form $a(u, v)$ with domain $D(a)$ is a complex valued functional on $D(a) \times D(a)$ which is linear in u and conjugate linear in v. $D(a)$ is to be a subspace of a Hilbert space (cf. [22, Chapter XII]). An operator A is *associated* with a bilinear form $a(u, v)$ if the statements $u \in D(A)$ and $Au = f$ are equivalent to $u \in D(a)$ and $a(u, v) = (f, v)$ for all $v \in D(a)$. Every self-adjoint operator is associated with a bilinear form.

Our assumptions are

1. H is a self-adjoin opertator on a Hilbert space $\mathscr{H}$. Let $\{E(\lambda)\}$ denote its spectral family and $\{E(I)\}$ its spectral measure. For $\zeta \in \varrho(\mathscr{H})$, set $R(\zeta) = (\zeta - H)^{-1}$. All norms and scalar products without subscripts will be those of H.

2. There are an open subset $\wedge$ of the real line, a Hilbert space $\mathscr{C}$ and a unitary operator F from $E(\wedge)\mathscr{H}$ *onto* $L^2(\wedge, \mathscr{C})$ such that

$$FE(I)F^{-1} = \chi_I, \quad I \subset \wedge. \tag{3.1}$$

3. There are a Hilbert space $\mathscr{K}$ and closed operators A, B from H to K such that

$$\tilde{D} = D(|H|^{1/2}) \subset D(A) \cap D(B). \tag{3.2}$$

4. A is injective.

5. There is a $\zeta_0 \in \varrho(H)$ such that $BR(\zeta)[R(\zeta_0)A^*]^a$ is a compact operator on $\mathscr{K}$ for each $\zeta \in \varrho(H)$.

6. There are continuous functions $Q_\pm(\lambda)$ from $\wedge$ to $B(\mathscr{K})$ such that $[BR(\zeta)A^*]^a$ converges in norm to $Q_\pm(\lambda)$ as $\zeta \to \lambda$ for each $\lambda \in \wedge$.

7. There is a function $M(\lambda)$ on $\wedge$ with values in $B(\mathscr{K})$ which is integrable over any compact subset of $\wedge$ and such that

$$\frac{d}{d\lambda}(E(\lambda)A^*u, A^*v) = (M(\lambda)u, v)_{\mathscr{K}} \quad \text{a.e.} \quad u, v \in D(A^*) \tag{3.3}$$

8. There is a self-adjoint operator H_1 on $\mathscr{H}$ such that $D(|H_1|^{1/2}) = \tilde{D}$ and its associated bilinear form satisfies

$$h_1(u, v) = h(u, v) + (Bu, Av)_{\mathscr{K}}, \quad u, v \in \tilde{D}, \tag{3.4}$$

where $h(u, v)$ is the bilinear form associated with H. We let $\{E_1(\lambda)\}$ and $\{E_1(I)\}$ denote the spectral family and measure of H_1, respectively. For $\zeta \in \varrho(H_1)$ we put $R_1(\zeta) = (\zeta - H_1)^{-1}$.

The main theorem of this section is:

THEOREM 3.1. *Under Assumptions* 1–8, *there is a closed set* $e \subset \wedge$ *of Lebesgue measure* 0 *and unitary operators* $W_\pm$ *from* $E(\wedge)\mathscr{H}$ *onto* $E_1(\wedge - e)\mathscr{H}$ *such that:*

(a) $H_1 W_\pm = W_\pm H$ *on* $E(\wedge)\mathscr{H}$

(b) $S = W_+^* W_-$ *is unitary on* $E(\wedge)\mathscr{H}$ *and commutes with* H.

(c) $W_\pm u = \lim_{t \to \pm\infty} e^{itH_1} e^{-itH} u, \quad u \in E(\wedge)\mathscr{H}$

(d) *The invariance principle holds.*

We give the proof by means of a series of lemmas. Lemmas 3.2–3.5 were proved in [7, I]. We make the following definitions.

$Q(\zeta) = [BR(\zeta) A^*]^a, \quad G(\zeta) = I + Q(\zeta).$

$Q_1(\zeta) = [BR_1(\zeta) A^*]^a, \quad G_1(\zeta) = I - Q_1(\zeta).$

$\Pi^\pm = \{\zeta \mid \operatorname{Im} \zeta \gtrless 0\}$

$\Pi_I^\pm = \Pi^\pm \cup I$

$G_\pm(\zeta) = G(\zeta) \quad \text{in} \quad \Pi^\pm$

$G_\pm(\lambda) = I + Q_\pm(\lambda), \quad \lambda \in \wedge.$

$e_\pm = \{\lambda \in \wedge \mid G_\pm(\lambda) \text{ is not injective}\}$

$e = e_+ \cup e_-$

$G_{1\pm}(\zeta) = G_1(\zeta) \quad \text{in} \quad \Pi^\pm$

$G_{1\pm}(\lambda) = G_\pm(\lambda)^{-1} \quad \text{in} \quad \wedge - e.$

LEMMA 3.2. *The set e is closed and of Lebesgue measure* 0.

LEMMA 3.3. *For* $\zeta \in \varrho(H) \cap \varrho(H_1)$

$$BR(\zeta) = G(\zeta)\, BR_1(\zeta) \tag{3.5}$$

and

$$[R(\delta) A^*]^a = [R_1(\zeta) A^*]^a\, G(\zeta). \tag{3.6}$$

LEMMA 3.4. $G_\pm(\zeta)$ *is continuous in* $\Pi_\wedge^\pm$, *and* $G_{1\pm}(\zeta)$ *is continuous in* $\Pi_{\wedge - e}^\pm$. *Also*

$$G_{1\pm}(\zeta) = G_\pm(\zeta)^{-1} \quad \text{in} \quad \Pi_{\wedge - e}^\pm. \tag{3.7}$$

LEMMA 3.5. *If I is bounded, then* $[E(I) A^*]^a$ *is in* $B(\mathscr{K}, \mathscr{H})$.

LEMMA 3.6. *Set*

$T = T_\lambda = FE(\wedge) A^*.$

Then T maps $D(A^*)$ into $L^2(\wedge, \mathscr{C})$. Also

$$\frac{d}{d\lambda}(E(\lambda)A^*u, A^*v) = (T_\lambda u, T_\lambda v)_{\mathscr{C}} \quad \text{a.e.,} \quad u, v \in D(A^*). \tag{3.8}$$

Proof. Let I be any subset of $\wedge$. Then

$$\begin{aligned}(E(I)A^*u, A^*v) &= (E(\wedge)E(I)E(\wedge)A^*u, E(\wedge)A^*v) \\ &= \int_\wedge (FE(I)E(\wedge)A^*u, FE(\wedge)A^*v)_{\mathscr{C}}\, d\lambda \\ &= \int_\wedge \chi_I(\lambda)(FE(\wedge)A^*u, FE(\wedge)A^*v)_{\mathscr{C}}\, d\lambda \\ &= \int_I (Tu, Tv)_{\mathscr{C}}\, d\lambda.\end{aligned} \tag{3.9}$$

Since this is true for any subset I, we get (2.8).

