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The Signature mod 8

by David Frank¹)

Let Γ_i be the group of exotic *i*-dimensional spheres, and let bP_{i+1} be the subgroup of those exotic spheres which bound π -manifolds. There is an exact sequence

$$0 \to bP_{4k} \to \Gamma_{4k-1} \to \pi'_{4k-1} \to 0$$
,

where π'_i is the cokernel of the *J*-homomorphism $J: \pi_i(SO) \to \pi_i = \pi_{i+t}(S^t)$, t large. In studying the group bP_{4k} , it was important for Milnor to know

PROPOSITION A. Let M^{4k} be a smooth, compact, oriented manifold with boundary an exotic sphere. If M is a π -manifold, then the signature of M is divisible by 8.

In showing that the above exact sequence was split, it was important for Brumfiel [2] and the author [3] to know the stronger

PROPOSITION B. Let M^{4k} be a smooth, compact, oriented manifold with boundary an exotic sphere. If M is a spin manifold, and if all decomposable Pontrjagin numbers of M are zero, then the signature of M is divisible by 8.

In this paper we will prove a very general theorem which includes Propositions A and B. Let BSG be the classifying space for stable oriented spherical fibrations. If we kill the second homotopy group of BSG, we obtain a space BSpinG, the classifying space for stable spherical fibrations with a spin structure. Let v_{2k} denote the universal Wu class in either BSG or BSpinG. We first show.

LEMMA 1. There is a class x_{2k} in H^{2k} (BSpinG; Z_4) whose mod 2 reduction is v_{2k} in H^{2k} (BSpinG; Z_2).

The corresponding statement is of course false in BSG.

We now use the Pontrjagin square cohomology operations. There is a family of such operations; we are interested only in the operations

$$P: H^{2k}(-; Z_4) \to H^{4k}(-; Z_8).$$

In particular, consider the universal characteristic class $P(x_{2k})$ in $H^{4k}(BSpinG; Z_8)$. If M is a spin Poincaré complex of dimension 4k with fundamental homology class [M], we may consider the characteristic number $P(x_{2k})[M]$, which is an integer modulo 8. Let $\sigma(M)$ denote the signature of M.

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THEOREM 2. $P(x_{2k})[M] = \sigma(M)$ modulo 8 for any spin Poincaré complex M^{4k} . For oriented Poincaré complexes, we can show

THEOREM 3. There is a characteristic class y_{4k} in $H^{4k}(BSG; Z_8)$ such that $y_{4k}[M] = \sigma(M)$ modulo 8 for any oriented Poincaré complex.

Theorem 3 is the best possible 'Hirzebruch Signature Theorem' for Poincaré complexes; the integer 8 cannot be replaced by a larger integer.

The classes y_{4k} are related to the k-invariants of the fibration $F/\text{Top} \to BST\text{op} \to BSG$.

1. The Spin Case

We begin by considering the Wu class v_{2k} in $H^{2k}(BSpinG; Z_2)$. We wish to show v_{2k} is the mod 2 reduction of a Z_4 cohomology class. Equivalently, we can show $Sq^1(v_{2k})=0$, where Sq^i is the *i*-th Steenrod Square. (In fact, $v_{2k}=0$ for k odd, but this is irrelevant to what follows.)

We recall one definition of the Wu classes v_i . Let γ be the universal spherical fibration on BSG(m), m large, let MSG(m) be the Thom space of γ , and let U be the Thom class. Then define

$$v_i = T^{-1}(\chi(Sq^iU)),$$

where T is the Thom isomorphism $(T: H^*(BSG(m)) \to H^*(MSG(m)))$ and χ is the anti-automorphism of the Steenrod Algebra. If M^n is a Poincaré complex, let $h: M \to BSG(m)$ be the classifying map of the stable Spivak normal fibration of M. Define $v_i(M)$ as $h^*(v_i)$. It follows from [6, Ch. III] that

$$(v_i(M) \cup x) [M] = Sq^i(x) [M]$$

for all x in $H^{n-i}(M; \mathbb{Z}_2)$. Thus this is an acceptable definition of the Wu classes.

