Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 48 (1973)

Artikel: Corners and Arithmetic Groups
Autor: Borel, A. / Serre, J-P.
DOI: https://doi.org/10.5169/seals-37166

Corners and Arithmetic Groups

by A. BOREL and J-P. SERRE

TABLE OF CONTENTS

Introduction

Let Γ be a torsion-free arithmetic subgroup of a semi-simple **Q**-group G. It operates properly and freely on the symmetric space X of maximal compact subgroups of the group $G(\mathbf{R})$ of real points of G and the quotient X/Γ is a manifold. If the **Q**-rank $r_{\mathbf{Q}}(G)$ of G is not zero, X/Γ is not compact and the main purpose of this paper is to provide a suitable compactification $\bar{X}/\Gamma$ for it. Topologically $\bar{X}/\Gamma$ is a compact manifold with interior X/Γ, whose boundary has the homotopy type of the quotient by Γ of the Tits building of parabolic **Q**-subgroups of G. However, from the differential-geometric point of view, $\bar{X}/\Gamma$ comes naturally equipped with a structure of (real analytic) manifold "with corners" (with boundary if $r_{\mathbf{Q}}(G)=1$), which is of interest in its own right, and which we shall therefore not smooth out to a boundary in the general case. As the notation suggests, we shall in fact first enlarge X to a space $\bar{X}$, whose construction involves the **Q**-structure of G, but not Γ. It is a manifold with (countably many) corners, on which $G(\mathbf{Q})$ operates so that the action is proper for any arithmetic subgroup of $G(\mathbf{Q})$. In the classical case where $G=\mathbf{SL}_2$, $\Gamma\subset\mathbf{SL}_2(\mathbf{Z})$ and X is the open unit disc, our $\bar{X}$ is the union of X with countably many lines, one for each cusp point on the unit circle, and $\bar{X}/\Gamma$ is a compact surface whose boundary consists of finitely many circles, one for each Γ-equivalence class of parabolic points. Thus, the cusp points, which are classically added to X, are here blown-up to lines, and this allows us to get a space on which Γ still acts *properly*. In this case, and more generally if $G=\mathbf{SL}_n$, our construction is equivalent to one of C. L. Siegel [27; §§1, 12].

The space $\bar{X}$ is set-theoretically the union of X and of euclidean spaces $e(P)$, one for each proper parabolic $\mathbf{Q}$-subgroup P of G (the codimension of $e(P)$ being equal to the parabolic rank $\mathrm{park}(P)$ of P). For a given P, the set of $e(Q)$'s $(Q\supset P)$ is organized into a corner $\bar{X}(P)$, isomorphic to $\mathbf{R}_+^k\times\mathbf{R}^{n-k}$, where $\mathbf{R}_+$ is the closed half-line of positive real numbers, $n=\dim X$ and $k=\mathrm{park}(P)$. By definition, the $\bar{X}(P)$'s form an open cover of $\bar{X}$. Now we wish to consider the closure $\overline{e(P)}$ of $e(P)$ in $\bar{X}$ as a space obtained from $e(P)$ by a similar construction, using the parabolic $\mathbf{Q}$-subgroups of P. However $e(P)$ is a homogeneous space under the group $P(\mathbf{R})$, which is not semi-simple and the isotropy groups of $P(\mathbf{R})$ are bigger than the maximal compact subgroups of $P(\mathbf{R})$. This led us to enlarge our framework, drop the assumption that G is semi-simple, or even reductive, and replace X by a suitable generalization of the above symmetric space, which we call a space of type S or $S-\mathbf{Q}$. Our construction has then a hereditary character well-suited for proofs by induction on $\dim G$, besides allowing us to handle directly a general arithmetic group. The price to pay is the appearance of some technical complications, mainly in §§1, 2, 3; in first approximation, it may be best for the reader not to dwell too much on them, and to keep in mind the case of a semi-simple G.

The properties of $\bar{X}$ and $\bar{X}/\Gamma$ are applied to the cohomology of Γ. It is shown that $H^i(\Gamma, \mathbf{Z}[\Gamma])=0$ except in dimension $m=\dim X-r_{\mathbf{Q}}(G)$, where it is a free module I, and that we have an isomorphism

$$H^i(\Gamma; A) = H_{m-i}(\Gamma; I\otimes A), \quad (i\in\mathbf{Z}), \tag{1}$$

for any Γ-module A. In particular, the cohomological dimension of Γ is m. If X/Γ is compact, then $I\cong\mathbf{Z}$ and (1) is just Poincaré duality. If X/Γ is not compact, then the rank of I is infinite and I is in a natural way a $G(\mathbf{Q})$-module which is a direct analogue of the Steinberg module of a finite Chevalley group.

We now give some more details on the contents of the various paragraphs. Let G be an affine algebraic group over a subfield k of $\mathbf{R}$. §1 is technical. It introduces a normal k-subgroup ${}^0G$ of G which is more or less a supplement to a maximal k-split torus of the radical of G, and discusses Cartan involutions of reductive groups. In particular, it is shown that if G is semi-simple, K a maximal compact subgroup of $G(\mathbf{R})$, and P a parabolic $\mathbf{R}$-subgroup of G, then $P(\mathbf{R})$ has a unique Levi subgroup stable under the Cartan involution θ_K associated to K (see 1.9 for a more general statement).

§2 is devoted to spaces of type $S-k$, or more generally of type S, but in this introduction we limit ourselves to the former. The homogeneous space X of $G(\mathbf{R})$ is of type $S-k$ if: (i) the isotropy groups $H_x(x\in X)$ are of the form $K\cdot S(\mathbf{R})$, where S is a maximal k-split torus of the radical $R(G)$ of G and K a maximal compact subgroup of $G(\mathbf{R})$ normalizing $S(\mathbf{R})$; (ii) there is given a map $x\mapsto L_x$ of X to Levi subgroups of $G(\mathbf{R})$ such that $L_{x.g}=g^{-1}\cdot L_x\cdot g$ and $L_x\supset H_x(x\in X;\, g\in G(\mathbf{R}))$. Condition (i) determines completely the homogeneous space structure of X (see 2.1), but there is a choice involved in (ii) (unless G is reductive). If P is a parabolic k-subgroup of G, then X is canonically of type $S-k$ under $P(\mathbf{R})$, the choice of the Levi subgroups being given by Corollary 1.9 mentioned above.

§3 introduces the notion of *geodesic action* on a space of type S. For simplicity, assume here G to be connected and semi-simple. Let P be a parabolic $\mathbf{R}$-subgroup of G, and Z the center of the quotient P/R_uP of P by its unipotent radical. For $x\in X$, denote by $Z(\mathbf{R})_x$ the unique lifting of $Z(\mathbf{R})$ in the Levi subgroup of P associated to x as above. There is then an action of $Z(\mathbf{R})$ on X, which commutes with $P(\mathbf{R})$, and is given by $x\circ z=x\cdot z_x$ $(z\in Z(\mathbf{R}), x\in X)$. The orbits of $Z(\mathbf{R})$ are totally geodesic flat submanifolds of X. If A is the identity component of the group of real points of the biggest $\mathbf{R}$-split torus of Z, then X becomes a principal A-bundle under this action. For a simple example, see 3.5.

§4 reviews some facts on parabolic k-subgroups. If P is such a group, let A_P be the identity component of the group of real points of the greatest k-split torus of the center of P/R_uP. There is a canonical isomorphism $A_P\to(\mathbf{R}_+^*)^d$, where $d=\dim A_P$, which is provided by a suitable set of simple k-roots. We let $\bar{A}_P$ be the closure of A_P in $\mathbf{R}^d$. It is therefore isomorphic to the positive quadrant $(\mathbf{R}_+)^d$. In §§4.3, 4.5 we discuss various decompositions of A_P or $\bar{A}_P$.

§5 defines the *corner* $X(P)$ associated to P. Take the simplest case, where $k=\mathbf{R}$ and P is minimal. We have then the familiar Iwasawa decomposition $G(\mathbf{R})=K\cdot A\cdot N$, where A is the identity component of the group of real points of a maximal $\mathbf{R}$-split torus of G stable under the Cartan involution θ_K. Then $R_uP(\mathbf{R})=N$, $P(\mathbf{R})$ is the normalizer of $A\cdot N$ in $G(\mathbf{R})$, and the projection $P\to P/R_uP$ maps A isomorphically onto A_P. There is then a canonical isomorphism $X\to A\times N$ and $X(P)$ is defined to be $\bar{A}\times N$. More intrinsically, in the general case, $X(P)$ is the associated bundle $X\times^{A_P}\bar{A}_P$ with typical fibre $\bar{A}_P$ associated to X, viewed as a principal A_P-bundle under the geodesic action of A_P. The stratification of $\bar{A}_P$ in orbits of A_P yields a stratification of $X(P)$ into locally closed subspaces. In particular, $X\times^{A_P}\{o\}=X/A_P$ is the face $e(P)$ associated to P. If Q is a k-parabolic subgroup of G containing P, then $X(Q)$ may be identified to an open submanifold of $X(P)$. In §6, it is shown that if G is reductive and P minimal, the Siegel sets in X, with respect to P [3, §12] allow one to describe the topology of $X(P)$ around $e(P)$. More precisely any point $y\in e(P)$ has a fundamental set of neighborhoods which are the closures of a suitable family of Siegel sets (6.2). The space X, and hence also $X(P)$, is a trivial bundle, and to any $x \in X$ is associated a trivialization of $X(P)$, whose cross-sections are orbits of the group ${}^0P(\mathbf{R})$.

§7 defines $\bar{X}$, and shows that it is a Hausdorff manifold with corners, which is paracompact if k is countable (7.8). The main point is the Hausdorff property, which is derived from 6.5, itself an immediate consequence of a known property of Siegel sets [3, Prop. 12.6]. It is also shown that the closure of $e(P)$ in $\bar{X}$ can be identified to the space $\overline{e(P)}$ associated by this construction to $e(P)$, viewed as a space of type $S-k$ under P (7.3).

§8 describes the homotopy type of $\partial\bar{X}$ and the cohomology with compact supports of $\bar{X}$.

In the last three paragraphs, $k=\mathbf{Q}$, and Γ is an arithmetic subgroup of G. It is shown that Γ acts properly on $\bar{X}$ and that $\bar{X}/\Gamma$ is compact (9.3). The proof is mainly an appeal to the main theorems of reduction theory [3, §§13, 15]. In turn, the properties of $\bar{X}$ and $\bar{X}/\Gamma$ yield various strengthenings and generalizations of some results of reduction theory, in particular with regard to the "Siegel property" and related facts, which are discussed in §10. Finally §11 gives the applications to the cohomology of Γ already mentioned.

In what follows, the notion of "manifold with corners" is taken for granted. Although this notion has already occurred at various places, there was a lack of foundational material on it, and we are grateful to A. Douady and L. Hérault, who have been willing to supply it; their paper is included here as an appendix.

The main results of this work have been announced in a Comptes Rendus Note [7].

The first named author gave a set of lectures on this topic at the University of Utrecht, in the Spring of 1971. We thank very much Mr. van der Hout, who wrote them up and whose Notes were helpful to us in the preparation of the present paper.

§0. Notation and Conventions

0.1. Let G be a group. If L is a subgroup of G, then $\mathscr{Z}_G(L)=\{g\in G \mid g\cdot x\cdot g^{-1} = x (x\in L)\}$ is the centralizer of L in G and $\mathscr{N}_G(L)=\{g\in G \mid g\cdot L\cdot g^{-1}=L\}$ the normalizer of L in G. The center $\mathscr{Z}_G(G)$ of G is denoted $\mathscr{C}(G)$. If $x\in G$ and $A\subset G$, then ${}^xA=x\cdot A\cdot x^{-1}$ and $A^x=x^{-1}\cdot A\cdot x$. Let N, M be subgroups of G. Then $N\lhd G$ means that N is normal in G, and $G=M\rhd\!\!\!<N$ that N is normal in G and G is the semi-direct product of M and N.

0.2. Algebraic groups will be affine and defined over fields of characteristic zero (mostly subfields of $\mathbf{R}$); we follow the notation and conventions of [4]. If G is an $\mathbf{R}$-group, then $G(\mathbf{R})$, endowed with the topology associated to the one of $\mathbf{R}$, is a real Lie group. The symbol $L(\ )$ will denote the Lie algebra both for algebraic groups and real Lie groups, as the case may be. Similarly, G^0 will denote the connected component of the identity in the Zariski topology if G is an algebraic group or in the ordinary topology if G is a real Lie group.

By definition, a parabolic subgroup of an algebraic group G is one of G^0, i.e. a closed subgroup P of G^0 such that G^0/P is a projective variety [4]. If G is defined over $\mathbf{R}$, and H is an open subgroup of $G(\mathbf{R})$, then a parabolic subgroup of H is by definition the intersection of H with a parabolic subgroup of G defined over $\mathbf{R}$.

0.3. As usual, the radical (resp. unipotent radical) of a connected k-group G is denoted RG (resp. R_uG). Then the greatest connected k-split subgroup ([4], §15) of RG is normal in G (because $G(k)$ is Zariski dense in G, [4], §18) and is the semi-direct product of R_uG by any maximal k-split subtorus of RG. It will be denoted R_dG and called the *split* or *k-split radical of G.*

If G is not connected, then, by definition, its radical, unipotent radical and split radical are those of G^0; they are also denoted RG, R_uG and R_dG respectively.

0.4. Let k be a field of characteristic zero. A k-group H is said to be *reductive* if H^0 is so, i.e. if $R_uH=\{e\}$. Let G be a k-group. Any reductive k-subgroup of G is contained in a maximal one. The maximal ones are called the Levi k-subgroups of G and are conjugate under $R_uG(k)$. If L is one of them, then $G=L\rtimes R_uG$ ([22], [6]).

Sometimes, the set of k-points of a Levi k-subgroup of G will be called a Levi subgroup of $G(k)$.

Recall that if $k=\mathbf{R}$, every compact subgroup of $G(\mathbf{R})$ is the group of real points of a reductive k-group; hence every compact subgroup is contained in a Levi $\mathbf{R}$-subgroup.

0.5. If G is a $\mathbf{Q}$-group, an *arithmetic subgroup Γ of G* is a subgroup of $G(\mathbf{Q})$ which is commensurable with $\varrho(G)\cap \mathbf{GL}_n(\mathbf{Z})$ for any injective $\mathbf{Q}$-morphism $\varrho: G\to \mathbf{GL}_n$ ([3], §7).

In this paper, k is a subfield of $\mathbf{R}$, G a k-group, U the unipotent radical of G and $\mathfrak{P}$ or $\mathfrak{P}(G)$ the set of parabolic k-subgroups of G. From §9 on, we assume $k=\mathbf{Q}$.

I. CARTAN INVOLUTIONS. GEODESIC ACTION. PARABOLIC SUBGROUPS

§1. The Group ${}^0G$. Cartan Involutions

1.1. Assume G to be connected, defined over k. We put

$$ {}^0G = \bigcap_{a\in X(G)_k} \ker a^2, \tag{1}$$

where, as usual, $X(G)_k$ is the group of k-morphisms of G into $\mathbf{GL}_1$. The group ${}^0G$ is normal in G, defined over k. If $a \in X(G)_k$, its restriction to ${}^0G$ is of order $\leqslant 2$, hence is trivial on $({}^0G)^0$, therefore

$$({}^0G)^0 = \left(\bigcap\nolimits_{a \in X(G)_k} \ker a\right)^0. \tag{2}$$

Any character is trivial on U, hence

$${}^0G = {}^0L \rtimes U \tag{3}$$

for any Levi k-subgroup L of G.

1.2. PROPOSITION. *Assume G to be connected. Let S be a maximal k-split torus of RG and $A = S(\mathbf{R})^0$. Then $G(\mathbf{R}) = A \rtimes {}^0G(\mathbf{R})$. The group ${}^0G(\mathbf{R})$ contains every compact subgroup of $G(\mathbf{R})$ and also, if $k = \mathbf{Q}$, every arithmetic subgroup of G.*

Let $a \in X(G)_k$ and let M be a compact subgroup of $G(\mathbf{R})$. Then $a(M)$ is a compact subgroup of $\mathbf{R}^*$, hence contained in $\{\pm 1\}$. Similarly $a(M) \subset \{\pm 1\}$ if $k = \mathbf{Q}$ and M is arithmetic. In both cases $M \subset \ker a^2$, which yields the second assertion.

To prove the first one we may assume, by 1.1 (3), that G is reductive. The group G is the almost direct product of S and of $({}^0G)^0$, as follows from 1.1 (2) and ([3], Prop. 10.7), hence $G(\mathbf{R})^0 \subset A \cdot {}^0G(\mathbf{R})$. Moreover $A \cap {}^0G(\mathbf{R})$ is finite, hence reduced to $\{e\}$ since A is torsion-free. On the other hand, $G(\mathbf{R})$ has finitely many connected components, hence is generated by $G(\mathbf{R})^0$ and a compact subgroup H [21]; since $H \subset {}^0G(\mathbf{R})$ by the above, this gives $G(\mathbf{R}) = A \cdot {}^0G(\mathbf{R})$, whence the proposition.

We shall on occasion use a slight variant of 1.2:

1.2′. PROPOSITION. *We keep the previous notation. Let S', S'' be k-tori in RG such that S' is k-split, $S' \cdot S''$ is a torus and $S' \cap S''$ is finite. There exists then a normal k-subgroup N of G containing S'' and ${}^0G$ such that $G(\mathbf{R}) = S'(\mathbf{R})^0 \rtimes N(\mathbf{R})$.*

Dividing out by U reduces us to the case where G is reductive. The tori S' and S'' belong to the center of G. Using the decomposition of a torus T in anisotropic and split factors T_a and T_d ([4], §8), we can write $\mathscr{C}(G)^0 = V \cdot S'$ where $V \supset \mathscr{C}(G)^0_a \cdot S''_d$ and $V \cap S'$ is finite. Let Y be the set of elements in $X(G)_k$ which are trivial on V, and let

$$N = \bigcap\nolimits_{a \in Y} \ker a^2.$$

Let $S = \mathscr{C}(G)^0_d$. It follows from ([3], 10.7) that the restriction map $X(G)_k \to X(S)$ is injective, with finite cokernel, and that $X(G) \to X(S')$ maps Y injectively onto a subgroup of finite index. From this we see that $N \supset {}^0G$, $G = N \cdot S'$ and $N \cap S'$ is finite; therefore $G(\mathbf{R})^0 = S'(\mathbf{R})^0 \rtimes N(\mathbf{R})^0$. Since $N(\mathbf{R})$ contains ${}^0G(\mathbf{R})$, it meets every connected component of $G(\mathbf{R})$ by 1.2, hence $G(\mathbf{R}) = S'(\mathbf{R})^0 \rtimes N(\mathbf{R})$.

