Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 48 (1973)

Artikel: The Signature mod 8

Autor: Frank, David

DOI: https://doi.org/10.5169/seals-37171

The Signature mod 8

by DAVID FRANK[1])

Let Γ_i be the group of exotic i-dimensional spheres, and let bP_{i+1} be the subgroup of those exotic spheres which bound π-manifolds. There is an exact sequence

$$0 \to bP_{4k} \to \Gamma_{4k-1} \to \pi'_{4k-1} \to 0,$$

where π'_i is the cokernel of the J-homomorphism $J: \pi_i(SO) \to \pi_i = \pi_{i+t}(S^t)$, t large. In studying the group bP_{4k}, it was important for Milnor to know

PROPOSITION A. *Let M^{4k} be a smooth, compact, oriented manifold with boundary an exotic sphere. If M is a π-manifold, then the signature of M is divisible by* 8.

In showing that the above exact sequence was split, it was important for Brumfiel [2] and the author [3] to know the stronger

PROPOSITION B. *Let M^{4k} be a smooth, compact, oriented manifold with boundary an exotic sphere. If M is a spin manifold, and if all decomposable Pontrjagin numbers of M are zero, then the signature of M is divisible by* 8.

In this paper we will prove a very general theorem which includes Propositions A and B. Let BSG be the classifying space for stable oriented spherical fibrations. If we kill the second homotopy group of BSG, we obtain a space BSpinG, the classifying space for stable spherical fibrations with a spin structure. Let v_{2k} denote the universal Wu class in either BSG or BSpinG. We first show.

LEMMA 1. *There is a class x_{2k} in H^{2k}* (BSpinG; Z_4) *whose* mod 2 reduction is *v_{2k} in H^{2k}*(BSpinG; Z_2).

The corresponding statement is of course false in BSG.

We now use the Pontrjagin square cohomology operations. There is a family of such operations; we are interested only in the operations

$$P: H^{2k}(—; Z_4) \to H^{4k}(—; Z_8).$$

In particular, consider the universal characteristic class $P(x_{2k})$ in H^{4k}(BSpinG; Z_8). If M is a spin Poincaré complex of dimension $4k$ with fundamental homology class $[M]$, we may consider the characteristic number $P(x_{2k})[M]$, which is an integer modulo 8. Let $\sigma(M)$ denote the signature of M.

[1]) Supported by NSF grant No. P 029431000.

THEOREM 2. *$P(x_{2k})[M]=\sigma(M)$ modulo 8 for any spin Poincaré complex M^{4k}.*
For oriented Poincaré complexes, we can show

THEOREM 3. *There is a characteristic class y_{4k} in $H^{4k}(\mathrm{BSG}; Z_8)$ such that $y_{4k}[M]=\sigma(M)$ modulo 8 for any oriented Poincaré complex.*

Theorem 3 is the best possible 'Hirzebruch Signature Theorem' for Poincaré complexes; the integer 8 cannot be replaced by a larger integer.

The classes y_{4k} are related to the k-invariants of the fibration $F/\mathrm{Top}\to\mathrm{BSTop}\to\to\mathrm{BSG}$.

1. The Spin Case

We begin by considering the *Wu* class v_{2k} in $H^{2k}(\mathrm{BSpinG}; Z_2)$. We wish to show v_{2k} is the mod 2 reduction of a Z_4 cohomology class. Equivalently, we can show $Sq^1(v_{2k})=0$, where Sq^i is the i-th Steenrod Square. (In fact, $v_{2k}=0$ for k odd, but this is irrelevant to what follows.)

We recall one definition of the *Wu* classes v_i. Let γ be the universal spherical fibration on $\mathrm{BSG}(m)$, m large, let $\mathrm{MSG}(m)$ be the Thom space of γ, and let U be the Thom class. Then define

$$v_i = T^{-1}(\chi(Sq^i U)),$$

where T is the Thom isomorphism ($T: H^*(\mathrm{BSG}(m))\to H^*(\mathrm{MSG}(m))$ and χ is the anti-automorphism of the Steenrod Algebra. If M^n is a Poincaré complex, let $h: M\to\to\mathrm{BSG}(m)$ be the classifying map of the stable Spivak normal fibration of M. Define $v_i(M)$ as $h^*(v_i)$. It follows from [6, Ch. III] that

$$(v_i(M)\cup x)[M] = Sq^i(x)[M]$$

for all x in $H^{n-i}(M; Z_2)$. Thus this is an acceptable definition of the *Wu* classes.