LEMMA 3.7. *$R(T)$ is dense in $L^2(\wedge, \mathscr{C})$.*

Proof. Let $g(\lambda)$ be any function in $L^2(\wedge, \mathscr{C})$. By Assumption 2 there is a w in $\mathscr{H}$ such that $g = Fw$. Since A is injective (Assumption 4), $R(A^*)$ is dense in $\mathscr{H}$. Thus there is a sequence $\{u_k\}$ of elements in $D(A^*)$ such that $A^*u_k \to w$ in $\mathscr{H}$. Thus $E(\wedge)A^*u_k \to E(\wedge)w$ in $\mathscr{H}$ and $FE(\wedge)A^*u_k \to FE(\wedge)w$ in $L^2(\wedge, \mathscr{C})$. □

LEMMA 3.8. *For each bounded subset I of $\wedge$ and each bounded Borel measurable function $f(\lambda)$,*

$$[A f(H) E(I) A^*]^a = \int_I f(\lambda) M(\lambda)\, d\lambda. \tag{3.10}$$

Proof. We have

$$\begin{aligned}(f(H)E(I)A^*u, A^*v) &= \int_I f(\lambda)\frac{d}{d\lambda}(E(\lambda)A^*u, A^*v)\, d\lambda \\ &= \int_I f(\lambda)(M(\lambda)u, v)_{\mathscr{K}}\, d\lambda, \qquad u, v \in D(A^*).\end{aligned}$$

Since $D(A^*)$ is dense in $\mathscr{K}$, we get (2.10).

LEMMA 3.9.

$$(M(\lambda)u, v)_{\mathscr{K}} = (T_\lambda u, T_\lambda v)_{\mathscr{C}} \quad \text{a.e.,} \quad u, v \in D(A^*).$$

Proof. Apply Lemma 3.6 to Assumption 7. □

Set

$$\delta_\varepsilon(\mu)=\frac{\varepsilon}{\pi}(\mu^2+\varepsilon^2)^{-1}$$

Thus

$$\delta_\varepsilon(H-\lambda)=\frac{\varepsilon}{\pi}R(\lambda+i\varepsilon)R(\lambda-i\varepsilon).$$

LEMMA 3.10. *For almost all* $\lambda\in\wedge$,

$$[A\delta_\varepsilon(H-\lambda)A^*]^a\to M(\lambda)$$

in norm as $\varepsilon\to 0$.

Proof. Let I be any compact interval in $\wedge$. Then for $\lambda\in I$

$$\begin{aligned}[A\delta_\varepsilon(H-\lambda)A^*]^a&=[A\delta_\varepsilon(H-\lambda)E(I)A^*]^a\\&\quad+[A\delta_\varepsilon(H-\lambda)E(R-I)A^*]^a.\end{aligned}$$

The first term on the right equals

$$\frac{\varepsilon}{\pi}\int_I\frac{1}{(\mu-\lambda)^2+\varepsilon^2}M(\mu)\,d\mu$$

by (3.10). This converges a.e. in norm to $M(\lambda)$. The second term equals

$$\frac{\varepsilon}{\pi}AR(\lambda+i\varepsilon)E(R-I)[R(\lambda-i\varepsilon)E(R-I)A^*]^a.$$

Since $AR(\lambda+i\varepsilon)$ is bounded on $E(R-I)$ for λ an interior point of I, this term tends to 0 in norm as $\varepsilon\to 0$ for such λ. □

LEMMA 3.11. *For* $u, v\in D(A^*)$

$$(\delta_\varepsilon(H_a-\lambda)A^*u, A^*v)=([A\delta_\varepsilon(H-\lambda)A^*]^a G_{1\pm}(\zeta)u, G_{1\pm}(\zeta)v)_{\mathscr{K}} \tag{3.11}$$

Proof. The left hand side of (3.11) equals

$$\frac{\varepsilon}{\pi}(R_1(\zeta)A^*u, R_1(\zeta)A^*v),$$

where $\zeta=\lambda+i\varepsilon$. By (3.6) and (3.7) this equals

$$\frac{\varepsilon}{\pi}([R(\zeta)A^*]^a G_{1\pm}(\zeta)u, [R(\zeta)A^*]^a G_{1\pm}(\zeta)v). \square$$

LEMMA 3.12. *If* $I\subset\subset\wedge$, *then* $[\chi_I T]^a$ *is in* $B(\mathscr{H}, L^2(\wedge, \mathscr{C}))$.
Proof. By Assumption 2 we have

$$\chi_I T=\chi_I FE(\wedge)A^*=FE(I)A^*.$$

Thus

$$[\chi_I T]^a=F[E(I)A^*]^a,$$

and the latter operator is bounded by Lemma 3.5. $\square$

LEMMA 3.13. *There is a linear map* $\tilde{T}$ *of* $\mathscr{H}$ *into* $L^2_{\mathrm{loc}}(\wedge, \mathscr{C})$ *such that*

1) $\tilde{T}$ *is an extension of* T
2) $D(\tilde{T})=\mathscr{H}$
3) $\chi_I\tilde{T}=F[E(I)A^*]$, $\quad I\subset\subset\wedge$
4) $\chi_I\tilde{T}\in B(\mathscr{H}, L^2(I, \mathscr{C}))$, $\quad I\subset\subset\wedge$. (3.12)

Proof. For any $I\subset\subset\wedge$, define $\chi_I\tilde{T}$ by (3.12). That this defines $\tilde{T}$ a.e. and does not depend on the choice of I follows from

$$\begin{aligned}\chi_J\chi_I\tilde{T}&=\chi_J F[E(I)A^*]^a=F[E(J)E(I)A^*]^a\\&=\chi_I F[E(J)A^*]^a.\end{aligned}$$

The other properties follow from the definition and Lemma 3.12. $\square$

LEMMA 3.14.

$$(M(\lambda)u, v)_{\mathscr{H}}=(\tilde{T}_\lambda u, \tilde{T}_\lambda v)_{\mathscr{C}} \quad \text{a.e.,} \quad u, v\in\mathscr{H}.$$

Proof. By Lemma 3.9, this holds a.e. for $u, v\in D(A^*)$. Thus for any $I\subset\subset\wedge$,

$$\int_I (M(\lambda)u, v)_{\mathscr{H}}\, d\lambda=\int_I (\tilde{T}_\lambda u, \tilde{T}_\lambda v)_{\mathscr{C}}\, d\lambda \tag{3.13}$$

holds for all $u, v\in D(A^*)$. Since both sides of (3.13) are continuous on $\mathscr{H}\times\mathscr{H}$ and $D(A^*)$ is dense in $\mathscr{H}$, we see that (3.13) holds for $u, v\in\mathscr{H}$. Since this is true for any $I\subset\subset\wedge$, the lemma follows. $\square$

LEMMA 3.15. *Set*

$$T_{\pm}=T_{\pm\lambda}=\tilde{T}_{\lambda}G_{1\pm}(\lambda).$$

Then

$$\frac{d}{d\lambda}(E_1(\lambda)\,A^*u,\,A^*v)=(T_{\pm\lambda}u,\,T_{\pm\lambda}v)_{\mathscr{C}}\quad \text{a.e.,}\quad u,v\in D(A^*). \tag{3.14}$$

