# Quadratic Spaces with Few Isometries
# (Quadratic Forms and Linear Topologies VI)

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## Introduction

What sort of metric automorphisms do always exist on infinite dimensional quadratic spaces? Clearly, we always have the symmetries about (nondegenerate) hyperplanes, the identity 1 of the space, $-1$, and of course finite products of these isometries; they form an invariant subgroup $\mathfrak{J}$ in the full orthogonal group of the space. In the finite dimensional case $\mathfrak{J}$ is already the full orthogonal group. In the infinite case however, $\mathfrak{J}$ usually represents only a negligible part of the orthogonal group associated with the space. In this note we shall show that there are quadratic spaces of arbitrarily large dimension whose full orthogonal groups equal $\mathfrak{J}$. In §1 we shall describe how to define such spaces over prescribed (non denumerable) base fields.

The spaces $E$ which we shall investigate below share the following property on subspaces $F$,

$$F \subset E \ \& \ \dim F \geqslant \aleph_0 \rightarrow \dim F^\perp < \dim E . \tag{*}$$

In particular, if such a space E is decomposed orthogonally, $E = E_1 \oplus E_2$, then one of the summands $E_i$ necessarily is of finite dimension. Spaces with such few orthogonal splittings are an extreme counterpart to quadratic spaces admitting orthogonal bases. For subspaces $F$ of spaces wich admit orthogonal bases we invariably have $\dim E/F^\perp = \dim F$ which sharply contrasts (*). We see in particular that $\dim E \neq \aleph_0$ for all $E$ satisfying (*). The construction given in §1 yields spaces which actually satisfy the stronger property on subspaces $F$,

$$F \subset E \ \& \ \dim F \geqslant \aleph_0 \rightarrow \dim F^\perp \leqslant \aleph_0 . \tag{**}$$

The notion which stands in the center of our discussion of spaces with small orthogonal group $\mathfrak{O}$ in the sense indicated above ($\mathfrak{O} = \mathfrak{J}$) is that of a locally algebraic isometry (§2). An isometry $T$ on $E$ is called locally algebraic if $T$ admits for every $x \in E$ a polynomial $f_x(T)$ (with coefficients in the base field of $E$) that annihilates $x$, $f_x(T)x = 0$. If $f_x$ does not depend on $x$ we call $T$ algebraic. Theorem 3 of §2 says that the spaces constructed in §1 admit locally algebraic isometries only; in other words, there are infinite dimensional (**)-spaces $E$ with property ($\lambda$): 'Every

isometry on $E$ is locally algebraic'. By means of somewhat complicated examples one can however show that (**) does not, in general, imply ($\lambda$) (the converse implication is seen not to be true either by Theorem 3 of §2). Spaces with property ($\lambda$) and which, in addition, satisfy (*) absolutely (i.e. which preserve (*) under extensions of the base field) are seen to have trivial quotient $\mathfrak{O}/\mathfrak{J}$ (Corollary 1 of Theorem 3 in §2).

In [3] it is shown that certain spaces constructed in §1 satisfy Witt's cancellation theorem: If $E = E_1 \oplus E_2 = F_1 \oplus F_2$ are orthogonal decompositions of $E$ with $E_1$ and $F_1$ isometric, then $E_2$ and $F_2$ must be isometric; a rare thing indeed to happen in the infinite dimensional case.

## Notations.

Generally speaking, forms $\Phi : E \times E \to k$ are additive in each argument and satisfy $\Phi(\lambda x, y) = \lambda \Phi(x, y)$, $\Phi(x, \lambda y) = \Phi(x, y)\lambda^\alpha$ with respect to some fixed involution $\alpha$ ($=$ antiautomorphism of period 2) of the division ring $k$. We shall however always assume below that $k$ is *commutative*. We shall furthermore assume $\Phi$ to be $\varepsilon$-*hermitean*, i.e. $\Phi(y, x) = \varepsilon \Phi(x, y)^\alpha$ with $\varepsilon = +1$ (hermitean) or $\varepsilon = -1$ (antihermitean). If $\alpha$ is the identity, then $k$ is necessarily commutative and we speak of symmetric and anti-symmetric forms respectively. In any case, '$x \perp y$', defined as usual to be '$\Phi(x, y) = 0$', is a symmetric relation. $E^\perp$ is called the *radical* of $E$ (rad $E$). If rad $E = (0)$ we call $\Phi$ non-degenerate and – in analogy with algebras – the space $(E, \Phi)$ *semisimple*. $\Phi$ is said to be *tracevalued* if for every $x \in E$ there is a $\xi \in k$ such that $\Phi(x, x) = \xi + \varepsilon \xi^\alpha$. We shall always assume $\Phi$ to be tracevalued, a non trivial requirement only when chark $k = 2$ ([1] §4, No. 2). We shall make use of Witt's theorem in §2 below ([1] §4, No. 3): Let $E$ be a space with a non degenerate form $\Phi$ which is hermitean or anti-hermitean, and tracevalued if it is hermitean. Then any isometry ($=$ vectorspace isomorphism that preserves $\Phi$) between finite-dimensional subspaces can be extended to a isometric automorphism of $E$.