Now note that $Sq^1(v_{2k}\cdot U)=Sq^1(v_{2k})\cdot U$, since $Sq^1(U)=0$. Thus $Sq^1(v_{2k})=0$ if $Sq^1(v_{2k}\cdot U)=0$. But

$$\begin{aligned} Sq^1(v_{2k}\cdot U) &= Sq^1(\chi Sq^{2k})\,U \\ &= (\chi Sq^1)(\chi Sq^{2k})\,U \\ &= \chi(Sq^{2k}Sq^1)\,U. \end{aligned}$$

Now

$$Sq^{2k}Sq^1 = Sq^2Sq^{2k-1} + aSq^1Sq^{2k},\ a\in Z_2,$$

by an Adem relation, so

$$\chi(Sq^{2k}Sq^1) = \chi(Sq^{2k-1})\,\chi(Sq^2) + a\chi(Sq^{2k})\,\chi(Sq^1).$$

Since the last expression is zero on the Thom class U of BSpinG(m), we have shown that $Sq^1(v_{2k})=0$ in BSpinG. Thus, as claimed in Lemma 1, there is a class x_{2k} in $H^{2k}(\text{BSpinG}; Z_4)$ whose mod 2 reduction is v_{2k}. If M^{4k} is a spin Poincaré complex, we wish to relate the class $x_{2k}(M)$ to the signature of M. In fact, we will prove Theorem 2 by showing

PROPOSITION 4. *Let M^{4k} be an oriented Poincaré complex and let x be any class in $H^{2k}(M; Z_4)$ whose* mod 2 *reduction is $v_{2k}(M)$. Then $P(x)[M]=\sigma(M)$* mod 8.

To prove Proposition 4, we first show

LEMMA 5. *Let M^{4k} be an oriented Poincaré complex and let a and b be classes in $H^{2k}(M; Z_4)$ whose* mod 2 *reduction is $v_{2k}(M)$. Then $P(a)=P(b)$.*

Proof. If t is a cohomology class with Z_{2^n} coefficients, we denote by t' the corresponding class with $Z_{2^{n+1}}$ coefficients (determined by the inclusion homomorphism from Z_{2^n} to $Z_{2^{n+1}}$). Now if a mod $2=b$ mod $2=v_{2k}(M)$, then $a=b+d'$, d in $H^{2k}(M{:}\ Z_2)$. Then

$$\begin{aligned} P(a) &= P(b+d') \\ &= P(b)+P(d')+(b \cup d')'. \end{aligned}$$

Thus we must show

$$P(d')+(b \cup d')' = 0.$$

But

$$\begin{aligned} (b \cup d')' &= ((b \bmod 2) \cup d)'' \\ &= (v_{2k}(M) \cup d)'' \\ &= (d \cup d)'', \quad \text{by definition of the } Wu \text{ class} \\ &= P(d'). \end{aligned}$$

(The last equality follows immediately from the cochain definition of the Pontrjagin Square.) This verifies Lemma 5.

Thus $P(x)[M]$ is independent of the choice of x (provided x mod $2=v_{2k}(M)$). A convenient choice for x is given by

LEMMA 6. *If M^{4k} is an oriented Poincaré complex, then $v_{2k}(M)$ is the reduction of an integral cohomology class.*

Proof. (E. Thomas) Let K be the subgroup of $H^{2k}(M; Z_2)$ consisting of all classes which are the mod 2 reduction of an integral class. Let L be the subgroup of $H^{2k}(M; Z_2)$ consisting of all classes whose cup product with the mod 2 reduction of every torsion class in $H^{2k}(M; Z)$ is zero. Clearly $K \subseteq L$. But an easy counting argument, using Poincaré duality, shows that $\dim K = \dim L$, so $K = L$. Since $v_{2k}(M)$ is in L, it is in K, and Lemma 6 is proved.

We now prove Theorem 2. Let z in $H^{2k}(M; Z)$ be a class whose mod 2 reduction is $v_{2k}(M)$. By a well-known property of bilinear forms (see [4]), $(z \cup z)[M] = \sigma(M)$ mod 8. Let x in $H^{2k}(M; Z_4)$ be the mod 4 reduction of z. Then $P(x)[M] = (z \cup z)$ $[M]$ mod 8 $= \sigma(M)$ mod 8, which proves Proposition 4 and Theorem 2.

2. The Oriented Case

Let Ω_{4k}^{PD} be the cobordism group of oriented $4k$-dimensional Poincaré complexes. There is an exact sequence ([5], [8], [9])

$$0 \to Z \xrightarrow{i} \Omega_{4k}^{PD} \xrightarrow{j} \pi_{4k}\mathrm{MSG} \to 0.$$

The infinite cyclic group is generated by the closed Milnor manifold of signature 8. Let $\sigma: \Omega_{4k}^{PD} \to Z_8$ be the signature homomorphism reduced mod 8. Since $\sigma i = 0$, there is a homomorphism $\bar{\sigma}: \pi_{4k}\mathrm{MSG} \to Z_8$ such that $\bar{\sigma} j = \sigma$.

