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The Ω-Spectrum for Brown-Peterson Cohomology. Part I

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Introduction

BP denotes the spectrum for the Brown-Peterson cohomology, $BP^*(\cdot)$, associated with the prime p [1, 3, 11]. The spectrum can be given as an Ω-spectrum $BP = \{BP_k\}$, [2, 16], i.e. $\Omega BP_k \simeq BP_{k-1}$ and BP_k is $k-1$ connected for $k>0$. We have $BP^k(\cdot) \simeq [\cdot, BP_k]$, the unstable homotopy classes of maps. The usual way of viewing $BP^*(\cdot)$ is $BP^*(\cdot) \simeq \{\cdot, BP\}^*$, the stable homotopy classes of maps of the suspension spectrum of a space into BP. We will study the Brown-Peterson cohomology theory from an unstable point of view by studying the BP_k.

Interest in the Brown-Peterson theory stems from the fact that it is a "small" cohomology theory which determines the complex cobordism theory localized at the prime p and that all of the nice properties of complex cobordism carry over to $BP^*(\cdot)$, such as knowledge of the operation ring. Historically, everything about the Brown-Peterson theory has been as nice as could be hoped for. We will push on further in that direction. $Z_{(p)}$ is the integers localized at p, i.e., rationals with denominator prime to p.

MAIN THEOREM (3.3). *The $Z_{(p)}$ (co)homology of the zero component of BP_k has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. (k can be less than zero.)* #

Using the main result of [12], the above theorem determines the Hopf algebra structure of the (co)homology. (see section 3) We begin by reviewing Larry Smith's result on the Eilenberg-Moore spectral sequence for stable Postnikov systems. [14] We combine this with Brown and Peterson's original construction of BP([3]) to calculate $H^*(BP_{2k+1}, Z_p)$ assuming a technical lemma which we prove in section 2. In section 3 we prove the main theorem and some miscellaneous items such as lifting our result to MU.

In Part II we determine the homotopy type of the BP_k using the main theorem here.

This paper is a part of work done for my Ph.D. thesis at M.I.T. under the supervision of Professor Frank Peterson. It is my pleasure to thank Prof. Peterson for his advice, encouragement, and understanding through the last several years. I am very grateful for the quite considerable influence which he has had on my attitudes and tastes in mathematics. Thanks are also due to Larry Smith and Dave Johnson for comments on a preliminary version of this paper, in particular for pointing out a mistake in the original proof for the prime 2.

Section 1

For the remainder of the paper all coefficient rings are assumed to be $Z_p = Z/pZ$ unless stated otherwise. In this section we show $H^*(\mathrm{BP}_{2k+1})$ is an exterior algebra on odd dimensional generators. $H^*(\mathrm{BP}_{2k+1})$ is a Hopf algebra, so for odd primes having odd dimensional generators is equivalent to being an exterior algebra. The general reference for Hopf algebras is [10]. We quote what we need from [14].

Let K be a product of Eilenberg-MacLane spaces. We will be concerned with the situation

$$\begin{array}{ccc} Y & \to & PK \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \underset{f}{\to} & K \end{array} \tag{A}$$

where all spaces are infinite loop spaces and all maps are infinite loop maps. π is the fibration induced by f from the path space PK over K. All cohomologies are thus cocommutative Hopf algebras and $H^*(K)\backslash\backslash f^*$ and $H^*(X)//f^*$, the kernel and cokernel of f^* in the category of Hopf algebras are defined.

There is a natural map $PH \to QH$, where P and Q denote the primitives and indecomposibles respectively of a Hopf algebra H. When this is onto, H is called primitive.

LEMMA 1.1 ([14, p. 69]). *$H' \subset H$ a subHopf algebra over Z_p, H primitive, then H' is primitive.* #

If V is a graded module, let s^qV be the graded module $(s^qV)_{n+q} = V_n$. Let V^- denote the elements of odd degree. From [14, p. 95] we have a filtration of $H^*(Y)$ of diagram A such that

$$E_0H^*(Y) \simeq H^*(X)//f^* \otimes E[\ldots] \otimes \frac{P[s^{-1}((Q(H^*(K)\backslash\backslash f^*))^-)]}{[s^{-1}((Q(H^*(K)\backslash\backslash f^*))^-)]^p} \tag{1.2}$$

as Hopf algebras. E and P denote exterior and polynomial algebras generated by odd and even dimensional elements respectively. $E[\ldots]$ is determined by $H^*(K)\backslash\backslash f^*$.