Proof. Let $\varepsilon\to 0$ in (3.11). The left hand side converges to the left side of (3.14) a.e. By Lemmas 3.4 and 3.10, the right hand side converges to

$$(M(\lambda)\,G_{1\pm}(\lambda)\,u,\quad G_{1\pm}(\lambda)\,v)_{\mathscr{H}}$$

a.e. in $\wedge - e$. We now apply Lemma 3.14. □

Since $R(A^*)$ is dense in $\mathscr{H}$, elements of the form

$$u=\sum_{k=1}^{l} E_1(I_k)\,A^*u_k,\qquad I_k\subset\subset \wedge - e,\ u_k\in D(A^*) \tag{3.15}$$

are dense in $E_1(\wedge - e)\,\mathscr{H}$. For such an element, set

$$u_{\pm}(\lambda)=\sum_{k=1}^{l}\chi_{I_k}(\lambda)\,T_{\pm\lambda}u_k. \tag{3.16}$$

Clearly, this is an element of $L^2(\wedge,\mathscr{C})$ with the square of its norm equal to

$$\sum\int_{I_j\cap I_k}(T_{\pm\lambda}u_j,\,T_{\pm\lambda}u_k)_{\mathscr{C}}\,d\lambda$$
$$=\sum(E_1(I_j)\,A^*u_j,\,E_1(I_k)\,A^*u_k)=\|u\|^2$$

by (3.14). Thus the mappings $u\to u_{\pm}$ can be extended to isometries $F_{\pm}$ of $E_1(\wedge - e)\,\mathscr{H}$ into $L^2(\wedge,\mathscr{C})$.

LEMMA 3.16. *The operators $F_{\pm}$ are surjective.*

Proof. Suppose $h(\lambda)$ is in $L^2(\wedge,\mathscr{C})$ and

$$\int_{\wedge}(F_{\pm}u,\,h)_{\mathscr{C}}\,d\lambda=0,\qquad u\in E_1(\wedge - e)\,\mathscr{H}.$$

In particular,

$$\int_{\wedge}(F_{\pm}E_1(I)A^*u,\,h)_{\mathscr{C}}\,d\lambda=0$$

holds for any $I \subset\subset \wedge - e$ and $u \in D(A^*)$. By (3.15) and (3.16), this means

$$\int_I (T_\pm u, h)_{\mathscr{C}}\, d\lambda = 0, \qquad I \subset\subset \wedge - e, u \in D(A^*)$$

If $I \subset\subset \wedge - e$, then $\chi_I T_\pm$ is a continuous map from $\mathscr{H}$ to $L^2(I, \mathscr{C})$ (Lemmas 3.4 and 3.13). Since $D(A^*)$ is dense in $\mathscr{H}$, this implies

$$\int_I (T_\pm u, h)_{\mathscr{C}}\, d\lambda = 0, \qquad I \subset\subset \wedge - e, \qquad u \in \mathscr{H}.$$

Since this is true for any such interval I, we have

$$(T_{\pm\lambda} u, h(\lambda))_{\mathscr{C}} = 0 \quad \text{a.e.}, \quad u \in \mathscr{H},$$

or

$$(\tilde{T}_\lambda G_{1\pm}(\lambda)\, u, h(\lambda))_{\mathscr{C}} = 0 \quad \text{a.e.}, \quad u \in \mathscr{H}$$

Since $G_{1\pm}(\lambda)$ is surjective for $\lambda \in \wedge - e$, this implies

$$(\tilde{T}_\lambda w, h(\lambda))_{\mathscr{C}} = 0 \quad \text{a.e.}, \quad w \in \mathscr{H}.$$

In particular, we have

$$(T_\lambda v, h(\lambda))_{\mathscr{C}} = 0 \quad \text{a.e.}, \quad v \in D(A^*).$$

Thus

$$\int_\wedge (T_\lambda v, h(\lambda))_{\mathscr{C}}\, d\lambda = 0, \qquad v \in D(A^*).$$

Since $R(T)$ is dense in $L^2(\wedge, \mathscr{C})$ (Lemma 3.7), this implies that $h(\lambda) = 0$ a.e. □

LEMMA 3.17. *Elements of the form* (3.16) *are dense in* $L^2(\wedge, \mathscr{C})$.
Proof. By Lemma 3.16 every element in $L^2(\wedge, \mathscr{C})$ is the limit of such elements.

LEMMA 3.18. $F_\pm E_1(J)\, F_\pm^{-1} = \chi_J$ *on* $L^2(\wedge, \mathscr{C})$ *for each* $J \subset \wedge - e$.
Proof. By Lemma 3.17, it suffices to verify this on elements of the form

$$g(\lambda) = \chi_I(\lambda)\, T_{\pm\lambda} u, \qquad I \subset\subset \wedge - e, \qquad u \in D(A^*).$$

By (3.15) and (3.16),

$$F_\pm^{-1} g = E_1(I)\, A^* u.$$

Thus

$$F_\pm E_1(J)\,F_\pm^{-1}g = F_\pm E_1(I\cap J)\,A^*u$$
$$= \chi_{I\cap J}(\lambda)\,T_{\pm\lambda}u = \chi_J(\lambda)\,g(\lambda). \quad \square$$

LEMMA 3.19. *Set*

$$\tilde{M}(\lambda) = G_{1\pm}(\lambda)^* M(\lambda)\,G_{1\pm}(\lambda).$$

Then $\tilde{M}(\lambda)$ is in $B(\mathscr{K})$ for each $\lambda \in \wedge - e$ and it is integrable over each set $I \subset\subset \wedge - e$. Moreover, for each bounded, Borel measurable function $f(\lambda)$,

$$[Af(H_1)\,\delta_\varepsilon(H_1-\lambda)\,A^*]^a \to f(\lambda)\,\tilde{M}(\lambda)$$

in norm a.e. as $\varepsilon \to 0$.

Proof. The first statement is obvious. To prove the second, suppose $I \subset\subset \wedge - e$. For λ in the interior of I we write

$$[Af(H_1)\,\delta_\varepsilon(H_1-\lambda)\,A^*]^a = [Af(H_1)\,\delta_\varepsilon(H_1-\lambda)\,E(I)\,A^*]^a$$
$$+ [Af(H_1)\,\delta_\varepsilon(H_1-\lambda)\,E(R-I)\,A^*]^a.$$

The first term equals

$$\frac{\varepsilon}{\pi}\int_I \frac{1}{(\mu-\lambda)^2+\varepsilon^2}\,f(\mu)\,\tilde{M}(\mu)\,d\mu$$

by Lemma 3.15. This converges to $f(\lambda)\,\tilde{M}(\lambda)$ in norm a.e. as $\varepsilon \to 0$. As in the proof of Lemma 3.10, the second term goes to 0 in norm. $\square$

LEMMA 3.20. *$W_\pm E(I) = E_1(I)\,W_\pm$ on $E(\wedge)\,\mathscr{H}$ for each $I \subset \wedge - e$, where $W_\pm = F_\pm^* F$.*

Proof. By Lemma 3.18,

$$F_\pm E_1(I) = \chi_I F_\pm.$$

Thus

$$E_1(I)\,F_\pm^* = F_\pm^* \chi_I.$$

Hence by Assumption 2,

$$W_\pm E(I) = F_\pm^* F E(I) = F_\pm^* \chi_I F$$
$$= E_1(I)\,F_\pm^* F = E_1(I)\,W_\pm. \quad \square$$

