

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 48 (1973)  
  
**Artikel:** The Signature mod 8  
**Autor:** Frank, David  
**DOI:** <https://doi.org/10.5169/seals-37171>

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## The Signature mod 8

by DAVID FRANK<sup>1)</sup>

Let  $\Gamma_i$  be the group of exotic  $i$ -dimensional spheres, and let  $bP_{i+1}$  be the subgroup of those exotic spheres which bound  $\pi$ -manifolds. There is an exact sequence

$$0 \rightarrow bP_{4k} \rightarrow \Gamma_{4k-1} \rightarrow \pi'_{4k-1} \rightarrow 0,$$

where  $\pi'_i$  is the cokernel of the  $J$ -homomorphism  $J: \pi_i(SO) \rightarrow \pi_i = \pi_{i+t}(S^t)$ ,  $t$  large. In studying the group  $bP_{4k}$ , it was important for Milnor to know

**PROPOSITION A.** *Let  $M^{4k}$  be a smooth, compact, oriented manifold with boundary an exotic sphere. If  $M$  is a  $\pi$ -manifold, then the signature of  $M$  is divisible by 8.*

In showing that the above exact sequence was split, it was important for Brumfiel [2] and the author [3] to know the stronger

**PROPOSITION B.** *Let  $M^{4k}$  be a smooth, compact, oriented manifold with boundary an exotic sphere. If  $M$  is a spin manifold, and if all decomposable Pontrjagin numbers of  $M$  are zero, then the signature of  $M$  is divisible by 8.*

In this paper we will prove a very general theorem which includes Propositions A and B. Let BSG be the classifying space for stable oriented spherical fibrations. If we kill the second homotopy group of BSG, we obtain a space BSpinG, the classifying space for stable spherical fibrations with a spin structure. Let  $v_{2k}$  denote the universal Wu class in either BSG or BSpinG. We first show.

**LEMMA 1.** *There is a class  $x_{2k}$  in  $H^{2k}(\text{BSpinG}; \mathbb{Z}_4)$  whose mod 2 reduction is  $v_{2k}$  in  $H^{2k}(\text{BSpinG}; \mathbb{Z}_2)$ .*

The corresponding statement is of course false in BSG.

We now use the Pontrjagin square cohomology operations. There is a family of such operations; we are interested only in the operations

$$P: H^{2k}(-; \mathbb{Z}_4) \rightarrow H^{4k}(-; \mathbb{Z}_8).$$

In particular, consider the universal characteristic class  $P(x_{2k})$  in  $H^{4k}(\text{BSpinG}; \mathbb{Z}_8)$ . If  $M$  is a spin Poincaré complex of dimension  $4k$  with fundamental homology class  $[M]$ , we may consider the characteristic number  $P(x_{2k})[M]$ , which is an integer modulo 8. Let  $\sigma(M)$  denote the signature of  $M$ .

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<sup>1)</sup> Supported by NSF grant No. P 029431000.

**THEOREM 2.**  $P(x_{2k})[M] = \sigma(M)$  modulo 8 for any spin Poincaré complex  $M^{4k}$ . For oriented Poincaré complexes, we can show

**THEOREM 3.** *There is a characteristic class  $y_{4k}$  in  $H^{4k}(\text{BSG}; \mathbb{Z}_8)$  such that  $y_{4k}[M] = \sigma(M)$  modulo 8 for any oriented Poincaré complex.*

Theorem 3 is the best possible ‘Hirzebruch Signature Theorem’ for Poincaré complexes; the integer 8 cannot be replaced by a larger integer.

The classes  $y_{4k}$  are related to the  $k$ -invariants of the fibration  $F/\text{Top} \rightarrow \text{BStop} \rightarrow \text{BSG}$ .

## 1. The Spin Case

We begin by considering the  $Wu$  class  $v_{2k}$  in  $H^{2k}(\text{BSpinG}; \mathbb{Z}_2)$ . We wish to show  $v_{2k}$  is the mod 2 reduction of a  $\mathbb{Z}_4$  cohomology class. Equivalently, we can show  $Sq^1(v_{2k}) = 0$ , where  $Sq^i$  is the  $i$ -th Steenrod Square. (In fact,  $v_{2k} = 0$  for  $k$  odd, but this is irrelevant to what follows.)