Now note that $Sq^1(v_{2k} \cdot U) = Sq^1(v_{2k}) \cdot U$, since $Sq^1(U) = 0$. Thus $Sq^1(v_{2k}) = 0$ if $Sq^1(v_{2k} \cdot U) = 0$. But

$$Sq^{1}(v_{2k} \cdot U) = Sq^{1}(\chi Sq^{2k}) U$$

= $(\chi Sq^{1}) (\chi Sq^{2k}) U$
= $\chi (Sq^{2k}Sq^{1}) U$.

Now

$$Sq^{2k}Sq^1 = Sq^2Sq^{2k-1} + aSq^1Sq^{2k}, a \in \mathbb{Z}_2,$$

by an Adem relation, so

$$\chi(Sq^{2k}Sq^{1}) = \chi(Sq^{2k-1})\chi(Sq^{2}) + a\chi(Sq^{2k})\chi(Sq^{1}).$$

522 DAVID FRANK

Since the last expression is zero on the Thom class U of BSpinG(m), we have shown that $Sq^1(v_{2k})=0$ in BSpinG. Thus, as claimed in Lemma 1, there is a class x_{2k} in $H^{2k}(BSpinG; Z_4)$ whose mod 2 reduction is v_{2k} . If M^{4k} is a spin Poincaré complex, we wish to relate the class $x_{2k}(M)$ to the signature of M. In fact, we will prove Theorem 2 by showing

PROPOSITION 4. Let M^{4k} be an oriented Poincaré complex and let x be any class in $H^{2k}(M; \mathbb{Z}_4)$ whose mod 2 reduction is $v_{2k}(M)$. Then $P(x)[M] = \sigma(M)$ mod 8.

To prove Proposition 4, we first show

LEMMA 5. Let M^{4k} be an oriented Poincaré complex and let a and b be classes in $H^{2k}(M; Z_4)$ whose mod 2 reduction is $v_{2k}(M)$. Then P(a) = P(b).

Proof. If t is a cohomology class with Z_{2^n} coefficients, we denote by t' the corresponding class with $Z_{2^{n+1}}$ coefficients (determined by the inclusion homomorphism from Z_{2^n} to $Z_{2^{n+1}}$). Now if a $\text{mod} 2 = b \mod 2 = v_{2k}(M)$, then a = b + d', d in $H^{2k}(M: Z_2)$. Then

$$P(a) = P(b + d')$$

= $P(b) + P(d') + (b \cup d')'$.

Thus we must show

$$P(d') + (b \cup d')' = 0.$$

But

$$(b \cup d')' = ((b \mod 2) \cup d)''$$

$$= (v_{2k}(M) \cup d)''$$

$$= (d \cup d)'', \text{ by definition of the } Wu \text{ class}$$

$$= P(d').$$

(The last equality follows immediately from the cochain definition of the Pontrjagin Square.) This verifies Lemma 5.

Thus P(x)[M] is independent of the choice of x (provided $x \mod 2 = v_{2k}(M)$). A convenient choice for x is given by

LEMMA 6. If M^{4k} is an oriented Poincaré complex, then $v_{2k}(M)$ is the reduction of an integral cohomology class.

Proof. (E. Thomas) Let K be the subgroup of $H^{2k}(M; \mathbb{Z}_2)$ consisting of all classes which are the mod 2 reduction of an integral class. Let L be the subgroup of $H^{2k}(M; \mathbb{Z}_2)$ consisting of all classes whose cup product with the mod 2 reduction of every

torsion class in $H^{2k}(M; Z)$ is zero. Clearly $K \subseteq L$. But an easy counting argument, using Poincaré duality, shows that $\dim K = \dim L$, so K = L. Since $v_{2k}(M)$ is in L, it is in K, and Lemma 6 is proved.

