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The Signature mod 8

by DAVID FRANK¹⁾

Let Γ_i be the group of exotic i -dimensional spheres, and let bP_{i+1} be the subgroup of those exotic spheres which bound π -manifolds. There is an exact sequence

$$0 \rightarrow bP_{4k} \rightarrow \Gamma_{4k-1} \rightarrow \pi'_{4k-1} \rightarrow 0,$$

where π'_i is the cokernel of the J -homomorphism $J: \pi_i(SO) \rightarrow \pi_i = \pi_{i+t}(S^t)$, t large. In studying the group bP_{4k} , it was important for Milnor to know

PROPOSITION A. *Let M^{4k} be a smooth, compact, oriented manifold with boundary an exotic sphere. If M is a π -manifold, then the signature of M is divisible by 8.*

In showing that the above exact sequence was split, it was important for Brumfiel [2] and the author [3] to know the stronger

PROPOSITION B. *Let M^{4k} be a smooth, compact, oriented manifold with boundary an exotic sphere. If M is a spin manifold, and if all decomposable Pontrjagin numbers of M are zero, then the signature of M is divisible by 8.*

In this paper we will prove a very general theorem which includes Propositions A and B. Let BSG be the classifying space for stable oriented spherical fibrations. If we kill the second homotopy group of BSG, we obtain a space BSpinG, the classifying space for stable spherical fibrations with a spin structure. Let v_{2k} denote the universal Wu class in either BSG or BSpinG. We first show.

LEMMA 1. *There is a class x_{2k} in $H^{2k}(BSpinG; \mathbb{Z}_4)$ whose mod 2 reduction is v_{2k} in $H^{2k}(BSpinG; \mathbb{Z}_2)$.*

The corresponding statement is of course false in BSG.

We now use the Pontrjagin square cohomology operations. There is a family of such operations; we are interested only in the operations

$$P: H^{2k}(-; \mathbb{Z}_4) \rightarrow H^{4k}(-; \mathbb{Z}_8).$$

In particular, consider the universal characteristic class $P(x_{2k})$ in $H^{4k}(BSpinG; \mathbb{Z}_8)$. If M is a spin Poincaré complex of dimension $4k$ with fundamental homology class $[M]$, we may consider the characteristic number $P(x_{2k})[M]$, which is an integer modulo 8. Let $\sigma(M)$ denote the signature of M .

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THEOREM 2. $P(x_{2k})[M] = \sigma(M)$ modulo 8 for any spin Poincaré complex M^{4k} . For oriented Poincaré complexes, we can show

THEOREM 3. There is a characteristic class y_{4k} in $H^{4k}(\text{BSG}; \mathbb{Z}_8)$ such that $y_{4k}[M] = \sigma(M)$ modulo 8 for any oriented Poincaré complex.

Theorem 3 is the best possible ‘Hirzebruch Signature Theorem’ for Poincaré complexes; the integer 8 cannot be replaced by a larger integer.

The classes y_{4k} are related to the k -invariants of the fibration $F/\text{Top} \rightarrow \text{BTop} \rightarrow \text{BSG}$.

1. The Spin Case

We begin by considering the Wu class v_{2k} in $H^{2k}(\text{BSpinG}; \mathbb{Z}_2)$. We wish to show v_{2k} is the mod 2 reduction of a \mathbb{Z}_4 cohomology class. Equivalently, we can show $Sq^1(v_{2k}) = 0$, where Sq^i is the i -th Steenrod Square. (In fact, $v_{2k} = 0$ for k odd, but this is irrelevant to what follows.)

We recall one definition of the Wu classes v_i . Let γ be the universal spherical fibration on $\text{BSG}(m)$, m large, let $\text{MSG}(m)$ be the Thom space of γ , and let U be the Thom class. Then define

$$v_i = T^{-1}(\chi(Sq^i U)),$$

where T is the Thom isomorphism ($T: H^*(\text{BSG}(m)) \rightarrow H^*(\text{MSG}(m))$) and χ is the anti-automorphism of the Steenrod Algebra. If M^n is a Poincaré complex, let $h: M \rightarrow \text{BSG}(m)$ be the classifying map of the stable Spivak normal fibration of M . Define $v_i(M)$ as $h^*(v_i)$. It follows from [6, Ch. III] that

$$(v_i(M) \cup x)[M] = Sq^i(x)[M]$$

for all x in $H^{n-i}(M; \mathbb{Z}_2)$. Thus this is an acceptable definition of the Wu classes.