$H^*(K)$ is primitive because it is generated by cohomology operations on fundamental classes, therefore, $H^*(K)\backslash\backslash f^*$ is primitive by 1.1. So for $x \in Q(H^*(K)\backslash\backslash f^*)$ we have $x' \to x$, $x' \in P(H^*(K)\backslash\backslash f^*)$ and thus $x' \to x'' \in PH^*(K)$. For x of odd degree, x' and thus x'', are determined uniquely by x. Let $i: \Omega K \to Y$ be the inclusion of the fibre.

LEMMA 1.3 ([14, p. 86 and p. 110]). *$i^*(s^{-1}(x)) = s^*(x'')$, s^* the cohomology suspension, $s^*: H^*(K) \to H^*(\Omega K)$.* #

Note that if x is of odd degree then $s^*(x'')\neq 0$ by the following lemma.

LEMMA 1.4. *$a\in PH^m(K)$, if $s^*(a)=0$, then $a=P^t x_{2t}$ or $a=\beta P^k x_{2k+1}$ where $P^i\in A$ is the i-th reduced p-th power, A is the Steenrod algebra and x_i is of degree i. ($p=2, P^i=Sq^{2i}$ and $\beta=Sq^1$).#*

Proof. It is enough to consider $K=K(Z_{(p)}, n)$ and $a=P^I i_n$ where $P^I\in A$ is an Adem basis element. The kernel of $s^*: QH^*(K(Z_{(p)}, n))\to PH^*(K(Z_{(p)}, n-1))$ is of the type $\beta P^k x_{2k+1}$. The proof is an argument on the excess of I and can be found in [13]. The kernel of $PH^*(K)\to QH^*(K)$ is of the type $P^t x_{2t}=(x_{2t})^p$. The degree of $P^t x_{2t}=2pt$ and the degree of $\beta P^k x_{2k+1}=2pk+2$ so the two terms cannot occur in the same dimension.#

Brown and Peterson [3] construct BP by a series of fibrations which we now describe. Let $\mathscr{R}$ be the set of sequences of non-negative integers $(r_1, r_2, \ldots)$ which are almost all zero. Define $d(R)=\sum 2r_i(p^i-1)$, $l(R)=\sum r_i$ and let Δ_i be the R with $r_i=1$ and zeros everywhere else. Let V_j be the graded abelian group, free over $Z_{(p)}$, generated by $R\in\mathscr{R}$ with $l(\mathscr{R})=j$ and graded by $d(R)$. Then we have the generalized Eilenberg-MacLane spectrum $K(V_j)=\vee_{l(R)=j} S^{d(R)}K(Z_{(p)})$. BP=inverse limit X^j where we have the fibrations

$$\begin{array}{ccc} K(V_j) & \xrightarrow{i_j} & X^j \\ & & \downarrow \\ & & X^{j-1} \xrightarrow{k_{j-1}} SK(V_j) \end{array} \qquad (*)$$

induced by k_{j-1}. We have an $A/A(Q_0)$ resolution for $A/A(Q_0, Q_1, \ldots)=H^*(\mathrm{BP})$, $d_j: M_j\to M_{j-1}$ with $H^*(K(V_j))=M_j$ and $(i_j)^*\cdot(k_j)^*=d_{j+1}$. The Q_i are the Milnor primitives [8]. (For $p=2$, $Q_i=P^{\Delta_{i+1}}$ in the Milnor basis.) For an $A/A(Q_0)$ generator $i_R\in H^*(K(V_j))$, $d_j(i_R)=\sum_i Q_i i_{R-\Delta_i}$.

The spectrum $K(V_j)$ can be given as an Ω-spectrum, $\{K(V_j, k)=\times_{l(R)=j} K\times \times(Z_{(p)}, d(R)+k)\}$. The entire diagram $(*)$ can be turned into Ω-spectra and maps of Ω-spectra. From this we get a sequence of fibrations with BP_k=inverse limit X^j.

$$\begin{array}{ccc} K(V_j, k) & \xrightarrow{i_j} & X^j \\ & & \downarrow \\ & & X^{j-1} \xrightarrow{k_{j-1}} K(V_j, k+1). \end{array} \qquad (**)$$