1.3. LEMMA. *Let L, L' be two Levi k-subgroups of G, and N a normal k-subgroup of G containing U. Then $L \cap N$ is a Levi k-subgroup of N. The groups L and L' are conjugate by an element of $U(k) \cap \mathscr{Z}_G(L \cap L')$.*

We have $G = L \rtimes U$, hence $N = (N \cap L) \rtimes U$. Moreover $U \subset R_u N$, and since N is normal, $R_u N \subset U$, whence $R_u N = U$, and the first assertion.

Let $u \in U(k)$ be such that ${}^u L = L'$ (0.4) and let $x \in L \cap L'$. Then $u \cdot x \cdot u^{-1} = x \cdot v$ for some $v \in U$. Since $u \cdot x \cdot u^{-1}$ and x belong to L', we have $v \in L' \cap U = \{e\}$, hence u centralizes $L \cap L'$.

1.4. We recall now a few standard facts about maximal compact subgroups or consequences thereof.

Let H be a real Lie group with finitely many connected components. Then any compact subgroup of H is contained in a maximal one. If K is a maximal one, then H is diffeomorphic to the direct product of K with a euclidean space. Moreover $K/K^0 = H/H^0$. Any two maximal compact subgroups are conjugate by an element of H^0 [21].

If N is a closed normal subgroup of H, with finitely many connected components, then the maximal compact subgroups of N are the intersections of N with the maximal compact subgroups of H. If M is a closed subgroup of H with finitely many connected components such that all maximal compact subgroups of H are conjugate by elements of M (e.g. if $H = K \cdot M$), then similarly the maximal compact subgroups of M are the intersections of M with the maximal compact subgroups of H. (In both cases, by taking a maximal compact subgroup K of H containing a maximal one of M, we see that $M \cap K$ is compact maximal in M for at least one K. It is then so for all maximal compact subgroups of H by conjugacy.)

Let $H \to H'$ be a surjective morphism of Lie groups whose kernel N has finitely many connected components. Then the maximal compact subgroups of H' are the images of the maximal compact subgroups of H. (This is well-known if H and N are connected and the reduction to that case is immediate.)

1.5. PROPOSITION. *Let P be a parabolic k-subgroup of G, S a maximal k-split torus of $R_d G$ and $A = S(\mathbf{R})^0$. Let K be a maximal compact subgroup of $G(\mathbf{R})$. Then $K \cap P$ is a maximal compact subgroup of $P(\mathbf{R})$ and $G(\mathbf{R}) = K \cdot P(\mathbf{R}) = K \cdot A \cdot {}^0P(\mathbf{R})$. If $K \cdot a \cdot {}^0P(\mathbf{R}) = K \cdot a' \cdot {}^0P(\mathbf{R})$ $(a, a' \in A)$, then $a = a'$. The map which assigns to $g \in G(\mathbf{R})$ the element $a = a(g) \in A$ such that $g \in K \cdot a \cdot {}^0P(\mathbf{R})$ is real analytic.*

The equality $G(\mathbf{R}) = K \cdot P(\mathbf{R})$ is well-known and follows from the Iwasawa decomposition (see e.g. [8], 14.7). We have then $G(\mathbf{R}) = K \cdot A \cdot {}^0P(\mathbf{R})$ by 1.2 applied to P. The group $K \cap P$ is a maximal compact subgroup of $P(\mathbf{R})$ by 1.4 and is contained in ${}^0P(\mathbf{R})$ by 1.2, hence we can identify $(K \cap P)\backslash P(\mathbf{R})$ and $A \times (K \cap P)\backslash {}^0P(\mathbf{R})$. Composing the obvious maps

$$G(\mathbf{R}) \to K\backslash G(\mathbf{R}) \cong (K \cap P)\backslash P(\mathbf{R}) \cong A \times (K \cap P)\backslash {}^0P(\mathbf{R}) \xrightarrow{pr_1} A,$$

we get a real analytic map $f: G(\mathbf{R}) \to A$. It is clear that $f(kap) = a$ $(k \in K, a \in A, p \in {}^0P(\mathbf{R}))$, which proves the uniqueness and analyticity of a.

1.6. PROPOSITION. *Let G be reductive and let K be a maximal compact subgroup of $G(\mathbf{R})$. There exists one and only one involutive automorphism θ_K of $G(\mathbf{R})$ whose fixed point set is K and which is "algebraic," i.e. the restriction to $G(\mathbf{R})$ of an involutive automorphism of algebraic groups of the Zariski-closure of $G(\mathbf{R})$ in G. Let $\mathfrak{p}$ be the (-1)-eigenspace of θ_K in $L(G(\mathbf{R}))$. Then $L(G(\mathbf{R})) = L(K) \oplus \mathfrak{p}$ and $(k, X) \mapsto K \cdot \exp X$ is an isomorphism of analytic manifolds of $K \times \mathfrak{p}$ onto $G(\mathbf{R})$. Let N be a normal $\mathbf{R}$-subgroup of G. Then $\theta_K(N(\mathbf{R})) = N(\mathbf{R})$.*

By a result of G. D. Mostow [22] (see also [5], § 1), we may arrange that $G \subset \mathbf{GL}_n$, $K \subset \mathbf{O}(n, \mathbf{R})$, and $G(\mathbf{R})$ is stable under $\theta: g \mapsto \check{g} = {}^t g^{-1}$, and then the latter automorphism has all the properties required from θ_K. There remains to prove the uniqueness. Let then θ' be an involutive automorphism of $G(\mathbf{R})$ whose fixed point set is K and which is algebraic in the above sense. Since K meets every connected component of $G(\mathbf{R})$, it suffices to show that θ and θ' coincide on $G(\mathbf{R})^0$, and hence that they define the same automorphism of $L(G(\mathbf{R}))$. For this, it is enough to prove that the (-1)-eigenspaces $\mathfrak{p}$ and $\mathfrak{p}'$ of θ and θ' are equal, and we may also assume G to be connected.

The group G is then the almost direct product of its derived group G', which is semi-simple, and of the identity component S of its center, which is a torus. Both $G'(\mathbf{R})$ and $S(\mathbf{R})$ are stable under θ', θ hence

$$\mathfrak{p}' = L(G') \cap \mathfrak{p}' \oplus L(S) \cap \mathfrak{p}', \qquad \mathfrak{p} = L(G') \cap \mathfrak{p} \oplus L(S) \cap \mathfrak{p}.$$

The group $G' \cap K$ is a maximal compact subgroup of $G'(\mathbf{R})$ (1.4), hence $L(G') \cap \mathfrak{p}'$ and $L(G') \cap \mathfrak{p}$ are both equal to the orthogonal complement of $L(K \cap G')$ in $L(G'(\mathbf{R}))$ with respect to the Killing form. The group S is the almost direct product of its greatest $\mathbf{R}$-anisotropic torus S_a and its greatest $\mathbf{R}$-split torus S_d and $S_a(\mathbf{R})^0$ is the greatest connected compact subgroup of $S(\mathbf{R})$ ([3], 10.8). We have then, using 1.4,

$$L(S(\mathbf{R})) = L(K \cap S) \oplus L(S_d(\mathbf{R})).$$

But θ, θ' are the restriction to $G(\mathbf{R})$ of an $\mathbf{R}$-automorphism of G, hence they must leave both $S_a(\mathbf{R})$ and $S_d(\mathbf{R})$ stable, whence

$$L(S) \cap \mathfrak{p} = L(S) \cap \mathfrak{p}' = L(S_d(\mathbf{R})).$$

Let now N be a normal $\mathbf{R}$-subgroup of G. It is then also reductive. By [22] (see also [5]) there exists a maximal compact subgroup K_1' of $G(\mathbf{R})$ such that the associated involution θ'_1 leaves $N(\mathbf{R})$ stable. There exists $g \in G(\mathbf{R})$ such that ${}^gK = K'_1$. But then θ'_1 is conjugate to θ_K under $\operatorname{Int} g$, whence the last assertion.

1.7. DEFINITION. The automorphism θ_K in 1.6 will be called the *Cartan involution* of $G(\mathbf{R})$ with respect to K.

If G is semi-simple, then θ_K is the usual Cartan involution, and the uniqueness is obvious.

1.8. PROPOSITION. *Let H be a $\mathbf{R}$-subgroup of G containing U. Assume that all maximal compact subgroups of $G(\mathbf{R})$ are conjugate under $H(\mathbf{R})$. Let K be a maximal compact subgroup of $G(\mathbf{R})$ and L a Levi subgroup of $G(\mathbf{R})$ containing K. Let θ_K be the Cartan involution of L with respect to K and $L_1 = (H \cap L) \cap \theta_K(H \cap L)$. Then L_1 is the unique Levi subgroup of $H(\mathbf{R})$ contained in L and stable under θ_K.*

The group L_1 is stable under θ_K, hence is reductive ([5], 1.5), and contains every subgroup of $H' = H \cap L$ stable under θ_K, whence the uniqueness assertion. Let now L' be a Levi subgroup of H'. Since $H(\mathbf{R}) = H' \cdot U(\mathbf{R})$, the group L' is a Levi subgroup of $H(\mathbf{R})$. By a result of Mostow ([22], see also 1.9 in [5]), L admits a Cartan involution θ' leaving L' stable. Let K' be its fixed point set. In view of the assumption on H there exists $h \in H(\mathbf{R})$ such that $K = {}^hK'$. Since ${}^hK' \subset L \cap {}^hL$, there exists by 1.3 an element $u \in H(\mathbf{R}) \cap \mathscr{Z}_H(K)$ such that ${}^{uh}L = L$. We have therefore $K = {}^{uh}K'$ and, by the uniqueness of Cartan involutions (1.6),

$$\theta_K \circ \operatorname{Int} uh = (\operatorname{Int} uh) \circ \theta'.$$

As a consequence ${}^{uh}L'$ is a Levi subgroup of $H(\mathbf{R})$ stable under θ_K, hence contained in L_1, hence equal to L_1 since L_1 is reductive; thus L_1 has all the required properties.

1.9. COROLLARY. *Let P be a parabolic $\mathbf{R}$-subgroup of G, K a maximal compact subgroup of $G(\mathbf{R})$ and L a Levi subgroup of $G(\mathbf{R})$ containing K. Then $L \cap P$ contains one and only one Levi subgroup of $P(\mathbf{R})$ stable under θ_K.*

Since $G(\mathbf{R}) = K \cdot P(\mathbf{R})$ by 1.5, this is a special case of 1.8.

§ 2. Homogeneous Spaces of Type S

2.1. LEMMA. *Let R be a solvable connected normal $\mathbf{R}$-subgroup of G containing U.*

(i) *Let K be a maximal compact subgroup of $G(\mathbf{R})$. Then R has a maximal $\mathbf{R}$-torus normalized by K.*

(ii) *If S is a maximal $\mathbf{R}$-torus of R, then $\mathscr{N}_G(S)$ contains a maximal compact subgroup of $G(\mathbf{R})$.*

(iii) *The subgroups of $G(\mathbf{R})$ of the form $K \cdot S(\mathbf{R})$, where S is a maximal torus of R*

defined over $\mathbf{R}$ *and* K *a maximal compact subgroup of* $G(\mathbf{R})\cap\mathcal{N}_G(S)$, *form one conjugacy class of subgroups of* $G(\mathbf{R})$. *Let* $H=K\cdot S(\mathbf{R})$ *be one of them and* $\bar{H}$ *the Zariski closure of* H. *Then* $\bar{H}$ *is reductive*, $H=\bar{H}(\mathbf{R})$, $S=R\cap\bar{H}$ *and* $S(\mathbf{R})=R\cap H$.

(iv) *Let* L *be a Levi* $\mathbf{R}$-*subgroup of* G *containing* H, *and* θ_K *the Cartan involution of* $L(\mathbf{R})$ *with respect to* K. *Then* $L\cap R=S$ *and* θ_K *leaves* $H(\mathbf{R})$ *and* $S(\mathbf{R})$ *stable.*

(i) Let L be a Levi $\mathbf{R}$-subgroup of G containing K. Then $R=(R\cap L)\ltimes U$, and $R\cap L$ is a maximal torus of R which is normalized by L, hence by K.

(ii) In view of (i) this assertion is true for at least one maximal $\mathbf{R}$-torus of R. Since such tori are conjugate under $R(\mathbf{R})$ ([8], 11.4), it is then true for all of them.

(iii) The first assertion follows from the conjugacy of maximal $\mathbf{R}$-tori of R and of maximal compact subgroups (1.4).

The Zariski-closure $\bar{K}$ of K is reductive and normalizes S, hence $\bar{K}^0$ centralizes S. We have $\bar{H}=\bar{K}\cdot S$, $\bar{H}^0=\bar{K}^0\cdot S$, hence $\bar{H}^0$, and therefore $\bar{H}$, is reductive. Moreover, $K\cdot S(\mathbf{R})$ contains $\bar{H}(\mathbf{R})^0$; but K is a maximal compact subgroup of $G(\mathbf{R})$, hence *a fortiori* of $\bar{H}(\mathbf{R})$, and consequently intersects every connected component of $\bar{H}(\mathbf{R})$; therefore $\bar{H}(\mathbf{R})=K\cdot\bar{H}(\mathbf{R})^0=H$. We have $S(\mathbf{R})\subset R\cap H$, hence $S\subset R\cap\bar{H}$. The group $R\cap\bar{H}$ is normal in $\bar{H}$, hence reductive. Since S is maximal reductive in R, it follows that $S=R\cap\bar{H}$, whence also

$$S(\mathbf{R})\subset R\cap H\subset(R\cap\bar{H})(\mathbf{R})=S(\mathbf{R}),$$

which ends the proof of (iii).

(iv) Any subspace of the Lie algebra of $L(\mathbf{R})$ which contains $L(K)$ is stable under θ_K, hence $L(H(\mathbf{R}))$ and $H(\mathbf{R})^0$ are stable under θ_K. Since $H(\mathbf{R})$ is generated by K and $H(\mathbf{R})^0$ (1.4), it is also stable under θ_K. The group $L\cap R$ is a normal $\mathbf{R}$-subgroup of L, hence its group of real points is stable under θ_K by 1.6. It is reductive, contains $R\cap\bar{H}=S$, hence is equal to S.

2.2. *Remark.* Notation being as above, assume G to be connected. Then L is also connected, hence centralizes the torus $L\cap R$. It follows that, in 2.2(iii), K *centralizes* S and K is the unique maximal compact subgroup of $K\cdot S(\mathbf{R})$.

2.3. DEFINITION. *A space of type* S *for* G or $G(\mathbf{R})$ is a pair consisting of a right homogeneous space X under $G(\mathbf{R})$ and of a family $(L_x)_{x\in X}$ of Levi subgroups of $G(\mathbf{R})$ satisfying the two following conditions:

SI. There exists a connected normal solvable $\mathbf{R}$-subgroup R_X of G containing U, such that the isotropy groups $H_x (x\in X)$ of $G(\mathbf{R})$ in X are of the form $K\cdot S(\mathbf{R})$, where S is a maximal $\mathbf{R}$-torus of R_X and K is a maximal compact subgroup of $G(\mathbf{R})$ normalizing S (cf. 2.1).

SII. We have $H_x\subset L_x$ and $L_{x\cdot g}=(L_x)^g$ for all $x\in X$ and $g\in G(\mathbf{R})$.

We shall often say simply that X is a homogeneous space of type S under $G(\mathbf{R})$. Note that, by 2.1, R_X is completely determined by the action of $G(\mathbf{R})$ on X. From §4 on, we shall be concerned only with the case where $R_X = R_d G$ is the k-split radical of G, in which case we shall say that X is of type $S-k$. If moreover $G = {}^0G$, then the isotropy subgroups are just the maximal compact subgroups of $G(\mathbf{R})$. As explained in the introduction, this is our main case of interest.

2.4. *Remarks.* (1) The condition SI of 2.3 implies that X is diffeomorphic to a euclidean space, $G(\mathbf{R})^0$ is transitive on X and the isotropy groups are conjugate under $G(\mathbf{R})^0$.

(2) Let X be a homogeneous space under $G(\mathbf{R})$ for which the isotropy groups $H_x (x \in X)$ are reductive. Then it is always possible to find a family $\{L_x\}_{x \in X}$ satisfying SII. Indeed, choose $y \in X$, a Levi subgroup $L \supset H_y$, and put $L_{y \cdot g} = L^g$ for $g \in G(\mathbf{R})$. Since $L^g = L$ for $g \in H$, the group L^g depends only on $y \cdot g$, and SII is then satisfied.

2.5. EXAMPLES. (1) Let G be semi-simple. Then $R_X = \{e\}$, and the isotropy groups are the maximal compact subgroups of $G(\mathbf{R})$. Since they are equal to their normalizers, X may be identified with the symmetric space of maximal compact subgroups of $G(\mathbf{R})$.

(2) Let G be reductive and let $C = \mathscr{C}(G^0)^0$ be its radical. The group R_X is an $\mathbf{R}$-subtorus of G, normal in G. The isotropy groups are then the subgroups $K \cdot R_X(\mathbf{R})$, where K runs through the maximal compact subgroups of $G(\mathbf{R})$. The group $K \cap RG$ is maximal compact in $RG(\mathbf{R})$. By standard facts on tori, there exists an $\mathbf{R}$-split subtorus D of RG normal in G, such that $RG(\mathbf{R}) = (K \cdot R_X(\mathbf{R}) \cap RG) \times A$, with $A = D(\mathbf{R})^0$. The group A operates properly and freely on X. On X/A, the isotropy groups of $G(\mathbf{R})$ are the groups $K \cdot RG(\mathbf{R})$, (K maximal compact), hence X/A may be identified with the space of maximal compact subgroups of $\mathscr{D}G^0(\mathbf{R})$. If G is connected, then A is central. Note that when G is reductive, SII is vacuously fulfilled since we must have $L_x = G(\mathbf{R})$ for all $x \in X$.