Now note that $Sq^1(v_{2k} \cdot U) = Sq^1(v_{2k}) \cdot U$, since $Sq^1(U) = 0$. Thus $Sq^1(v_{2k}) = 0$ if $Sq^1(v_{2k} \cdot U) = 0$. But

$$\begin{aligned} Sq^1(v_{2k} \cdot U) &= Sq^1(\chi Sq^{2k}) U \\ &= (\chi Sq^1)(\chi Sq^{2k}) U \\ &= \chi(Sq^{2k} Sq^1) U. \end{aligned}$$

Now

$$Sq^{2k} Sq^1 = Sq^2 Sq^{2k-1} + a Sq^1 Sq^{2k}, \quad a \in \mathbb{Z}_2,$$

by an Adem relation, so

$$\chi(Sq^{2k} Sq^1) = \chi(Sq^{2k-1}) \chi(Sq^2) + a \chi(Sq^{2k}) \chi(Sq^1).$$

Since the last expression is zero on the Thom class U of $\text{BSpinG}(m)$, we have shown that $Sq^1(v_{2k})=0$ in BSpinG . Thus, as claimed in Lemma 1, there is a class x_{2k} in $H^{2k}(\text{BSpinG}; \mathbb{Z}_4)$ whose mod 2 reduction is v_{2k} . If M^{4k} is a spin Poincaré complex, we wish to relate the class $x_{2k}(M)$ to the signature of M . In fact, we will prove Theorem 2 by showing

PROPOSITION 4. *Let M^{4k} be an oriented Poincaré complex and let x be any class in $H^{2k}(M; \mathbb{Z}_4)$ whose mod 2 reduction is $v_{2k}(M)$. Then $P(x)[M] = \sigma(M) \bmod 8$.*

To prove Proposition 4, we first show

LEMMA 5. *Let M^{4k} be an oriented Poincaré complex and let a and b be classes in $H^{2k}(M; \mathbb{Z}_4)$ whose mod 2 reduction is $v_{2k}(M)$. Then $P(a) = P(b)$.*

Proof. If t is a cohomology class with \mathbb{Z}_{2^n} coefficients, we denote by t' the corresponding class with $\mathbb{Z}_{2^{n+1}}$ coefficients (determined by the inclusion homomorphism from \mathbb{Z}_{2^n} to $\mathbb{Z}_{2^{n+1}}$). Now if $a \bmod 2 = b \bmod 2 = v_{2k}(M)$, then $a = b + d'$, d in $H^{2k}(M; \mathbb{Z}_2)$. Then

$$\begin{aligned} P(a) &= P(b + d') \\ &= P(b) + P(d') + (b \cup d')'. \end{aligned}$$

Thus we must show

$$P(d') + (b \cup d')' = 0.$$

But

$$\begin{aligned} (b \cup d')' &= ((b \bmod 2) \cup d)'' \\ &= (v_{2k}(M) \cup d)'' \\ &= (d \cup d)'', \quad \text{by definition of the Wu class} \\ &= P(d'). \end{aligned}$$

(The last equality follows immediately from the cochain definition of the Pontrjagin Square.) This verifies Lemma 5.

Thus $P(x)[M]$ is independent of the choice of x (provided $x \bmod 2 = v_{2k}(M)$). A convenient choice for x is given by

LEMMA 6. *If M^{4k} is an oriented Poincaré complex, then $v_{2k}(M)$ is the reduction of an integral cohomology class.*

Proof. (E. Thomas) Let K be the subgroup of $H^{2k}(M; \mathbb{Z}_2)$ consisting of all classes which are the mod 2 reduction of an integral class. Let L be the subgroup of $H^{2k}(M; \mathbb{Z}_2)$ consisting of all classes whose cup product with the mod 2 reduction of every

torsion class in $H^{2k}(M; \mathbb{Z})$ is zero. Clearly $K \subseteq L$. But an easy counting argument, using Poincaré duality, shows that $\dim K = \dim L$, so $K = L$. Since $v_{2k}(M)$ is in L , it is in K , and Lemma 6 is proved.