Let $(E, \Phi)$ be an $\varepsilon$-hermitean $k$-vectorspace with respect to the involution $\alpha$. Assume that the division ring $k'$ contains $k$ and admits an extension (involution) of $\alpha$ to $k'$. We know that the abelian group $E' = k' \otimes_k E$ may be regarded as a vector-space over $k$ and as a vectorspace over $k'$. The form $\Phi' : E' \times E' \to k'$, defined by $\Phi'(\sum \lambda_i \otimes x_i, \sum \mu_j \otimes y_i) = \sum \lambda_i \Phi(x_i, y_j)\mu_j^\alpha$ for $\lambda_i, \mu_j \in k'$ is $\varepsilon$-hermitean. We say that $\Phi$ satisfies (*), or (**), *absolutely*, if the form $\Phi'$ possesses these properties for all extensions $k'$ of $k$. $(E', \Phi')$ is called the $k'$-*ification* of $(E, \Phi)$ or the space obtained from $(E, \Phi)$ by extending the ring of scalars.

A space $(E, \Phi)$ is called anisotropic if it contains no isotropic elements, i. e. no vectors $x \neq 0$ with $\Phi(x, x) = 0$.

Unless stated otherwise, $(E, \Phi)$ will be assumed to be of infinite dimension.

## §1. The Existence of Spaces with Property **

In this short section we shall describe the construction of infinite dimensional spaces $(E, \Phi)$ where $\Phi$ is an $\varepsilon$-hermitean form satisfying (**) absolutely.

Let $\alpha$ be an involution of the commutative field $k$, $\operatorname{card} k > \aleph_0$. Let $X \subset k$ be a maximal subset of algebraically independent elements over the prime field $k_0$ so that $k$ is an algebraic extension of $k_0(X)$. Let $\varepsilon = +1$ or $-1$. Since $\alpha$ is of period 2, there is a subset $Y \subset X$ with card $Y = \operatorname{card} X$ ($= \operatorname{card} k$) and for every $\eta \in Y$ either $\varepsilon \eta^\alpha = \eta$ or $\varepsilon \eta^\alpha \notin Y$. Let then $(e_\iota)_{\iota \in I}$ be a basis of a $k$-vectorspace with $\operatorname{card} k \geqslant \operatorname{card} I > \aleph_0$. We define an $\varepsilon$-hermitean form $\Phi$ on $E \times E$ as follows: Pick an ordering on $I$. For all $\iota < \kappa$ in $I$ set $\Phi(e_\iota, e_\kappa) = \varepsilon \Phi(e_\kappa, e_\iota)^\alpha = \eta_{\iota\kappa} \in Y$ such that all elements $\eta_{\iota\kappa}$ ($\iota < \kappa$) are different. Furthermore $\Phi(e_\iota, e_\iota) = \varepsilon \Phi(e_\iota, e_\iota)^\alpha \in k$ such that no $\Phi(e_\iota, e_\iota)$ equals a $\Phi(e_\iota, e_\kappa)$ with $\iota \neq \kappa$. We assert that $\Phi$ satisfies (**).

*Proof.* Let $U$ and $V$ be subspaces of $E$ with $\dim V > \dim U = \aleph_0$, $(u_i)_{i \in \mathbb{N}}$ and $(v_\iota)_{\iota \in J}$ bases of $U$ and $V$ respectively. $u_i = \sum \alpha_{ik} e_\kappa$, $v_\iota = \sum \beta_{\iota\kappa} e_\kappa$ where the first sum extends over the finite set $M_i = \{\kappa \in I \mid \alpha_{i\kappa} \neq 0\}$, the second over the finite set $N_\iota = \{\kappa \in I \mid \beta_{\iota\kappa} \neq 0\}$. Set $M = \bigcup_\mathbb{N} M_i$, $N = \bigcup_J N_\iota$. Thus $\operatorname{card} N > \operatorname{card} M = \aleph_0$. Our assertion is proved if we can exhibit a pair $u, v \in U \times V$ with $\Phi(u, v) \neq 0$. Such a pair is found as follows.