Now the spectrum MSG is a product of Eilenberg-MacLane spectra. (We need this only at the prime 2: see [1].) Therefore there is a cohomology class t_{4k} in $H^{4k}(\mathrm{MSG}; Z_8)$ such that for any g in $\pi_{4k}\mathrm{MSG}$,

$$\bar{\sigma}(g) = g^*(t_{4k})[S^{4k}].$$

Let y_{4k} be the class in $H^{4k}(\mathrm{BSG}; Z_8)$ corresponding to t_{4k} under the Thom isomorphism. Then if M^{4k} is an oriented Poincaré complex and $h: M \to \mathrm{BSG}$ the classifying map for the normal spherical fibration, let c_M denote the cobordism class of M in Ω_{4k}^{PD}. Then $j(c_M) \in \pi_{4k}\mathrm{MSG}$, and $j(c_M)_*[S^{4k}]$, the Hurewicz image of $j(c_M)$, corresponds to $[h_* M]$ under the Thom isomorphism. Hence

$$\begin{aligned} y_{4k}[M] &= h^*(y_{4k})[M] = \langle y_{4k}, h_*[M] \rangle \\ &= \langle t_{4k}, j(c_M)_*[S^{4k}] \rangle \\ &= \langle j(c_M)^*(t_{4k}), [S^{4k}] \rangle \\ &= \bar{\sigma}(j(c_M)) = \sigma(c_M) \\ &= \sigma(M) \bmod 8, \end{aligned}$$

which proves Theorem 3.

3. Proposition B

We show how the techniques of this paper imply Proposition B. Let M^0 be a smooth, compact, spin manifold of dimension $4k$ with boundary an exotic sphere.

Let M be the closed topological manifold formed by attaching a $4k$-disk along the boundary of M^0. Also, let BSpin be the classifying space for stable vector bundles with a spin structure and $h: M^0 \to \text{BSpin}$ be the classifying map for the stable normal bundle.

According to E. Thomas [7], all 2-torsion in $H^*(\text{BSpin}; Z)$ is of order 2. Let v_{2k} be the *Wu* class in $H^{2k}(\text{BSpin}; Z_2)$. Then $Sq^1(v_{2k})=0$ means that v_{2k} is the mod 2 reduction of an integer class z_{2k} in $H^{2k}(\text{BSpin}; Z)$. Define $z_{2k}(M)$ as $h^*(z_{2k})$. Then $(z_{2k}(M))^2[M]=\sigma(M) \bmod 8$. Let q_{2k} in $H^{2k}(\text{BSpin}; Q)$ be the class corresponding to z_{2k} under the inclusion of Z in Q. Then $(q_{2k}(M))^2[M]=\sigma(M) \bmod 8$ and q_{2k} is a polynomial (with rational coefficients) in the Pontrjagin classes. Since $(q_{2k})^2$ is decomposable, this proves Proposition B.

REFERENCES

[1] BROWDER, W., LIULEVICIUS, A., and PETERSON, F. P., *Cobordism theories*, Ann. of Math. *84* (1966), 91–101.

[2] BRUMFIEL, G., *On the homotopy groups of BPL and PL/O*, Ann. of Math. *88* (1968), 291–311.

[3] FRANK, D., *The signature defect and the homotopy of BPL and PL/O*, Comment. Math. Helv. **48** (1973), 525.

[4] HIRZEBRUCH, F., NEUMANN, W. D., and KOH, S. S., *Differentiable manifolds and quadratic forms*, Lecture Notes in Pure and Applied Mathematics, Vol. 4, Marcel Dekker, New York, 1971.

[5] LEVITT, N., *Poincaré duality cobordism*, Ann. of Math. *96* (1972), 211–244.

[6] STEENROD, N. E. and EPSTEIN, D. B. A., *Cohomology Operations*, Ann. of Math. Studies *50*, Princeton, 1962.

[7] THOMAS, E., *On the cohomology groups of the classifying space of the stable spinor group*, Bol. Soc. Mat. Mexicana (2) *7* (1962), 57–69.

[8] JONES, L. *Patch spaces*, Ann. of Math. *97* (1973), 306–343.

[9] QUINN, F., *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc. *78* (1972), 262–267.

State University of New York at Stony Brook.

Received May 5, 1973.