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Some Non-Linear Equivariant Sphere Bundles

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1. Introduction

Let $\varphi: G \to O_m$ be a real m-dimensional representation of a compact Lie group G. Assume that $\pi: T \to B$ is a smooth G bundle such that the action takes place on the fibres, and each fibre is equivariantly diffeomorphic to S^{m-1} where the action of G on S^{m-1} is given by the representation φ.

Is π smoothly equivalent to the sphere bundle of a G vector bundle with fibre representation φ?

If yes, π is called linear, otherwise it is called *non-linear*. If π is linear it bounds a smooth equivariant disk bundle with fibre action induced by φ. Topologically, of course, π is always the boundary of a disk bundle with fibre action induced by φ: The mapping cylinder of π serves as the total space of the required equivariant disk bundle.

For G the trivial group, examples of non-linear sphere bundles over spheres were found by S. P. Novikov [15] and P. Antonelli, D. Burghelea, P. J. Kahn [1]. Let G be one of the groups O_n, U_n, Sp_n, and let ϱ_n be the standard representation of G, of real dimension n, $2n$, or $4n$, respectively. It is not difficult to show that any sphere bundle with fibre representation ϱ_n is linear. We consider sphere bundles with fibre representation $\varrho_n \oplus \varrho_n$. We prove that for G the orthogonal group O_n, $n \geqslant 3$, any G sphere bundle with fibre representation $\varrho_n \oplus \varrho_n$ is linear (Corollary 4.4). On the other hand, for G the unitary or symplectic group of n dimensions, $n \geqslant 3$, we will construct many non-linear G sphere bundles with fibre representation $\varrho_n \oplus \varrho_n$ and base space a sphere (Theorem 4.5). It is not clear whether or not these sphere bundles are smoothly linear if one forgets the action of G.

The methods used in this work are quite different from those of [15; 1]. The total space of an equivariant sphere bundle with action induced by $\varrho_n \oplus \varrho_n (n \geqslant 3)$ on the fibres, is a G manifold with two orbit types and orbit space a manifold with boundary. The construction of our non-linear bundles relies on the classification of these G manifolds by W. C. Hsiang and W. Y. Hsiang [10] and K. Jänich [11].

Our results have some consequences, naturally, concerning the homotopy type of the topological group of all equivariant self-diffeomorphisms of the unit sphere in the representation space of $\varrho_n \oplus \varrho_n$, $n \geqslant 3$. In the orthogonal case, this group has the homotopy type of O_2 (Theorem 4.3), whereas in the unitary case it does not have the homotopy type of a finite CW complex (Theorem 4.8).

We finally deal with the problem of classifying equivariantly the total spaces of the non-linear bundles over spheres constructed here. It turns out that in most cases these total spaces are products of a homotopy sphere and the fibre (Theorem 5.2 and Proposition 5.4).

2. Λ_n manifolds ocer $\Sigma^k \times D^{d+1}$

As we simultaneously deal with orthogonal, unitary, and symplectic actions, the following notation will be convenient (cf. [7]). Λ_n is the orthogonal group O_n, the unitary group U_n, or the symplectic group Sp_n. ϱ_n is the corresponding standard representation of real dimension n, $2n$, or $4n$, respectively. Let $\pi: T \to B$ be a smooth Λ_n sphere bundle over B, with fibre action $\varrho_n \oplus \varrho_n$. The fibre is S^{2dn-1} where $d=1, 2,$ or 4 depending on the group acting. π factors through the orbit map $T \to T'$, and we have a commutative diagram:

$$\begin{array}{ccc} T & \longrightarrow & T' = T/\Lambda_n \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & B & \end{array}$$

S^{2dn-1} and T are Λ_n manifolds with orbit types (Λ_{n-1}) and (Λ_{n-2}), the slice representations corresponding to the orbit types are $\varrho_{n-1} \oplus$ trivial and trivial, respectively. The orbit space of S^{2dn-1} is D^{d+1}, hence $T' \to B$ is a D^{d+1} bundle. To find and distinguish bundles $\pi: T \to B$, it is therefore important to classify Λ_n manifolds with orbit space a D^{d+1} bundle over B such that over each fibre of this bundle we have S^{2dn-1} with action $\varrho_n \oplus \varrho_n$. We use [10] and [11] to do this for a special case.

THEOREM 2.1. *Let k be a positive integer; $k > 1$ if $\Lambda_n = O_n$ or Sp_n. Let Σ^k be a smooth manifold homeomorphic* to S^k. *For every $n \geqslant 3$, there is a* $1-1$ *correspondence between equivariant diffeomorphism classes of smooth Λ_n manifolds over $\Sigma^k \times D^{d+1}$ satisfying the conditions*

(i) *for each $p \in \Sigma^k$, the union of the orbits over $p \times D^{d+1}$ is equivariantly diffeomorphic to S^{2dn-1} with action induced by $\varrho_n \oplus \varrho_n$,*

(ii) *the principal orbit bundle is trivial, and elements of* $\operatorname{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$.

G_{d+1} is the H-space of degree one mappings of S^d onto itself, and $\pi_k SO_{d+1} \to \pi_k G_{d+1}$ is induced by inclusion. Lateron we will see that a Λ_n manifold corresponding to a non-zero element of $\operatorname{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$ is the total space of a non-linear Λ_n sphere bundle over Σ^k.