2.6. LEMMA. *Let X be a homogeneous space of type S under $G(\mathbf{R})$ and $\mathscr{H} = \{H_x\}_{x \in X}$ the set of isotropy groups of $G(\mathbf{R})$ on X. Let G' be an $\mathbf{R}$-subgroup of G containing U such that $G'(\mathbf{R})$ is transitive on X. Then X satisfies* SI *under G', the corresponding solvable group R'_X being equal to $(G' \cap R_X)^0 \cdot R_u G'$.*

Let K be a maximal compact subgroup of G such that $K \cap G'$ is maximal compact in $G'(\mathbf{R})$ and let $x \in X$ be such that $K \subset H_x$; thus $H'_x = H_x \cap G'$ contains a maximal compact subgroup of $G'(\mathbf{R})$; by conjugacy, this is then true for all $x \in X$. The group $R' = (G' \cap R_X)^0 \cdot R_u G'$ is a normal connected solvable $\mathbf{R}$-subgroup of G' and $R_u R' = R_u G'$. Let S be a maximal $\mathbf{R}$-torus of R_X. Then $R_X = S \rtimes U$, hence $R_X \cap G' = (S \cap G') \rtimes U$, and $R' = S' \rtimes R_u G'$, where $S' = (S \cap G')^0$ is a maximal $\mathbf{R}$-torus of R', and we have $S'=(S\cap R')^0$. Choose $x\in X$ such that $S(\mathbf{R})\subset H_x$. Then

$$S(\mathbf{R})\cap G'=S(\mathbf{R})\cap H_x\cap G'=S(\mathbf{R})\cap H'_x,$$

which shows that

$$S'(\mathbf{R})\lhd H'_x,\qquad S'(\mathbf{R})\subset S(\mathbf{R})\cap H'_x,\qquad S'(\mathbf{R})^0=(S(\mathbf{R})\cap H'_x)^0. \tag{1}$$

Let K' be a maximal compact subgroup of H'_x. We have already seen that it is maximal compact in $G'(\mathbf{R})$; by (1), it normalizes $S'(\mathbf{R})$. We wish to show that $H'_x=K'\cdot S'(\mathbf{R})$, which will prove the lemma. For this, it is enough, by the last assertion of 1.4, to prove that $H'_x/S'(\mathbf{R})$ is compact.

Let $\bar{H}$ be the Zariski-closure of H_x in G. The group S being normal in $\bar{H}$, the set $M=(\bar{H}\cap G')\cdot S$ is an $\mathbf{R}$-subgroup of G. Since $H_x=\bar{H}(\mathbf{R})$ by 2.1, $H'_x\cdot S(\mathbf{R})$ is an open subgroup of finite index in $M(\mathbf{R})$. In particular, it is a closed subgroup of $G(\mathbf{R})$. As a consequence, $H'_x\cdot S(\mathbf{R})/S(\mathbf{R})$ may be identified with a closed subgroup of $H_x/S(\mathbf{R})$. Since the latter is compact, so is the former. But $H'_x/(H'_x\cap S(\mathbf{R}))$ is isomorphic (as a Lie group) to $H'_x\cdot S(\mathbf{R})/S(\mathbf{R})$, hence is compact, too. By (1), $H'_x/(H'_x\cap S(\mathbf{R}))$ and $H'_x/S'(\mathbf{R})$ have a common finite covering, hence $H'_x/S'(\mathbf{R})$ is compact.

2.7. *Restriction to subgroups. Examples.* Let X and G' be as in 2.6. Then X satisfies SI with respect to $G'(\mathbf{R})$, and, by 2.4(2), there is then at least one way to make X of type S under G'. We shall now indicate some cases in which this can be done in a canonical manner. In the sequel, it will always be understood that X will be viewed of type S under G' with the choice of the Levi subgroups of $G'(\mathbf{R})$ given below. These will often be denoted $L_{x,G'}$ $(x\in X)$.

(1) *G' is normal in G.* We define $L_{x,G'}$ as $G'\cap L_x$. This applies in particular to G^0.

(2) *G is connected.* By 2.2, H_x has a unique maximal compact subgroup, say K_x. Since $G'(\mathbf{R})$ operates transitively on X, it follows that all maximal compact subgroups of $G(\mathbf{R})$ are conjugate under $G'(\mathbf{R})$. We then define $L_{x,G'}$ as the unique Levi subgroup of $G'\cap L_x$ which is stable under the Cartan involution of L_x with respect to K_x (cf. Prop. 1.8).

(3) *$G'=P$ is parabolic.* We apply (1) to G^0 and (2) to $P\subset G^0$.

Remark. Assume G' to satisfy one of the above three conditions. Let $g\in G(\mathbf{R})$ and $G''=G'^g$. Then G'' satisfies the same condition, and we have

$$L_{x\cdot g,G''}=(L_{x,G'})^g\quad (x\in X). \tag{4}$$

2.8. Let X be a space of type S under G. It is the total space of a principal fibration with structure group $U(\mathbf{R})$, where $U(\mathbf{R})$ operates as a subgroup of $G(\mathbf{R})$. Let V be a (necessarily connected) $\mathbf{R}$-subgroup of U normal in G. Let $\pi:G\to G'=G/V$ and

$\sigma: X \to X' = X/V(\mathbf{R})$ be the natural projections. Let $x, y \in X$ be such that $\sigma(x) = \sigma(y)$. Then $x \in y \cdot V(\mathbf{R})$, hence $L_y = L_x^v$ for some $v \in V(\mathbf{R})$ and $\pi(L_x) = \pi(L_y)$. It is then immediate that X' is of type S under G', with $R_{X'} = \pi(R_X)$, and $L_{\sigma(x)} = \pi(L_x)$ for all $x \in X$. Assume V to be defined over k. Then $\pi(R_d G) = R_d G'$, therefore X' is of type $S-k$ if X is so.

§3. Geodesic Action

In this section, X is a space of type S under $G(\mathbf{R})$; for $x \in X$, H_x is the isotropy subgroup of x and L_x the Levi subgroup of $G(\mathbf{R})$ associated to x.

3.1. LEMMA. *Let P be a parabolic $\mathbf{R}$-subgroup of G, Z the center of $P/R_u P$ and $\pi: P \to R/R_u P$ the canonical projection. Let Y be the greatest compact subgroup of $Z(\mathbf{R})$. Then, for $x \in X$, $Z_0 = Z \cap \pi(H_x \cap P)$ is generated by $\pi(R_X(\mathbf{R}))$ and by Y. In particular it is independent of $x \in X$.*

We have $R_X \cap R_u P = U$, therefore π defines an injective homomorphism of R_X/U into $P/R_u P$, whose image is a torus which is normal, hence central; we have thus $\pi(R_X) \subset Z$. We have $\pi(R_X) = \pi(S)$ where S is any maximal torus of R_X, hence, by 2.1,

$$\pi(R_X)(\mathbf{R}) = \pi(H_x \cap R_X) \qquad \text{for any } x \in X.$$

On the other hand, $H_x = K \cdot (H_x \cap R_X)$, where K is a suitable maximal compact subgroup of $G(\mathbf{R})$, hence

$$H_x \cap P = (K \cap P) \cdot (H_x \cap R_X)$$
$$Z_0 = Z \cap \pi(H_x \cap P) = (\pi(K \cap P) \cap Z) \cdot \pi(H_x \cap R_X).$$

The group $K \cap P$ is maximal compact in $P(\mathbf{R})$ (1.5), hence $\pi(K \cap P)$ is maximal compact in $(P/R_u P)(\mathbf{R})$ (1.4), and then $\pi(K \cap P) \cap Z$ is maximal compact in $Z(\mathbf{R})$ (1.4). Therefore $\pi(K \cap P) \cap Z = Y$, whence the lemma.

Remark. Let S_1 be the $\mathbf{R}$-torus of Z which is generated by the greatest $\mathbf{R}$-anisotropic torus of Z and by the image of R_X. We have then $S_1(\mathbf{R})^0 = (Z_0)^0$. Let S_2 be an $\mathbf{R}$-torus in Z such that Z^0 is the almost direct product of S_1 and S_2. Then $S_2(\mathbf{R})^0 \cap \cap Z_0 = \{e\}$ and $S_2(\mathbf{R})^0$ maps isomorphically onto $Z(\mathbf{R})/Z_0$ under the natural projection. The torus S_2 splits over $\mathbf{R}$, and it follows from 1.2′ that there exists a normal $\mathbf{R}$-subgroup N of P containing R_X and all compact subgroups of $P(\mathbf{R})$ and such that $P(\mathbf{R}) = S_2(\mathbf{R})^0 \ltimes N(\mathbf{R})$.

3.2. *Definition of the geodesic action.* Let P, Z, Z_0 be as before. We shall define here an action of $Z(\mathbf{R})$ on X which commutes with $P(\mathbf{R})$, is trivial for Z_0 and defines a proper and free action of $Z(\mathbf{R})/Z_0$.

By 2.7(3), X is canonically of type S under P. For $x\in X$, let L'_x be the Levi subgroup of $P(\mathbf{R})$ associated to x; it is contained in L_x. Let $Z_x=\mathscr{C}(L'_x)$. It follows from 1.8 and the definition of L'_x (2.7) that Z_x is the unique lifting of $Z(\mathbf{R})$ in $P(\mathbf{R})$ which is stable under the Cartan involution of L_x with respect to a maximal compact subgroup of H_x. Given $z\in Z(\mathbf{R})$, let z_x be its lifting in Z_x. Fix $x\in X$. Let $y\in X$ and $g\in P(\mathbf{R})$ be such that $y=x\cdot g$. Let us put

$$y \mathbf{o}_x z = x\cdot z_x\cdot g. \tag{1}$$

Let $g'\in P(\mathbf{R})$ be such that $x\cdot g'=y$. Then $g'=h\cdot g$ for some $h\in H_x\cap P\subset L'_x$; the element h then commutes with z_x, hence $x\cdot z_x\cdot g'=x\cdot z_x\cdot g$. This shows that the right hand side of (1) depends only on x, y, z and justifies the notation of the left hand side.

LEMMA. *We have*

(i) $y\cdot p\, \mathbf{o}_x z = (y\, \mathbf{o}_x z)\cdot p, \quad (x, y\in X;\ p\in P(\mathbf{R}),\ z\in Z(\mathbf{R}))$.
(ii) $y\, \mathbf{o}_x z = y\, \mathbf{o}_{x'} z, \quad (x, x', y\in X;\ z\in Z(\mathbf{R}))$.

Let $g\in P(\mathbf{R})$ be such that $y=x\cdot g$. For $p\in P(\mathbf{R})$, we have $y\cdot p=x\cdot g\cdot p$, hence

$$(y\cdot p)\, \mathbf{o}_x z = x\cdot z_x\cdot g\cdot p = (y\, \mathbf{o}_x z)\cdot p, \tag{2}$$

which gives (i).

Let now $h\in G(\mathbf{R})$ and set $P'=P^h$. Then $\mathrm{Int}\, h$ induces an $\mathbf{R}$-isomorphism of Z onto $Z'=\mathscr{C}(P'/R_uP')$. Let us also denote by z^h the image of $z\in Z$ under this map. It is clear, by "transport de structure," that we have

$$(y\, \mathbf{o}_x z)\cdot h = y\cdot h\, \mathbf{o}_{x\cdot h} z^h \quad (x, y\in X;\ z\in Z(\mathbf{R})). \tag{3}$$

Let $x'\in X$, and choose $h\in P(\mathbf{R})$ such that $x\cdot h=x'$. By (3), applied to $y\cdot h^{-1}$, we have, taking (i) into account:

$$y\, \mathbf{o}_x z = (y\cdot h^{-1}\, \mathbf{o}_x z)\cdot h = y\, \mathbf{o}_{x'} z^h;$$

but, since $h\in P(\mathbf{R})$, we have $z^h=z$, whence (ii).

In particular, this shows that the actions $\mathbf{o}_x$ and $\mathbf{o}_{x'}$ are the same. We may therefore omit the reference point, and get an action

$$\gamma : (x, z)\mapsto x\, \mathbf{o}\, z \quad (x\in X;\ z\in Z(\mathbf{R}))$$

of $Z(\mathbf{R})$ on X. By (1), with $g=e$:

$$x\, \mathbf{o}\, z = x\cdot z_x, \quad (x\in X;\ z\in Z(\mathbf{R})), \tag{4}$$

which shows that

$$(x \circ z) \circ z' = x \circ (z \cdot z') \quad (x \in X;\ z, z' \in Z(\mathbf{R})).$$

We can now write (3) in the form

$$(x \circ z) \cdot h = x \cdot h \circ z^h, \quad (x \in X,\ z \in Z(\mathbf{R}),\ h \in G(\mathbf{R})), \tag{5}$$

where $\circ$ refers to the geodesic action of $Z(\mathbf{R})$ on the left-hand side and of ${}^hZ(\mathbf{R})$ on the right-hand side.

3.3. DEFINITION. The action γ defined above is called the *geodesic action* of $Z(\mathbf{R})$ on X.

The reader will note that this action depends only on the structure of type S of X under P. If P is reductive, then Z is a subgroup of P, and the geodesic action is just the ordinary action.

3.4. PROPOSITION. *The geodesic action commutes with $P(\mathbf{R})$. The group Z_0 operates trivially and $Z(\mathbf{R})/Z_0$ operates freely.*

(In fact, the action of $Z(\mathbf{R})/Z_0$ is a principal bundle action, see 3.6.)

The first assertion follows from 3.2(i). If $z \in Z_0$ then, by 3.1, $z_x \in H_x$ for all $x \in X$, hence z acts trivially by 3.2(4).

It also follows from 3.1 that if $z \in Z(\mathbf{R})$, $z \notin Z_0$, then $z_x \notin H_x$ for any $x \in X$, hence $Z(\mathbf{R})/Z_0$ acts freely on X.

Remark. The group Z_0 contains the maximal compact subgroup of $Z(\mathbf{R})$. We may therefore write $Z(\mathbf{R}) = Z_0 \times A$, where A is the identity component of the group of real points of an $\mathbf{R}$-split torus of Z. By 3.4, and 3.2(4), the orbit $x \circ Z(\mathbf{R})$ of $x \in X$ may be identified with $x \cdot A_x$, i.e. with the orbit of x under the ordinary action of the identity component of the group of real points of an $\mathbf{R}$-split torus. If G is reductive, and hence X is a symmetric space with negative curvature, then $x \cdot A_x$ is a totally geodesic flat submanifold, isometric to a euclidean space, and the orbits of 1-dimensional subgroups of A_x are geodesics, whence the terminology.

3.5. EXAMPLE. Let $G = \mathbf{SL}_2$ and X be the upper half-plane. Let $G(\mathbf{R})$ act on X by

$$z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (dz + b)(cz + a)^{-1}.$$

Let P be the group of upper triangular matrices in G. The group Z_0 has two elements ± 1. Let $A = Z(\mathbf{R})/Z_0$. The stability group of i is $\mathbf{SO}(2, \mathbf{R})$ and the Cartan involution associated to it is $g \mapsto {}^tg^{-1}$. For $x = i$, $L_x = Z_x$ is the group of diagonal matrices in $G(\mathbf{R})$. For $a \in A$, let $\alpha(a)$ be the square of the upper left entry of the lifting of a in A_i.

Then, for $z=x+iy$, we have

$$z \circ a = x + i\alpha(a)\cdot y.$$

Hence the orbits of A are the vertical lines and on each such line, identified to $\mathbf{R}_+^*$ by the y coordinate, $z \mapsto z \circ a$ is the multiplication by $\alpha(a)$.

If we take as model of X the open unit disc, then the choice of a parabolic $\mathbf{R}$-subgroup P of G corresponds to that of the fixed point P_0 of $P(\mathbf{R})$ on the unit circle. The orbits of $Z(\mathbf{R})/Z_0$ under the geodesic action are then the geodesics abutting to P_0.

3.6. *Bundles defined by the geodesic action.* Let P, Z, Z_0 be as before. Let T be an $\mathbf{R}$-split torus in $\mathscr{C}(P/R_u P)$ whose intersection with Z_0 is finite and A be the image of $T(\mathbf{R})^0$ into $Z(\mathbf{R})/Z_0$. By 1.2′, there exists a normal k-subgroup M of P containing R_X and all maximal compact subgroups of $P(\mathbf{R})$, such that $P(\mathbf{R})=T(\mathbf{R})^0 \ltimes M(\mathbf{R})$. Since $P(\mathbf{R})$ commutes with A, the latter operating by geodesic action, we have an action of $A\times M(\mathbf{R})$ onto X defined by

$$x \mapsto (x \circ a)\cdot m, \qquad (a\in A,\ m\in M(\mathbf{R});\ x\in X).$$

3.7. PROPOSITION. *The above action of $A\times M(\mathbf{R})$ on X is transitive. For $x\in X$, $H_x\cap P\subset M(\mathbf{R})$ and the isotropy group of x in $A\times M(\mathbf{R})$ is $\{e\}\times(H_x\cap P)$.*

Let $x\in X$. Then $(x\circ A)\cdot M(\mathbf{R})=x\cdot A_x\cdot M(\mathbf{R})=x\cdot P(\mathbf{R})=X$, which proves the first assertion. The space X is of type S under $P(\mathbf{R})$ (2.7) and in particular the isotropy group $H_x\cap P$ of x under $P(\mathbf{R})$ is generated by a maximal compact subgroup K_x' of $P(\mathbf{R})$ and a subgroup of $R_X(\mathbf{R})$. Since both of these groups are contained in $M(\mathbf{R})$, we have $H_x\cap P\subset M(\mathbf{R})$ for all $x\in X$. Let now $a\in A$ and $m\in M(\mathbf{R})$ be such that $(x\circ a)\cdot m=x$. We then have $a_x\cdot m\in H_x\cap P$ hence $a_x\cdot m\in M(\mathbf{R})$ and $a_x=e$, whence the proposition.

3.8. COROLLARY. *Let $x\in X$. The map $(a\cdot m)\mapsto(x\circ a)\cdot m$ $(a\in A;\ m\in M(\mathbf{R}))$ induces an isomorphism $\mu_x: A\times(H_x\cap P)\backslash M(\mathbf{R})\xrightarrow{\sim} X$ of $(A\times M(\mathbf{R}))$-homogeneous spaces. The space X is a trivial principal A-bundle, and the orbits of $M(\mathbf{R})$ are cross-sections of this fibration.*

By 3.7, μ_x is an analytic bijective map of $(A\times M(\mathbf{R}))$-spaces. Since these are homogeneous spaces, it is then an isomorphism. This proves the first assertion; the second one is an obvious consequence.

Remark. Since Z_0 contains the greatest compact subgroup of $Z(\mathbf{R})$, we may in particular, in 3.6, choose T such that $A=Z(\mathbf{R})/Z_0$; hence 3.8 applies to the geodesic action of $Z(\mathbf{R})/Z_0$ on X.

3.9. *Structure of type S for X/A.* We keep the previous notation and let $\sigma: X\to$

$\rightarrow X' = X/A$ be the canonical projection. In view of 3.4, $P(\mathbf{R})$ commutes with σ, whence a transitive action of $P(\mathbf{R})$ on X'. For $x \in X$, the fibre $F_x = \sigma^{-1}(\sigma(x))$ is $x \cdot A_x$ (3.2), hence the isotropy group of $\sigma(x)$ is $(H_x \cap P) \cdot A_x$. The condition S I is then fulfilled, with respect to P, if we let $R_{X'}$ be the subgroup of P generated by R_X and the inverse image of T. In particular if $R_X = R_d G$ and T is the greatest k-split torus of $\mathscr{C}(P/R_u P)$, then $R_{X'} = R_d P$.