LEMMA 3.21. *Let B_0 be the set of complex valued Borel measurable functions with compact supports in $\wedge - e$. Then for each $f \in B_0$ and each $u \in D(A^*)$*

$$AR_1(\lambda + i\varepsilon) f(H_1) A^* u \to h(\lambda, f, u) \tag{3.17}$$

in $L^2(R, \mathscr{K})$ as $\varepsilon \to 0$, and

$$\int_{-\infty}^{\infty} \|h(\lambda, f, u)\|_{\mathscr{K}}^2 \, d\lambda = 2\pi \int_0^{\infty} d\xi \left\| \int e^{-i\xi\mu} f(\mu) \tilde{M}(\mu) u d_\mu \right\|_{\mathscr{K}}^2. \tag{3.18}$$

Proof. Let $I \subset\subset \wedge - e$ contain the support of f. By lemmas 3.14 and 3.15

$$AR_1(\lambda + i\varepsilon) f(H_1) E_1(I) A^* u = \int_I \frac{f(\mu)}{\lambda + i\varepsilon - \mu} \tilde{M}(\mu) u d_\mu.$$

Since $f(x) \tilde{M}(x) u \in L^2(R, \mathscr{K})$, this converges in $L^2(R, \mathscr{K})$ to an element $h(\lambda, f, u)$. The identity (3.18) follows from the theory of Fourier transforms. □

LEMMA 3.22. *For $g \in B_0$, $u, v \in D(A^*)$,*

$$\begin{aligned} \frac{d}{d\lambda}(E(\lambda) A^* u, g(H_1) A^* v) &= g(\lambda)(\tilde{M}(\lambda) G_+(\lambda) u, v)_{\mathscr{K}} \\ &+ \frac{1}{2\pi i}([G_+(\lambda) - G_-(\lambda)] u, h(\lambda, g, v))_{\mathscr{K}}. \end{aligned} \tag{3.19}$$

Proof. Put $\zeta = \lambda + i\varepsilon$. Then by (3.6)

$$\begin{aligned} &2\pi i(\delta_\varepsilon(H - \lambda) A^* u, g(H_1) A^* v) \\ &= ([R(\zeta) - R(\bar{\zeta})] A^* u, g(H_1) A^* v) \\ &= (\{[R_1(\zeta) A^*]^a G(\zeta) - [R_1(\bar{\zeta}) A^*]^a G(\bar{\zeta})\} u, g(H_1) A^* v) \\ &= 2\pi i([\delta_\varepsilon(H_1 - \lambda) A^*]^a G(\zeta) u, g(H_1) A^* v) \\ &+ ([G(\zeta) - G(\bar{\zeta})] u, AR_1(\zeta) g(H_1) A^* v)_K. \end{aligned}$$

Take the limit as $\varepsilon \to 0$ and use Lemma 3.21. □

LEMMA 3.23. *For $f, g \in B_0$ and $u, v \in D(A^*)$, we have*

$$\begin{aligned} &([W_+ - I] f(H) A^* u, g(H_1) A^* v) \\ &= -\frac{1}{2\pi i} \int_\wedge f(\lambda)([G_+(\lambda) - G_-(\lambda)] u, h(\lambda, g, v))_{\mathscr{K}} \, d\lambda. \end{aligned} \tag{3.20}$$

Proof. Multiply (3.19) by $f(\lambda)$ and integrate over $\wedge$. The left side becomes $(f(H)A^*u, g(H_1)A^*v)$. The first term on the right becomes

$$\begin{aligned}\int_\wedge f(\lambda)\overline{g(\lambda)}\,(T_\lambda u, T_{+\lambda}v)_{\mathscr{C}}\,d\lambda
&=\int_\wedge (Ff(H)E(\wedge)A^*u, g(\lambda)T_{+\lambda}v)_{\mathscr{C}}\,d\lambda\\
&=\int_\wedge (Ff(H)A^*u, F_+g(H_1)A^*v)_{\mathscr{C}}\,d\lambda\\
&=(W_+f(H)A^*u, g(H_1)A^*v),\end{aligned}$$

by Lemmas 3.14, 3.15 and 3.18. □

LEMMA 3.24. *Let $\varphi(\lambda)$ be a real valued Borel measurable function on $\wedge$ such that*

$$\int_0^\infty \Big|\int_I e^{-it\varphi(\lambda)-i\xi\lambda}\,d\lambda\Big|^2\,d\xi\to 0 \quad \text{as} \quad t\to\infty \tag{3.21}$$

for each $I\subset\subset\wedge-e$. Then

$$e^{it\varphi(H_1)}\,e^{-it\varphi(H)}\,E(\wedge)\,u\to W_+E(\wedge)\,u \quad \text{as} \quad t\to\infty$$

for each $u\in\mathscr{H}$.

Proof. Suppose $I\subset\subset\wedge$ and $J\subset\subset\wedge-e$. Put $f(\lambda)=\chi_I e^{-it\varphi(\lambda)}$ and $g(\lambda)=\chi_J e^{-it\varphi(\lambda)}$.

By Lemma 3.23,

$$\begin{aligned}&([W_+-I]\,e^{-it\varphi(H)}\,E(I)\,A^*u, e^{-it\varphi(H_1)}\,E_1(J)\,A^*v)\\
&\quad=-\frac{1}{2\pi i}\int_I e^{-it\varphi(\lambda)}([G_+(\lambda)-G_-(\lambda)]\,u, h(\lambda,g,v))_{\mathscr{K}}\,d\lambda\end{aligned}$$

holds for all $u, v\in D(A^*)$. In absolute value the square of this is

$$\leqslant\frac{1}{4\pi^2}\int_I \|[G_+(\lambda)-G_-(\lambda)]\|_{\mathscr{K}}^2\,d\lambda\int_I \|h(\lambda,g,v)\|_{\mathscr{K}}^2\,d\lambda.$$

By (3.18)

$$\int_{-\infty}^{\infty} \|h(\lambda, g, v)\|_{\mathscr{K}}^2 \, d\lambda = 2\pi \int_0^{\infty} d\xi \, \|\int_J e^{-i\xi\mu - it\varphi(\mu)} \tilde{M}(\mu) \, v d\mu\|_{\mathscr{K}}^2 \to 0$$

by (3.21). Since linear combinations of elements of the form $E(I)\,A^*u$ are dense in $E(\wedge)\,\mathscr{H}$ and linear combinations of elements of the form $E_1(J)\,A^*v$ are dense in $E_1(\wedge - e)\,\mathscr{H}$, we have

$$([W_+ - I]\, e^{-it\varphi(H)}u, \, e^{-it\varphi(H_1)}v) \to 0$$

as $t \to \infty$ for $u \in E(\wedge)\,\mathscr{H}$ and $v \in E_1(\wedge - e)\,\mathscr{H}$. By Lemma 2.20, this implies

$$(e^{it\varphi(H_1)} e^{-it\varphi(H)}\, u, v) \to (W_+ u, v) \tag{3.22}$$

for all such u and v. Thus for such u we have

$$\begin{aligned} \|[e^{it\varphi(H_1)}\, e^{-it\varphi(H)} - W_+]\, u\|^2 &= \|e^{it\varphi(H_1)} e^{-it\varphi(H)} u\|^2 \\ &+ \|W_+ u\|^2 - 2Re\,(e^{it\varphi(H_1)}\, e^{-it\varphi(H)}\, u, \, W_+ u) \\ &= 2\,\|u\|^2 - 2Re\,(e^{it\varphi(H_1)}\, e^{-it\varphi(H)}\, u, \, W_+ u). \end{aligned}$$

Since $R(W_+) = E_2(\wedge - e)\,\mathscr{H}$, this tends to 0 as $t \to \infty$. □

The conclusions of Theorem 3.1 are contained in those of Lemmas 3.2–3.24. We now make an important observation due to Kato and Kuroda [2].