(i) $X$ contains a denumerable subset $A$ such that $\{\alpha_{i\kappa} \mid i \in \mathbb{N}, \kappa \in M_i\}$ is contained in the algebraic closure in $k$ of the subfield $k_0(A)$.

(ii) There is a $\varrho_0 \in N \backslash M$ such that

$$A \cap \{\Phi(e_v, e_{\varrho_0}), \Phi(e_{\varrho_0}, e_v) \mid v \in I \backslash \{\varrho_0\}\} = \emptyset.$$

Let $\varrho_0 \in N_{v_0}$.

(iii) $X$ contains a finite subset $B$ such that $\{\beta_{v_0\mu} \mid \mu \in N_{v_0}\}$ is contained in the algebraic closure in $k$ of $k_0(B)$. Since $M$ is infinite, there is a $\kappa_0 \in M$ such that $\Phi(e_{\kappa_0}, e_{\varrho_0})$, $\Phi(e_{\varrho_0}, e_{\kappa_0}) \notin B$. Let $\kappa_0 \in M_{\iota_0}$.

(iv) Notice that $\kappa_0 \neq \varrho_0$. If $\kappa_0 < \varrho_0$ we let

$$C = \{\Phi(e_\kappa, e_\varrho) \mid (\kappa, \varrho) \in M_{\iota_0} \times N_{v_0} \backslash \{(\kappa_0, \varrho_0)\}\};$$

if $\varrho_0 < \kappa_0$ we let

$$C = \{\Phi(e_\varrho, e_\kappa) \mid (\varrho, \kappa) \in N_{v_0} \times M_{\iota_0} \backslash \{(\varrho_0, \kappa_0)\}\}.$$

Thus, if $\kappa_0 < \varrho_0$ we see by (ii), (iii), (iv) that $\eta_{\kappa_0\varrho_0} = \Phi(e_{\kappa_0}, e_{\varrho_0}) \notin A \cup B \cup C$; similarly, if $\varrho_0 < \kappa_0$ we have $\eta_{\varrho_0\kappa_0} = \Phi(e_{\varrho_0}, e_{\kappa_0}) \notin A \cup B \cup C$. Thus, if $k_1$ is the algebraic closure in $k$ of $k_0(A \cup B \cup C)$ we see that $\eta_{\kappa_0\varrho_0} \notin k_1$ if $\kappa_0 < \varrho_0$ and $\eta_{\varrho_0\kappa_0} \notin k_1$ when $\varrho_0 < \kappa_0$. In

the first case we consider

$$\Phi(u_{\iota_0}, v_{\nu_0}) = \sum_{(\kappa,\rho)\neq(\kappa_0,\rho_0)} \alpha_{\iota_0\,\kappa}\beta_{\nu_0\varrho}\,\phi(e_\kappa, e_\varrho) + \alpha_{\iota_0\kappa_0}\beta_{\nu_0\varrho_0}\,\eta_{\kappa_0\varrho_0}.$$

If we had $\Phi(u_{\iota_0}, v_{\nu_0})=0$ then we had a nontrivial linear equation for $\eta_{\kappa_0\varrho_0}$ with coefficients in $k_1$, so $\eta_{\kappa_0\varrho_0}\in k_1$. If $\varrho_0<\kappa_0$ we conclude in the same manner that $\Phi(v_{\nu_0}, u_{\iota_0})$ $\neq 0$. Clearly our proof remains valid if we pass to the form $\Phi'$ on the $k'$-ification $E'=k'\otimes_k E$ of $E$ with respect to some overfield $k'$ of $k$ (admitting an extension of $\alpha$). This proves our assertions. We note our result as

THEOREM 1. *For* $\varepsilon = +1$ *and for* $\varepsilon = -1$ *there exist $\varepsilon$-hermitean forms $\Phi$ over any commutative field $k$ with given involution and* $\mathrm{card}\,k > \aleph_0$ *which satisfy* (**) *absolutely; we may choose the dimension of $\Phi$ to be* $\mathrm{card}\,k$.

We had $\mathrm{card}\,k \geqslant \dim E$ for the spaces $E$ in the above construction. We do not know if this is necessarily so for spaces with property (**). It is easy to see that (**) does imply $(\mathrm{card}\,k)\aleph_0 \geqslant \dim E$. Thus, at least in the special cases where $\mathrm{card}\,k$ is a beth (e.g. when $k=\mathbb{R}$ or $\mathbb{C}$), (**) does imply $\mathrm{card}\,k \geqslant \dim E$.