We suppress the k in the notation for X^j, i_j and k_j. Note that k can be less than zero. We have $(i_j)^*\cdot(k_j)^*\cdot s^*=s^*\cdot(i_j)^*\cdot(k_j)^*$ where the i_j and k_j on the right are for BP_k and on the left for BP_{k-1}. This is because k_j for BP_{k-1} is the loop map of the k_j for BP_k. Similarly for i_j. The iterated cohomology suspension gives a map $s^*: M_j\to H^*\times \times(K(V_j, k))$ which has as its image the primitives, $PH^*(K(V_j, k))$. In general we will

denote the iterated suspension by s^* and it should be clear when we mean only one. We have the following commutative diagram.

$$\begin{array}{ccc} M_j & \xleftarrow{d_{j+1}} & M_{j+1} \\ {\scriptstyle s^*}\downarrow & & {\scriptstyle s^*}\downarrow \\ H^*(K(V_j,k)) & \xleftarrow[(i_j)^*\cdot(k_j)^*]{} & H^*(K(V_{j+1},k+1)). \end{array}$$

We will often use $s^*(d_{j+1})$ for $(i_j)^*\cdot(k_j)^*$. It is given by the same formula $\sum Q_i i_{R-\Delta_i}$. In the next section we prove the following lemma.

LEMMA 1.5(j). *For k odd, if $a\in PH^{2i+1}(K(V_j,k+1))$ such that $(k_{j-1})^*(a)=0$, then there exists $b\in PH^*(K(V_{j+1},k+1))$ such that $(i_j)^*\cdot(k_j)^*(b)=s^*(a)\neq 0$.* #

We use this to prove the next proposition.

PROPOSITION 1.6(j). *For k odd, $H^*(X^j)//(k_j)^*$ has no even dimensional* generators. *(For $p=2$ it is an exterior algebra.)* #

Proof. For $j=0$, $X^0=K(Z_{(p)},k)$ and all generators of $H^*(X^0)$ are in the image of $s^*: M_0=A/A(Q_0)\to H^*(K(Z_{(p)},k))$. So if x is an even dimensional generator of $H^*(K(Z_{(p)},k))$ and k is odd, then there is an odd dimensional $x'\in M_0$ with $s^*(x')=x$. We have the exact sequence

$$M_1 \xrightarrow{d_1} A/A(Q_0)=M_0 \xrightarrow{\varepsilon} A/A(Q_0,Q_1,\ldots)\to 0.$$

Thus there exists $x''\in M_1$ with $d_1(x'')=x'$ as $\varepsilon(x')=0$ because $\varepsilon(x')$ is an odd dimensional element in $A/A(Q_0,Q_1,\ldots)$ which only has even degree elements. So $s^*(x'')\times$ $\times\in H^*(K(V_1,k+1))$ and $(k_0)^*(s^*(x''))=s^*(d_1)\cdot s^*(x'')=s^*(d_1x'')=s^*(x)=x$ and the even dimensional generator $x\in H^*(X^0)$ goes to zero in $H^*(X^0)//(k_0)^*$. (For $p=2$ and x an odd dimensional generator, then $x^2=Sq^{\deg x}x$ is killed by the same argument, so we have an exterior algebra.)

By induction, assume proposition 1.6(j-1). By 1.2 we have:

$$E_0H^*(X^j)\simeq H^*(X^{j-1})//(k_{j-1})^*\otimes E[\ldots]$$
$$\otimes\frac{P[s^{-1}(Q(H^*(K(V_j,k+1))\backslash\backslash(k_{j-1})^*)^-)]}{[s^{-1}(Q(H^*(K(V_j,k+1))\backslash\backslash(k_{j-1})^*)^-)]^p}.$$

Now by our induction assumption, all even dimensional generators look like $s^{-1}(x)$ where $x\in QH^*(K(V_j,k+1))\backslash\backslash(k_{j-1})^{*-}$. These elements map injectively to the cohomology of the fibre, see 1.3 and the remark after it. As discussed above (before 1.3), x can be represented by an $a\in PH^*(K(V_j,k+1))$ with $(k_{j-1})^*(a)=0$. Now, as a is of odd degree, from 1.5(j), there exists b such that $(i_j)^*\cdot(k_j)^*(b)=s^*(a)\neq 0$. But by 1.3, $(i_j)^*(s^{-1}(x))=s^*(a)$ and $(i_j)^*$ is injective on these even degree indecomposibles

giving that $(k_j)^*(b)=s^{-1}(x)+$decomposibles. Therefore, the generator $s^{-1}(x)$ goes to a decomposible in $H^*(X^j)//(k_j)^*$ and we are done. #