Proof of Theorem 2.1. Let T be a Λ_n manifold over $\Sigma^k \times D^{d+1}$ with the properties stated in the theorem. T is a so-called special Λ_n manifold [11], also [10], and is classified by an equivalence class of pairs (P, σ). Our notation follows [11; 12; 7]. P is the compactified principal bundle of the principal orbit bundle of T, i.e. $\Sigma^k \times D^{d+1} \times \Lambda_2 \to \Sigma^k \times D^{d+1}$ by (ii). σ is a reduction of the structure group Λ_2 of ∂P to the subgroup $\Lambda_1 \times \Lambda_1$ (cf. [7, 3.2]), i.e. a cross-section σ: $\Sigma^k \times S^d \to \Sigma^k \times S^d \times (\Lambda_2/\Lambda_1 \times \Lambda_1)$ of the bundle $\partial P/\Lambda_1 \times \Lambda_1$. As $\Lambda_2/\Lambda_1 \times \Lambda_1$ is diffeomorphic to S^d, σ is given by a map f: $\Sigma^k \times S^d \to S^d (= \Lambda_2/\Lambda_1 \times \Lambda_1)$. Because of condition (i), $f \mid p \times S^d$ has degree ± 1 [7, 3.2]. Thus f is a fibre homotopy trivialization of the trivial d-sphere bundle over Σ^k. By taking a suitable identification of $\Lambda_2/\Lambda_1 \times \Lambda_1$ with S^d, f becomes an oriented fibre homotopy trivialization. On the other hand, by Jänich's construction, any such fibre homotopy trivialization gives rise to a Λ_n manifold as in Theorem 2.1. Now f: $\Sigma^k \times S^d \to S^d$ is nothing but a map $\Sigma^k \to G_{d+1}$ which we also denote by f. It is the class represented by f in $\operatorname{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$ which corresponds to the equivariant diffeomorphism class of T. To prove the $1-1$ correspondence we have to analyze Jänich's equivalence relation of pairs (P, σ) in our particular case. Two pairs (P, σ) and (P', σ') are equivalent (i.e. the corresponding Λ_n manifolds equivariantly diffeomorphic over $\Sigma^k \times D^{d+1}$) if and only if there is a bundle isomorphism of P and P' carrying σ to σ' [11, 3.1]. If P and P' are identified with the trivial bundle $\Sigma^k \times D^{d+1} \times \Lambda_2 \to \Sigma^k \times D^{d+1}$, (P, σ) and (P, σ') are equivalent if and only if there is a bundle automorphism of the above trivial bundle carrying σ to σ'. Such a bundle automorphism is given by

$$H: \Sigma^k \times D^{d+1} \times \Lambda_2 \to \Sigma^k \times D^{d+1} \times \Lambda_2$$
$$H(x, y, z) = (x, y, z\eta(x, y))$$

where $\eta: \Sigma^k \times D^{d+1} \to \Lambda_2$. ($P$ is a right principal bundle.) Therefore equivalence of (P, σ) and (P, σ') means the existence of a commutative diagram

$$\begin{array}{ccc}
\Sigma^k \times S^d \times \Lambda_2 & \xrightarrow{H \mid \cdots} & \Sigma^k \times S^d \times \Lambda_2 \\
\downarrow & & \downarrow \\
\Sigma^k \times S^d \times S^d & \xrightarrow{h} & \Sigma^k \times S^d \times S^d \\
& {}_{\sigma}\nwarrow \quad \nearrow_{\sigma'} & \\
& \Sigma^k \times S^d &
\end{array} \qquad (*)$$

where H is defined by $\eta: \Sigma^k \times D^{d+1} \to \Lambda_2$ as above and h is induced by H via the identification of $\Lambda_2/\Lambda_1 \times \Lambda_1$ with S^d. We shall need two facts: If σ and σ' are homotopic reductions, then (P, σ) and (P, σ') are equivalent [9, p. 23]. $\Lambda_2/\Lambda_1 \times \Lambda_1$ can be identified with S^d in such a way that the action of Λ_2 on $\Lambda_2/\Lambda_1 \times \Lambda_1$ corresponds to the orthogonal action of Λ_2 on S^d via a homomorphism $\tau: \Lambda_2 \to O_{d+1}$ with kernel the center of Λ_2. This is well-known (e.g. [2]).

Now suppose (P, σ) is equivalent to (P, σ'), the equivalence given by $\eta: \Sigma^k \times D^{d+1} \to \Lambda_2$. Let σ (σ') be given by a map f (f'): $\Sigma^k \to G_{d+1}$. Change η by a homotopy such that it is constant on all disks $p \times D^{d+1}$, $p \in \Sigma^k$. This changes σ' by a homotopy, using the diagram (*), and changes neither the equivalence class of (P, σ') nor the homotopy class of f'. So we may assume that we have a diagram (*) with η a map from Σ^k to Λ_2. $\sigma' = h\sigma$, h being defined by $\eta: \Sigma^k \to \Lambda_2$. If $\bar{\eta}: \Sigma^k \to O_{d+1}$ is the composition of η with the above homomorphism $\tau: \Lambda_2 \to O_{d+1}$, this means $f' = \bar{\eta} \cdot f$, or $[f'] - [f] = [\bar{\eta}]$. Thus $[f]$, $[f'] \in \pi_k G_{d+1}$ differ by an element in the image of $\pi_k SO_{d+1}$.