THEOREM 2. *Let* $k=k_0(X)$ *be a purely transcendental extension of $k_0$ and* $\mathrm{card}\,X > \aleph_0$. *If – in the notation of the preceding construction – $\Phi$ is chosen symmetric with* $\Phi(e_\iota, e_\kappa)=\xi_{\iota\kappa}\in X(\iota,\kappa\in I$ *and* $\mathrm{card}\,I > \aleph_0)$ *such that* $\xi_{\iota\kappa}=\xi_{\nu\mu}$ *if and only if* $\{\iota,\kappa\}$ $=\{\nu,\mu\}$, *then $\Phi(x,x)$ is a square in $k$ only when $x=0$.*

This result is proved in [3]; it guarantees the existence of *anisotropic* forms with property (**) over all fields of a certain type. In the special case where $k_0$ is assumed orderable the part of theorem 2 ruling out isotropic vectors follows directly from Jacobi's diagonalization formula (for finite spaces). It is clear that after extending the base field $\Phi$ may admit isotropic vectors. The fact that $k$ is a purely transcendental extension of some $k_0$ is not however crucial for the existence of an anisotropic $\Phi$ over $k$ satisfying (**). We give an example of such a form over $\mathbb{R}$ by specifying a subspace of an infinite separable Hilbertspace $(H, \Phi)$ over the reals: Note that the collection of all sets $M$ of linearly independent vectors $x, y, \cdots$ with $\{\Phi(x,y) \mid x, y\in M\}$ algebraically independent over $\mathbb{Q}$ is inductively ordered by inclusion. Let $M_0$ be a maximal element by Zorn's lemma. If $\mathrm{card}\,M_0 > \aleph_0$, then the restriction of $\Phi$ to the span of $M_0$ satisfies (**) as we have demonstrated above. Assume by way of contradiction that $\mathrm{card}\,M_0 \leqslant \aleph_0$. Let $(x_i)_{i\in J}$ be the elements of $M_0$ in some ordering, and let $A=\{\Phi(x_i, x_j) \mid i, j\in J\}$. Introduce an orthonormal basis $(e_i)_{i\in J}$ in the span $X$ of the $x_i$ $(i\in J)$, $e_i=\sum \alpha_{ij}x_j$ with $(\alpha_{ij})$ triangular. Then $(\alpha_{ij})^{-1}=(\beta_{ij})$ is triangular and $\alpha_{ij}, \beta_{ij}\in\overline{\mathbb{Q}(A)}$ (real closure). Since $\mathrm{card}\,A \leqslant \aleph_0$ we can pick a family $(t_i)_{i\in J}$, the $t_i$ in $\mathbb{R}$ and algebraically independent over $\overline{\mathbb{Q}(A)}$ with $\sum_J t_i^2 = t < \infty$. The closure $\bar{X}$ of $X$ in $H$ (in the normtopology of $\Phi$) contains a vector $x$ with $\Phi(x, e_i)=\lambda_i t_i$ for any choice of $\lambda_i$ with, say, $0<\lambda_i<1$. We

have $\Phi(x, x_i) = \sum \beta_{ij}\lambda_j t_j$. It follows that the set $\{\Phi(x, x_j) \mid j \in J\}$ is algebraically independent over $(\mathbb{Q}A)$ for $\lambda_i$ rational. If we can arrange for $\Phi(x, x) = \sum (\lambda_i t_i)^2$ to be outside $\overline{\mathbb{Q}(A \cup (t_i)_J)}$ we have the desired contradiction: $M_0 \cup \{x\}$ contradicting the maximality of $M_0$. Now if $J$ should be finite, then $\tilde{X} = X$ and we may, if necessary, pass from $x$ to a vector $x + y$ with $y \in X^\perp$ and $\Phi(y, y) = \alpha - \Phi(x, x)$ and suitably chosen $\alpha$. If card $J = \aleph_0$, then by varying the *rational* $\lambda_i$ in the open unit interval we can arrange for $\Phi(x, x)$ to be any real number of the open interval $[0, t]$. Clearly then, there is a choice with $\Phi(x, x)$ outside the denumerable $\overline{\mathbb{Q}(A \cup (t_i)_J)}$. Q.E.D. We can do the same for hermitean forms over a complex Hilbert space. Thus

THEOREM 3. *There exist (infinite) positive definite symmetric (hermitean) forms over* $\mathbb{R}(\mathbb{C})$ *which satisfy* (**) *absolutely.*