COROLLARY 1.7. *For k odd, $H^*(\mathrm{BP}_k)$ is an exterior algebra on odd dimensional generators.* #

Proof. Because $K(V_j, k)$ is highly connected for high j we have $H^*(\mathrm{BP}_k)=$direct limit $H^*(X^j)//(k_j)^*$. Because we are working with Hopf algebras, odd dimensional generators for odd primes means we have an exterior algebra. The direct limit is achieved in a finite number of stages so we have the result using 1.6. #

Section 2

We will now prove lemma 1.5(j). We have already seen that that $s^*(a)\neq 0$. (1.4)

Let A be the mod p Steenrod algebra. We define a filtration: $A=F^0A\supset F^1A\supset F^2A\supset \ldots$ by giving a basis for F^sA. Given an Adem basis element, $\beta^{\varepsilon_0}P^{i_1}\beta^{\varepsilon^1}\ldots P^{i_n}\beta^{\varepsilon_n}$, it is basis element for F^sA if $s\leqslant\sum\varepsilon_i$. Also, we give a basis for B_s by taking all Adem basis elements with $s=\sum\varepsilon_i$. For $p=2$, $P^i=Sq^{2i}$. We do not define B_s for $p=2$ using the Adem basis.

For our purposes it is usually more convenient to work in the Adem basis, however, the Milnor basis is a necessary excursion for $p=2$. For odd primes, a Milnor basis element Q^IP^R $(Q^I=Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\ldots)$ is a basis element for F^sA if $s\leqslant\sum\varepsilon_i$. For $p=2$, a Milnor basis element P^R is a basis element for F^sA if $R=(r_1, r_2,\ldots)$ has s or more odd r_i. Again, a basis element for B_s has $s=\sum\varepsilon_i$ ($p=2$, s odd r_i).

CLAIM 1. i) *The two definitions of F^sA and B_s are the same.*

ii) *If $a\in F^sA$ and $b\in F^tA$, then $ab\in F^{s+t}A$.*

iii) $F^sA=B_s\oplus F^{s+1}A$.

Sketch proof. Milnor's $Q_i=P^{\Delta i}\beta-\beta P^{\Delta i}$. For odd primes $P^{\Delta i}$ is in the algebra of reduced p-th powers and so can be written in the Adem basis without any β's, similarly for all P^R in the Milnor basis. The Adem relations for p odd preserve the number of β's exactly, so we see that $Q_i\in B_1\subset F^1A$. If we were to rewrite a Milnor basis element Q^IP^R in the Adem basis we would still have $\sum\varepsilon_i$ β's.

The proof of the second part just uses the fact that the Adem relations never decrease the number of Bocksteins.

The proof for $p=2$ is slightly more complicated and is left for the reader. iii) is elementary. #

Given $a\in PH^*(K(V_j, k))$, (any k), it can be written as $a=\sum_{l(R)=j}a_Ri_R$ where i_R is the fundamental class of $K(Z_{(p)}, d(R)+k)$ and $a_R\in A$. If it can be written like this with each $a_R\in F^nA$, then we say a is with n Bocksteins (*w.n*β's). If $n=1$, we just say *w.*β's. If a is with n Bocksteins but not with $n+1$ β's we say a is with exactly n β's. As

discussed above, Q_i is with exactly one Bockstein. Therefore by the definition of d_j and the above claim, if a is with $n\beta$'s, then $s^*(d_j)(a)$ is with $n+1$ β's. Recall that by our notation $s^*(d_j)=(i_{j-1})^*\cdot(k_{j-1})^*$.

CLAIM 2. *If $a=s^*(d_j)(b)$ and a is with 2 Bocksteins, then there is a b' with β's such that $s^*(d_j)(b')=a$.* #

Proof. First for odd primes; write $b=\sum_{l(R)=j} a_R i_R$ with $a_R\in A$. $A=B_0\oplus F^1A$, so write $a_R=b_R+c_R$ with $b_R\in B_0$ and $c_R\in F^1A$. $b_RQ_i\in B_1$ and $c_RQ_i\in F^2A$ so $s^*(d_j)(\sum b_R i_R)=0$. Let $b'=\sum c_R i_R$.