We recall one definition of the  $Wu$  classes  $v_i$ . Let  $\gamma$  be the universal spherical fibration on  $\text{BSG}(m)$ ,  $m$  large, let  $\text{MSG}(m)$  be the Thom space of  $\gamma$ , and let  $U$  be the Thom class. Then define

$$v_i = T^{-1}(\chi(Sq^i U)),$$

where  $T$  is the Thom isomorphism ( $T: H^*(\text{BSG}(m)) \rightarrow H^*(\text{MSG}(m))$ ) and  $\chi$  is the anti-automorphism of the Steenrod Algebra. If  $M^n$  is a Poincaré complex, let  $h: M \rightarrow \text{BSG}(m)$  be the classifying map of the stable Spivak normal fibration of  $M$ . Define  $v_i(M)$  as  $h^*(v_i)$ . It follows from [6, Ch. III] that

$$(v_i(M) \cup x)[M] = Sq^i(x)[M]$$

for all  $x$  in  $H^{n-i}(M; \mathbb{Z}_2)$ . Thus this is an acceptable definition of the  $Wu$  classes.

Now note that  $Sq^1(v_{2k} \cdot U) = Sq^1(v_{2k}) \cdot U$ , since  $Sq^1(U) = 0$ . Thus  $Sq^1(v_{2k}) = 0$  if  $Sq^1(v_{2k} \cdot U) = 0$ . But

$$\begin{aligned} Sq^1(v_{2k} \cdot U) &= Sq^1(\chi Sq^{2k}) U \\ &= (\chi Sq^1)(\chi Sq^{2k}) U \\ &= \chi(Sq^{2k} Sq^1) U. \end{aligned}$$

Now

$$Sq^{2k} Sq^1 = Sq^2 Sq^{2k-1} + a Sq^1 Sq^{2k}, \quad a \in \mathbb{Z}_2,$$

by an Adem relation, so

$$\chi(Sq^{2k} Sq^1) = \chi(Sq^{2k-1}) \chi(Sq^2) + a \chi(Sq^{2k}) \chi(Sq^1).$$

Since the last expression is zero on the Thom class  $U$  of  $B\text{Spin}G(m)$ , we have shown that  $Sq^1(v_{2k})=0$  in  $B\text{Spin}G$ . Thus, as claimed in Lemma 1, there is a class  $x_{2k}$  in  $H^{2k}(B\text{Spin}G; Z_4)$  whose mod 2 reduction is  $v_{2k}$ . If  $M^{4k}$  is a spin Poincaré complex, we wish to relate the class  $x_{2k}(M)$  to the signature of  $M$ . In fact, we will prove Theorem 2 by showing

**PROPOSITION 4.** *Let  $M^{4k}$  be an oriented Poincaré complex and let  $x$  be any class in  $H^{2k}(M; Z_4)$  whose mod 2 reduction is  $v_{2k}(M)$ . Then  $P(x)[M] = \sigma(M) \bmod 8$ .*

To prove Proposition 4, we first show

**LEMMA 5.** *Let  $M^{4k}$  be an oriented Poincaré complex and let  $a$  and  $b$  be classes in  $H^{2k}(M; Z_4)$  whose mod 2 reduction is  $v_{2k}(M)$ . Then  $P(a) = P(b)$ .*

*Proof.* If  $t$  is a cohomology class with  $Z_{2^n}$  coefficients, we denote by  $t'$  the corresponding class with  $Z_{2^{n+1}}$  coefficients (determined by the inclusion homomorphism from  $Z_{2^n}$  to  $Z_{2^{n+1}}$ ). Now if  $a \bmod 2 = b \bmod 2 = v_{2k}(M)$ , then  $a = b + d'$ ,  $d$  in  $H^{2k}(M; Z_2)$ . Then

$$\begin{aligned} P(a) &= P(b + d') \\ &= P(b) + P(d') + (b \cup d')'. \end{aligned}$$

Thus we must show

$$P(d') + (b \cup d')' = 0.$$

But

$$\begin{aligned} (b \cup d')' &= ((b \bmod 2) \cup d)'' \\ &= (v_{2k}(M) \cup d)'' \\ &= (d \cup d)'', \text{ by definition of the } Wu \text{ class} \\ &= P(d'). \end{aligned}$$

(The last equality follows immediately from the cochain definition of the Pontrjagin Square.) This verifies Lemma 5.

Thus  $P(x)[M]$  is independent of the choice of  $x$  (provided  $x \bmod 2 = v_{2k}(M)$ ). A convenient choice for  $x$  is given by

**LEMMA 6.** *If  $M^{4k}$  is an oriented Poincaré complex, then  $v_{2k}(M)$  is the reduction of an integral cohomology class.*

*Proof.* (E. Thomas) Let  $K$  be the subgroup of  $H^{2k}(M; Z_2)$  consisting of all classes which are the mod 2 reduction of an integral class. Let  $L$  be the subgroup of  $H^{2k}(M; Z_2)$  consisting of all classes whose cup product with the mod 2 reduction of every

torsion class in  $H^{2k}(M; Z)$  is zero. Clearly  $K \subseteq L$ . But an easy counting argument, using Poincaré duality, shows that  $\dim K = \dim L$ , so  $K = L$ . Since  $v_{2k}(M)$  is in  $L$ , it is in  $K$ , and Lemma 6 is proved.