LEMMA 3.25. *Assumption 6 is implied by 6′. There is a function $N(\lambda)$ from $\wedge$ to $B(\mathscr{K})$ which is locally Holder continuous and such that*

$$\frac{d}{d\lambda}(E(\lambda)\,A^*u, B^*v) = (N(\lambda)\,u, v)_{\mathscr{K}}, \qquad u \in D(A^*), \qquad v \in D(B^*). \tag{3.23}$$

Proof. As in the case of Lemma 3.8 we get

$$[Bf(H)\,E(I)\,A^*] = \int_I f(\lambda)\,N(\lambda)\,d\lambda \tag{3.24}$$

for each Borel $I \subset \wedge$ and each bounded Borel measurable function $f(\lambda)$. Now suppose $I \subset\subset \wedge$. Then

$$\begin{aligned} Q(\zeta) &= B[R(i)\,A^*]^a + (i-\zeta)\,BR(i)\,E(R-I)\,[R(i)\,A^*]^a \\ &+ (i-\zeta)^2\,BR(i)\,R(\zeta)\,E(R-I)\,[R(i)\,A^*]^a \\ &+ (i-\zeta)\,BR(\zeta)\,E(I)\,[R(i)\,A^*]^a. \end{aligned}$$

All but the last term on the right can be easily extended to be continuous in $\Pi_I^{\pm}$. By (3.24)

$$BR(\zeta)\,E(I)\,[R(i)\,A^*]^a=\int_I \frac{1}{(\zeta-\mu)\,(i-\mu)}\,N(\lambda)\,d\lambda\,.$$

Since $N(\lambda)$ is Holder continuous in I, Privalov's theorem implies that this too can be extended to be continuous in $\Pi_I^{\pm}$. Thus $Q(\zeta)$ has a continuous extension to $\Pi_\wedge^{\pm}$. □

4. The Application

We now turn to the proofs of the theorems of Section 2. We take H in Theorem 3.1 to be P_0, the closure in L^2 of $P(D)$ on C_0^∞. We know that

$$\sigma(P_0)=\{P(\xi),\ \xi\in E^n\}$$

([14, p. 65]). One checks easily that this set is merely the interval $[\lambda_0, \infty)$, where

$$\lambda_0=\min_{\xi\in E^n} P(\xi)\,.$$

We take $\wedge=\sigma(P_0)-\Gamma$. Thus $E(\wedge)$ is the identity on L^2. Assumption 2 of Theorem 3.1 is easily verified (cf. [8] or [7, II]). Next, let $\mathscr{K}$ be the direct sum of N^2 copies of L^2, with each summand having indices j, k each running from 1 to N. Elements of $\mathscr{K}$ will be denoted by $\{u_{jk}\}$. We define

$$Av=\{\bar{g}_{jk}P_j(D)\,v\},\ Bu=\{h_{jk}P_k(D)\,u\}.$$

If we define A and B first on C_0^∞, we see that they are closable. By considering the closures, we obtain closed operators from $H=L^2$ to $\mathscr{K}$. Note that

$$A^*\{u_{jk}\}=\sum \bar{P}_j(D)\,g_{jk}u_{jk} \tag{4.1}$$

and

$$B^*\{v_{jk}\}=\sum \bar{P}_k(D)\,\bar{h}_{jk}v_{jk}. \tag{4.2}$$

In particular we have

$$(Q(x, D)\,u,\,v)=(Bu,\,Au)_{\mathscr{K}} \tag{4.3}$$

For s real, let $H^{s,2}$ denote the completion of C_0^∞ with respect to the norm given by

$$\|\varphi\|_{s,2}^2=\int (1+|\xi|^2)^s\,|F\varphi|^2\,d\xi\,,$$

where F denotes the Fourier transform. When s is a positive integer, the norm of $H^{s,2}$ is equivalent to the sum of the L^2 norms of all derivatives up to order s. By (2.3)

$$(P_0 u, u) \geqslant a \|u\|_{m,2}^2 - K_1 \|u\|^2,$$

where K_1 is some constant independent of u. Moreover, hypothesis (II) implies that there is a constant K_2 such that

$$\|Au\|^2 + \|Bu\|^2 \leqslant \tfrac{1}{2} a \|u\|_{m,2}^2 + K_2 \|u\|^2$$

(cf. [14, p. 140]). Thus the symmetric bilinear form

$$h(u, v) = (P_0 u, v) + (Bu, Av)_{\mathscr{H}}, \qquad u, v \in H^{m,2} \tag{4.4}$$

is bounded from below. The operator associated with it is self-adjoint (cf. Theorem 7.6, chapter 1 of [14]). Since C_0^∞ is dense in $H^{m,2}$, this operator is unique, and Lemma 2.1 is proved. This also shows that Assumption 3 of Theorem 3.1 is satisfied. It also follows from hypothesis (II) that $h_{jk} P_k(D)$ is P_0-compact for each j and k ([14, p. 112]). Thus B is P_0-compact and Assumption 5 of Section 3 is verified. We have also verified Assumption 8.

Next we note that we can always arrange things so that A is injective. In fact, if we set $P_0(\xi) = 1$, $h_{0k}(x) \equiv 0$ and let $g_{0k}(x)$ be functions which do not vanish anywhere and satisfy hypotheses (I)–(III), then

$$Q(x, D) = \sum_{j,k=0}^{N} \bar{P}_j(D)\, g_{jk}(x)\, h_{jk}(x)\, P_k(D),$$

and nothing is changed. Now $\mathscr{H}$ becomes $(N+1)^2$ copies of L^2 and A is injective. It therefore remains only to verify that Assumptions 6 and 7 are satisfied.