For $x \in X$, let $L_{x,P}$ be the Levi subgroup of P assigned to x by the condition S II. If $y \in F_x$, then L_y is conjugate to L_x by an element $a \in A_x$. But $A_x \subset L_x$, hence $L_x = L_y$ and therefore, by construction (2.7), $L_{y,P} = L_{x,P}$. Thus $x \mapsto L_{x,P}$ is constant along the fibres of σ, and S II is clearly satisfied for the action of $P(\mathbf{R})$ on X' if we put $L_{x'} = L_{x,P}$ for any $x \in \sigma^{-1}(x')$. This choice will be understood in the sequel. Therefore X' is canonically of type S under P; by the end remark of the previous paragraph, it is also of type $S-k$ if X is so under G.

The group $Z(\mathbf{R})$ is commutative, therefore the geodesic action of $Z(\mathbf{R})$ goes over to an action on X', trivial on A, and commuting with $P(\mathbf{R})$. In view of the definition of $L_{x'}$ ($x' \in X'$), it is clearly the geodesic action of $Z(\mathbf{R})$ on X', for the structure of space of type S under P just defined.

3.10. Assume G to be connected, and let G' be as in 2.6. Then, by 2.7(2), X is canonically of type S under G' and for $x \in X$ the Levi subgroup L'_x of $G'(\mathbf{R})$ associated to x is contained in L_x. The group $\mathscr{C}(G/U)$ is reductive, therefore the canonical homomorphism $G'/U \rightarrow G'/R_u G'$ maps $\mathscr{C}(G/U) \cap (G'/U)$ isomorphically onto an $\mathbf{R}$-subgroup of $\mathscr{C}(G'/R_u G')$. Let $x \in X$. Since $L'_x \subset L_x$, an element $z \in \mathscr{C}(G/U)(\mathbf{R}) \cap (G'/U)$ and its image in $\mathscr{C}(G'/R_u G')$ have the same lifting associated to x. Consequently, the geodesic actions of z on X, with respect to the structures of type S under G and G', are the same.

If G' is parabolic, then G'/U is parabolic in G/U, hence contains $\mathscr{C}(G/U)$, which is then identified to a subgroup of $\mathscr{C}(G'/R_u G')$, and this identification is compatible with the geodesic actions on X under G and G'. Returning to the situation of 3.2, we may in particular apply this to two $\mathbf{R}$-parabolic subgroups $P \subset Q$ of G, which play the role of G' and G in the preceding discussion, and get:

3.11. PROPOSITION. *Let $P \subset Q$ be two parabolic* $\mathbf{R}$*-subgroups of* G. *Then* $\mathscr{C}(Q/R_u Q)$ *may be canonically identified with a subgroup of* $\mathscr{C}(P/R_u P)$, *and the geodesic action of* $\mathscr{C}(Q/R_u Q)(\mathbf{R})$ *on X is the restriction of the geodesic action of* $\mathscr{C}(P/R_u P)(\mathbf{R})$ *on X.*

II. CORNERS

§4. Parabolic k-Subgroups

4.1. In this section, we review some standard facts on parabolic subgroups and fix some notation. We recall that if H is a k-group, $\mathfrak{P}(H)$ is the set of parabolic k-subgroups of H. Let R be a connected normal solvable k-subgroup of G and $\pi: G \to G/R$ the canonical projection. Then $P \mapsto \pi(P)$ induces a bijection of $\mathfrak{P}(G)$ onto $\mathfrak{P}(G/R)$ whose inverse map is given by $Q \mapsto \pi^{-1}(Q)$. Assume now R to be k-split [4, §15], which is automatically the case if R is unipotent. Then $\pi(k): G(k) \to (G/R)(k)$ is surjective, hence the bijection $\mathfrak{P}(G) \to \mathfrak{P}(G/R)$ preserves conjugacy classes over k. In particular the classification of parabolic k-subgroups up to conjugacy over k is "the same" in G, G/U or G/R_dG; this reduces us to the case of reductive groups. Let S be a maximal k-split torus of G^0/U, ${}_k\Phi = \Phi(S, G/U)$ the set of k-roots of G^0/U with respect to S, and Δ a basis of ${}_k\Phi$ ([8], §5). By [8, §5], the conjugacy classes in $\mathfrak{P}(G)$ with respect to $G^0(k)$ are in $1-1$ correspondence with the subsets of Δ. The class corresponding to $J \subset \Delta$ is represented by the standard parabolic subgroup P_J: the image P_J/U of P_J in G/U is the semi-direct product of its unipotent radical U_J by the centralizer $Z(S_J)$ of S_J, where $S_J = (\bigcap_{\alpha \in J} \ker \alpha)^0$, and its split radical is $S_J \cdot U_J$. Given $P \in \mathfrak{P}$, the only I such that P is conjugate to P_I under $G^0(k)$ will be denoted $I(P)$ and called the type of P.

4.2. Let $P \in \mathfrak{P}(G)$. The quotient $S_P = R_dP/(R_uP \cdot R_dG)$ is a k-split torus, and is also the greatest k-split torus in $C_P = \mathscr{C}(P/(R_uP \cdot R_dG))$. We let A_P be the identity component of $S_P(\mathbf{R})$. Let $P' \in \mathfrak{P}(G)$ be conjugate to P under G^0, and let $x \in G^0$ be such that ${}^xP' = P$. Then $\mathrm{Int}\,x$ induces an isomorphism of $C_{P'}$ onto C_P. If $P'^y = P$ with $y \in G^0$, then, since P' is its own normalizer in G^0, $y \in x \cdot P'$. Clearly, $\mathrm{Int}\,p'$ $(p' \in P')$ induces the trivial automorphism of $C_{P'}$, hence $\mathrm{Int}\,y$ induces the same isomorphism of $C_{P'}$ onto C_P as $\mathrm{Int}\,x$. Since we may take $y \in G^0(k)$, this isomorphism is defined over k, hence defines a canonical isomorphism

$$\sigma_{P',P}: S_{P'} \xrightarrow{\sim} S_P. \tag{1}$$

Let in particular $P' = P_I$ be standard. Then $S_{P'} = S_I/S_\Delta$. The elements of $\Delta - I$ define a basis of $X^*(S_I/S_\Delta) \otimes \mathbf{Q}$, where $X^*(\ )$ denotes the group of rational characters [4], which is carried over onto a basis of $X^*(S_P) \otimes \mathbf{Q}$, to be denoted in the same way. We thus have a canonical isomorphism:

$$A_P \xrightarrow{\sim} (\mathbf{R}_+^*)^{\Delta - I}. \tag{2}$$

4.3. Let Q be a parabolic k-subgroup containing P and let $J = I(Q)$. The inclusion

$R_d Q \subset R_d P$ induces an injective morphism of S_Q into S_P which maps S_Q onto $(\bigcap_{\alpha \in J-I} \ker \alpha)^0$. Let $A_{P,Q} = (\bigcap_{\alpha \in \Delta - J} \ker \alpha) \cap A_P$. Then the product decomposition

$$A_P = A_{P,Q} \times A_Q \tag{3}$$

corresponds to the factorization

$$(\mathbf{R}_+^*)^{\Delta - I} = (\mathbf{R}_+^*)^{J-I} \times (\mathbf{R}_+^*)^{\Delta - J}; \tag{4}$$

The group A_P associated to P, viewed as subgroup of Q, is A_P/A_Q, i.e. is the isomorphic image of $A_{P,Q}$ under the canonical projection. Thus, our A_P above can be written A_{P,G^0}, and then (3) takes the form

$$A_{P,G^0} = A_{P,Q} \times A_{Q,G^0}. \tag{5}$$

For a subset L of $\Delta - I$, let

$$A_{P(L)} = (\bigcap_{\alpha \in L} \ker \alpha) \cap A_P, \tag{6}$$

where $P(L)$ is the parabolic subgroup of type $I \cup L$ containing P. In particular

$$A_{P(\emptyset)} = A_P, \; A_{P(\Delta - I)} = A_{G^0} = \{e\}.$$

4.4. The isomorphism 4.2(2) yields an open embedding of A_P into $\mathbf{R}^{\Delta - I}$. The closure of A_P is $\mathbf{R}_+^{\Delta - I}$ and will be denoted $\bar{A}_P$ (or $\bar{A}_{P,G}$ if we wish to emphasize the ambient group). The elements of $\Delta - I$ are then coordinates on $\bar{A}_P$, taking all positive values (zero included), and they identify $\bar{A}_P$ to the positive quadrant in $\mathbf{R}^{\Delta - I}$. The action of A_P on itself by means of translations extends to one on $\bar{A}_P$, given by coordinate-wise multiplication.

4.5. For every $L \subset \Delta - I$, let o_L be the point with coordinates

$$\alpha(o_L) = \begin{cases} 0 & \alpha \notin L \\ 1 & \alpha \in L. \end{cases}$$

In particular:

$$o_\emptyset = (0, 0, \ldots, 0), \qquad o_{\Delta - I} = (1, 1, \ldots, 1) = e.$$

Then $A_P \cdot o_L = \bar{A}_P(L)$ is the face of $\bar{A}_P$ given by

$$\bar{A}_P(L) = \{x \in \bar{A}_P \mid \alpha(x) = 0 \, (\alpha \notin L), \alpha(x) \neq 0 \, (\alpha \in L)\}. \tag{1}$$

In particular:

$$A_P \cdot o_\emptyset = \bar{A}_P(\emptyset) = \{o_\emptyset\}, \qquad \bar{A}_P(\Delta - I) = A_P \cdot o_{\Delta - I} \cong A_P,$$

and we have the orbit space decomposition:

$$\bar{A}_P = \coprod_{L \subset \Delta - I} \bar{A}_P(L). \tag{2}$$

The isotropy group of o_L is $A_{P(L)}$; we have

$$A_P \cdot o_L = A_{P,P(L)} \cdot o_L \cong A_{P,P(L)} \quad (L \subset \Delta - I), \tag{3}$$

and the orbit map $a \mapsto a \cdot o_L$ extends to a diffeomorphism:

$$\bar{A}_{P,P(L)} \xrightarrow{\sim} Cl(\bar{A}_P(L)). \tag{4}$$

In this formula, the right-hand side is the closure of $\bar{A}_P(L)$ in $\bar{A}_P$ and the left-hand side is defined as in 4.3, but with G replaced by $P(L)$.

§5. The Corner $X(P)$ associated to a Parabolic Subgroup

5.0. From now on, X is a homogeneous space of type $S-k$ for G (2.3). Thus we have $R_X = R_d G$. This latter condition determines uniquely the isotropy groups H_x $(x \in X)$ of $G(\mathbf{R})$ on X; it involves the k-structure of G, as the case of a torus already shows. Note that X is also of type $S-k$ under G^0 or under the group ${}^0(G^0)$ of 1.1. If $G^0 = {}^0(G^0)$, the isotropy groups in X are the maximal compact subgroups of $G(\mathbf{R})$. By 2.1 and 2.4(2), G always has a homogeneous space of type $S-k$.

5.1. We keep the previous notation. By 3.8 and the definition of A_P (4.2), X is a principal A_P-bundle under the geodesic action. By definition, the *corner* $X(P)$ *associated to* P is the total space of the associated bundle with typical fibre $\bar{A}_P$:

$$X(P) = X \times^{A_P} \bar{A}_P. \tag{1}$$

Thus $X(P)$ is the quotient of $X \times \bar{A}_P$ under the equivalence relation: $(x, z) \sim (x', z')$ if and only if there exists $a \in A_P$ such that $x = x' \circ a$ and $z' = a \cdot z$. The space $X(P)$ is endowed with a natural (real analytic) structure of manifold with corners coming from that of the fibres (the components of the boundary being the $e(Q)$ described below).

In view of 4.5(2), we have

$$X(P) = \coprod_{L \subset \Delta - I} X(P, L), \tag{2}$$

where

$$X(P, L) = X \times^{A_P} A_P \cdot o_L. \tag{3}$$

By 4.5(3)

$$X(P, L) = X/A_{P(L)} \times^{A_{P,P(L)}} A_{P,P(L)} \simeq X/A_{P(L)}, \tag{4}$$

in particular

$$X(P, \emptyset) = X/A_P, \quad X(P, \Delta - I) = X. \tag{5}$$

Let us put

$$e_X(Q) = e(Q) = X/A_Q, \quad (Q \in \mathfrak{P}),$$

in particular $e(G^0) = X$. By 3.9, $e(Q)$ is canonically of type $S-k$ under Q. The equality (2) can be written

$$X(P) = \coprod_{Q \in \mathfrak{P}, Q \supset P} e(Q). \tag{6}$$

We have a principal fibration

$$X \to e(Q) \quad \text{with structural group} \quad A_Q. \tag{7}$$

Let J be a subset of L. Then $P(J) \subset P(L) = Q$. Replacing X, G, P by $e(Q), Q, P(J)$, we then have also a principal fibration

$$\nu_{P(J),Q}: e(Q) \to e(P(J)) \quad \text{with structural group} \quad A_{P(J),Q}. \tag{8}$$

We have the factorization $A_{P(J)} = A_{P(J),Q} \times A_Q$. The group $A_{P(J)}$ operates by geodesic action on the fibration (7), and the action induced on $e(Q)$ is the geodesic action of $A_{P(J),Q}$ which underlies (8).

The group $P(\mathbf{R})$ operates on X, and commutes with A_P. The action of $P(\mathbf{R})$ extends then to $X(P)$, leaving the faces $e(Q)$ stable. Since A_P is commutative, its action on X also extends to $X(P)$, leaving each $e(Q)$ stable, and it still commutes with $P(\mathbf{R})$. Moreover, $P(\mathbf{R})$ operates on the fibrations (7), (8).

5.2. Let V be a normal unipotent k-subgroup of G, and $\pi: G \to G' = G/V$ the canonical projection. The group $V(\mathbf{R})$ operates properly and freely on X, and $X' = X/V(\mathbf{R})$ is canonically of type $S-k$ under G' (2.8). Moreover, if $\sigma: X \to X'$ is the canonical projection, then $L_{\sigma(x)} = \pi(L_x)$ $(x \in X)$. If $P \in \mathfrak{P}$ and $P' = \pi(P)$, then $A_{P'}$ is canonically identified with A_P and one checks that the geodesic actions of A_P on X and X' commute with σ. As a consequence, $X(P)$ is a principal $V(\mathbf{R})$-bundle over $X'(P')$, and the projection $\tau: X(P) \to X'(P')$ extends σ. For every $Q \supset P$, $Q \in \mathfrak{P}$ the restriction of τ to $e_X(Q)$ is the projection of a principal $V(\mathbf{R})$-fibration with base $e_{X'}(Q)$.

5.3. PROPOSITION. *Let $P \subset Q$ be two parabolic k-subgroups of G and $I_0 = I(Q) - I(P)$. The inclusion $X(Q) \hookrightarrow X(P)$ is an isomorphism of manifolds with corners of $X(Q)$ onto an open subset of $X(P)$. We have $Cl_{X(P)}\, e(Q) = \coprod_{Q \supset R \supset P, R \in \mathfrak{P}} e(R) = e(Q)(P)$,* *where $e(Q)$ is viewed as a space of type $S-k$ under Q (5.1), and $e(Q)(P)$ is the corner of $e(Q)$ associated to the parabolic k-subgroup P of Q.*

The canonical factorization (4.3(3))

$$A_P = A_{P,Q} \times A_Q$$

yields an embedding

$$A_{PQ} \times \bar{A}_Q \to \bar{A}_P \tag{1}$$

which, for every $L \subset \Delta - I(Q)$, induces a homeomorphism

$$A_{PQ} \times \bar{A}_Q(L) \xrightarrow{\sim} \bar{A}_P(L \cup I_0), \quad (L \subset \Delta - I(Q)). \tag{2}$$

We have clearly

$$X \times^{A_Q} \bar{A}_Q = X \times^{A_{PQ} \times A_Q} (A_{PQ} \times \bar{A}_Q), \tag{3}$$

$$X \times^{A_Q} \bar{A}_Q(L) = X \times^{A_{PQ} \times A_Q} (A_{PQ} \times \bar{A}_Q(L)). \tag{4}$$

In view of (1), (2), this yields

$$X(Q) = X \times^{A_P} (A_{PQ} \times \bar{A}_Q), \tag{5}$$

$$e(Q(L)) = X(Q, L) = X \times^{A_P} (\bar{A}_P(L \cup I_0)), \quad (L \subset \Delta - I(Q)). \tag{6}$$

The inclusion $X(Q) \hookrightarrow X(P)$ is then defined by the inclusion (1) of the typical fibres in the right-hand sides of (5) and 5.1(1). By (2) and (6), its restriction to $X(Q, L)$ is an isomorphism of $X(Q, L)$ onto $X(P, L \cup I_0)$. Taking 5.1(4) into account, we get canonical isomorphisms

$$e(Q(L)) = X/A_{Q(L)} \cong X(Q, L) \to e(P(L \cup I(Q)) \cong X/A_{P(L \cup I_0)}$$
$$(L \subset \Delta - I(Q)). \tag{7}$$

Thus $X(Q, L)$ is endowed with structures of space of type S under both Q and P (5.1). It is immediate from the definitions that the latter structure can also be associated to P viewed as a subgroup of Q, hence the inclusion commutes with the geodesic action of A_P.

Let $J \subset \Delta - I(P)$. The closure of $\bar{A}_P(J)$ is the set of points of $\bar{A}_P$ on which the elements of $\Delta - I(P) - J$ are zero. Therefore

$$Cl(\bar{A}_P(J)) = \coprod_{L \subset J} \bar{A}_P(L). \tag{8}$$

But

$$Cl(e(P(J))) = X \times^{A_P} Cl(\bar{A}_P(J)) \tag{9}$$

whence

$$Cl(e(P(J)) = \coprod_{L \subset J} X(P, L) = \coprod_{P(J) \supset R \supset P, R \in \mathfrak{P}} e(R). \tag{10}$$

Let now $J = I_0$, i.e. $P(J) = Q$. We have $A_P = A_{PQ} \times A_Q$ and A_Q acts trivially on $Cl(\bar{A}_P(J))$ whence

$$Cl(e(Q)) = (X/A_Q) \times^{A_{PQ}} Cl(\bar{A}_P(I_0)). \tag{11}$$

In view of 4.5(4), this can be written

$$Cl(e(Q)) = (X/A_Q) \times^{A_{PQ}} \overline{\bar{A}_{PQ}} \tag{12}$$

or, taking 4.3 into account

$$Cl(e(Q)) = e(Q)(P). \tag{13}$$

Together with (10) and 4.3, this proves the second assertion of 5.3.