We now prove Theorem 2. Let z in $H^{2k}(M; \mathbb{Z})$ be a class whose mod 2 reduction is $v_{2k}(M)$. By a well-known property of bilinear forms (see [4]), $(z \cup z)[M] = \sigma(M)$ mod 8. Let x in $H^{2k}(M; \mathbb{Z}_4)$ be the mod 4 reduction of z . Then $P(x)[M] = (z \cup z)[M]$ mod 8 = $\sigma(M)$ mod 8, which proves Proposition 4 and Theorem 2.

2. The Oriented Case

Let Ω_{4k}^{PD} be the cobordism group of oriented $4k$ -dimensional Poincaré complexes. There is an exact sequence ([5], [8], [9])

$$0 \rightarrow Z \xrightarrow{i} \Omega_{4k}^{PD} \xrightarrow{j} \pi_{4k} \text{MSG} \rightarrow 0.$$

The infinite cyclic group is generated by the closed Milnor manifold of signature 8. Let $\sigma: \Omega_{4k}^{PD} \rightarrow \mathbb{Z}_8$ be the signature homomorphism reduced mod 8. Since $\sigma i = 0$, there is a homomorphism $\bar{\sigma}: \pi_{4k} \text{MSG} \rightarrow \mathbb{Z}_8$ such that $\bar{\sigma}j = \sigma$.

Now the spectrum MSG is a product of Eilenberg-MacLane spectra. (We need this only at the prime 2: see [1].) Therefore there is a cohomology class t_{4k} in $H^{4k}(\text{MSG}; \mathbb{Z}_8)$ such that for any g in $\pi_{4k} \text{MSG}$,

$$\bar{\sigma}(g) = g^*(t_{4k})[S^{4k}].$$

Let y_{4k} be the class in $H^{4k}(\text{BSG}; \mathbb{Z}_8)$ corresponding to t_{4k} under the Thom isomorphism. Then if M^{4k} is an oriented Poincaré complex and $h: M \rightarrow \text{BSG}$ the classifying map for the normal spherical fibration, let c_M denote the cobordism class of M in Ω_{4k}^{PD} . Then $j(c_M) \in \pi_{4k} \text{MSG}$, and $j(c_M)_*[S^{4k}]$, the Hurewicz image of $j(c_M)$, corresponds to $[h_* M]$ under the Thom isomorphism. Hence

$$\begin{aligned} y_{4k}[M] &= h^*(y_{4k})[M] = \langle y_{4k}, h_*[M] \rangle \\ &= \langle t_{4k}, j(c_M)_*[S^{4k}] \rangle \\ &= \langle j(c_M)^*(t_{4k}), [S^{4k}] \rangle \\ &= \bar{\sigma}(j(c_M)) = \sigma(c_M) \\ &= \sigma(M) \text{ mod } 8, \end{aligned}$$

which proves Theorem 3.

3. Proposition B

We show how the techniques of this paper imply Proposition B. Let M^0 be a smooth, compact, spin manifold of dimension $4k$ with boundary an exotic sphere.

Let M be the closed topological manifold formed by attaching a $4k$ -disk along the boundary of M^0 . Also, let BSpin be the classifying space for stable vector bundles with a spin structure and $h: M^0 \rightarrow \text{BSpin}$ be the classifying map for the stable normal bundle.

According to E. Thomas [7], all 2-torsion in $H^*(\text{BSpin}; \mathbb{Z})$ is of order 2. Let v_{2k} be the Wu class in $H^{2k}(\text{BSpin}; \mathbb{Z}_2)$. Then $Sq^1(v_{2k}) = 0$ means that v_{2k} is the mod 2 reduction of an integer class z_{2k} in $H^{2k}(\text{BSpin}; \mathbb{Z})$. Define $z_{2k}(M)$ as $h^*(z_{2k})$. Then $(z_{2k}(M))^2 [M] = \sigma(M) \bmod 8$. Let q_{2k} in $H^{2k}(\text{BSpin}; \mathbb{Q})$ be the class corresponding to z_{2k} under the inclusion of \mathbb{Z} in \mathbb{Q} . Then $(q_{2k}(M))^2 [M] = \sigma(M) \bmod 8$ and q_{2k} is a polynomial (with rational coefficients) in the Pontrjagin classes. Since $(q_{2k})^2$ is decomposable, this proves Proposition B.

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