A simple computation shows that

$$\pi F[\delta_\varepsilon (P_0 - \lambda) w] = \frac{\varepsilon}{[P(\xi) - \lambda]^2 + \varepsilon^2} Fw.$$

Put

$$\begin{aligned} \psi_{\lambda,\varepsilon}(\xi) &= \frac{\varepsilon}{[P(\xi)-\lambda]^2+\varepsilon^2}, \quad && |P(\xi)-\lambda| < 1 \\ &= 0, && |P(\xi)-\lambda| \geqslant 1, \end{aligned}$$

and

$$\varphi_{\lambda,\varepsilon}(\xi) = \frac{\varepsilon}{[P(\xi)-\lambda]^2+\varepsilon^2} - \psi_{\lambda,\varepsilon}(\xi).$$

Then by definition,

$$K_\lambda(x) = \lim \bar{F}(\psi_{\lambda,\varepsilon}),$$

when $\bar{F}$ denotes the inverse Fourier transform. Since $|\varphi_{\lambda,\varepsilon}| \leqslant \varepsilon$, we have

$$(\varphi_{\lambda,\varepsilon} u, v) \to 0 \quad \text{as} \quad \varepsilon \to 0, u, v \in L^2.$$

Thus if u, v are functions in L^2 with compact supports, then

$$\begin{aligned}\pi(\delta_\varepsilon(P_0 - \lambda) u, v) &= ([\psi_{\lambda,\varepsilon} + \varphi_{\lambda,\varepsilon}] Fu, Fv) \\ &= (\bar{F}(\psi_{\lambda,\varepsilon}) * u, v) + (\varphi_{\lambda,\varepsilon} Fu, Fv) \to (K_\lambda * u, v) \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Thus if $\{E(\lambda)\}$ denotes the spectral family of P_0, we have

$$\pi \frac{d}{d\lambda}(E(\lambda) u, v) = (K_\lambda * u, v). \tag{4.6}$$

Thus Assumption 7 will be verified if we can show that the operator $AK_\lambda * A^*$ can be extended to a bounded operator on $\mathscr{H}$ which is locally Hölder continuous with respect to the norm topology. Likewise by Lemma 3.25, Assumption 6 will be satisfied if we can do the same for the operator $BK_\lambda \cdot A^*$. We note that hypothesis (III) implies just that for both of these operators. In fact we have

$$\begin{aligned}&|(K_\lambda * [P_j(D) g_{jk} u], \quad \bar{P}_t(D) \bar{h}_{st} v)|^2 \\ &\quad \leqslant \int\int |g_{jk}(g)|^2 \, |P_j(D) P_t(D) K_\lambda(x-y)| \, |v(x)|^2 \, dx \, dy \\ &\quad \times \int\int |h_{s,t}(x)|^2 \, |P_j(D) P_t(D) K_\lambda(x-y)| \, |u(y)|^2 \, dx \, dy \\ &\quad \leqslant C \|u\|^2 \, \|v\|^2.\end{aligned} \tag{4.7}$$

with the other inequalities similarly verified. This completes the proof of Theorem 3. A different variation of the above inequality proves Theorem 2.4. In fact we have

$$\begin{aligned}&|(K_\lambda * [P_j(D) g_{jk} u], \bar{P}_t(D) \bar{h}_{st} v)|^2 \\ &\quad \leqslant \int\int |g_{jk}(y) \, h_{st}(x)|^2 \, |P_j(D) P_t(D) K_\lambda(x-y)|^2 \, dx \, dy \\ &\quad \times \int\int |u(y) \, v(x)|^2 \, dx \, dy \leqslant C \|u\|^2 \, \|v\|^2,\end{aligned} \tag{4.8}$$

with similar calculations for the other inequalities. This proves Theorem 2.4.

We postpone the proof of Theorem 2.5 until the next section, and turn our attention to Theorems 2.6 and 2.7. First we need some lemmas. Let S^{n-1} denote the unit sphere $|x|=1$ in E^n, and let $(\ ,\ )_S$ denote the scalar product in $L^2(S^{n-1})$.

LEMMA 4.1. *If* $s>\frac{1}{2}$, *then*

$$\gamma_\lambda h(\omega)=h(\lambda\omega), \qquad \omega\in S^{n-1}, \tag{4.9}$$

is a mapping from $(0, \infty)$ *to* $B(H^{s,2}, L^2(S^{n-1}))$ *which is locally Holder continuous.*

This was proved by Kuroda [6, Lemma 4.1].

LEMMA 4.2. *Let I be any open interval in* $\wedge$. *Then there are a finite number of functions* γ_k *each mapping I into* $B(H^{s,2}, L^2(S^{n-1}))$ *and locally Hölder continuous and such that*

$$\frac{d}{d\lambda}(E(\lambda)\,u, v)=\Sigma\,(\gamma_k Fu, \gamma_k Fv)_S, \qquad u, v\in C_0^\infty. \tag{4.10}$$

Proof. Let $r_1,\ldots, r_l$ be the distinct positive roots of $p'(r^2)=0$ in ascending order. Let $r_0=0$, $r_{l+1}=\infty$ and $J_k=[r_{k-1}, r_k]$. Let I_k be the range of $p(r^2)$ on J_k. By the implicit function theorem there is a function $f_k(t)\in C^\infty(I_k)$ such that

$$p(f_k(t)^2)=t, \qquad t\in I_k \tag{4.11}$$

$$f_k(p(r^2))=r, \qquad r\in J_k. \tag{4.12}$$

Now

$$\begin{aligned}\pi(\delta_\varepsilon(P_0-\lambda)\,u, v)&=\int\frac{\varepsilon}{[p(|\xi|^2)-\lambda]^2+\varepsilon^2}\,Fu\overline{Fv}\,d\xi\\ &=\int_0^\infty\frac{\varepsilon}{[p(r^2)-\lambda)^2+\varepsilon^2}\,(Fu\,(r\omega), Fv\,(r\omega))_S\,r^{n-1}\,dr\\ &=\sum_k\int_{I_k}\frac{\varepsilon}{(t-\lambda)^2+\varepsilon^2}\,(Fu\,(f_k(t)\,\omega), Fv\,(f_k(t)\,\omega)_S\\ &\quad\times f_k(t)^{n-1}\,|f_k'(t)|\,dt.\end{aligned}$$

Taking the limit as $\varepsilon\to 0$, we obtain

$$\frac{d}{d\lambda}(E(\lambda)\,u, v)=\sum_{\lambda\in I_k}(\gamma_k Fu, \gamma_k Fv)_S, \tag{4.12}$$

where

$$\gamma_k h(\omega)=f_k(\lambda)^{(n-1)/2}\,|f_k'(\lambda)|^{1/2}\,h(f_k(\lambda)\,\omega). \tag{4.13}$$

By Lemma 4.1, γ_k is a mapping from I_k to $B(H^{s,2}, L^2(S^{n-1}))$ which is locally Hölder continuous. Since the endpoints of the I_k are contained in Γ, an arbitrary open interval $I \subset \wedge$ is either contained in I_k or does not intersect it. This gives the lemma.

For a locally square integrable function h we put

$$\tilde{h}(x) = \int |h(y)|^2 e^{-|x-y|^2} dy.$$

LEMMA 4.3. *If*

$$\sup_x (1+|x|^2)^s \tilde{h}(x) < \infty, \tag{4.14}$$

then the operator S given by

$$Su = (1+|x|^2)^{1/2s} \int h(y)\, u(y)\, e^{-|x-y|^2} dy \tag{4.15}$$

is bounded on L^2.