Conversely, given $f, f': \Sigma^k \to G_{d+1}$, defining reductions σ, σ', assume there is $\bar{\eta}: \Sigma^k \to SO_{d+1}$ such that $[f'] - [f] = [\bar{\eta}]$ in $\pi_k G_{d+1}$. $\bar{\eta}$ can be lifted to $\eta: \Sigma^k \to \Lambda_2$. (Here, if $\Lambda_n = O_n$ or Sp_n, $k > 1$ is used; see Remark 1 below.) This shows that the reduction defined by $\bar{\eta} \cdot f$ can be obtained from σ by an automorphism of P (namely the one defined by η). As $\bar{\eta} \cdot f$ and f' are homotopic, (P, σ) and (P, σ') are equivalent, and the proof is complete.

Remark 1. For $k = 1$, the proof shows what has to be modified if $\Lambda_n = O_n$ or Sp_n. In the symplectic case, one gets a $1-1$ correspondence to the elements of $\pi_1 G_5$. In the orthogonal case, one gets a $1-1$ correspondence with the elements of $\operatorname{cok}(\pi_1 SO_2 \to \pi_1 G_2)$ where $\pi_1 SO_2 \to \pi_1 G_2$ is obtained by composing the double covering $SO_2 \to SO_2$ with the inclusion $SO_2 \subset G_2$.

Remark 2. The zero element of $\operatorname{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$ clearly corresponds to the 'trivial' Λ_n manifold $\Sigma^k \times S^{2dn-1}$ over $\Sigma^k \times D^{d+1}$.

Remark 3. The inclusion $SO_2 \subset G_2$ is a homotopy equivalence, so $\operatorname{cok}(\pi_k SO_2 \to \pi_k G_2) = 0$. $\pi_k SO_3 \to \pi_k G_3$ is a monomorphism [14], so $\operatorname{cok}(\pi_k SO_3 \to \pi_k G_3) \cong \pi_k(G_3, SO_3)$ which is isomorphic to $\pi_{k+2} S^2$ for $k \geqslant 3$.

3. The Orbit Space as a Bundle

It was proved in [7, 2.3] that the linear automorphisms of S^{2dn-1} compatible with the representation $\varrho_n \oplus \varrho_n$, form a group isomorphic to Λ_2 $(n \geqslant 3)$. The action of this group on the orbit space $S^{2dn-1}/\Lambda_n \cong D^{d+1}$ is what one would expect:

PROPOSITION 3.1. *The action of the group Λ_2 of equivariant linear automorphisms of S^{2dn-1} $(n \geqslant 3)$ induced on the orbit space $S^{2dn-1}/\Lambda_n \cong D^{d+1}$ is equivalent to the orthogonal action of Λ_2 on D^{d+1} given by a homomorphism $\tau: \Lambda_2 \to O_{d+1}$ with ker $\tau =$* center (Λ_2).

Proof. Let F be the real, complex, or quaternionic field, depending on whether the orthogonal, the unitary, or the symplectic group acts. Recall from [7, 2.4] how Λ_n and Λ_2 act on S^{2dn-1}. Write elements of S^{2dn-1} as n by 2 matrices over F. Then Λ_n acts by left multiplication, and Λ_2 acts by right multiplication. To prove Proposition 3.1 we may confine ourselves to the orbits of Λ_n over $B^{d+1} = \operatorname{int} D^{d+1}$ (i.e. principal orbits). An n by 2 matrix is on a principal orbit if and only if the two columns are linearly independent. If $\alpha=\sqrt{\frac{1}{2}}$, then the Λ_n orbit of the point

$$q=\begin{bmatrix}\alpha & 0\\ 0 & \alpha\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}$$

is a fixed point of the action of Λ_2. We are going to determine the orbit type of non-fixed points of the Λ_2 action on B^{d+1}. We first need nice representatives of the points in the orbit space B^{d+1}/Λ_2.

Clearly, any point of S^{2dn-1} over B^{d+1} is on a Λ_n orbit of a point of the form

$$\begin{bmatrix}r & t\\ 0 & s\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}, \tag{**}$$

$r\in\mathbf{R}$, $r>0$, $s\neq 0$. Applying a suitable element of Λ_2 (i.e. without changing the Λ_2 orbit) makes t real non-negative. Then we make s real positive by applying an appropriate element of Λ_n. So far we have shown that any point in the orbit space B^{d+1}/Λ_2 has a representative of the form (**) with $r, s, t\in\mathbf{R}$, $r>0$, $s>0$, $t\geqslant 0$. The following lemma guarantees that we may even assume $t=0$.

LEMMA 3.2. *Given real numbers* $r\neq 0$, $s\neq 0$, t, *there are orthogonal* 2 *by* 2 *matrices* M, N *such that*

$$M\begin{bmatrix}r & t\\ 0 & s\end{bmatrix}N$$

is a diagonal matrix.

(Lemma 3.2 is proved below.)