*Remark.* We briefly indicate how to construct spaces which satisfy (**) but not absolutely so. Let $k$ be nondenumerably infinite. Let $(f_i)_{i \in I}$, $(g_i)_{i \in I}$ be bases of $k$-vectorspaces $F$ and $G$ respectively, card $I = $ card $k$. Choose subsets $X$ and $Y$ of $k$ with $X \cap Y = \emptyset$ and $X \cup Y$ algebraically independent over the primefield $k_0$ of $k$. Define a symmetric bilinear form $\Phi$ on $E = F \oplus G$ as follows: $\Phi(f_i, f_\kappa) = -\Phi(g_i, g_\kappa) = \xi_{i\kappa}$. $\Phi(f_i, g_\kappa) = \Phi(f_\kappa, g_i) = \eta_{i\kappa}$ with $\xi_{i\kappa} \in X$, $\eta_{i\kappa} \in Y$ and $\xi_{i\kappa} = \xi_{v\mu}$ and $\eta_{i\kappa} = \eta_{v\mu}$ if and only if $\{i, \kappa\} = \{v, \mu\}$. If $k$ is assumed orderable, then the reader proves by the method illustrated above that $E = F \oplus G$ satisfies (**). However, over the extension $k(\sqrt{-1})\ E$ decomposes orthogonally, $E = H \oplus L$ with $H$ spanned by all $f_i + \sqrt{-1} \cdot g_i (i \in I)$ and $L$ spanned by all $f_i - \sqrt{-1} \cdot g_i (i \in I)$.

## §2. The Orthogonal Group

In this section we study the orthogonal group $\mathfrak{O}$ associated with certain infinite dimensional spaces $(E, \Phi)$ which satisfy (*). Here $\Phi$ will always be symmetric or anti-symmetric and tracevalued if it is symmetric.

Consider an isometry $T$ such that there is an orthogonal decomposition $E = E_0 \oplus E_1$ with $\dim E_1 < \infty$ and $T = \pm 1$ on $E_0$. Any isometry $T$ with $\mathrm{Ker}(T-1)$ or $\mathrm{Ker}(T+1)$ of finite codimension in $E$ admits such an orthogonal decomposition of $E$. The set $\mathfrak{J}$ of all such isometries $T$ is an invariant subgroup of the orthogonal group $\mathfrak{O}$ associated with the space $E$; it contains the subgroup $\mathfrak{J}_0$ of index $\leq 2$ of all $T$ which are the identity on almost all of $E$. For symmetric $\Phi$ and char $k \neq 2$ [2] gives a detailed account of $\mathfrak{J}_0$; in that case $\mathfrak{J}_0$ is generated by all symmetries about semisimple hyperplanes. We shall show that for prescribed natural $n > 1$ there are infinite spaces $(E, \Phi)$ with $\mathfrak{O}/\mathfrak{J}_0$ isomorphic to a product of $n$ copies of $\mathbb{Z}_2$ (characteristic not 2).

It is natural to expect, that spaces with few orthogonal splittings in the sense of (**) admit 'few' isometries. A confirmation of this expection is provided by the first two theorems.

THEOREM 1. *If $(E, \Phi)$ satisfies (\*), then every isometry on $E$ is determined modulo a factor from $\mathfrak{J}_0$ by its action on any subspace of denumerably infinite dimension.*

THEOREM 2. *If $(E, \Phi)$ satisfies (\*), and this absolutely so when the base field is not algebraically closed, then every locally algebraic isometry belongs to the group $\mathfrak{J}$ associated with $(E, \Phi)$.*

*Proof of Theorem 1.* Assume first that $E$ is semisimple. For $\lambda \neq 0$ an element of the basefield $k$ let $X(\lambda)$ be the eigenspace $\ker(T - \lambda 1)$ of the isometry $T$ of $E$. $X(\lambda) \perp X(\mu)$ if $\lambda \mu \neq 1$. Thus we cannot have $\dim X(\lambda) = \dim E$ unless $\lambda^2 = 1$ by (\*). $\mathrm{Im}(\lambda T - 1) \subset$ $\subset X(\lambda)^\perp$ and $\mathrm{Ker}(\lambda T - 1) = X(\lambda^{-1})$ so