5.4. *Canonical cross-sections.* Let $J \subset L$ be subsets of $\Delta - I(P)$. Put $Q = P(J)$, $R = P(L)$ and consider the fibration 5.1(8) with structural group $A_{Q,R}$

$$\nu_{Q,R}: X(P, L) \to X(P, J) \tag{1}$$

which can also be written

$$\nu_{Q,R}: e(R) \to e(Q). \tag{2}$$

The space $X(P, L)$ is of type S under Q, associated to $R_d R$, hence it is of type S under ${}^0Q(\mathbf{R})$, and the isotropy groups in ${}^0Q(\mathbf{R})$ are its maximal compact subgroups.

Let $y \in X(P, L)$ and M its isotropy group in ${}^0Q(\mathbf{R})$. By 3.6, the map $A_{Q,R} \times {}^0Q(\mathbf{R}) \to X(P, L)$ defined by $(a, q) \mapsto (y \circ a) \cdot q$ induces an isomorphism

$$\mu_y: A_{Q,R} \times X(P, J) \simeq X(P, L), \tag{3}$$

which commutes with ${}^0Q(\mathbf{R})$ acting in the usual way on $X(P, L)$ and $X(P, J)$. The images of the sets $\{a\} \times X(P, J)$ are the orbits of ${}^0Q(\mathbf{R})$ and will be called the canonical or standard cross-sections of the fibration (1).

Let now $x \in X$. For $Q \supset P$, denote x_Q its projection on $e(Q)$. The trivialization

$$\mu_x: A_P \times e(P) \simeq X \tag{4}$$

induces one of the associated bundle $X(P)$

$$\mu_x: \bar{A}_P \times e(P) \simeq X(P) \tag{5}$$

which commutes with A_P and ${}^0P(\mathbf{R})$. It is immediate from the definitions that, on the face $e(Q)$, this trivialization coincides with the trivialization given by (3) with $y=x_Q$, and $J=\emptyset$. If we replace G by Q, we get similarly an isomorphism

$$\mu_x: \bar{A}_{P,Q} \times e(P) \to Cl_{X(P)}\, e(Q) \cong e(Q)(P) \tag{6}$$

(cf. 5.3). We have then $X(P) \cong \bar{A}_{P,Q} \times \bar{A}_Q \times e(P)$, whence also an isomorphism

$$\mu_x: \bar{A}_Q \times e(Q)(P) \xrightarrow{\sim} X(P) \tag{7}$$

which commutes with A_Q and ${}^0P(\mathbf{R})$, acting in the obvious way. It is immediate from the definitions that the diagram

$$\begin{array}{ccc} \bar{A}_{P,Q} \times e(P) & \xrightarrow[\sim]{\mu_{xQ}} & e(Q)(P) \\ \downarrow & & \downarrow \\ \bar{A}_P \times e(P) & \xrightarrow[\sim]{\mu_x} & X(P), \end{array} \tag{6}$$

where the vertical arrows are the canonical injections, is commutative, and that all maps commute with the natural actions of ${}^0P(\mathbf{R})$ and A_P.

Together with 5.3, this shows the commutativity of the following diagram, where the vertical arrows are inclusions:

$$\begin{array}{ccc} \bar{A}_R \times e(R)(Q) & \overset{\mu_x}{\xrightarrow{\sim}} & X(Q) \\ \downarrow & & \downarrow \\ \bar{A}_R \times e(R)(P) & \overset{\mu_x}{\xrightarrow{\sim}} & X(P) \end{array} \tag{7}$$

5.5. PROPOSITION. *Let $Q, R \in \mathfrak{P}$ be such that $Q \cap R \in \mathfrak{P}$. Then the geodesic action of A_Q on X extends to a geodesic action on $e(R)$.*

Let $P = Q \cap R$. Then $e(Q)$ and $e(R)$ may be canonically identified to faces of $X(P)$ (5.3) and A_Q, A_R to subgroups of A_P. The action of A_Q is then defined by the extended geodesic action of A_P on $X(P)$.

Remark. Let $P' \in \mathfrak{P}$, $P' \subset P$. The canonical inclusions $X(P) \hookrightarrow X(P')$ and $A_Q \hookrightarrow A_P$ being compatible with the extended geodesic actions, it is clear that the above action of A_Q on $e(R)$ can also be defined using the corner $X(P')$.

5.6. Let $g \in G(k)$, $P \in \mathfrak{P}$, $P' = P^g$ and $I = I(P) = I(P')$. Then $\operatorname{Int} g^{-1}$ induces an isomorphism $\alpha: A_P \to A_{P'}$ compatible with the identifications of both groups with $(\mathbf{R}_+^*)^{\Delta - I}$ (4.2), hence it extends to an isomorphism of $\bar{A}_P$ onto $\bar{A}_{P'}$ also denoted $a \mapsto a^g$. We have

$$x \cdot p \cdot g = (x \cdot g) \cdot p^g \qquad (x \circ a) \cdot g = (x \cdot g) \circ a^g \qquad (a \in A_P,\ x \in X,\ p \in P(\mathbf{R})) \tag{1}$$

cf. 3.2(5); therefore $x \mapsto x \cdot g$ extends to an isomorphism of $X(P)$ onto $X(P')$, also denoted $x \mapsto x^g$, which also satisfies (1) with $x \in X(P)$. Let $Q \in \mathfrak{P}$, $Q \supset P$ and $Q' = Q^g$. Then $x \mapsto x^g$ induces an isomorphism $e(Q) \to e(Q^g)$ which is induced by passage to the quotient from the translation $x \mapsto x \cdot g$. Translation by g also gives rise to a commutative diagram of trivializations:

$$\begin{array}{ccc} \bar{A}_Q \times e(Q)(P) & \to & \bar{A}_{Q'} \times e(Q')(P') \\ \downarrow \mu_x & & \downarrow \mu_{x \cdot g} \\ X(P) & \to & X(P') \end{array} \tag{2}$$

The map μ_x (resp. μ_{xg}) commutes with $A_Q \times {}^0P(\mathbf{R})$ (resp. $A_{Q'} \times {}^0P'(\mathbf{R})$) (5.4). In particular, if $g \in Q$, then $Q = Q'$, $a^g = a$ $(a \in A_Q)$, and all maps in (2) commute with A_Q.

§ 6. Topology of $X(P)$ and Siegel Sets

6.1. Let $P \in \mathfrak{P}$, and $I = I(P)$. For $t > 0$, we put

$$\begin{aligned} A_{P,t} &= \{a \in A_P \mid \alpha(a) \leqq t, \quad (\alpha \in \Delta - I)\} \\ \bar{A}_{P,t} &= \{a \in \bar{A}_P \mid \alpha(a) \leqq t, \quad (\alpha \in \Delta - I)\}. \end{aligned} \tag{1}$$

Let $x \in X$. A *Siegel set* in X, with respect to P, x, is a set

$$\mathfrak{S} = \mathfrak{S}_{t,\omega} = (x \circ A_{P,t}) \cdot \omega \tag{2}$$

where ω is a relatively compact subset of ${}^0P(\mathbf{R})$.

Let x_P be the canonical projection of x on $e(P)$. Then, if $\mu_x : A_P \times e(P) \to X$ is the canonical isomorphism of 5.5, we have

$$\mu_x^{-1}(\mathfrak{S}_{t,\omega}) = A_{P,t} \times x_P \cdot \omega. \tag{3}$$

In particular, every point $y \in X$ has a neighborhood of this form. If $P = G^0$, then $A_P = \{e\}$, and the Siegel sets with respect to P are just relatively compact subsets.

Let S' be a maximal torus of $R_d P$ stable under the Cartan involution of L_x with respect to a maximal compact subgroup of H_x. Let $A' = S'(\mathbf{R})^0$. Then $P(\mathbf{R}) = A' \ltimes {}^0P(\mathbf{R})$ and there is a canonical projection $\sigma : A' \to A_P$. Let $y \in X$. There exists $p \in P(\mathbf{R})$ such that $y = x \cdot p$. Write $p = a' \cdot q$ with $a' \in A'$ and $q \in {}^0P(\mathbf{R})$. Then $y = (x \circ \sigma(a')) \cdot q$, and

$$(y \circ A_{P,t}) \cdot \omega = (x \circ \sigma(a') \cdot A_{P,t}) \cdot q \cdot \omega.$$

From this it is clear that any Siegel set with respect to x, P is contained into one with respect to y, P and conversely. Thus the choice of the origin matters little.

6.2. PROPOSITION. *For $Q \supset P$, $Q \in \mathfrak{P}$, let $J \subset \Delta - I$ such that $Q = P(J)$ and x_Q be* *the canonical projection of* x *onto* $X(P, J)$. *Let* $y \in e(P)$ *and* $p \in {}^0P(\mathbf{R})$ *be such that* $y = x_P \cdot p$. *Let* $\mathfrak{S} = \mathfrak{S}_{t,\omega}$ *be a Siegel set with respect to* P, x.

(i) *The closure* $\bar{\mathfrak{S}}$ *of* $\mathfrak{S}$ *in* $X(P)$ *is compact. We have*

$$\bar{\mathfrak{S}} \cap X(P, J) = (x_Q \circ A_{P,Q,t}) \cdot \bar{\omega},$$

where $A_{P,Q,t} = A_{P,Q} \cap A_{P,t}$. *In particular, the left-hand side is a Siegel set in* $X(P, J)$, *with respect to* P *and* x_Q, *and any such Siegel set can be obtained in this way.*

(ii) *Let* $t_i \to 0$ *and* ω_i *be a fundamental decreasing sequence of relatively compact neighborhoods of* e *in* ${}^0P(\mathbf{R})$, $(i = 1, 2, \ldots)$. *The closures in* $X(P)$ *of the sets* $(x \circ A_{P,t_i}) \cdot \cdot p \cdot \omega_i$ *form a fundamental system of neighborhoods of* y.

The canonical isomorphism μ_x extends to $\bar{A}_P \times e(P) \xrightarrow{\sim} X(P)$ and 6.1(3) implies

$$\mu_x^{-1}(\bar{\mathfrak{S}}) = \bar{A}_{P,t} \times x_P \cdot \bar{\omega}, \tag{1}$$

which proves (ii) and the first part of (i). The second part of (i) follows from (1) and 5.4(6).

6.3. It is clear from the definitions and the equalities

$$G(\mathbf{R}) = K \cdot G^0(\mathbf{R}), \qquad G^0(\mathbf{R}) = (G^0 \cap H_x) \cdot {}^0(G^0)(\mathbf{R})$$

(1.2, 1.4) that Siegel sets do not change if we replace G by G^0 or ${}^0(G^0)$. Similarly, let V be a normal k-subgroup of R_dG and let $X' = X/V(\mathbf{R})$, viewed as usual as a space of type $S-k$ for $G' = G/V$ (2.8). Then the image of a Siegel set in X under the canonical projection is a Siegel set in X', and any such set can be obtained in this way. In particular, in discussing properties of Siegel sets in X, we may always assume G to be connected, reductive and even to have no non-trivial central k-split torus. The most important ones will be deduced from reduction theory. For this, we have to relate the present Siegel sets to those considered in [3], which are subsets of reductive groups.

6.4. Assume then G to be reductive. Let P be a minimal parabolic k-subgroup of G and K a maximal compact subgroup of $G(\mathbf{R})$. Let L_P be the Levi subgroup of $P(\mathbf{R})$ stable under the Cartan involution θ_K (1.9), $S'_P = L_P \cap R_d(P)$ and $A'_P = (S'_P)^0$. The elements of Δ define a surjective homomorphism $A'_P \xrightarrow{\sim} (\mathbf{R}^*_+)^\Delta$ which goes over to the canonical isomorphism $A_P \xrightarrow{\sim} (\mathbf{R}^*_+)^\Delta$ under the natural projection $A'_P \to A_P$ and whose kernel is $A'_P \cap \mathscr{C}(G)$.

We now want to prove:

$$(H_x \cap R_dG)^0 = (H_x \cap A'_P)^0, \quad (x \in X \text{ fixed under } K). \tag{1}$$

The group $H_x \cap R_dG$ is contained in P, stable under θ_K (2.1(iv)), hence contained in

L_P (1.9), and we have

$$(H_x \cap R_d G)^0 \subset (H_x \cap L_P \cap R_d P)^0 = (H_x \cap A'_P)^0 \subset (H_x \cap R_d P)^0. \tag{2}$$

The group $H_x \cap R_d G$ is contained in $R_d P$, and $H_x = K \cdot (H_x \cap R_d G)$, hence

$$H_x \cap R_d P = (K \cap R_d P) \cdot (H_x \cap R_d G),$$
$$(H_x \cap R_d P)^0 = (K \cap R_d P(\mathbf{R})^0)^0 \cdot (H_x \cap R_d G)^0.$$

But $R_d P(\mathbf{R})^0$ is the semi-direct product of A'_P and $R_u P(\mathbf{R})$, hence is contractible, and has no compact subgroup $\neq \{e\}$. Therefore

$$(H_x \cap R_d P)^0 = (H_x \cap R_d G)^0$$

which, together with (2), proves (1).

Define $A'_{P,t}$ in the same way as $A_{P,t}$. A Siegel set of $G(\mathbf{R})$ (with respect to K, P, S') is then a set of the form

$$\mathfrak{S}' = \mathfrak{S}'_{t,\omega} = K \cdot A'_{P,t} \cdot \omega, \tag{3}$$

where ω is a relatively compact subset of ${}^0P(\mathbf{R})$. This is the definition of a standard normal Siegel set in [3, §12], except that we do not require S' to be defined over k, and ω to be a neighborhood of e. The maximal tori defined over $\mathbf{R}$ of $R_d P$ are conjugate under $P(\mathbf{R})$. Therefore, given P, we may always choose x so that S' is defined and split over k.

Let $x \in X$ be such that $K \subset H_x$. Then

$$x \cdot \mathfrak{S}'_{t,\omega} = (x \circ A_{P,t}) \cdot \omega = \mathfrak{S}_{t,\omega} \tag{4}$$

is a Siegel set of X, with respect to P, x, as defined in 6.1. By definition (see 2.3 and (1)), $H_x = K \cdot (H_x \cap R_d G)^0 = K \cdot (H_x \cap A'_P)^0$. Since $H_x \cap A'_P$ is the intersection of the kernels of the elements of Δ, we have

$$H_x \cdot \mathfrak{S}' = \mathfrak{S}', \quad \mathfrak{S}' = \pi_x^{-1}(x \cdot \mathfrak{S}'), \tag{5}$$

where $\pi_x : G(\mathbf{R}) \to X$ is the orbital map $g \mapsto x \cdot g$. Thus the Siegel sets in $G(\mathbf{R})$, with respect to K, P are the inverse images of the Siegel sets in X with respect to P, x, where x is fixed under K.

6.5. PROPOSITION. *Let G be reductive. Let P be a parabolic k-subgroup of G, P_0 be a minimal parabolic k-subgroup of G contained in P, and $g \in G^0(k)$. Let $x \in e(P)$, $\{x_n\}$ $(n = 1, 2, \ldots)$ a sequence of points of X which tends to x in $X(P_0)$ and such that $\{x_n \cdot g\}_{n \geq 1}$ is relatively compact in $X(P_0)$. Then $g \in P(k)$.*

Fix a point $x_0 \in X$. By 6.2, there exists a Siegel set $\mathfrak{S}=\mathfrak{S}_{t,\omega}$ in X, with respect to x_0, P_0, which contains x_n and $x_n \cdot g$ for all n's. Let then $a_n \in A_{P,t}$ and $p_n \in \omega$ be such that $x_n=(x_0 \circ a_n)\cdot p_n$, $(n=1,2,\ldots)$. Let I be such that $P=P_{0I}$. Then $e(P)=X(P_0, I)$. The isomorphism

$$\mu=\mu_{x_0}\colon \bar{A}_{P_0} \times e(P_0) \simeq X(P_0)$$

of 5.4 maps $A_{P_0,P} \times e(P_0)$ onto $e(P)$. Thus, if we write $\mu^{-1}(x)=(a,b)$ with $a \in \bar{A}_{P_0}$, $b \in e(P_0)$, we have $\alpha(a)=0$ for $\alpha \in \Delta - I$. As a consequence

$$\lim \alpha(a_n)=0 \qquad (\alpha \in \Delta - I). \tag{1}$$

We now fix a maximal compact subgroup K of the stability group of x_0, let π: $G(\mathbf{R}) \to X$ be the orbital map $g \mapsto x_0 \cdot g$ and $\mathfrak{S}'=\pi^{-1}(\mathfrak{S})$. Then $\mathfrak{S}'$ is a Siegel set in $G(\mathbf{R})$, with respect to K, P_0 (6.4), and we have $\mathfrak{S}'=K\cdot A'_{P,t}\cdot \omega$, in the notation of 6.4. Let then a'_n be an element of $A'_{P,t}$ which maps onto a_n under the natural projection. Our assumptions and (1) imply

$$\begin{aligned} & a'_n \cdot p_n,\ a'_n \cdot p_n \cdot g \in \mathfrak{S}' && (n=1,2,\ldots), \\ & \lim \alpha(a'_n)=0 && (\alpha \in \Delta - I). \end{aligned}$$

We have then $g \in P$ by Prop. 12.6 of [3].

§7. The Manifold with Corners $\bar{X}$

7.1. We shall denote by $\bar{X}$ or $\bar{X}(G)$ the disjoint union of the sets $e(P)$ $(P \in \mathfrak{P})$ (where, by definition, $e(G^0)=X$). For $P \in \mathfrak{P}$, we identify $X(P)$ with $\bigcup_{Q \supset P} e(Q)$ (see 5.1(6)). We have then

$$X(P) \cap X(Q)=X(R), \qquad (P, Q \in \mathfrak{P}), \tag{1}$$

where R is the smallest parabolic k-subgroup of G containing P and Q. By 5.3, the inclusion map $X(P') \to X(P)$ $(P \subset P' \in \mathfrak{P})$ is an isomorphism, of manifolds with corners, of $X(P')$ onto an open submanifold of $X(P)$. There exists therefore one and only one structure of manifold with corners on $\bar{X}$ such that the $X(P)$'s are open submanifolds with corners of X. The space $\bar{X}$ will always be endowed with that structure.