For prime 2 we have $Q_{i-1}=P^{\Delta i}$ and for a Milnor basis element P^R we have $P^RP^{\Delta i}=\sum_{r_{i+j}\text{ even}} P^{R-2^i\Delta_j+\Delta_{i+j}}$, thus $b_RP^{\Delta i}\in B_1$ and $c_RP^{\Delta i}\in F^2A$ and same proof works. #

PROPOSITION 2.1(j). *Given $a\in PH^*(K(V_j,k))$, a with β's such that $s^*(d_j)(a)=0$, then there exists $\bar{a}\in M_j$ such that $s^*(\bar{a})=a$ and $d_j(\bar{a})=0$.* #

Proof of 1.5(j) For k and a odd, then a is with β's in $PH^*(K(V_j,k+1))$ for dimensional reasons, i.e., all of the Steenrod algebra elements used are odd dimensional, and all odd dimensional elements have β's. $(k_{j-1})^*(a)=0$ implies $s^*(d_j)(a)=0$ and we can apply proposition 2.1(j) to get $\bar{a}$ such that $s^*(\bar{a})=a$ and $d_j(\bar{a})=0$. By exactness, there exists $\bar{b}\in M_{j+1}$ such that $d_{j+1}(\bar{b})=\bar{a}$. Then $b'=s^*(\bar{b})\in PH^*(K\times\times(V_{j+1},k+2))$ has $s^*(d_{j+1})(b')=s^*(d_{j+1})(s^*(\bar{b}))=s^*(d_{j+1}(\bar{b}))=s^*(\bar{a})=a$. So let $b=s^*(b')$, then $s^*(a)=s^*(d_{j+1})(b)$ which is what we want. #

PROPOSITION 2.2(j). *Given an a as in* 2.1(j), *then there exists $b\in PH^*(K\times\times(V_{j+1},k+1))$ such that $s^*(d_{j+1})(b)=a$.* #

Proof. See proof of 1.5(j). #

Remark. Proposition 2.2(j) is really the essential feature that makes everything work. It means that exactness still holds in the unstable range for primitives with β's.

We need proposition 2.2(j-1) in the induction argument for the proof of proposition 2.1(j).

Proof of 2.1(j). This follows at once from the next proposition, just lift a up one step at a time until it is in the stable range. #

PROPOSITION 2.3(j). *Given a with β's in $PH^*(K(V_j,k))$ (any k) such that $s^*(d_j)(a)=0$, then there exists $\bar{a}$ with β's in $PH^*(K(V_j,k+1))$ such that $s^*(\bar{a})=a$ and $s^*(d_j)(\bar{a})=0$. (For $j=0$, $s^*(d_0)(a)=0$ is a vacuous condition).* #

Proof. $j=0$, trivial. For $j=1$ the arguments is the same as for $j>1$ except easier, so assume $j>1$. Now, trivially, there exists a' with β's such that $s^*(a')=a$. (Let $a'=\tau(a)$.) Now $s^*(d_j)(a')\in\ker s^*$ by commutativity of the following diagram.

$$\begin{array}{ccc} PH^*(K(V_{j-1}, k)) & \xleftarrow{s^*(d_j)} & PH^*(K(V_j, k+1)) \quad a' \\ \downarrow s^* & & \downarrow s^* \quad \Big\downarrow \\ PH^{*-1}(K(V_{j-1}, k-1)) & \xleftarrow{s^*(d_j)} & PH^{*-1}(K(V_j, k)) \\ & & \\ 0 & \longleftarrow & a \end{array}$$

By 1.4, $s^*(d_j)(a')=0$, $P^n x_{2n}$, or $\beta P^t x_{2t+1}$ in $PH^*(K(V_{j-1}, k))$.

Case 1. If $s^*(d_j)(a')=0$ we are done.

Case 2. If $s^*(d_j)(a')=P^n x_{2n}$ then $s^*(d_{j-1})(P^n x_{2n})=0$ because $d_{j-1}\cdot d_j=0$. $0=s^*(d_{j-1})(P^n x_{2n})=s^*(d_{j-1})(x_{2n})^p=[s^*(d_{j-1})(x_{2n})]^p$. $H^*(K(V_{j-2}, k-1))$ is a free commutative algebra so this implies $s^*(d_{j-1})(x_{2n})=0$. Now a' is with β's so $s^*(d_j)(a')$ is with 2 β's. This gives us that $P^n x_{2n}$ is with 2 β's. If $x_{2n}=\sum_R(\sum_i \lambda_i b_i)i_R$ with $\lambda_i \neq 0 \in Z_p$ and b_i Adem basis elements, then $P^n x_{2n}=\sum_R(\sum_i \lambda_i P^n b_i)i_R$ and for dimensional reasons $P^n b_i$ is in Adem basis form. Since each $P^n b_i$ is with 2 β's, each b_i is with 2 β's and so x_{2n} is with 2 β's. x_{2n} is also in the kernel of $s^*(d_{j-1})$ so we can apply 2.2(j-1) to produce a $y_{2n} \in PH^{2n}(K(V_j, k+1))$ with $s^*(d_j)(y_{2n})=x_{2n}$. By claim 2 we can choose y_{2n} to be with β's. $a'-P^n y_{2n}$ is with β's and has $s^*(d_j)(a'-P^n y_{2n})=0$ and $s^*(a'-P^n y_{2n})=s^*(a')=a$, so we are done.