Assume $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is in the isotropy group of a point of B^{d+1} represented by

$$\begin{bmatrix}r & 0\\ 0 & s\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}. \quad \text{Then} \quad \begin{bmatrix}r & 0\\ 0 & s\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}ra & rb\\ sc & sd\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0\end{bmatrix}$$

is on the same Λ_n orbit as

$$\begin{bmatrix} r & 0 \\ 0 & s \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Therefore $r^2=r^2|a|^2+s^2|c|^2$ and $s^2=r^2|b|^2+s^2|d|^2$, so $r^2(1-|a|^2)=r^2|c|^2=s^2|c|^2$ and $s^2(1-|d|^2)=s^2|b|^2=r^2|b|^2$, i.e. $r=s$ or $c=0$ and $b=0$. For $r=s$ we have the fixed point q, otherwise the isotropy group is $\Lambda_1\times\Lambda_1$. Thus the positive dimensional orbits of Λ_2 on B^{d+1} are spheres $\Lambda_2/\Lambda_1\times\Lambda_1$ of dimension d, the orbit space B^{d+1}/Λ_2 is the half open interval $(0, \alpha]$, parametrized by r. Hence B^{d+1} is equivalent, as a Λ_2 space, to the representation space of τ composed with the standard orthogonal representation of O_2, SO_3, or SO_5, respectively.

Proof of Lemma 3.2. Left (right) multiplication by an orthogonal matrix does not change the inner product of the columns (rows) of a real 2 by 2 matrix. As the orthogonal group operates transitively on spheres, it is sufficient to find an orthogonal matrix $\begin{bmatrix} u & u' \\ -u' & u \end{bmatrix}$, $u'=\sqrt{1-u^2}$, such that the columns of

$$\begin{bmatrix} r & t \\ 0 & s \end{bmatrix} \begin{bmatrix} u & u' \\ -u' & u \end{bmatrix}$$

have inner product zero. This leads to an equation

$$u^4-u^2+\frac{r^2t^2}{(r^2-t^2-s^2)^2+4r^2t^2}=0$$

which does have a solution u in the unit interval.

COROLLARY 3.3. *If an equivariant linear S^{2dn-1} bundle $\pi: T\to B$ is defined by transition functions $t_j: X_j\to\Lambda_2$, then the orbit space T' is the total space of a D^{d+1} bundle over B with transition functions $\tau\circ t_j$.*

4. The Homotopy Type of the Equivariant Diffeomorphism Group of the Fibre

The following two (well-known) lemmas are used in the proof of the next theorem.

LEMMA 4.1. *The group of diffeomorphisms of D^2 is homotopy equivalent to O_2.*
Proof. The group of diffeomorphisms of S^1 is homotopy equivalent to O_2. This is

elementary. So the group of all diffeomorphisms of D^2 is homotopy equivalent to the group of all diffeomorphisms of D^2 being orthogonal on the boundary. The latter is homotopy equivalent to the product of O_2 and the group of all diffeomorphisms of D^2 leaving S^1 fixed. But the second factor is contractible [6, p. 132].

Lemma 4.2. *Let G be the group of equivariant diffeomorphisms of a manifold M with respect to some fixed smooth action of a Lie group on M. Then G has the homotopy type of a countable CW complex.*

Proof. If the action is trivial, the Lemma is obtained by combining [5, p. 277, 283] and [16, Theorem 14]. In [5], the diffeomorphisms close to the identity are identified with certain cross-sections of the tangent bundle of M. This gives the local structure of a locally convex topological vector space. Therefore it is sufficient to observe that the equivariant diffeomorphisms correspond to equivariant cross-sections, which form a linear subspace.

DEFINITION. The diffeomorphisms of S^{2dn-1} onto itself which are equivariant with respect to the diagonal action $\varrho_n \oplus \varrho_n$ of Λ_n, endowed with the C^∞ topology, form a topological group. We denote this group by $\mathrm{Diff}(\Lambda_n, S^{2dn-1})$, or briefly $D_n(\Lambda)$.

The group of all linear equivariant diffeomorphisms of S^{2dn-1} is a subgroup of $D_n(\Lambda)$ which is isomorphic to Λ_2 [7, 2.3].

THEOREM 4.3. *For $n \geqslant 3$, the inclusion $j: O_2 \subset D_n(O)$ is a homotopy equivalence.*

Proof. Every equivariant self-diffeomorphism of S^{2n-1} is homotopic to a linear one [7, 6.1]. So j induces an isomorphism for π_0. To prove that j induces isomorphisms for π_k, $k>0$, we use that the equivalence classes of bundles over S^{k+1} with structure group G are classified by $\pi_k G$ modulo the action of $\pi_0 G$ [19, 18.5]. Let $\pi: T \to S^{k+1}$ be an equivariant S^{2n-1} bundle (with structure group $D_n(O)$). The orbit space T' is a D^2 bundle over S^{k+1}, with structure group O_2 (Lemma 4.1).

Assume $k>1$. Then $T' \to S^{k+1}$ is a trivial bundle, so T is an O_n manifold over $S^{k+1} \times D^2$. The principal orbit bundle of T is a bundle over $S^{k+1} \times B^2$ with structure group O_2, so is also trivial. Thus by Theorem 2.1, the O_n manifold T corresponds to an element of cok $(\pi_{k+1} SO_2 \to \pi_{k+1} G_2)$. As this cokernel is zero, T is the 'trivial' O_n manifold over $S^{k+1} \times D^2$, i.e. $S^{k+1} \times S^{2n-1}$. Therefore every equivariant S^{2n-1} bundle over S^{k+1} is trivial, which means $\pi_k D(O) = 0 = \pi_k O_2$ $(k>1)$.