Proof. We have

$$(Su, v) = \int\int (1+|x|^2)^{s/2} e^{-|x-y|2} h(y)\, u(y)\, \overline{v(x)}\, dx\, dy.$$

The square of this is bounded by

$$\int\int (1+|x|^2)^s e^{-|x-y|^2} |h(y)\, v(x)|^2\, dx\, dy$$
$$\times \int\int e^{-|x-y|^2} |u(y)|^2\, dx\, dy \leqslant C\, \|v\|^2\, \|u\|^2.$$

COROLLARY 4.4. *If* (4.14) *holds, then the operator L given by*

$$Lu = F[e^{-|x|^2} * (hu)] \tag{4.16}$$

is bounded from L^2 to $H^{s,2}$

Proof. We have

$$\|Lu\|_{s,2} = \|(1+|x|^2)^{s/2} [e^{-|x|^2} * (hu)]\|.$$

Apply Lemma 4.3. □

LEMMA 4.5. *If* (4.14) *holds, then there exists a mapping T_λ of $\wedge$ to $B(L^2)$ which is locally Hölder continuous and such that*

$$\frac{d}{d\lambda}(E(\lambda)(hu), hv) = (T_\lambda u, v), \quad u, v \in L^2. \tag{4.17}$$

Proof. It clearly suffices to prove (4.12) for $u, v \in C_0^\infty$. Let I be a component of $\wedge$. For those I_k which contain I, set

$$\psi_k(\xi) = \exp\{\tfrac{1}{4}(f_k(\lambda)^2 - |\xi|^2)\} \tag{4.18}$$

(see the proof of Lemma 4.2). Then

$$F\psi_k = \exp[f_k(\lambda)^2 - |x|^2] \equiv \tilde{\psi}_k, \tag{4.19}$$

and

$$\psi_k(f_k(\lambda)\,\omega) = 1. \tag{4.20}$$

Thus

$$\begin{aligned}(\gamma_k F(hu), \gamma_k F(hv))_S &= (\gamma_k \psi_k F(hu), \gamma_k \psi_k F(hv))_S \\ &= (\gamma_k F[\tilde{\psi}_k * (hu)], \gamma_k F[\tilde{\psi}_k * (hv)])_S \\ &= (T_k u, v),\end{aligned} \tag{4.21}$$

where

$$T_k = e^{2f_k(\lambda)^2} L^* \gamma_k^* \gamma_k L. \tag{4.22}$$

By Lemma 4.1 and Corollary 4.4, this is a locally Hölder continuous map of I_k into $B(L^2)$. If we put

$$T_\lambda = \sum_{I \subset I_k} T_k \tag{4.23}$$

and apply Lemma 4.2, we obtain (4.17).

LEMMA 4.6. *If there is a $\beta < \frac{1}{2}(n-1)$ such that*

$$\sup_x \int \tilde{h}(y)\,(1 + |x-y|)^{-\beta}\, dy < \infty, \tag{4.24}$$

then (4.17) *holds for some locally Hölder continuous map T_λ from $\wedge$ to $B(L^2)$.*

Proof. By (4.21), (4.10) and (4.6)

$$\begin{aligned}\sum(\gamma_k F(hu), \gamma_k F(hv))_S &= \sum(\gamma_k F[\tilde{\psi}_k * (hu)], \gamma_k F[\tilde{\psi}_k * (hv)])_S \\ &= \frac{d}{d\lambda}(E(\lambda)[\tilde{\psi}_k * (hu)], \tilde{\psi}_k * (hv))_S \\ &= (K_\lambda * [\tilde{\psi}_k * (hu)], \tilde{\psi}_k * (hv))_S \\ &= \sum e^{2f_k(\lambda)^2} \int\!\!\int\!\!\int\!\!\int K_\lambda(x-y)\, e^{-|y-t|^2}\, h(z)\, u(z) \\ &\quad \times e^{-|x-t|^2}\, \overline{h(t)\, v(t)}\, dx\, dy\, dz\, dt.\end{aligned}$$

By (2.18) and (2.19), it suffices to show that the integral

$$\int\int\int\int (1+|x-y|)^{-\beta} e^{-|y-z|^2-|x-t|^2} |h(z)\, u(z)\, h(t)\, v(t)|\, dx\, dy\, dz\, dt$$

is bounded by a constant times $\|u\|\ \|v\|$. But its square is bounded by

$$\int\int\int\int (1+|x-y|)^{-\beta} e^{-|y-z|^2-|x-t|^2} |h(z)\, v(t)|^2\, dx\, dy\, dz\, dt$$
$$\times \int\int\int\int (1+|x-y|)^{-\beta} e^{-|y-z|^2-|x-t|^2} |h(t)\, u(z)|^2\, dx\, dy\, dz\, dt.$$

By (4.24), this is bounded by a constant times

$$\int\int e^{-|x-t|^2} |v(t)|^2\, dx\, dt \int\int e^{-|y-z|^2} |u(z)|^2\, dy\, dz$$

which equals a constant times $\|u\|^2\ \|v\|^2$. □

For h locally square integrable and $a>0$ set

$$\hat{h}_a(x) = \int_{|x-y|<a} |h(y)|^2\, dy \tag{4.25}$$

LEMMA 4.7. *For each $a>0$ there is a constant C_a depending only on a and n such that*

$$\|\tilde{h}\|_p \leqslant C_a \|\hat{h}_a\|_p, \qquad 1 \leqslant p \leqslant \infty. \tag{4.26}$$

Proof. For each k, let $z_1^{(k)}, \ldots, z_{N(k)}^{(k)}$ be points such that the set $ka<|x|<(k+1)\,a$ is covered by $N(k)$ balls of radius a and centers at the $z_j^{(k)}$. We know that there is a constant c_n depending only on n such that

$$N(k) \leqslant c_n (ka)^{n-1}.$$

Now

$$A_k(x) \equiv \int_{ka<|x-y|<(k+1)a} |h(y)|^2 e^{-|x-y|^2}\, dy \leqslant e^{-k^2a^2} \sum_{j=1}^{N(k)} \hat{h}_a(x+z_j^{(k)}).$$

Thus

$$\|\tilde{h}\|_p \leqslant \sum_{k=0}^{\infty} \|A_k\|_p \leqslant \sum_{k=0}^{\infty} e^{-k^2a^2} \sum_{j=1}^{N(k)} \|\hat{h}_a\|_p \leqslant c_n \|\hat{h}_a\|_p \sum_{k=0}^{\infty} (ka)^{n-1} e^{-k^2a^2}. \quad \square$$

LEMMA 4.8. *For each $a>0$ and $\alpha \geqslant 0$ there is a constant C depending only on* *a, α and n such that*

$$\|(1+|x|)^\alpha \tilde{h}\|_\infty \leqslant C\|(1+|x|)^\alpha \hat{h}_a\|_\infty. \tag{4.27}$$

Proof. We may assume $a\leqslant 1$. Set

$$A_k(x)=(1+|x|)^\alpha \int_{ka<|x-y|<(k+1)a} |h(y)|^2 e^{-|x-y|^2}\,dy.$$

Let c_n be as in the preceding proof. If $|x|\leqslant(2k+2)\,a$, then

$$A_k\leqslant(2k+3)^\alpha c_n (ka)^{n-1} e^{-k^2a^2}\hat{C},$$

where $\hat{C}$ denotes the right side of (4.27). On the other hand, if $|x|>(2k+2)\,a$, then $|x-y|<(k+1)\,a$ implies $|y|>\frac{1}{2}|x|$. Thus in this case

$$A_k\leqslant(1+|x|)^\alpha e^{-k^2a2} c_n (ka)^{n-1} (1+\tfrac{1}{2}|x|)^{-\alpha}\hat{C},$$

since the centers of the covering balls are a distance $>\frac{1}{2}|x|$ from the origin. Combining the two estimates, we get

$$A_k\leqslant\hat{C}\,c_n 2^\alpha(2k+3)^\alpha (ka)^{n-1} e^{-k^2a^2}$$

for any x. Since the left hand side of (4.27) is $\|\sum A_k\|_\infty$, the result follows. □