For every $P \in \mathfrak{P}$, the subspace $e(P)$ has an open neighborhood which meets only finitely many $e(Q)$'s $(Q \in \mathfrak{P})$, namely $X(P)$. Consequently *the $e(P)$'s $(P \in \mathfrak{P})$, or their closures in $\bar{X}$, form a locally finite cover of $\bar{X}$.*

By definition, we have

$$\bar{X}=\coprod_{P \in \mathfrak{P}} e(P)=\bigcup_{P \in \mathfrak{P}} X(P), \tag{2}$$

and the $X(P)$ $(P\in\mathfrak{P})$ form an open cover of $\bar{X}$. For $Q\in\mathfrak{P}$, we let

$$Y(Q)=\bigcup_{P\in\mathfrak{P}(Q)} X(P). \tag{3}$$

We have then also

$$Y(Q)=\coprod_{R\in\mathfrak{P},\, R\cap Q\in\mathfrak{P}} e(R). \tag{4}$$

7.2. (i) The space X is canonically of type S under G^0, and $\mathfrak{P}(G)=\mathfrak{P}(G^0)$ by definition. Therefore $\bar{X}(G)=\bar{X}(G^0)$.

(ii) Assume G to be reductive. Then $R_dG(\mathbf{R})$ operates trivially on X, and X is of type S under G/R_dG, and also under G^0/R_dG. Since $\mathfrak{P}(G)=\mathfrak{P}(G/R_dG)=\mathfrak{P}(G^0/R_dG)$, we also have natural identifications

$$\bar{X}(G)=\bar{X}(G/R_dG)=\bar{X}(G^0/R_dG).$$

(iii) Let V be a connected unipotent normal k-subgroup of G and $\pi: G\to G'=G/V$, $\sigma: X\to X'=X/V(\mathbf{R})$ the canonical projections; the latter is the projection map of a principal fibration with structural group $V(\mathbf{R})$. By 5.2, for every $P\in\mathfrak{P}(G)$, this fibration extends to a principal fibration

$$X(P)\to X'(P/V) \quad \textit{with structural group} \quad V(\mathbf{R})$$

commuting with A_P, which is therefore also compatible with the inclusions $X(Q)\hookrightarrow X(P)$ $(Q\supset P;\ Q\in\mathfrak{P})$. It follows that these principal fibrations match to give one for $\bar{X}$ over $\bar{X}'$.

7.3. PROPOSITION. (i) *The embedding $e(P)\to\bar{X}$ $(P\in\mathfrak{P})$ extends to an isomorphism of manifolds with corners of $\overline{e(P)}$ onto the closure of $e(P)$ in $\bar{X}$.*

(ii) *For $Q\in\mathfrak{P}$, the space $Y(Q)=\bigcup_{P\in\mathfrak{P}(Q)}X(P)$ is an open neighborhood of $\overline{e(Q)}$. For $x\in X$, the isomorphism $\mu_x: A_Q\times e(Q)\xrightarrow{\sim} X$ (see 5.4) extends to an isomorphism of $\bar{A}_Q\times\overline{e(Q)}$ onto $Y(Q)$, which commutes with A_Q, acting on $\bar{A}_Q\times\overline{e(Q)}$ via its natural action on $\bar{A}_Q$, and on each $e(Q')$, $(Q'\in\mathfrak{P}(Q))$, by geodesic action* (5.5).

(In (i), $\overline{e(P)}$ means the manifold with corners extending $e(P)$, where $e(P)$ is endowed with its canonical structure of space of type $S-k$ under P (5.1).)

(i) Let Z be the closure of $e(Q)$ in $\bar{X}$. Let $P\in\mathfrak{P}$. Since $X(P)$ is open, Z meets $e(P)$ only if $X(P)\cap e(Q)\neq\emptyset$, i.e. if $P\subset Q$. Therefore Z is the union of the spaces $X(P)\cap Z$ for $P\in\mathfrak{P}$, $P\subset Q$. By 5.3, $Z\cap X(P)$ may be canonically identified with $e(Q)(P)$, whence (i).

(ii) Since the $X(P)$'s are open in $\bar{X}$, the space $Y(Q)$ is an open neighborhood of $\overline{e(Q)}$. For $P\in\mathfrak{P}(Q)$, we have, by the above and (6), (7) of 5.4, an isomorphism

$$\mu_x: \bar{A}_Q\times e(Q)(P)\xrightarrow{\sim} X(P) \tag{1}$$

which commutes with A_Q, ${}^0P(\mathbf{R})$ and with the inclusions

$$e(Q)(P) \to e(Q)(P') \quad \text{and} \quad X(P) \to X(P') \quad (P' \subset P;\, P, P' \in \mathfrak{P}(Q)).$$

This proves that the maps μ_x, for $P \in \mathfrak{P}(Q)$, match and define an isomorphism $\mu_x : \bar{A}_Q \times \overline{e(Q)} \xrightarrow{\sim} Y(Q)$ commuting with A_Q, whence (ii). From now on, we identify $\overline{e(P)}$ with the closure of $e(P)$ in $\bar{X}(P \in \mathfrak{P})$.

7.4. COROLLARY. *Let $P, Q \in \mathfrak{P}$. Then $\overline{e(P)} \cap \overline{e(Q)}$ is equal to $\overline{e(P \cap Q)}$ if $P \cap Q \in \mathfrak{P}$ and is empty otherwise. In particular $\overline{e(P)} = \overline{e(Q)}$ if and only if $P = Q$.*

This follows from 7.3(i) and the definition.

7.5. COROLLARY. *Let $P, Q \in \mathfrak{P}$. Then $e(P) \cap \overline{e(Q)} \neq \emptyset \Leftrightarrow e(P) \subset \overline{e(Q)} \Leftrightarrow P \subset Q$.*

This follows again from the fact that, by 7.3(i), $\overline{e(Q)}$ is the union of the $e(Q')$ with $Q' \in \mathfrak{P}(Q)$.

7.6. PROPOSITION. *The action of $G(k) \cdot R_u G(\mathbf{R})$ on X extends to one on $\bar{X}$, which preserves the structure of manifold with corners of $\bar{X}$, and in particular permutes the faces $e(P)$ $(P \in \mathfrak{P})$. For $g \in G(k) \cdot R_u G(\mathbf{R})$ and $P \in \mathfrak{P}$, we have $e(P) \cdot g = e(P^g)$.*

This is clear from 5.6, in particular 5.6(1) and 7.2, or by "transport de structure."

7.7. COROLLARY. *Let $P, Q \in \mathfrak{P}$.*

(1) $\{g \in G(k) \mid P^g = Q\} = \{g \in G(k) \mid e(P) \cdot g \cap e(Q) \neq \emptyset\}$
$= \{g \in G(k) \mid e(P) \cdot g = e(Q)\}$.

(2) $\{g \in G(k) \mid P^g \cap Q \in \mathfrak{P}\} = \{g \in G(k) \mid \overline{e(P)} \cdot g \cap \overline{e(Q)} \neq \emptyset\}$.

(3) $Q(k) = \{g \in G^0(k) \mid \overline{e(Q)} \cdot g \cap \overline{e(Q)} \neq \emptyset\}$.

(1) and (2) follow from 7.4, 7.6. By (2) the right-hand side of (3) is $\{g \in G^0(k) \mid Q^g \cap Q \in \mathfrak{P}\}$, which is known to be $Q(k)$.

7.8. THEOREM. *The manifold with corners $\bar{X}$ is Hausdorff. If k is countable, then $\bar{X}$ is countable at infinity.*

To prove the first assertion, we proceed by induction on $\dim G$. If $\dim G = 0$, then $\bar{X}$ is reduced to a point, so we assume our assertion to be true for every k-group G' of dimension $< \dim G$.

Let $y, y' \in \bar{X}$ and let $\{V_n\}$ (resp. $\{V_n'\}$) $(n = 1, 2, \dots)$ be a fundamental sequence of neighborhoods of y (resp. y') such that $V_n \cap V_n' \neq \emptyset$ for all n. We have to prove that $y = y'$. Since a corner $X(P)$ $(P \in \mathfrak{P})$ is open, by definition, and Hausdorff, it suffices to show that y and y' belong to one. Assume first $U = R_u G \neq \{e\}$. Let $X' = X/U(\mathbf{R})$.

By 7.2, the projection $X \to X'$ extends to one σ of $\bar{X}$ onto $\bar{X}'$, and we have

$$\sigma(X(P)) = X'(P/U), \quad \sigma^{-1}(X'(P/U)) = X(P), \quad (P \in \mathfrak{P}).$$

Since $\bar{X}'$ is Hausdorff by induction, we have $\sigma(y) = \sigma(y') \in X'(P/U)$ for some $P \in \mathfrak{P}$, whence $y, y' \in X(P)$.

This reduces us to the case where G is reductive. Let $P, P' \in \mathfrak{P}$ be the parabolic k-subgroups such that $y \in e(P)$ and $y' \in e(P')$. By 7.1(1), $X(P) \cap X(P')$ is the union of the $e(Q)$, with $Q \in \mathfrak{P}$, $Q \supset P, P'$. Since these are finite in number, there exists $Q \in \mathfrak{P}$ such that $e(Q) \cap V_n \cap V_n' \neq \emptyset$ for all n's. The points y and y' then belong to the closure of $e(Q)$, which may be identified with $\overline{e(Q)}$ by 7.3. If $Q \neq G^0$, the induction assumption, applied to Q and $\overline{e(Q)}$, shows that $y = y'$. So assume $Q = G^0$, i.e. $V_n \cap V_n' \cap X \neq \emptyset$ for all n's, and let $x_n \in V_n \cap V_n' \cap X$ $(n = 1, 2, \ldots)$. Thus both y and y' are limit points of the sequence $\{x_n\}$. Let P_0 be a minimal parabolic k-subgroup of G contained in P, and $g \in G^0(k)$ be such that $P_0 \subset P'^g$. We have then

$$x_n \to y \in e(P) \subset X(P_0), \quad x_n \cdot g \to y' \cdot g \in e(P'^g) \subset X(P_0).$$

We may assume $x_n, x_n \cdot g \in X(P_0)$ for all n's. The $x_n \cdot g$ then form a relatively compact subset of $X(P_0)$, and we have $g \in P$ by 6.5. The relation $P'^g \cap P \supset P_0$ then yields

$$P' \cap P = (P'^g \cap P)^{g^{-1}} \supset P_0^{g^{-1}}$$

whence $e(P'), e(P) \subset X(P_0^{g^{-1}})$; this shows that y and y' are contained in one corner, and finally that $y = y'$ since, as remarked above, each corner is Hausdorff.

Assume now k to be countable. Then so is $G(k)$, and also $\mathfrak{P}$, since the latter is the union of finitely many orbits of $G^0(k)$. Since each $e(P)$ is a countable union of compact subsets, the second assertion follows.

7.9. COROLLARY. *Let $P \in \mathfrak{P}$ and $x \in X$. Then the closure in $\bar{X}$ of a Siegel set $\mathfrak{S}$ with respect to x, P is contained in $X(P)$ and is compact.*

The closure $A = Cl_{X(P)}(\mathfrak{S})$ of $\mathfrak{S}$ in $X(P)$ is compact by 6.2. Since $\bar{X}$ is Hausdorff, this implies that A is compact and closed in $\bar{X}$, hence $A = Cl_{\bar{X}}(\mathfrak{S})$.

§8. Homotopy Type of $\partial\bar{X}$

8.1. *Retracts*

We recall some basic facts about *absolute retracts* (AR) and *absolute neighborhood retracts* (ANR) in the category of metric spaces. Proofs can be found in [24], [16], [17] and [19], App. II.

A metric space X is an AR if and only if the following equivalent conditions are satisfied:

(AR_1) X is a retract of any metric space which contains it as a closed subspace.

(AR_2) For any continuous map $f: A \to X$, where A is a closed subspace of a metric space Y, there exists a continuous map $F: Y \to X$ which extends f.

Similarly, the fact that X is an ANR can be characterized by the equivalent conditions:

(ANR_1) For any embedding of X as a closed subspace of a metric space Z, there is a neighborhood of X in Z of which X is a retract.

(ANR_2) For any continuous map $f: A \to X$, where A is a closed subspace of a metric space Y, there is a neighborhood U of A in Y, and a continuous map $F: U \to X$, such that F extends f.

The property of being ANR is *local* ([24], p. 8); every metrizable manifold (with boundary) is an ANR ([24], p. 3).

If X and Y are ANR's, every weak homotopy equivalence $f: X \to Y$ is a homotopy equivalence ([24], th. 15).

If X is an ANR, then (cf. [24], p. 5):

X is an AR $\Leftrightarrow X$ is contractible $\Leftrightarrow$ all $\pi_i(X)$ are 0.

8.1.1. LEMMA. *Let Y be a metric space, X a closed subspace of Y and $f: X \to Z$ a continuous map of X into a topological space Z. Assume X is an* ANR *and Z is contractible. Then f can be extended to a continuous map $F: Y \to Z$.*

By (ANR_1) we can choose a neighborhood U of X in Y of which X is a retract. The map f can be extended to a continuous map $f': U \to Z$; since Z is contractible, f' is homotopic to a constant map. Since a constant map can be extended to Y, the same is true for f, by the "homotopy extension theorem," cf. [12], p. 1–05.

8.2. *Nerves*

We need a variant of Weil's theorem ([30], p. 141) comparing a space with the nerve of one of its covers.

Let Y be a space, and $(Y_i)_{i \in I}$ a locally finite cover of Y by closed non-empty subsets. Let T be the *nerve* of that cover; it is a simplicial complex, whose set of vertices is I; a simplex s of T is a finite subset of I such that $Y_s = \bigcap_{i \in s} Y_i$ is non-empty. We denote by S the set of simplices of T, and by $|T|$ (resp. by $|s|$, for $s \in S$) the geometrical realization of T (resp. s); we put on $|T|$ the *weak topology*: a subset U of $|T|$ is open if and only if $U \cap |s|$ is open in $|s|$ for any $s \in S$ ([19], p. 41). We make the following assumptions:

(1) T has *finite dimension*, i.e. there exists an integer N such that $\mathrm{Card}(s) < N$ for all $s \in S$.

(2) All the Y_s, $s \in S$, are *absolute retracts*, cf. 8.1.

8.2.1. THEOREM. *The spaces Y and $|T|$ have the same homotopy type.*

We prove a more precise statement. Identify $|T|$ as usual with the subspace of $\mathbf{R}^{(I)}$ made of those $(x_i)_{i\in I}$ with $0\leqslant x_i\leqslant 1$, $\sum x_i=1$, and $\{i \mid x_i>0\}\in S$. If $i\in I$, call $|T_i|$ the subspace of $|T|$ made of those (x_i) such that $x_i\geqslant x_j$ for all $j\in I$. If $s\in S$, we put $|T_s|=\bigcap_{i\in s}|T_i|$; it is the *star* of the barycenter of s in the first barycentric subdivision $|T^1|$ of $|T|$, see below; it is contractible. The refined form of th. 8.2.1. is

8.2.2. THEOREM. (i) *There exist continuous maps $f\colon Y\to|T|$ and $g\colon|T|\to Y$ such that $f(Y_i)\subset|T_i|$ and $g(|T_i|)\subset Y_i$ for every $i\in I$; they are unique, up to homotopy.*

(ii) *If f and g are chosen as in* (i), *$f\circ g$ and $g\circ f$ are homotopic to the identity.*

Proof of (i).

(i_1) *Construction of $f\colon Y\to|T|$.*

If $n>0$, call $S(n)$ the set of $s\in S$ with $\mathrm{Card}(s)\geqslant n$, and put $Y_n=\bigcup_{s\in S(n)} Y_s$. We have

$$\emptyset = Y_N \subset Y_{N-1} \subset\cdots\subset Y_2 \subset Y_1 = Y\,.$$

We use decreasing induction on n to construct a continuous map

$$f_n\colon Y_n \to |T|$$

such that $f_n(Y_s)\subset|T_s|$ for all $s\in S(n)$. To get f_n from f_{n-1}, we have to define $f_{n,s}\colon Y_s\to\to|T_s|$ for every s with $\mathrm{Card}(s)=n$, and $f_{n,s}$ is known on all Y_t with $t\supset s$, $t\neq s$. Using assumption (2) together with Lemma 3.2 of [24], one sees that the union of those Y_t is an ANR, which is closed in Y_s; the existence of $f_{n,s}$ then follows from Lemma 8.1.1 since $|T_s|$ is contractible. This completes the induction process, hence the construction of $f_1=f$.

The uniqueness of f (up to homotopy) is proved in a similar way; one uses the fact that, if Z is an AR (resp. an ANR), the same is true for $Z\times[0,1]$.

(i_2) *Construction of $g\colon|T|\to Y$.*

Let T^1 be the first barycentric subdivision of T. The set of vertices of T^1 is S. A subset σ of S is a simplex of T^1 if and only if it is totally ordered by inclusion; we then denote by $s(\sigma)$ (resp. $t(\sigma)$) its smallest (resp. biggest) element. We identify the topological spaces $|T^1|$ and $|T|$ in the usual way; a vertex s of $|T^1|$ corresponds to the barycenter of the simplex $|s|$ of $|T|$; moreover, $|s|$ is the union of the simplices $|\sigma|$ with $t(\sigma)\subset s$. The star of s in $|T^1|$ is $|T_s|=\bigcap_{i\in s}|T_i|=\bigcup_{s(\sigma)\supset s}|\sigma|$.

If σ is a simplex of T^1, put $Y_\sigma=Y_{s(\sigma)}$. One checks easily that the condition $g(|T_i|)\subset Y_i$ for all $i\in I$ is equivalent to $g(|\sigma|)\subset Y_\sigma$ for all σ's. Since the Y_σ's are contractible, the existence of g follows from the "aspherical carrier theorem" ([19], p. 75–76); the same argument proves the uniqueness of g, up to homotopy.

Proof of (ii).

(ii_1) *The maps $g\circ f$, $\mathrm{Id}_Y\colon Y\to Y$ are homotopic.*

The proof is analogous to (i_1). Using decreasing induction on n, one constructs homotopies

$$F_n: Y_n \times [0, 1] \to Y$$

between $g \circ f$ and Id_Y, such that $F_n(Y_s \times [0, 1]) \subset Y_s$ for all $s \in S(n)$. To get F_n from F_{n+1}, we have to define $F_{n,s}: Y_s \times [0, 1] \to Y_s$ for every s with $\mathrm{Card}(s) = n$, and $F_{n,s}$ is known on the union of $Y_s \times \{0\}$, $Y_s \times \{1\}$ and all $Y_t \times [0, 1]$ with $t \supset s$, $t \neq s$; since Y_s is an AR, the corresponding extension problem is solvable by (AR_2).