The case $k=1$ is slightly more complicated. The principal orbit bundle of T is a bundle over $\mathrm{int}(T') \simeq S^2$. If $\pi: T \to S^2$ is given by an element $t \in \pi_1 D_n(O)$, the principal orbit bundle of T is given by some element $t_0 \in \pi_1 O_2$ such that $(j_* t_0)^{-1} t \in \pi_1 D_n(O)$ defines an equivariant S^{2n-1} bundle over S^2 with trivial principal orbit bundle. So we may assume that T already has trivial principal orbit bundle. If T' is a non-trivial D^2 bundle over S^2, $\partial T'$ is a lens space $L(q)$ $(q \geqslant 1)$. The principal bundle of the principal orbit bundle of T is int $T' \times O_2 \to \text{int } T'$, the reduction of the structure group to $O_1 \times O_1$ according to Jänich's classification is a cross-section of the bundle $\partial T' \times \times O_2/O_1 \times O_1 \to \partial T'$, which is of degree ± 1 on any fibre of $\partial T' \to S^2$. So it is given by a map

$$\begin{array}{ccc} \partial T' & \to & O_2/O_1 \times O_1 \\ \| & & \| \\ L(q) & \to & S^1 \end{array}$$

which is of degree ± 1 on any fibre of $L(q) \to S^1$. As every map $L(q) \to S^1$ is null homotopic, this is impossible. Therefore the bundle $T' \to S^2$ is trivial. Now we can apply Theorem 2.1. As $\text{cok}(\pi_2 SO_2 \to \pi_2 G_2) = 0$, π is equivalent to the trivial bundle $S^2 \times S^{2n-1}$ over S^2. This proves that j induces a surjective map $\pi_1 O_2/\pi_0 O_2 \to \pi_1 D_n(O)/\pi_0 D_n(O)$. As the total spaces of two different linear equivariant S^{2n-1} bundles over S^2 have different principal orbit bundles, $\pi_1 O_2/\pi_0 O_2 \to \pi_1 D_n(O)/\pi_0 D_n(O)$ is injective. Then $j_* : \pi_1 O_2 \to \pi_1 D_n(O)$ is an isomorphism because $\pi_0 O_2 = \pi_0 D_n(O) = \mathbf{Z}_2$.

So far we have shown that j is a weak homotopy equivalence. But O_2 and $D_n(O)$ have the homotopy type of CW complexes (Lemma 4.2). Hence j is a homotopy equivalence [18, p. 405].

COROLLARY 4.4. *Any O_n equivariant S^{2n-1} bundle with fibre action $\varrho_n \oplus \varrho_n$, $n \geqslant 3$, is a linear bundle.*

THEOREM 4.5. *Let Λ_n be the group U_n or Sp_n, $n \geqslant 3$. Let $k \geqslant 3$. If T is a Λ_n manifold over $S^k \times D^{d+1}$ corresponding to a non-zero element of* $\text{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$ *in the classification of Theorem 2.1, then $\pi : T \to S^k$ is a non-linear Λ_n equivariant S^{2dn-1} bundle with fibre action $\varrho_n \oplus \varrho_n$.*

$\pi : T \to S^k$ is of course the composition of the orbit map with the projection on the first factor. Note that in the orthogonal case, the above cokernel is always zero.

Proof. If $\pi : T \to S^k$ is a linear bundle, it is equivariantly trivial. This follows from Corollary 3.3 and the isomorphism $\tau_* : \pi_{k-1} \Lambda_2 \to \pi_{k-1} SO_{d+1}$. But then, by Remark 2 of section 2, T corresponds to zero in $\text{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$. So we only have to make sure that $\pi : T \to S^k$ is a bundle, i.e. locally trivial. If $B = S^k$-point, $\pi^{-1} B$ is a Λ_n manifold over $B \times D^{d+1}$. The reduction of a structure group occuring in the classification by the Hsiangs and Jänich, is a map $B \to G_{d+1}$, so is homotopic to a constant map. As homotopic reductions yield equivariantly equivalent Λ_n manifolds [9, p. 23], $\pi^{-1} B$ is equivariantly diffeomorphic over B to $B \times S^{2dn-1}$.

COROLLARY 4.6. *If* $\text{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1}) \neq 0$ *for some* $k \geqslant 3$, *then* $\text{cok}(\pi_{k-1} \Lambda_2 \to \pi_{k-1} D^n(\Lambda)) \neq 0$ *for every* $n \geqslant 3$.

Proof. By Theorem 4.5, there is a bundle over S^k with structure group $D_n(\Lambda)$ which is non-linear, i.e. the structure group of which cannot be reduced to Λ_2. So the corresponding element of $\pi_{k-1}D_n(\Lambda)$ is not in the image of $\pi_{k-1}\Lambda_2$.