We now prove Theorem 2. Let  $z$  in  $H^{2k}(M; Z)$  be a class whose mod 2 reduction is  $v_{2k}(M)$ . By a well-known property of bilinear forms (see [4]),  $(z \cup z)[M] = \sigma(M) \bmod 8$ . Let  $x$  in  $H^{2k}(M; Z_4)$  be the mod 4 reduction of  $z$ . Then  $P(x)[M] = (z \cup z)[M] \bmod 8 = \sigma(M) \bmod 8$ , which proves Proposition 4 and Theorem 2.

## 2. The Oriented Case

Let  $\Omega_{4k}^{PD}$  be the cobordism group of oriented  $4k$ -dimensional Poincaré complexes. There is an exact sequence ([5], [8], [9])

$$0 \rightarrow Z \xrightarrow{i} \Omega_{4k}^{PD} \xrightarrow{j} \pi_{4k} \text{MSG} \rightarrow 0.$$

The infinite cyclic group is generated by the closed Milnor manifold of signature 8. Let  $\sigma: \Omega_{4k}^{PD} \rightarrow Z_8$  be the signature homomorphism reduced mod 8. Since  $\sigma i = 0$ , there is a homomorphism  $\bar{\sigma}: \pi_{4k} \text{MSG} \rightarrow Z_8$  such that  $\bar{\sigma} j = \sigma$ .

Now the spectrum MSG is a product of Eilenberg-MacLane spectra. (We need this only at the prime 2: see [1].) Therefore there is a cohomology class  $t_{4k}$  in  $H^{4k}(\text{MSG}; Z_8)$  such that for any  $g$  in  $\pi_{4k} \text{MSG}$ ,

$$\bar{\sigma}(g) = g^*(t_{4k})[S^{4k}].$$

Let  $y_{4k}$  be the class in  $H^{4k}(\text{BSG}; Z_8)$  corresponding to  $t_{4k}$  under the Thom isomorphism. Then if  $M^{4k}$  is an oriented Poincaré complex and  $h: M \rightarrow \text{BSG}$  the classifying map for the normal spherical fibration, let  $c_M$  denote the cobordism class of  $M$  in  $\Omega_{4k}^{PD}$ . Then  $j(c_M) \in \pi_{4k} \text{MSG}$ , and  $j(c_M)_*[S^{4k}]$ , the Hurewicz image of  $j(c_M)$ , corresponds to  $[h_* M]$  under the Thom isomorphism. Hence

$$\begin{aligned} y_{4k}[M] &= h^*(y_{4k})[M] = \langle y_{4k}, h_*[M] \rangle \\ &= \langle t_{4k}, j(c_M)_*[S^{4k}] \rangle \\ &= \langle j(c_M)^*(t_{4k}), [S^{4k}] \rangle \\ &= \bar{\sigma}(j(c_M)) = \sigma(c_M) \\ &= \sigma(M) \bmod 8, \end{aligned}$$

which proves Theorem 3.

## 3. Proposition B

We show how the techniques of this paper imply Proposition B. Let  $M^0$  be a smooth, compact, spin manifold of dimension  $4k$  with boundary an exotic sphere.

Let  $M$  be the closed topological manifold formed by attaching a  $4k$ -disk along the boundary of  $M^0$ . Also, let  $B\text{Spin}$  be the classifying space for stable vector bundles with a spin structure and  $h: M^0 \rightarrow B\text{Spin}$  be the classifying map for the stable normal bundle.

According to E. Thomas [7], all 2-torsion in  $H^*(B\text{Spin}; Z)$  is of order 2. Let  $v_{2k}$  be the  $Wu$  class in  $H^{2k}(B\text{Spin}; Z_2)$ . Then  $Sq^1(v_{2k})=0$  means that  $v_{2k}$  is the mod 2 reduction of an integer class  $z_{2k}$  in  $H^{2k}(B\text{Spin}; Z)$ . Define  $z_{2k}(M)$  as  $h^*(z_{2k})$ . Then  $(z_{2k}(M))^2[M] = \sigma(M) \bmod 8$ . Let  $q_{2k}$  in  $H^{2k}(B\text{Spin}; Q)$  be the class corresponding to  $z_{2k}$  under the inclusion of  $Z$  in  $Q$ . Then  $(q_{2k}(M))^2[M] = \sigma(M) \bmod 8$  and  $q_{2k}$  is a polynomial (with rational coefficients) in the Pontrjagin classes. Since  $(q_{2k})^2$  is decomposable, this proves Proposition B.

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Received May 5, 1973.