THEOREM 4.9. *Suppose there are constants $a>0$, $\alpha\geqslant 0$ and $p\leqslant\infty$ satisfying* (1.5) *such that*

$$(1+|x|)^\alpha \hat{h}_a\in L^p \tag{4.28}$$

Then there is a mapping T_λ from $\wedge$ to $N(L^2)$ which is locally Hölder continuous and such that (4.17) *holds.*

Proof. Let c be a number such that

$$\alpha>c>1-\frac{2n}{(n+1)\,p},$$

and set

$$\begin{aligned} t&=p, \quad p<2n/(n+1)\\ &=\max[(1-c)\,p, 0], \quad p\geqslant 2n/(n+1). \end{aligned}$$

Thus $t<2n/(n+1)$ and consequently

$$\|(1+|x|)^{-\beta}\|_{t'}<\infty, \quad t'=t/(t-1). \tag{4.29}$$

Put $\theta = t/p$ and $s = a/2(1-\theta)$. Then $s > \frac{1}{2}$. Let I be a component of $\wedge$, and define T_λ by (4.23). Then T_λ maps C_0^∞ into the distributions and

$$\sum_{I \subset I_k} (\gamma_k F(hu), \gamma_k F(hu))_S = (T_\lambda u, v), \qquad u, v \in C_0^\infty. \tag{4.30}$$

If (4.24) holds, then T_λ can be extended to a bounded operator on L^2, and there is a $b > 0$ such that

$$[T_\lambda] \equiv \|T_\lambda\| + \sup_{\lambda'} |\lambda - \lambda'|^{-b} \, \|T_\lambda - T_{\lambda'}\| \leqslant \text{constant} \, \|(1+|x|)^{-\beta} * \tilde{h}\|_\infty$$

(Lemma 4.6). By Hölder's inequality and (4.29), this is bounded by a constant times $\|\tilde{h}\|_t$. By Lemma 4.7 this implies

$$[T_\lambda] \leqslant \text{constant} \, \|\hat{h}_a\|_t. \tag{4.31}$$

For the case $t = p$, we are finished. For then $a = 0$ satisfies (1.5), and the result follows from (4.31). To take care of the other case, note that by Lemmas 4.5 and 4.8 there is a $b > 0$ such that

$$[T_\lambda] \leqslant \text{constant} \, \|(1+|x|^2)^s \, \hat{h}_a\|_\infty. \tag{4.32}$$

If we apply a general interpolation theorem due to Stein and Weiss [20] to (4.31) and (4.32), we obtain

$$\begin{aligned}[T_\lambda] &\leqslant \text{constant} \, \|(1+|x|^2)^{(1-\theta)s} \, \hat{h}_a\|_{t/\theta} \\ &\leqslant \text{constant} \, \|(1+|x|)^a \, \hat{h}_a\|_p.\end{aligned}$$

5. The Remaining Proofs

In this section we shall prove Lemma 2.2 and Theorem 2.5.

Proof of Lemma 2.2. Let I be an open interval containing λ and such that $\bar{I} \subset \wedge$. Set

$$\Omega = \{\xi \in E^n \mid P(\xi) \in I\}, \qquad \Sigma = \{\xi \in E^n \mid P(\xi) = \lambda\}.$$

Then there is a C^∞ diffeomorphism φ of $\bar{I} \times \Sigma$ onto Ω (cf. [7, II, Proposition 2.2]). Let $J(t, \omega)$ denote the Jacobian of this transformation. Then

$$\begin{aligned} K_\lambda(x) &= \lim_{\varepsilon \to 0} \int_\Omega \frac{\varepsilon e^{ix\xi} \, d\xi}{[P(\xi) - \lambda]^2 + \varepsilon^2} \\ &= \lim_{\varepsilon \to 0} \int_I \int_\Sigma \frac{\varepsilon J(t, \omega) \exp\{i\varphi(t, \omega) x\} \, dt \, d\omega}{(t-y)^2 + \varepsilon^2} \\ &= \pi \int_\Sigma J(\lambda, \omega) \exp\{i\varphi(\lambda, \omega) x\} \, d\omega, \end{aligned} \tag{5.1}$$

and the convergence is uniform for bounded x. The last formula shows that K_λ is infinitely differentiable. □

Proof of Theorem 2.5. Let the intervals I_k, J_k and the functions f_k be given as in Lemma 4.2. Suppose λ is any point in $\wedge$. If $\lambda \in I_k$, set $t_k = f_k(\lambda)$. By Bochner's formula [21, p. 235], $K_\lambda(x)$ is the limit of

$$\begin{aligned}
&2\pi(2\pi/|x|)^\gamma \int_0^\infty \frac{\varepsilon r^{\gamma+1} J_\gamma(r|x|)\, dr}{[p(r^2)-\lambda]^2+\varepsilon^2} \\
&\quad = 2\pi(2\pi/|x|)^\gamma \sum_k \int_{J_k} \\
&\quad = \pi(2\pi/|x|)^\gamma \sum_k \int_{I_k} \frac{\varepsilon f_k(s)^\gamma J_\gamma(f_k(s)|x|)\, ds}{[(s-\lambda)^2+\varepsilon^2]\, p'(f_k(s)^2)} \\
&\quad \to \pi^2(2\pi/|x|)^\gamma \sum \frac{t_k^\gamma J_\gamma(t_k|x|)}{p'(t_k^2)},
\end{aligned} \tag{5.2}$$

where the summation is taken over those k for which $\lambda \in I_k$. This proves (2.17). To prove the rest of the lemma, we note that a simple induction shows that $D^\mu K_\lambda$ is a finite sum of terms of the form

$$R(x)\, h(t)\, J_\nu(|x|t)/(|x|t)^\nu,$$

where $\nu - \frac{1}{2}n$ is an integer $\geqslant -1$, $R(x)$ is a polynomial of degree $\leqslant \nu + 1 - \frac{1}{2}n$, t is one of the t_k and h is infinitely differentiable. This immediately implies (2.18). To prove (2.19), set $f(s) = J_\nu(s)/s^\nu$. Then there is a constant c depending only on ν such that

$$|f(s)| + |f'(s)| \leqslant c/(1+s)^{\nu+1/2}, \qquad s \geqslant 0. \tag{5.3}$$

Moreover, for each positive integer k there is a constant C_k such that

$$|t-1|^k \leqslant C_k |t^k - 1|, \qquad t \text{ real}. \tag{5.4}$$

Thus

$$\begin{aligned}
|f(s_1) - f(s_2)|^k &\leqslant C_k |f(s_1)^k - f(s_2)^k| \\
&= kC_k |f(\tilde{s})^{k-1} f'(\tilde{s})\, (s_1 - s_2)| \\
&\leqslant C|s_1 - s_2|\, (1+\tilde{s})^{k(\nu+1/2)},
\end{aligned}$$

where $\tilde{s}$ is some value between s_1 and s_2. Thus

$$|f(Rt_1) - f(Rt_2)| \leqslant C|t_1 - t_2|^{1/k} (1+R)^{(1/k)-\nu-1/2}$$

This implies (2.19). □

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