COROLLARY 4.7. *Neither of the inclusions $U_2 \subset D_n(U)$, $Sp_2 \subset D_n(Sp)$ is a homotopy equivalence.*

Proof. This follows from $\operatorname{cok}(\pi_3 SO_3 \to \pi_3 G_3) \cong \mathbf{Z}_2$ and $\operatorname{cok}(\pi_6 SO_5 \to \pi_6 G_5) \cong \mathbf{Z}_2$.

THEOREM 4.8. *For any $n \geqslant 3$, $D_n(U) = \operatorname{Diff}(U_n, S^{4n-1})$, the group of all self-diffeomorphisms of S^{4n-1} which are equivariant with respect to the action $\varrho_n \oplus \varrho_n$ of U_n, does not have the homotopy type of a finite CW complex.*

Proof. As $\pi_3(G_3, SO_3) \cong \operatorname{cok}(\pi_3 SO_3 \to \pi_3 G_3) \cong \mathbf{Z}_2$, $\pi_2 D_n(U)$ is non-zero. But according to [3, Theorem 6.11], a topological group of the homotopy type of a finite CW complex, has zero 2-dimensional homotopy group.

Remark. We do not know whether or not $D_n(Sp)$ has the homotopy type of a finite CW complex. The above method does not work in the symplectic case since $\operatorname{cok}(\pi_3 SO_5 \to \pi_3 G_5) = 0$.

5. Classifying the Total Spaces

In view of the exact homotopy sequence

$$\cdots \to \pi_k SO_{d+1} \to \pi_k G_{d+1} \to \pi_k(G_{d+1}, SO_{d+1}) \xrightarrow{\partial} \pi_{k-1} SO_{d+1} \to \cdots,$$

$\operatorname{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$ is isomorphic to $\ker \partial \subset \pi_k(G_{d+1}, SO_{d+1})$. This kernel can be calculated to be non-zero in many cases, giving many examples of non-linear bundles by Theorem 4.5. It turns out, however, that the total spaces of these bundles in most cases are equivariantly diffeomorphic to a product of a homotopy sphere and S^{2dn-1}. Before going into this question, we prove a rather technical lemma.

LEMMA 5.1. *Let Σ^k be a homotopy k-sphere, $k \geqslant 5$, $F: S^k \times D^{d+1} \to \Sigma^k \times D^{d+1}$ a diffeomorphism. Then F is strongly diffeotopic to a diffeomorphism G such that $G \mid \Delta \times D^{d+1}: \Delta \times D^{d+1} \to \Delta' \times D^{d+1}$ has the form $G(x, y) = (g(x), y)$, where Δ, Δ' are k-disks in S^k, Σ^k, respectively, and $g: \Delta \to \Delta'$ is a diffeomorphism.*

Proof. If $p \in S^k$, $p' \in \Sigma^k$, then the map $F': p \times (D^{d+1}, S^d) \to \Sigma^k \times (D^{d+1}, S^d)$, defined by restricting F, is homotopic to a map $F'': p \times (D^{d+1}, S^d) \to \Sigma^k \times (D^{d+1}, S^d)$ such that $\operatorname{im} F'' = p' \times (D^{d+1}, S^d)$ and $\pi_2 \circ F'' = id$. This homotopy may be assumed to be composed by two homotopies, the first one moving a neighborhood of the boundary close to the boundary and leaving the complement of a neighborhood of the boundary fixed, the second one moving only the complement of a neighborhood of the boundary in the complement of a neighborhood of the boundary.

As $F'' \mid p \times S^d : p \times S^d \to \Sigma^k \times S^d$ is $(k-1)$-connected, we can replace the first homotopy by a strong diffeotopy of $\Sigma^k \times D^{d+1}$ which is the identity outside a neighborhood of the boundary. This is done using [8, p. 47] and the product structure of small neighborhoods of the boundary. To replace the second homotopy by a strong diffeotopy leaving a neighborhood of the boundary fixed, one has to extend Haefliger's existence theorem for diffeotopies [8, p. 47] to relative homotopies not affecting a neighborhood of the boundaries. Using the composition of the two diffeotopies, we have realized the homotopy between F' and F'' by a strong diffeotopy of $\Sigma^k \times D^{d+1}$. Now if Δ, Δ' are k-disks, $p \in \Delta \subset S^k$, $p' \in \Delta' \subset \Sigma^k$, $g: \Delta \to \Delta'$ a diffeomorphism such that $g(p) = p'$, then $F'' \mid \Delta \times D^{d+1}$ and $g \times id: \Delta \times D^{d+1} \to \Delta' \times D^{d+1}$ are tubular maps for $p' \times D^{d+1}$ in $\Sigma^k \times D^{d+1}$. (To be precise, we can give Δ and Δ' linear structures such that p is the origin in Δ and g is a linear isomorphism.) As D^{d+1} is contractible, there is another strong diffeotopy of $\Sigma^k \times D^{d+1}$ carrying $F \mid \Delta \times D^{d+1}$ to $g \times id$. Combining all the diffeotopies yields G.

Levine [13] constructed a homomorphism $\omega_3 : \Theta^{k+d+1,k} \to \pi_k(G_{d+1}, SO_{d+1})$. ($\Theta^{m,k}$ is the group of k-dimensional knots which are homotopy spheres in S^m, $k \geqslant 5$.) $\omega_3(\varkappa)$ is the obstruction for a knot $\varkappa$ to bound a framed manifold in S^{k+d+1}. $\omega_3(\varkappa) \in \ker \partial$ if and only if $\varkappa$ has trivial normal bundle.