$$\dim E/X(\lambda^{-1}) \leqslant \dim X(\lambda)^\perp. \tag{1}$$

Assume that for some subspace $U$ of $E$ we have $T|_U = 1_U$, $\dim U = \aleph_0$. Since $T$ preserves $\Phi$ we conclude that $\mathrm{Im}(T - 1)$ is contained in $U^\perp$ and thus of dimension smaller than $\dim E$. Hence we must have $\dim X(1) = \dim E$ and therefore $\dim X(1)^\perp < \infty$ by (\*). Hence $\dim E/X(1) < \infty$ by (1) and therefore $\dim X(1)^\perp \leqslant \dim E/X(1)$ as $E$ is semisimple. Together with (1) $\dim X(1)^\perp = \dim E/X(1) < \infty$. From this we conclude that there exists a subspace $H \subset X(1)$ of finite codimension in $E$ with $E = H \oplus H^\perp$. Since $T$ is the identity on $H$ we have $T \in \mathfrak{J}_0$. If $E$ is not semisimple, then $\mathrm{rad}\, E$ is of finite dimension. Let $E_0$ be a linear complement of $\mathrm{rad}\, E$ in $E$. We can find $T_0$ in $\mathfrak{J}_0$ such that $T_0 T(E_0)) \subset E_0$. Since radicals are mapped onto themselves under isometries we must have $T_0 T(E_0) = E_0$. By what we have already proved it follows that the restriction of $T_0 T$ to $E_0$ is determined modulo $\mathfrak{J}_0$ by its action on $U \cap E_0 \cap \mathrm{Ker}(T_0 - 1)$. Hence the same holds for $T$. Q.E.D.

*Proof of Theorem 2. Case 1:* there is a $\lambda$ with $\dim X(\lambda) = \dim E$. Hence $\lambda^2 = 1$ and $T \in \mathfrak{J}$ by Theorem 1.

*Case 2:* $\dim X(\lambda^{-1}) < \dim E$ for all $\lambda \in k \backslash \{0\}$. Thus $\dim X(\lambda)^\perp = \dim E$ by (1) and so $\dim X(\lambda) < \infty$ for all $\lambda \in k \backslash \{0\}$ by (\*). For every member $x$ of a Basis $\mathscr{B}$ of $E$ we let $f_x$ be the annihilating polynomial. $f_x$ splits into linear factors over the algebraic closure $k'$ of $k$, $f_x = \prod (Z - \lambda_i)$. Every linear factor provides an eigenvalue $\lambda_i \in k'$ of $T' : E'$ $= k' \otimes E \to E'$. Since $E'$ satisfies (\*) by the assumptions of the theorem we see that the number $l$ of different $\lambda_i$ must be less than $\dim E$. Hence there are only $l < \dim E$ different annihilating polynomials $f_x$ ($x \in \mathscr{B}$). We conclude that there is at least one $f_x$ annihilating a subspace $G \subset E$ of dimension $\dim G = \dim E$. Let $f_x = \prod (Z - \lambda_i)$ be the splitting of this very polynomial. If some of the $\lambda_i$ equal $\pm 1$ we let $G_0$ be the image of $G$ under the map $\prod_{\lambda_i = \pm 1} (T - \lambda_i 1)$. We have $\dim G_0 = \dim G$ in the present case. Let $g$ be the product of the remaining linear factors $(Z - \lambda)$. Since $\dim G_0 = \dim E$ and since $g(T)$ annihilates $G_0$ and hence also $G_0' = k' \otimes G_0$, we conclude that the dimension of $\ker(T - \lambda)$ must equal $\dim E$ for at least one $\lambda \neq \pm 1$. This is a contradiction as $G_0'$ satisfies property (\*).

COROLLARY. *If $(E, \Phi)$ is as in Theorem 2, then the set of all locally algebraic isometries on $E$ is a group. It coincides with the set of all algebraic isometries on $E$ and it is generated by all $T \in \mathfrak{O}$ with $E/\mathrm{Ker}(T-1)$ or $E/\mathrm{Ker}(T+1)$ finite dimensional; hence it is a normal subgroup of $\mathfrak{O}$.*

LEMMA. *Assume that $E_1, \ldots, E_n$ all satisfy (*) and that $\dim E_i > \dim E_{i+1}$ ($i = 1, \ldots, n-1$). If $T$ is any endomorphism of the orthogonal sum $E = E_1 \oplus \cdots \oplus E_n$ that preserves orthogonality then the $E_i$ are left almost invariant under $T$: $\dim(E_i + T(E_i))/E_i$ is finite for all $i$.*

*Proof.* Let $F_1 = E_2 \oplus \cdots \oplus E_n$. $\mathrm{Dim}\, E_1 > \dim F_1$ so that there is a subspace $V_1$ of $E_1$ with $T(V_1) \subset E_1$ and $\dim V_1 = \dim E_1$. By the assumptions of the lemma $\dim T(V_1) = \dim E_1$. Call $K_1$ the projection of $T(F_1)$ onto $E_1$ (for the decomposition $E = E_1 \oplus F_1$). $T(V_1) \perp K_1$ hence $K_1$ and $(F_1 + T(F_1))/F_1$ are finite dimensional. Setting $F_2 = E_3 \oplus \cdots \oplus E_n$ we have $\dim E_2 > \dim F_2$. As $F_1 + T(F_1))/F_1$ is finite dimensional we conclude that there exists $V_2 \subset E_2$ with $T(V_2) \subset E_2$ and $\dim V_2 = \dim E_2$. It is now clear how the argument may be repeated in order to conclude that there exist spaces $V_i \subset E_i$ with $T(V_i) \subset E_i$ and $\dim V_i = \dim E_i$. Let then $K_{ij}$ be the projection of $T(E_i)$ on $E_j$. $K_{ij} \perp V_j$ for all $i \neq j$. Since $\dim T(V_i) = \dim E_i$ by the choice of the $V_i$ and by the assumptions of the lemma, we conclude that $K_{ij}$ is finite dimensional for all pairs $i \neq j$. This is what the lemma asserts.