Case 3. If $s^*(d_j)(a')=\beta P^t x_{2t+1}$ the proof is similar to case 2. We sketch the differences. βP^t is injective on $PH^{2t+1}(K(V_{j-2}, k-1))$ because it is for any product of Eilenberg-MacLane spaces [13]. So we get x_{2t+1} is in the kernel of $s^*(d_{j-1})$. If $\sum_R(\sum_i \lambda_i b_i i_R))=x_{2t+1}$ b_i Adem basis elements, then each $\beta P^t b_i$ is also in Adem basis form and since $\beta P^t x_{2t+1}$ must be with 2 β's, each b_i is with one and so x_{2t+1} must be with β's. Use 2.2(j-1) again to produce y_{2t+1} with $s^*(d_j)(y_{2t+1})=x_{2t+1}$. Now $a'-\beta P^t x_{2t+1}$ has the desired property. #

Section 3

Our first objective is to compute the (co)homology of BP_{2k}. The bar construction ([4]) gives a spectral sequence of Hopf algebras: (k odd)

$$\mathrm{Tor}^{H^*(BP_k)}(Z_p, Z_p) => E_0 H_*(\text{zero component of } BP_{k+1}).$$

Now $H_*(BP_k)$ is an exterior algebra on odd dimensional generators $QH_*(BP_k)$. (Cor. 1.7) A standard computation (see [14]) gives: $\mathrm{Tor}^{H^*(BP_k)}(Z_p, Z_p)=\Gamma(s^1(QH_* \times \times(BP_k)))$ where Γ denotes the Hopf algebra dual to the polynomial algebra. Now all elements in $\Gamma(s^1(QH_*(BP_k)))$ are of even degree and the differentials change degree by one, so our spectral sequence collapses and we have: $H^*(\text{zero component of } BP_{k+1})=[E_0 H_*(\text{zero component of } BP_{k+1})]^*=[\mathrm{Tor}^{H^*(BP_{k+1})}(Z_p, Z_p)]^*=[\Gamma(s^1(QH_*(BP_k)))]^*$ =polynomial algebra.

We will now show $H_*(BP_{k-1})$ is a polynomial algebra for k odd. Using the

Eilenberg-Moore spectral sequence ([6, 14]) we have $\mathrm{Tor}_{H_*(\mathrm{BP}_k)}(Z_p, Z_p) => E_0H^*(\mathrm{BP}_{k-1})$ if BP_k is simply connected. Assume it is, then the same argument just given shows $H_*(\mathrm{BP}_{k-1})$ is a polynomial algebra. The only modification is:

$$\mathrm{Tor}_{H^*(\mathrm{BP}_k)}(Z_p, Z_p) = \Gamma(s^{-1}(QH^*(\mathrm{BP}_k))).$$

If BP_k, k odd, is not simply connected, then it is easy to see that one can get a splitting $\mathrm{BP}_k \simeq (\times S^1)_{(p)} \times X$ where X is simply connected. This is because BP_k is an H-space with $Z_{(p)}$ free homotopy. Its k-invariants are therefore torsion and primitive, but $(\times S^1)_{(p)}$ has no torsion in $Z_{(p)}$ cohomology. Thus we have a spectral sequence of Hopf algebras:

$$\mathrm{Tor}_{H^*(X)}(Z_p, Z_p) = E_0H^*(\text{zero component of } \mathrm{BP}_{k-1})$$

and our argument goes through. We have proved the following proposition.