(ii_2) *The maps* $f \circ g$, $\mathrm{Id}_{|T|}: |T| \to |T|$ *are homotopic.*

Both maps send each simplex $|\sigma|$ of $|T^1| = |T|$ into $|T_{s(\sigma)}|$, which is contractible. We then apply the aspherical carrier theorem, as above.

8.3. *The* $\overline{e(P)}$*'s.*

We go back to the hypotheses and notation of §7; we assume moreover that the ground field k is *countable*. The manifold with corners $\bar{X}$ is Hausdorff and countable at infinity (th. 7.8), hence metrizable ([24], th. 1); the $\overline{e(P)}$'s, for $P \in \mathfrak{P}$, make up a locally finite closed cover of $\bar{X}$ (7.1, 7.3).

8.3.1. LEMMA. *For every* $P \in \mathfrak{P}$, $\overline{e(P)}$ *is an absolute retract.*

Note first that, from the topological point of view, "corners" and "boundaries" are the same thing, hence $\overline{e(P)}$ is a metrizable manifold with boundary; by 8.1, it is an ANR. Moreover, it is known that a metrizable manifold with boundary has the same homotopy type as its "interior" (this follows for instance from the collar theorem of M. Brown [10] – in the present case, we may also use the differentiable structure of $\overline{e(P)}$ to get a differentiable collar, cf. the Appendix to the present paper). By 3.9, the interior $e(P)$ of $\overline{e(P)}$ is a homogeneous space of type $S-k$ under $P(\mathbf{R})$, hence is homeomorphic to some euclidean space. This shows that $\overline{e(P)}$ is contractible; by 8.1, it is an AR.

8.3.2. *Remark.* Instead of the (global) collar theorem, one may use a local deformation argument to prove that

$$\pi_i(e(P)) \to \pi_i(\overline{e(P)})$$

is surjective for all i, hence that all $\pi_i(\overline{e(P)})$ are 0. The fact that $\overline{e(P)}$ is an AR then follows from 8.1.

8.4. *Comparison between* $\partial\bar{X}$ *and the Tits building of* G.

Recall that G^0 is the biggest element of $\mathfrak{P}$, and that $X = e(G^0)$; all other $e(P)$'s are contained in the boundary $\partial\bar{X}$ of $\bar{X}$. We denote by I the set of maximal elements of $\mathfrak{P} - \{G^0\}$; by abuse of language, an element of I is called a *maximal parabolic subgroup* of G.

8.4.1. THEOREM. *The $\overline{e(P)}$'s, for $P \in I$, make up a locally finite closed cover of $\partial\bar{X}$. This cover has properties* (1) *and* (2) *of* 8.2. *Its nerve is the Tits building T of G.*

(Recall cf. [29], that the *Tits building* of G is the simplicial complex whose set of vertices is I, and whose simplices are the non-empty subsets s of I such that $P_s = \bigcap_{P \in s} P$ is a parabolic subgroup of G. It is canonically isomorphic to the building attached to the Tits system of $G^0(k)/RG(k)$ constructed in [8], cf. Bourbaki, LIE IV, §2, exerc. 10.)

The cover of $\partial\bar{X}$ given by the $\overline{e(P)}$ is locally finite (7.1). If s is a finite non-empty subset of I, we know (cf. 7.4) that $\bigcap_{P \in s} \overline{e(P)}$ is non-empty if and only if $P_s = \bigcap_{P \in s} P$ is parabolic, i.e. if and only if s is a simplex of the Tits building T. If this is the case, we have $\mathrm{Card}(s) \leqslant l$, where l is the rank of the corresponding BN-pair (i.e. the k-rank of the semi-simple group G^0/RG, or equivalently the semi-simple k-rank of G^0/R_uG, cf. [8], def. 4.23); moreover, by 7.4, the intersection of the $\overline{e(P)}$, for $P \in s$, is $\overline{e(P_s)}$, which is an AR by 8.3.1. All the assertions of 8.4.1 are now obvious.

8.4.2. COROLLARY. *The spaces $\partial\bar{X}$ and $|T|$ have the same homotopy type.*

This follows from 8.2. More precisely, 8.2.2 gives homotopy equivalences

$$f: \partial\bar{X} \to |T| \quad \text{and} \quad g: |T| \to \partial\bar{X}$$

which are canonical and inverse to each other, up to homotopy, and allow us to identify the homology groups

$$H_i(\partial\bar{X}, \mathbf{Z}) \quad \text{and} \quad H_i(T, \mathbf{Z}) \qquad (i = 0, 1, \ldots)$$

of $\partial\bar{X}$ and T. By *transport de structure*, this identification is compatible with the action of $G(k)$ on both groups.

8.4.3. *Remark.* For each $x \in X$, the geodesic action (3.2) allows one to construct an *explicit* homotopy equivalence $g_x: |T| \to \partial\bar{X}$ of the type required in 8.2.2. We sketch the construction:

Let T^1 be the first barycentric subdivision of T and σ a simplex of T^1; let t be a point of $|\sigma|$. If s is a vertex of σ (hence a simplex of T), we denote by t_s the s-coordinate of t. Choose now a maximal simplex s_0 of T containing all the $s \in \sigma$, so that $P = P_{s_0}$ is a minimal parabolic subgroup of G, contained in all P_s for $s \in \sigma$. If Δ is the corresponding basis of the k-roots (cf. 4.1), the elements s of σ may be identified with subsets of Δ. For every $\alpha \in \Delta$, put

$$a_\alpha(t) = \textstyle\sum_{\alpha \notin s} t_s,$$

where the sum extends to those $s \in \sigma$ which do not contain α. We have $a_\alpha(t) \in [0, 1]$, and one of them at least is 0. Let $a(t)$ be the element of $\bar{A}_P$ whose coordinates are $a_\alpha(t)$, cf. 4.3. Using the natural map $X \times \bar{A}_P \to \bar{X}$, we get a point $x \cdot a(t)$ of $\bar{X}$, which belongs to $\partial \bar{X}$ and does not depend on the choice of s_0. We now define $g_{x,\sigma}$ on $|\sigma|$ by the formula

$$g_{x,\sigma}(t) = x \cdot a(t).$$

One checks that the $g_{x,\sigma}$ are compatible with each other, and define a continuous map $g_x: |T| \to \partial \bar{X}$ having the required property (there is also a natural extension of g_x to a map $\bar{g}_x: C(|T|) \to \bar{X}$, where $C(|T|)$ is the cone on $|T|$).

It would be interesting to have a similar explicit construction for one of the maps $f: \partial \bar{X} \to |T|$.

8.5. *Homotopy type of* $|T|$.

We keep the notations of 8.3 and 8.4. In particular, l denotes the k-rank of the semi-simple group G^0/RG; the dimension of the Tits building T is $l-1$. The following result is known (cf. [28], [14]):

8.5.1. THEOREM. *The space* $|T|$ *has the homotopy type of a bouquet of* $(l-1)$-*dimensional spheres with the weak topology.*

(When $l=0$, this means that T is empty.)

We just outline the proof.

Assume $l \geqslant 1$, and choose an $(l-1)$-dimensional simplex s of T. Let Σ be the set of "apartments" of T containing s (see, e.g., Bourbaki LIE IV, §2, exerc. 10). It is known that any apartment A is isomorphic to the Coxeter complex of the Weyl group W of G, hence is a subdivision of an $(l-1)$-sphere. This allows us to identify $|A|$ with the sphere S_{l-1}. Now, form the bouquet

$$Bo_\Sigma = \bigvee_{A \in \Sigma} |A|,$$

of the spheres $|A|$, with $A \in \Sigma$, choosing for base-point a point of $|s|$. The inclusion maps $|A| \to |T|$ define a continuous map

$$i: Bo_\Sigma \to |T|$$

and the refined form of th. 8.5.1 is:

8.5.2. THEOREM. *The map* $i: Bo_\Sigma \to |T|$ *is a homotopy equivalence.*

This is proved by remarking first that each apartment A contains a unique $(l-1)$-simplex s_A which is *opposite* to s (the corresponding notion for parabolic subgroups being the one defined in [8], n° 4.8). Moreover, the $(l-1)$-simplices which are not opposite to s make up a *contractible* subcomplex T' of T, and T' contains all the faces of the s_A's. Pinching $|T'|$ to a point, one thus gets a map

$$j:|T| \to Bo_\Sigma$$

which is a homotopy inverse of i (for more details, see [14]).

8.5.3. *Remark.* Note that Bo_Σ, i and j all depend on the choice of s, i.e. of the choice of a minimal parabolic subgroup P of G. Hence, one can only assert that the homotopy equivalences i and j between $|T|$ and Bo_Σ are compatible (up to homotopy) with the action of $P(k)$ on both spaces. Note also that Σ can be identified with the set of maximal k-split tori in P/R_uG; in particular, $P(k)$ acts transitively on Σ. When moreover G is reductive, Σ may be identified with $P(k)/Z(S)(k)$, where S is a maximal k-split torus of P, and $Z(S)$ its centralizer in G^0; in particular, writing P as a semi-direct product $Z(S)\cdot R_uP$, one sees that $R_u(P)(k)$ acts *transitively and freely* on Σ; if $l \geqslant 1$, this implies that $\mathrm{Card}(\Sigma)=\aleph_0$.

8.6. *Homology and cohomology of* $\bar{X}$ *and* $\partial\bar{X}$.
Putting 8.4 and 8.5 together we get:

8.6.1. THEOREM. *The boundary* $\partial\bar{X}$ *of* $\bar{X}$ *is empty if* $l=0$. *If* $l \geqslant 1$, *it has the homotopy type of a bouquet of an infinite number of* $(l-1)$*-spheres.*

For $l=1$, this means that $\partial\bar{X}$ has an infinite number of components, and that each component is contractible.

8.6.2. COROLLARY. *The space* $\partial\bar{X}$ *is* $(l-2)$*-connected, i.e.* $\pi_i(\partial\bar{X})=0$ *for* $i \leqq l-2$.

In particular, $\partial\bar{X}$ is connected if $l \geqslant 2$ and simply connected if $l \geqslant 3$. When $l=2$, $\pi_1(\partial\bar{X})$ is a free (non-abelian) group with an infinite basis.

Denote by $\tilde{H}_i(\partial\bar{X})$ the *reduced* homology groups of $\partial\bar{X}$, defined by:

$\tilde{H}_i(\partial\bar{X}) = H_i(\partial\bar{X}, \mathbf{Z})$ if $i \geqslant 1$

$\tilde{H}_0(\partial\bar{X}) = \mathrm{Ker}: H_0(\partial\bar{X}, \mathbf{Z}) \to \mathbf{Z}$.

Th. 8.6.1 implies:

8.6.3. COROLLARY. *If* $l \geqslant 1$, *the only non-zero* $\tilde{H}_i(\partial\bar{X})$ *is* $\tilde{H}_{l-1}(\partial\bar{X})$; *it is free abelian of infinite rank.*

On the other hand, Lemma 8.3.1, applied to $P=G^0$, gives:

8.6.4. LEMMA. *The space* $\bar{X}$ *is contractible. We have*

$$H_0(\bar{X}) = H^0(\bar{X}) = \mathbf{Z} \quad \textit{and} \quad H_i(\bar{X}) = H^i(\bar{X}) = 0 \quad \textit{for} \quad i \neq 0.$$

Denote now by $H_c^i(\bar{X})$ the i-th cohomology group of $\bar{X}$ *with compact carriers* [13], the coefficient group being $\mathbf{Z}$.

8.6.5. THEOREM. *The groups* $H_c^i(\bar{X})$ *are* 0 *for* $i \neq d-l$, *where* $d = \dim X$. *The group* $H_c^{d-l}(\bar{X})$ *is free abelian; its rank is* 1 *if* $l=0$ *and* $\aleph_0$ *if* $l \geqslant 1$.

Let $\Omega_X = H_c^d(X)$ be the *orientation group* of X; it is a free abelian group of rank 1 whose bases correspond to the two orientations of X. If $l=0$, $\bar{X}$ is equal to X, hence is an orientable manifold, and Poincaré duality gives a canonical isomorphism

$$H_c^i(\bar{X}) = H_{d-i}(\bar{X}) \otimes \Omega_X,$$

whence the required result since $H_j(\bar{X})=0$ for $j>0$ and $H_0(\bar{X})=\mathbf{Z}$.

Assume now $l \geqslant 1$. Since $\bar{X}$ is contractible, the homology exact sequence yields an isomorphism between $\tilde{H}_j(\partial\bar{X})$ and the relative homology group $H_{j+1}(\bar{X}, \partial\bar{X})$. On the other hand, Poincaré duality for manifolds with boundary (see below) gives isomorphisms

$$H_c^i(\bar{X}) = H_{d-i}(\bar{X}, \partial\bar{X}) \otimes \Omega_X.$$

By 8.6.3, this gives $H_c^i(\bar{X})=0$ for $i \neq d-l$, and

$$H_c^{d-l}(\bar{X}) = \tilde{H}_{l-1}(\partial\bar{X}) \otimes \Omega_X = \tilde{H}_{l-1}(T) \otimes \Omega_X, \tag{8.6.6}$$

which is free abelian of infinite rank, see above.

8.6.7. *Remark.* Poincaré duality for non-compact manifolds with boundary is well-known, but not easy to find in the literature. One can for instance prove it by the sheaf-theoretic method of Cartan's seminar ([13], p. 20–04 and 20–05). Another possibility is to apply Poincaré duality to the manifolds (with empty boundary) X and $\partial\bar{X}$ and to use the exact sequence

$$\cdots \to H_c^i(X) \to H_c^i(\bar{X}) \to H_c^i(\partial\bar{X}) \to H_c^{i+1}(X) \to \cdots$$

The details may be left to the reader.

8.6.8. *Remark.* The isomorphism

$$H_c^{d-l}(\bar{X}) = \tilde{H}_{l-1}(T) \otimes \Omega_X, \quad \text{valid for } l \geqslant 1, \tag{8.6.6}$$

is canonical, hence compatible with the natural action of $G(k)$ on both groups. Using 8.5.3, this gives information on the action of $P(k)$ on $H_c^{d-l}(\bar{X})$, where P is a minimal parabolic subgroup of G. Let us assume for simplicity that G is reductive, and put $B=P(k)$, $H=Z(S)(k)$, where S is a maximal k-split torus of P. One then finds that *the* $\mathbf{Z}[B]$*-module* $H_c^{d-l}(\bar{X})$ *is isomorphic to the induced module* $\mathbf{Z}[B]\otimes_{\mathbf{Z}[H]}\Omega_X$. In particular, if we write B as a semi-direct product $B=H\ltimes U$, we see that *the* $\mathbf{Z}[U]$*-module* $H_c^{d-l}(\bar{X})$ *is free of rank* 1; this is analogous to what happens in the *Steinberg representation* of a finite group endowed with a BN-pair, cf. [28].

III. THE QUOTIENT OF $\bar{X}$ BY AN ARITHMETIC SUBGROUP

From now on $k=\mathbf{Q}$ *and* Γ *is an arithmetic subgroup of* $G(\mathbf{Q})$ (0.5).

§9. The quotient $\bar{X}/\Gamma$

9.1. LEMMA. *Let* Y *be a locally compact space,* Z *a closed subspace with empty interior,* L *a discrete group which operates continuously on* Y *and leaves* Z *stable. Assume the following condition to be fulfilled:*

(*) *For any compact subsets* C, D *of* Y

$$\{g\in L \mid C\cdot g\cap D\cap(Y-Z)\neq\emptyset\}\quad \textit{is finite.}$$

Then L *operates properly on* Y.

Let C', D' be compact subsets of Y and C, D compact neighborhoods in Y of C' and D' respectively. Let $g\in L$ be such that $C'\cdot g\cap D'\neq\emptyset$. Then $C\cdot g\cap D$ is a neighborhood of some point in Y, hence it meets $Y-Z$, and we have

$$E=\{g\in L \mid C'\cdot g\cap D'\neq\emptyset\}\subset\{g\in L \mid C\cdot g\cap D\cap(Y-Z)\neq\emptyset\}.$$

Therefore E is finite by (*), which proves the lemma.

9.2. In this section, we consider the following situation: V is a real Lie group, T a locally compact principal V-bundle, L a discrete group operating continuously on T. Let H be the group of homeomorphisms of T, $\sigma:L\to H$ the natural homomorphism and identify V with a subgroup of H. Assume that $\sigma(L)$ normalizes V and that σ is injective on $N=\sigma^{-1}(\sigma(L)\cap V)$. We identify N with a subgroup of V. Let $\pi:T\to T'=T/V$ be the natural projection. It follows from our assumptions that L commutes with π and that the action of L on T induces one of $L'=L/N$ on T' by passage to the quotient.

LEMMA. *We keep the previous notation and assumptions.*

(i) *Assume* L' *to act properly on* T' *and* N *to be discrete in* V. *Then* L *acts properly on* T.

(ii) *Assume moreover* V/N *and* T'/L' *to be compact. Then* T/L *is compact.*

(i) Let C, D be compact subsets of T, and $E=\{g\in L \mid C\cdot g\cap D\neq\emptyset\}$. Then $\pi(E)$ is finite, hence E consists of finitely many subsets of the form $N\cdot g\cap E$, with $g\in E$. But, for $g\in L$:

$$N\cdot g\cap E=\{n\cdot g\mid n\in N \quad \text{and} \quad C\cdot n\cap D\cdot g^{-1}\neq\emptyset\},$$

and the latter set is finite since V acts properly on T and N is discrete in V.

(ii) Our assumptions imply the existence of compact subsets $C\subset T$ and $D\subset N$ such that $T'=\pi(C)\cdot L'$ and $V=D\cdot N$. We have then

$$T=C\cdot V\cdot L=(C\cdot D)\cdot L$$

with $C\cdot D=\bigcup_{d\in D}(C\cdot d)$ compact since both C and D are.

We now prove one of the main results of this paper:

9.3. THEOREM. *The group Γ operates properly on $\bar{X}$. The quotient $\bar{X}/\Gamma$ is compact.*

Let Γ' be a subgroup of finite index of Γ. If our assertions are true for Γ', then they are true for Γ. We may therefore replace Γ by $\Gamma\cap G^0$. Moreover, X is canonically of type $S-\mathbf{Q}$ under G^0, and $\bar{X}(G)=\bar{X}(G^0)$ (7.2). Thus we may (and do) assume G to be connected.