We now consider the orthogonal sum of finitely many spaces $(E_i, \Phi_i)$ of the kind constructed in §1. For the sake of simplicity we choose $\Phi_i$ symmetric: For $i = 1, 2, \ldots n$ let $(e_i^i)_{i \in J(i)}$ be a basis of $E_i$, $\Phi_i(e_i^i, e_v^i) = \xi_{iv}^i$ where $\xi_{iv}^i = \xi_{\mu\kappa}^i$ if and only if $\{i, v\} = \{\mu, \kappa\}$ and where, for every fixed $i$, the set $X^i$ of all $\xi_{iv}^i$ $\{i, v \in J(i)\}$ is algebraically independent over the prime field $k_0$ of the basefield $k$. We shall *not* assume that the sets $X^1, \ldots, X^n$ are disjoint. For these symmetric spaces we prove

THEOREM 3. *Assume that $\dim E_i > \dim E_{i+1} > \aleph_0$ ($i = 1, \ldots, n-1$). Then every isometry of the orthogonal sum $E = E_1 \oplus \cdots \oplus E_n$ is locally algebraic.*

*Proof.* For the sake of simplicity we omit the superscript 1 when mentioning $e_i^1$ and $x_{iv}^1$; furthermore let $J(1) = J$. Let us study the action of $T$ on $E_1$ for $T$ an isometry of $E$:

$$Te_i = \sum_J \alpha_{i\mu} e_\mu + g_i, \quad \text{where} \quad g_i \in E_2 \oplus \cdots \oplus E_n$$

By the previous lemma, $G = k(g_i)_{i \in J}$ is of finite dimension. Let $Q \subset J$ be such that $(g_i)_{i \in Q}$ is a basis of $G$. We introduce the finite sets $M(i) = \{\mu \in J \mid \alpha_{i\mu} \neq 0\}$. Let $M = \bigcup_{i \in J} [M(i) \backslash \{i\}]$. We show that $M$ is finite. Assume by way of contradiction that $M$ is infinite. There is a denumerably infinite subset $S \subset J$ and a map $\kappa$ that assigns to every $i \in S$ a $\kappa(i) \in J$ with $\kappa(i) \in M(i) \backslash \{i\}$ and $\kappa(i) \neq \kappa(v)$ for all $i \neq v$ in $S$. There is a

subset $A$ of $X^1 \cup \cdots \cup X^n$ with $\mathrm{card}\, A \leqslant \aleph_0$ such that $\alpha_{\iota\kappa} \in \overline{k_0(A)}$ (the algebraic closure of $k_0(A)$ in $k$) for all $\kappa \in M(\iota)$, $\iota \in S$ and $\Phi(g_\iota, g_\kappa) \in \overline{k_0(A)}$ for all $\iota \in S$, $\kappa \in Q$. Let $N = \bigcup_{\iota \in S} M(\iota)$. $\mathrm{card}\, N = \aleph_0$. There is a $\nu \in J$ and for it a $\mu_0 \in M_\nu$ such that $\mu_0 \notin N \cup S$ and

$$\{\xi_{\iota\mu_0} \in X \mid \iota \in N\} \cap A = \emptyset \tag{2}$$

For $Te_\nu = \sum_{\mu \in M(\nu)} \alpha_{\nu\mu} \cdot e_\mu + \sum_{\kappa \in Q} \beta_{\nu\kappa} g_\kappa$ we have

$$(Te_\iota, Te_\nu) = \xi_{\iota\nu} = \alpha_{\iota\kappa(\iota)}\alpha_{\nu\mu_0}\xi_{\kappa(\iota)\mu_0} + \sum_{\kappa\mu} \alpha_{\iota\kappa}\alpha_{\nu\mu}\xi_{\kappa\mu} + \sum_{\kappa \in Q} \beta_{\mu\kappa}\Phi(g_\iota, g_\kappa) \tag{3}$$