PROPOSITION 3.1. *The* mod$p$ *(co)homology of the zero component of* BP_k *is a polynomial algebra on even dimensional generators for k even, and an exterior algebra on odd dimensional generators for k odd.* (Note that for k odd, BP_k is connected.)#

PROPOSITION 3.2. *The* $Z_{(p)}$ *(co)homology of* BP_k *has no torsion.*#

Proof. For k even this is trivial because $H^*(\mathrm{BP}_k)$ has no elements in odd degrees. For k odd we view the Bockstein spectral sequence as a spectral sequence of Hopf algebras. The differentials are the higher order Bocksteins. Let β_s be the first non-trivial differential and let x be the minimum degree generator that β_s acts non-trivially on. $\beta_s(x)$ is an even dimensional primitive, contradiction, so all differentials are zero.#

We can now prove the main theorem.

THEOREM 3.3. *The* $Z_{(p)}$ *(co)homology of* BP_{2k+1} *is an exterior algebra and the* $Z_{(p)}$ *(co)homology of the zero component of* BP_{2k} *is a polynomial algebra.*#

Proof. We will do the case for polynomial algebras, the exterior case being similar. From 3.2 we know the (co)homology is free over $Z_{(p)}$ and so we can lift the mod$p$ generators (3.1) up to it. These lifted elements generate the $Z_{(p)}$ (co)homology ring because there is no torsion and their mod$p$ reductions generate the Z_p (co)homology. By considering the rank we can see there can be no relations and we have a polynomial algebra.#

We can now lift our result to MU. Normally the spectrum MU is given by $\{\mathrm{MU}(n)\}$, the Thom complexes, and maps $S^2\mathrm{MU}(n) \to \mathrm{MU}(n+1)$. [9, 15] However, if $M_n = \lim(k \to \infty)\Omega^{2k-n}\mathrm{MU}(k)$, then $\Omega M_n \simeq M_{n-1}$ and for finite complexes $\mathrm{MU}^n(X) = \lim(k \to \infty)[S^{2k-n}X, \mathrm{MU}(k)] = \lim(k \to \infty)[X, \Omega^{2k-n}\mathrm{MU}(k)] = [X, M_n]$. Thus, $\{M_n\} = \mathrm{MU}$ as an Ω-spectrum.

COROLLARY 3.4. *The integer (co)homology of the zero component of M_n has no torsion and is a polynomial algebra over Z for n even and an exterior algebra for n odd.* #

Proof. From [3] we have $\mathrm{MU}_{(p)} \simeq \vee_i S^{2n_i}\mathrm{BP}$ and so $(M_n)_{(p)} \simeq \prod_i \mathrm{BP}_{n+2n_i}$. By 3.3 for n even $H_*(M_n, Z)\otimes Z_{(p)} \simeq H_*(M_n, Z_{(p)}) \simeq H_*((M_n)_{(p)}, Z_{(p)}) \simeq$ polynomial algebra over $Z_{(p)}$. Thus the integer homology has no torsion, and localized at every prime it is a polynomial algebra, so it is a polynomial algebra over Z. Similarly for n odd. Since there is no torsion, the same thing works for cohomology. #

Remark 1. A completely analogous theorem is true for MSO if the ring $Z(1/2)$ is used.

Remark 2. There are several ways to determine the number of generators for 3.1, 3.3, and 3.4. The spaces BP_n and M_n are just products of rational Eilenberg-MacLane spaces when localized at **Q**. (This is because their k-invariants are torsion.) Because there is no torsion, the number of generators is the same as for the rationals. As examples we have $\pi^S_*(\mathrm{BP}) = Z_{(p)}[x_{2(p-1)}, \ldots, x_{2(p^i-1)}, \ldots]$ so for $2n>0$, $H^*(\mathrm{BP}_{2n}, Z_{(p)}) \simeq Z_{(p)}[s^{2n}\pi^S_*(\mathrm{BP})]$ and $\pi^S_*(\mathrm{MU}) = Z[x_2, \ldots, x_{2i}, \ldots]$ so for $2n>0$, $H^*(M_{2n}, Z) \simeq Z[s^{2n}\pi^S_*(\mathrm{MU})]$.

We have shown that both the cohomology and homology of the zero component of BP_{2n} are polynomial algebras. This is a very strong statement, in fact, it determines the Hopf algebra structure of the (co)homology.

DEFINITION. *A connected bicommutative Hopf algebra is called bipolynomial if both it and its dual are polynomial algebras.* #

There is a bipolynomial Hopf algebra $B_{(p)}[x, 2n]$ over $Z_{(p)}$ (or Z_p) which has generators $a_k(x)$ of degree $2p^k n$ [7]. It is isomorphic as Hopf algebras to its own dual.