THEOREM 5.2. *Let Σ^k be a homotopy k-sphere, $k \geqslant 5$. Let T be a U_n or Sp_n manifold over $S^k \times D^{d+1}$, corresponding to an element $x \in \ker \partial \subset \pi_k(G_{d+1}, SO_{d+1})$. Then T is equivariantly diffeomorphic to $\Sigma^k \times S^{2dn-1}$ if and only if there is a knot $\varkappa$ diffeomorphic to Σ^k with $\omega_3(\varkappa) = -x$.*

We first prove the following auxiliary

PROPOSITION 5.3. *Let $\varkappa$ be a knot diffeomorphic to Σ^k, of codimension $d+1$, with trivial normal bundle, and T the U_n or Sp_n manifold over $\Sigma^k \times D^{d+1}$ corresponding to $\omega_3(\varkappa) \in \ker \partial \subset \pi_k(G_{d+1}, SO_{d+1})$. Then T is equivariantly diffeomorphic to $S^k \times S^{2dn-1}$.*

Proof. Recall how $\omega_3(\varkappa)$ is defined if $\varkappa$ has trivial normal bundle [13, 3.1]. Let $h: \Sigma^k \times D^{d+1} \to X$ be a tubular map for $\varkappa$. $S^{k+d+1} - \operatorname{int} X$ is diffeomorphic to $D^{k+1} \times S^d$ by a diffeomorphism $g: D^{k+1} \times S^d \to S^{k+d+1} - \operatorname{int} X$ such that $g \mid S^k \times S^d : S^k \times S^d \to \partial X$ extends to a diffeomorphism $h_0 : S^k \times D^{d+1} \to X$ [17, Theorem 4.1]. If π_2 is the projection on the second factor, then $\pi_2 g^{-1} h : \Sigma^k \times S^d \to S^d$ defines an element of $\pi_k G_{d+1}$ whose image in $\pi_k(G_{d+1}, SO_{d+1})$ is $\omega_3(\varkappa)$.

By the diffeomorphism $h^{-1} h_0 : S^k \times D^{d+1} \to \Sigma^k \times D^{d+1}$, T can be lifted to a Λ_n manifold T' over $S^k \times D^{d+1}$, which can be detected by an element $y \in \pi_k(G_{d+1}, SO_{d+1})$ according to the classification in Theorem 2.1. As T' was obtained by lifting from T, y is represented by the composition $(\pi_2 g^{-1} h) \circ (h^{-1} h_0) = \pi_2$. So T' is equivariantly diffeomorphic to $S^k \times S^{2dn-1}$. But T is equivariantly diffeomorphic to T'.

Proof of Theorem 5.2. First assume the existence of κ diffeomorphic to Σ^k such that $\omega_3(\varkappa) = -x$. By Proposition 5.3, the Λ_n manifold T' over $(-\Sigma^k)\times D^{d+1}$ corresponding to $x\in\pi_k(G_{d+1}, SO_{d+1})$, is equivariantly diffeomorphic to $S^k\times S^{2dn-1}$. Hence there is a commutative diagram

$$\begin{array}{ccc} S^k \times S^{2dn-1} & \xrightarrow{E} & T' \\ \downarrow & & \downarrow \\ S^k \times D^{d+1} & \xrightarrow{F} & (-\Sigma^k) \times D^{d+1} \end{array}$$

where the equivariant diffeomorphism E induces a diffeomorphism F of the orbit spaces. Let $S^k = D^k_+ \bigcup_{id} D^k_-$, $D^k_\pm$ k-disks, matching the boundaries by the identity of S^{k-1}, $-\Sigma^k = D^k_+ \bigcup_s D^k$ matching the boundaries by an autodiffeomorphism s of S^{k-1}. Applying Lemma 5.1, F may be assumed to map $D^k_+\times D^{d+1}$ onto $D^k_+\times D^{d+1}$ by the identity. This means that the fibre homotopy trivialization $(-\Sigma^k)\times S^d \to S^d$ representing x (and defining the Λ_n manifold T') is just the second projection when restricted to $D^k_+\times S^d$. Now we cut our Λ_n manifolds $S^k\times S^{2dn-1}$ and T' in two pieces, according to the decomposition of S^k and $-\Sigma^k$ in two hemispheres. The two pieces are glued together after inserting a twist defined by the map s^{-1} on S^{k-1}. This defines a diagram

$$\begin{array}{ccc} \Sigma^k \times S^{2dn-1} & \xrightarrow{E'} & T'' \\ \downarrow & & \downarrow \\ \Sigma^k \times D^{d+1} & \xrightarrow{F'} & S^k \times D^{d+1} \end{array}$$

where E' is again an equivariant diffeomorphism. As the fibre homotopy trivializations that define the new Λ_n manifolds over $\Sigma^k\times D^{d+1}$ and $S^k\times D^{d+1}$ still are equal to the second projection when restricted to $D^k_+\times S^d$, we did not change the corresponding elements in $\pi_k(G_{d+1}, SO_{d+1})$. So we really have the product $\Sigma^k\times S^{2dn-1}$ on the left hand side (corresponding to $0\in\pi_k(G_{d+1}, SO_{d+1})$), and a Λ_n manifold corresponding to x on the right hand side (i. e. T). Therefore T is equivariantly diffeomorphic to T'', which is equivariantly diffeomorphic to $\Sigma^k\times S^{2dn-1}$ by E'.