The first sum in (3) extends over the set $[M(\iota) \times M(\nu)]\backslash\{\kappa(\iota), \mu_0\}$. There is a finite subset $B$ of $X^1 \cup \cdots \cup X^n$ such that $\alpha_{\nu\mu} \in \overline{k_0(B)}$ for all $\mu \in M(\nu)$ and $\beta_{\nu\kappa} \in \overline{k_0(B)}$ for all $\kappa \in Q$. Since $S$ is infinite, there is a $\sigma \in S$ with $\xi_{\kappa(\sigma)\mu_0} \notin B$. As $\kappa(\sigma) \neq \sigma$ by the choice of the map $\kappa$ and since $\mu_0 \neq \sigma$ we have $\xi_{\sigma\nu} \neq \xi_{\kappa(\sigma)\mu_0}$. Let $C = A \cup B \cup \{\xi_{\sigma\nu}, \xi_{\kappa\mu} \mid (\kappa, \mu) \in [M(\sigma) \times M(\nu)]\backslash(\kappa(\sigma), \mu_0)\}$. By (2) we have $\xi_{\kappa(\sigma)\mu_0} \notin A$, hence $\xi_{\kappa(\sigma)\mu_0} \notin C$. All quantities in equation (3) equated for $\iota = \sigma$ are contained in $\overline{k_0(C)}$ with the exception of $\xi_{\kappa(\sigma)\mu_0}$. The coefficient of $\xi_{\kappa(\sigma)\mu_0}$ in (3) is not zero. Hence we should have $\xi_{\kappa(\sigma)\mu_0} \in \overline{k_0(C)}$; so $\xi_{\kappa(\sigma)\mu_0}$ is algebraically dependent over $C$ which is a contradiction. We have thus shown that $M$ is finite. $G$ being finite dimensional, there is a subspace $F_1$ of $E$, spanned by finitely many $e_\iota^i$, $\iota \in J(i)$, $i = 1, \ldots, n$ such that $Te_\nu^1 \in k(e_\nu^1) + F_1$ for all $\nu \in J(1)$. In the same manner we find for $i = 2, \ldots, n$ finite dimensional spaces $F_i$ such that $Te_\nu^i \in k(e_\nu^i) + F_i$. Set $F = \sum_{i=1}^n F_i$. We have $Te_\mu^i \in k(e_\mu^i) + F$ for all $\mu \in J(i)$ and all $i = 1, \ldots, n$. In particular $T(F) \subset F$. Since $F$ is finite dimensional we conclude that $T$ is locally algebraic on all basis vectors $e_\mu$ and hence locally algebraic on each $x \in E$.   Q.E.D.

Let us look at the proof for one more moment. We have shown that there is a subspace $F$ of $E$, spanned by finitely many of the basisvectors $e_i^\iota$ such that $Te \in k(e) + F$ for all basis vectors $e = e_i^\iota$. Hence $F$ is the orthogonal sum of its projections onto the summands $E_i$ in the decomposition $E = E_1 + \cdots + E_n$. These projections, say $G_i$, are semissimple (as are all spans of collections of basisvectors of our particular bases $(e_i^\iota)_{\iota \in J(i)}$, $(i = 1, \ldots, n)$). Therefore $E_i = G_i \oplus (G_i^\perp \cap E_i)$. Since $T(F) = F$ it follows that the spaces $G_i^\perp \cap E_i$ are left invariant under $T$. If we extend $T^{-1}\mid_F$ to an isometry $T_0$ on $E$ by letting $T_0$ act as the identity on $F^\perp$ we have $T_0 \in \mathfrak{J}_0(E)$ and $T_0 \circ T$ leaves each summand $E_i$ of $E$ invariant. The restriction of $T_0 \circ T$ to $E_i$ is locally algebraic. Hence if $\mathrm{char}\, k \neq 2$ then we see by Theorem 2 that these restrictions are, up to a factor $\pm 1$, a product of finitely many symmetries. We have thus shown that we can find altogether finitely many symmetries $S$ on $E$ such that $T_0 \circ T \circ \prod S$ acts on each $E_i$ as $1_{E_i}$ or $-1_{E_i}$. Since $T_0 \in \mathfrak{J}_0(E)$ we obtain the

**COROLLARY.** *Let* $E = E_1 \oplus \cdots \oplus E_n$ *be as in Theorem* 3 *and* $\mathrm{char}\, k \neq 2$. *The*

*quotient group $\mathfrak{O}/\mathfrak{J}_0$ of the full orthogonal group of $E$ modulo the invariant subgroup $\mathfrak{J}_0$ is isomorphic to the direct product of $n$ copies of $\mathbb{Z}_2$. In particular, if $n=1$, then $\mathfrak{O}/\mathfrak{J}$ is trivial.*

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