In [12] we prove the following proposition.

PROPOSITION 3.5. *If H is a bipolynomial Hopf algebra over $Z_{(p)}$ (or Z_p), then $H \simeq \otimes_j B_{(p)}[x_j, 2d_j]$. (For $p=2$ and Z_2, replace $2d_j$ by d_j).* #

Using this and the counting argument of remark 2 we can just write down the Hopf algebra structure for BP_{2n}. As an example, we will do this for $n>0$. Let $\mathscr{R}_n$ be the set of sequences of non-negative integers $R=(r_1, r_2, \ldots)$ with almost all $r_i=0$. Let $d(R)=2n+\sum 2(p^i-1)\, r_i$ for our fixed prime p. We say R is prime if it cannot be written $R=pS+(n, 0, 0, \ldots)$, $S\in\mathscr{R}_n$.

PROPOSITION 3.6. *For $n>0$, $H^*(\mathrm{BP}_{2n}, Z_{(p)}) \simeq \underset{R \text{ prime}}{\otimes_{R\in\mathscr{R}_n}} B_{(p)}[x_R, d(R)]$.* #

If we work over the integers and let $B[x, 2d]$ be the bipolynomial Hopf algebra on generators $c_n(x)$ of degree $2dn$ with coproduct $c_n(x) \to \sum c_{n-j}(x)\oplus c_j(x)$ ([7]) then we have an analogous proposition. [12]

PROPOSITION 3.7. *If H is a bipolynomial Hopf algebra over Z, then $H \simeq \otimes_j B[x_j, 2d_j]$.* #

We can now apply this to $\mathrm{MU}=\{M_n\}$. Let I_n be the set of sequences of non-negative integers $I=(i_1, i_2, \ldots)$ with $i_1 \geqslant n$ and almost all $i_j=0$, $(n>0)$. Let $d(I)=\sum_j 2ji_j$. We say I is prime if it cannot be written $I=kJ$, where $k>1$ and $J \in I_n$.

PROPOSITION 3.8. *If $\{M_k\}$ is the Ω-spectrum for* MU, *then for $n>0$, $H^*(M_{2n}, Z) \simeq \otimes_{I \text{ prime}} \in_{I_m} B[x_I, d(I)]$ as Hopf algebras.* #

Proof. Just use 3.7 and the counting done in remark 2. #

Let S be the sphere spectrum and let $i: S \to \mathrm{BP}$ represent $1 \in \pi_0^S(\mathrm{BP})$. $S=\{QS^n\}$ as an Ω-spectrum where $QX=\lim \Omega^n S^n X$. i induces maps $i_n: QS^n \to \mathrm{BP}_n$. $H_*(QS^n)$ is given in terms of homology operations on the n dimensional generator [5].

PROPOSITION 3.9. *Let $n>0$, the kernel of $(i_n)_*: H_*(QS^n) \to H_*(BP_n)$ is generated by homology operations on the n-dimensional class which have Bocksteins in them.* #

PROPOSITION 3.10. *Let $n>0$, if $j_n: \mathrm{BP}_n \to K(Z_{(p)}, n)$ represents the generator of $H^n(\mathrm{BP}_n, Z_{(p)})$, then the kernel of $(j_n)^*: H^*(K(Z_{(p)}, n)) \to H^*(\mathrm{BP}_n)$ is generated by cohomology operations on the n-dimensional class which have Bocksteins in them.* #

Proof of 3.9. By 3.2, any homology operation which has Bocksteins in it goes to zero. Let u be a homology operation with no β's such that $ux_n \neq 0$ in $H_*(QS^n)$. As u has no β's, $u(s_*)^k x_n$ is a p-th power for some k. So $u(s_*)^k x_n = ux_{n+k} = (u'x_{n+k})^p$. Now by induction on the degree of u, $i_*(u'x_{n+k}) \neq 0$ in $H_*(\mathrm{BP}_{n+k})$ and $n+k$ is even since we have a p-th power. $H_*(\mathrm{BP}_{n+k})$ is a polynomial algebra and so $[i_*(u'x_{n+k})]^p \neq 0$ and is $= i_*[u'x_{n+k}]^p = i_* u(s_*)^k x_n = i_*(s_*)^k u x_n = (s_*)^k i_*(ux_n)$ and so $i_*(ux_n) \neq 0$. #

The proof of 3.10 is similar.

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