We now prove Theorem 2. Let z in $H^{2k}(M; Z)$ be a class whose mod 2 reduction is $v_{2k}(M)$. By a well-known property of bilinear forms (see [4]), $(z \cup z)[M] = \sigma(M)$ mod 8. Let x in $H^{2k}(M; Z_4)$ be the mod 4 reduction of z. Then $P(x)[M] = (z \cup z)[M]$ mod $8 = \sigma(M)$ mod 8, which proves Proposition 4 and Theorem 2.

2. The Oriented Case

Let Ω_{4k}^{PD} be the cobordism group of oriented 4k-dimensional Poincaré complexes. There is an exact sequence ([5], [8], [9])

$$0 \to Z \xrightarrow{i} \Omega_{4k}^{PD} \xrightarrow{j} \pi_{4k} MSG \to 0$$
.

The infinite cyclic group is generated by the closed Milnor manifold of signature 8. Let $\sigma: \Omega_{4k}^{PD} \to Z_8$ be the signature homomorphism reduced mod 8. Since $\sigma i = 0$, there is a homomorphism $\bar{\sigma}: \pi_{4k} \text{MSG} \to Z_8$ such that $\bar{\sigma}j = \sigma$.

Now the spectrum MSG is a product of Eilenberg-MacLane spectra. (We need this only at the prime 2: see [1].) Therefore there is a cohomology class t_{4k} in $H^{4k}(MSG; Z_8)$ such that for any g in $\pi_{4k}MSG$,

$$\bar{\sigma}(g) = g^*(t_{4k}) \left[S^{4k} \right].$$

Let y_{4k} be the class in $H^{4k}(BSG; Z_8)$ corresponding to t_{4k} under the Thom isomorphism. Then if M^{4k} is an oriented Poincaré complex and $h: M \to BSG$ the classifying map for the normal spherical fibration, let c_M denote the cobordism class of M in Ω_{4k}^{PD} . Then $j(c_M) \in \pi_{4k}MSG$, and $j(c_M)_* [S^{4k}]$, the Hurewicz image of $j(c_M)$, corresponds to $[h_*M]$ under the Thom isomorphism. Hence

$$y_{4k}[M] = h^*(y_{4k})[M] = \langle y_{4k}, h_*[M] \rangle$$

$$= \langle t_{4k}, j(c_M)_*[S^{4k}] \rangle$$

$$= \langle j(c_M)^*(t_{4k}), [S^{4k}] \rangle$$

$$= \bar{\sigma}(j(c_M)) = \sigma(c_M)$$

$$= \sigma(M) \mod 8,$$

which proves Theorem 3.

3. Proposition B

We show how the techniques of this paper imply Proposition B. Let M^0 be a smooth, compact, spin manifold of dimension 4k with boundary an exotic sphere.

524 DAVID FRANK

Let M be the closed topological manifold formed by attaching a 4k-disk along the boundary of M^0 . Also, let BSpin be the classifying space for stable vector bundles with a spin structure and $h: M^0 \to B$ Spin be the classifying map for the stable normal bundle.

According to E. Thomas [7], all 2-torsion in $H^*(BSpin; Z)$ is of order 2. Let v_{2k} be the Wu class in $H^{2k}(BSpin; Z_2)$. Then $Sq^1(v_{2k})=0$ means that v_{2k} is the mod 2 reduction of an integer class z_{2k} in $H^{2k}(BSpin; Z)$. Define $z_{2k}(M)$ as $h^*(z_{2k})$. Then $(z_{2k}(M))^2[M]=\sigma(M)$ mod 8. Let q_{2k} in $H^{2k}(BSpin; Q)$ be the class corresponding to z_{2k} under the inclusion of Z in Q. Then $(q_{2k}(M))^2[M]=\sigma(M)$ mod 8 and q_{2k} is a polynomial (with rational coefficients) in the Pontrjagin classes. Since $(q_{2k})^2$ is decomposable, this proves Proposition B.

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