Conversely, let T be equivariantly diffeomorphic to $\Sigma^k\times S^{2dn-1}$. As before, in the diagram

$$\begin{array}{ccc} T & \xrightarrow{E} & \Sigma^k \times S^{2dn-1} \\ \downarrow & & \downarrow \\ S^k \times D^{d+1} & \xrightarrow{F} & \Sigma^k \times D^{d+1} \end{array}$$

we may assume that $F \mid D^k_+\times D^{d+1}$ is the identity (with respect to a decomposition $\Sigma^k = D^k_+ \bigcup_t D^k_-$). Inserting an appropriate twist as above, we obtain a diagram

$$\begin{array}{ccc} T' & \xrightarrow{E'} & S^k \times S^{2dn-1} \\ \downarrow & & \downarrow \\ (-\Sigma^k) \times D^{d+1} & \xrightarrow{F'} & S^k \times D^{d+1} \end{array}$$

where T' is still a Λ_n manifold corresponding to $x \in \pi_k(G_{d+1}, SO_{d+1})$. Because of the above diagram, x is representable by $\pi_2 \circ F' \mid (-\Sigma^k) \times S^d$. On the other hand, according to our remark at the beginning of this proof, for the knot $-\varkappa$ given by $F'((-\Sigma^k) \times \times 0) \subset S^k \times D^{d+1} \subset S^{k+d+1}$, $\omega_3(-\varkappa)$ is also represented by $\pi_2 \circ F' \mid (-\Sigma^k) \times S^d$. Thus $\omega_3(\varkappa) = -x$. This completes the proof of Theorem 5.2.

By Theorem 5.2, the problem of deciding whether the total spaces of the bundles constructed in Theorem 4.5 are equivariantly diffeomorphic to $\Sigma^k \times S^{2dn-1}$, is largely reduced to homotopy theory. As $\operatorname{cok}(\pi_k SO_5 \to \pi_k G_5) = 0$ for $k = 3,4$, there are no such non-linear symplectic bundles in these dimensions. We do not know whether our non-linear unitary bundles have familiar total spaces for $k = 3,4$. Now assume $k \geqslant 5$. We have Levine's exact sequence [13]

$$\Theta^{k+d+1,k} \overset{\omega_3}{\to} \pi_k(G_{d+1}, SO_{d+1}) \to P_k \to \Theta^{k+d,k-1}.$$

According to Theorem 5.2, the total spaces of all bundles constructed in Theorem 4.5 are equivariantly diffeomorphic to some $\Sigma^k \times S^{2dn-1}$ if and only if $\ker(\partial : \pi_k(G_{d+1}, SO_{d+1}) \to \pi_{k-1} SO_{d+1}) \cong \operatorname{cok}(\pi_k SO_{d+1} \to \pi_k G_{d+1})$ is contained in $\operatorname{im} \omega_3$. As $\pi_k(G_{d+1}, SO_{d+1})$ is finite for all $k \geqslant 5$, $d = 2, 4$, ω_3 is certainly surjective unless $k \equiv 2 \bmod 4$. In the latter case, $P_k = \mathbf{Z}_2$, and ω_3 is an epimorphism if and only if a codimension 2 knot in S^{k+1} with Arf invariant 1 remains non-trivial after $(d-1)$-fold suspension. (Exactly then $P_k \to \Theta^{k+d,k-1}$ is injective.) As the Kervaire sphere is not diffeomorphic to the standard sphere in dimensions different from $2^r - 3$ [4, Corollary 2], ω_3 is surjective for all $k \neq 2^r - 2$. For $k = 6$, 14, using [20], ω_3 can be computed to be surjective in the unitary case $(d = 2)$. For $k = 6$, $d = 4$ (symplectic action), $\Theta^{k+d,k-1}$ is zero for dimensional reasons [13]. As $\ker \partial = \pi_6(G_5, SO_5) = \mathbf{Z}_2$, ω_3 is not surjective in this case, and we have spotted a non-linear symplectic S^{8n-1} bundle over S^6 whose total space is not equivariantly diffeomorphic to $S^6 \times S^{8n-1}$. We summarize:

PROPOSITION 5.4. *If $k \geqslant 5$, $k \neq 2^r - 2$, then the total spaces of the nonlinear equivariant S^{2dn-1} bundles over S^k, constructed in Theorem 4.5, are equivariantly diffeomorphic to a product of a homotopy k-sphere with trivial action and S^{2dn-1}. This is also true for $k = 6$, 14 in the unitary case. For $k = 6$, there is a non-linear symplectic S^{8n-1} bundle over S^6 whose total space is not equivariantly diffeomorphic to a product of a homotopy sphere with trivial action and S^{8n-1}.*

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Added in proof: G. Bredon has proved that Levine's homomorphism $P_k \to \theta^{k+2,k-1}$ is injective for all $k \equiv 2 \bmod 4$ (Classification of regular actions of classical groups with three orbit types, preprint, Cor. 8.2). Thus in Proposition 5.4, we may drop the hypothesis "$k \neq 2^r - 2$" in the unitary case.

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