We prove the theorem by induction on $\dim G$. Assume first that $V=R_uP\neq\{e\}$. Let $\sigma:G\to G'=G/V$ and $\pi:X\to X'=X/V(\mathbf{R})$ be the canonical projections. The group $\Gamma'=\sigma(\Gamma)$ is arithmetic in G' [2], $\Gamma\cap V$ is arithmetic in V, $V(\mathbf{R})/(\Gamma\cap V(\mathbf{R}))$ is compact [3; 8.4], X' is canonically of type $S-\mathbf{Q}$ under G' (2.8). The space $\bar{X}$ is a principal $V(\mathbf{R})$-bundle and $\bar{X}/V(\mathbf{R})=\bar{X}'$ (7.2(iii)). By induction assumption Γ' operates properly on $\bar{X}'$ and $\bar{X}'/\Gamma'$ is compact. Our conclusion then follows from 9.2. This reduces us to the case where G is *connected* and *reductive*.

We now prove that Γ acts properly. In view of 9.1, applied to $Y=\bar{X}$, $Z=\partial\bar{X}$ and $L=\Gamma$, it suffices to show that if C, D are compact subsets of $\bar{X}$, then

$$E=\{\gamma\in\Gamma\mid C\cdot\gamma\cap D\cap X\neq\emptyset\} \quad \textit{is finite}. \tag{1}$$

Fix $x\in X$ and let $P\in\mathfrak{P}$. The closure $Cl_{\bar{X}}(\mathfrak{S})$ in $\bar{X}$ of a Siegel set $\mathfrak{S}$ with respect to P, x is compact (7.9) and every point in the corner $X(P)$ has a neighborhood of this form (6.2). Since the corners $X(P)$, where P runs through the set $\mathfrak{P}_0$ of minimal parabolic k-subgroups of G, form an open cover of $\bar{X}$ (7.1), it suffices to consider the case where $C=Cl_{\bar{X}}(\mathfrak{S})$, $D=Cl_{\bar{X}}(\mathfrak{S}')$, where $\mathfrak{S}$ (resp. $\mathfrak{S}'$) is a Siegel set with respect to x and a minimal parabolic k-subgroup P (resp. P'). There exists $g\in G(k)$ such that $P'=P^g$ (4.1). Then $\mathfrak{S}'\cdot g^{-1}$ is a Siegel set with respect to P, x. Since any two Siegel sets are contained in a bigger one (see 6.1), we may assume that $\mathfrak{S}'=\mathfrak{S}\cdot g$. We may also assume the set ω occurring in the definition of $\mathfrak{S}$ in 6.1 to be compact. Then $\mathfrak{S}$ is closed in X, hence equal to $Cl_{\bar{X}}(\mathfrak{S})\cap X$. Under those conditions, (1) may therefore be written

$$E=\{\gamma\in\Gamma\mid\mathfrak{S}\cdot\gamma\cap\mathfrak{S}\cdot g\neq\emptyset\} \quad \textit{is finite}. \tag{2}$$

Let $\pi: G(\mathbf{R}) \to X$ be the map $g \mapsto x \cdot g$, and let $\mathfrak{S}' = \pi^{-1}(\mathfrak{S})$. By 6.4, $\mathfrak{S} = \pi(\mathfrak{S}')$ and $\mathfrak{S}'$ is a Siegel set in $G(\mathbf{R})$, with respect to a maximal compact subgroup K of H_x, P and a suitable maximal torus S' of $R_d P$. We have then $E = \{\gamma \in \Gamma \mid \mathfrak{S}'\gamma \cap \mathfrak{S}' \cdot g \neq \emptyset\}$. It follows from the end remark in 6.1 that we may change at will the origin x used to define the Siegel sets. In particular (6.4), we may assume x to be such that S' is defined and split over k. Then $\mathfrak{S}'$ is a Siegel set in the sense of [3, § 12], and the finiteness of E follows from Theorem 15.4 in [3].

In view of the relation between Siegel sets in X and in $G(\mathbf{R})$ (6.3). and of Theorem 13.1 in [3], there exists a Siegel set $\mathfrak{S}$ in X (with respect to some minimal parabolic $\mathbf{Q}$-subgroup P) and a finite subset C of $G(\mathbf{Q})$, such that $X = \mathfrak{S} \cdot C \cdot \Gamma$. By 7.9, the closure M of $\mathfrak{S} \cdot C$ in $\bar{X}$ is compact. Since Γ acts properly on $\bar{X}$, the family of sets $M \cdot \gamma$ $(\gamma \in \Gamma)$ is locally finite in $\bar{X}$, hence is closed in $\bar{X}$. On the other hand, it contains X, which is dense in $\bar{X}$. Therefore $M \cdot \Gamma = \bar{X}$ and M is mapped onto $\bar{X}/\Gamma$ under the natural projection. Hence $\bar{X}/\Gamma$ is compact.

9.4. PROPOSITION. *Let $\pi: \bar{X} \to \bar{X}/\Gamma$ be the natural projection. For $P \in \mathfrak{P}$, let $\Gamma_P = \mathscr{N}_G(P) \cap \Gamma$ and $e'(P) = \pi(e(P))$. Let D be a set of representatives for $\mathfrak{P}/\Gamma$.*

(i) *We have $e'(P) = e(P)/\Gamma_P$ and, for $Q \in \mathfrak{P}$,*

$$e'(P) \cap e'(Q) \neq \emptyset \Leftrightarrow e'(P) = e'(Q) \Leftrightarrow \exists \gamma \in \Gamma \quad \text{such that} \quad P^\gamma = Q. \tag{1}$$

The set D is finite and $\bar{X}/\Gamma = \coprod_{P \in D} e'(P)$.

$$\text{(ii)} \quad Cl_{\bar{X}/\Gamma}(e'(P)) = \pi(\overline{e(P)}). \tag{2}$$

If $\Gamma \subset G^0$, then $\Gamma_P = \Gamma \cap P$ and

$$\pi(\overline{e(P)}) \cong \overline{e(P)}/\Gamma_P = \coprod_{Q \in \mathfrak{P}(P)/\Gamma_P} e'(Q); \tag{3}$$

in particular $e'(Q)$ is in the closure of $e'(P)$ if and only if Q is conjugate under Γ to a subgroup of P.

(i) The first equality and (1) follow from 7.7 and imply the last equality of (i). Since the $e(P)$'s ($P \in \mathfrak{P}$) are permuted by Γ and form a locally finite family in $\bar{X}$ (7.1), it follows that the $e'(P)$ form a locally finite family in $\bar{X}/\Gamma$, parametrized by D. Since $\bar{X}/\Gamma$ is compact, this shows that D is finite. (The finiteness of $\mathfrak{P}/\Gamma$ also follows from [3; 15.6].)

(ii) We have $\overline{e(P)} \cdot \gamma = \overline{e(\gamma^{-1} \cdot P \cdot \gamma)}$ $(\gamma \in \Gamma)$, therefore (7.1) the $\overline{e(P)} \cdot \gamma\, (\gamma \in \Gamma)$ form a locally finite family in $\bar{X}$, and $\overline{e(P)} \cdot \Gamma$ is closed in $\bar{X}$. It is the inverse image of $\pi(\overline{e(P)})$, hence the latter is closed, and contains the closure of $e'(P)$. On the other hand, $e(P)$ is dense in $\overline{e(P)}$, hence $e'(P)$ is dense in $\pi(\overline{e(P)})$, which proves (2). The first equality in (3) follows from 7.7, and the second one from (i), applied to $\overline{e(P)}$ and Γ_P.

9.5. Clearly, any compact subgroup of $G(\mathbf{R})$ has a fixed point in X. Since Γ acts properly (9.3), it acts freely if and only if it is torsion-free. Assume this to be the case. Then $\pi:\bar{X}\to\bar{X}/\Gamma$ is a local homeomorphism and X/Γ inherits the structure of manifold with corners of $\bar{X}$. Let $y\in e(P)$, $y'=\pi(y)$ and U' a sufficiently small neighborhood of y'. Then U' is isomorphic under π, as a manifold with corners, with a suitable neighborhood U of $y\in X(P)$. In particular the faces of the corner are the $e'(Q)\cap \cap U'(Q\in\mathfrak{P};\, Q\supset P)$.

§10. Strong Separation Properties

10.1. *Distinguished neighborhoods of* $\overline{e(P)}$. We keep the notation of §9. For $x\in X$, $P\in\mathfrak{P}$ and $t>0$, we put

$$U_{x,P,t}=(x\circ A_{P,t})\cdot{}^0P(\mathbf{R}). \tag{1}$$

Since x is fixed under a maximal compact subgroup of ${}^0P(\mathbf{R})$, we also have

$$U_{x,P,t}=(x\circ A_{P,t})\cdot({}^0P(\mathbf{R}))^0. \tag{2}$$

In the notation of 5.4, (1) may be written

$$U_{x,P,t}=\mu_x(A_{P,t}\times e(P)) \tag{3}$$

and $U_{x,P,t}$ is closed in X; in view of 7.3, we have therefore

$$\bar{U}_{x,P,t}=\mu_x(\bar{A}_{P,t}\times\overline{e(P)}),\qquad U_{x,P,t}=\bar{U}_{x,P,t}\cap X, \tag{4}$$

and $\bar{U}_{x,P,t}$ is a neighborhood of $\overline{e(P)}$ in $\bar{X}$. Any neighborhood of $\overline{e(P)}$ containing some $\bar{U}_{x,P,t}$ will be called *distinguished.*

10.2. LEMMA. *The neighborhood $\bar{U}_{x,P,t}$ is stable under $A_{P,1}\times{}^0P(\mathbf{Q})$, where the semi-group $A_{P,1}$ acts by geodesic action and ${}^0P(\mathbf{Q})$ by ordinary action, and μ_x commutes with $A_{P,1}\times{}^0P(\mathbf{Q})$. If V is a neighborhood of $\overline{e(P)}$ stable under $\Gamma\cap P$, then V is distinguished.*

We have clearly $A_{P,t}\cdot A_{P,1}=A_{P,t}$ hence also $\bar{A}_{P,t}\cdot A_{P,1}=\bar{A}_{P,t}$. Together with 7.3 and 7.6 this implies the first assertion.

Let V be a neighborhood of $\overline{e(P)}$. Let C be a compact subset of $\overline{e(P)}$. As t varies, the sets $\mu_x(\bar{A}_{P,t}\times C)$ form a family of compact sets whose intersection is C. Therefore V contains one of them. By 9.3, we may choose C so that $\overline{e(P)}=C\cdot(\Gamma\cap P)$. If $V\cdot(\Gamma\cap P)=V$, we have then for a suitable t, taking 7.3 into account:

$$V=V\cdot(\Gamma\cap P)\supset\mu_x(\bar{A}_{P,t}\times C)\cdot(\Gamma\cap P)=\mu_x(\bar{A}_{P,t}\times C\cdot(\Gamma\cap P))=\bar{U}_{x,P,t}.$$

10.3. PROPOSITION. *We keep the previous notation. Let $\pi : \bar{X} \to \bar{X}/\Gamma$ be the natural projection and assume G to be connected. There exists $t>0$ such that the equivalence relations defined on $\bar{U}_{x,P,t}$ by Γ and $\Gamma \cap P$ are the same. For any such t, the isomorphism μ_x induces an isomorphism*

$$\mu'_x : \bar{A}_{P,t} \times \overline{e(P)}/(\Gamma \cap P) \xrightarrow{\sim} \pi(\bar{U}_{x,P,t}), \tag{1}$$

such that the following diagram

$$\begin{array}{ccc} \bar{A}_{P,t} \times \overline{e(P)} & \xrightarrow[\sim]{\mu_x} & \bar{U}_{x,P,t} \\ {\scriptstyle \mathrm{id.}\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \bar{A}_{P,t} \times \overline{e(P)}/(\Gamma \cap P) & \xrightarrow[\sim]{\mu'_x} & \pi(\bar{U}_{x,P,t}) \end{array} \tag{2}$$

is commutative. The geodesic action of $A_{P,1}$ on $\bar{U}_{x,P,t}$ commutes with π and induces an action on $\pi(\bar{U}_{x,P,t})$. All the maps in (2) *commute with $A_{P,1}$.*

By 9.4, the equivalence relations defined on $e(P)$ by Γ and $\Gamma \cap P$ are the same, and $\pi(\overline{e(P)}) = \overline{e(P)}/(\Gamma \cap P)$. By 9.3, there exists a compact subset C of $\overline{e(P)}$ such that $\overline{e(P)} = C \cdot (\Gamma \cap P)$. Since Γ operates properly (9.3), there is a neighborhood U of C in $\bar{X}$ such that, for any $\gamma \in \Gamma$, $U \cdot \gamma \cap U \neq \emptyset$ implies $C \cdot \gamma \cap C \neq \emptyset$, and hence $\gamma \in \Gamma \cap P$. Since C is compact, there exists $t>0$ such that $\mu_x(\bar{A}_{P,t} \times C) \subset U$. We wish to show that any such t satisfies our conditions.

Let $a, b \in \bar{U}_{x,P,t}$ and $\gamma \in \Gamma$ be such that $a \cdot \gamma = b$. Since $\Gamma \cap P$ commutes with μ_x (10.2), there exist $a', b' \in \mu_x(\bar{A}_{P,t} \times C)$ and $\sigma, \tau \in \Gamma \cap P$ such that $a = a' \cdot \sigma$, $b = b' \cdot \tau$. We have then $b' = a' \cdot \sigma \cdot \gamma \cdot \tau^{-1}$, hence $\sigma \cdot \gamma \cdot \tau^{-1} \in \Gamma \cap P$ and $\gamma \in \Gamma \cap P$.

This proves the first assertion. The other assertions then follow immediately from 7.3 and 10.2.

10.4. PROPOSITION. *Let $P, Q \in \mathfrak{P}$, $x, y \in X$ and $g \in G(\mathbf{Q})$. Then the following four conditions are equivalent:*

(i) $U_{x,P,t} \cdot g \cap U_{y,Q,t} \neq \emptyset$ *for all* $t>0$.
(ii) $\bar{U}_{x,P,t} \cdot g \cap \bar{U}_{y,Q,t} \neq \emptyset$ *for all* $t>0$.
(iii) $\overline{e(P)} \cdot g \cap \overline{e(Q)} \neq \emptyset$.
(iv) $P^g \cap Q = R$ *is parabolic.*

If they are fulfilled, the sets $\bar{U}_{x,P,t} \cdot g \cap \bar{U}_{y,Q,t}$ $(t>0)$ form a basis of ${}^0R(\mathbf{Q})$-invariant neighborhoods of $\overline{e(R)}$.

It follows from 10.2 that we may assume $x = y$. Moreover, replacing P by P^g, we may take $g = e$.

The equivalence of (iii) and (iv) follows from 7.7. Clearly, (iii) $\Rightarrow$ (ii). By 9.3, there exist compact subsets $C \subset \overline{e(P)}$ and $C' \subset \overline{e(Q)}$ such that

$$C \cdot (\Gamma \cap P) = \overline{e(P)} \quad \text{and} \quad C' \cdot (\Gamma \cap Q) = \overline{e(Q)}.$$

For $t>0$, let

$$D_t = \mu_x(\bar{A}_{P,t} \times C), \qquad D'_t = \mu_x(\bar{A}_{Q,t} \times C'). \tag{1}$$

We have then

$$\bar{U}_{x,P,t} = D_t \cdot (\Gamma \cap P), \qquad \bar{U}_{x,Q,t} = D'_t \cdot (\Gamma \cap Q). \tag{2}$$

The D_t (resp. D'_t) are compact and their intersection is C (resp. C'). Since Γ acts properly on $\bar{X}$ (9.3), there exists $t>0$ such that

$$\{\gamma \in \Gamma \mid D_t \cdot \gamma \cap D'_t = \emptyset\} = \{\gamma \in \Gamma \mid C \cdot \gamma \cap C' \neq \emptyset\} \tag{3}$$

and the set of such γ's is finite. Let in particular $\{\gamma_i\}_{1 \leqq i \leqq m}$ be those $\gamma \in \Gamma$ which satisfy (3) and are contained in $(\Gamma \cap P) \cdot (\Gamma \cap Q)$; for each of them choose a decomposition

$$\gamma_i = \sigma_i \cdot \tau_i^{-1} \quad (\sigma_i \in \Gamma \cap P;\ \tau_i \in \Gamma \cap Q;\ i = 1, \ldots, m). \tag{4}$$

If now $\sigma \in \Gamma \cap P$ and $\tau \in \Gamma \cap Q$ are such that $D_t \cdot \sigma \cap D'_t \cdot \tau \neq \emptyset$, then, for some $i \leqq m$, we have $\sigma \cdot \tau^{-1} = \sigma_i \cdot \tau_i^{-1}$ and hence

$$\sigma_i^{-1} \cdot \sigma = \tau_i^{-1} \cdot \tau \in \Gamma \cap P \cap Q.$$

This implies readily

$$\bar{U}_{x,P,t} \cap \bar{U}_{x,Q,t} = \bigcup_{1 \leqq i \leqq m} (D_t \cdot \sigma_i \cap D'_t \cdot \tau_i) \cdot (\Gamma \cap P \cap Q). \tag{5}$$

Assume now (ii). Then there exists $i (1 \leqq i \leqq m)$ such that $D_t \cdot \sigma_i \cap D'_t \cdot \tau_i \neq \emptyset$ for all $t>0$, hence such that $C \cdot \sigma_i \cap C' \cdot \tau_i \neq \emptyset$. This proves that (iii) holds. Clearly, (i) $\Rightarrow$ (ii). Assume again (ii) to hold. Then, (iii) holds and $\bar{U}_{x,P,t} \cdot g \cap \bar{U}_{y,Q,t}$ is a neighborhood in $\bar{X}$ of any point in $\overline{e(P)} \cdot g \cap \overline{e(Q)}$. Therefore

$$X \cap (\bar{U}_{x,P,t} \cdot g \cap \bar{U}_{x,Q,t}) \neq \emptyset;$$

by 10.1 (4) this is condition (i).

Assume R to be parabolic. Then $\overline{e(P)} \cap \overline{e(Q)} = \overline{e(R)}$ by 7.4, and the left-hand side of (5) is a closed neighborhood of $\overline{e(R)}$, which is stable under ${}^0R(\mathbf{Q}) \subset {}^0P(\mathbf{Q}) \cap {}^0Q(\mathbf{Q})$, hence distinguished (10.2). For each $i (1 \leqq i \leqq m)$ we have

$$\bigcup_{t>0} D_t \cdot \sigma_i \cap D'_t \cdot \tau_i = C \cdot \sigma_i \cap C' \tau_i,$$

hence, given $s>0$, there exists $t>0$ such that

$$D_t \cdot \sigma_i \cap D'_t \cdot \tau_i \subset \bar{U}_{x,R,s}, \quad (1 \leqq i \leqq m);$$

(5) then shows that the left-hand side of (5) is contained in $\bar{U}_{x,R,s}$, whence the last assertion.

10.5. COROLLARY. *There exists $t>0$ such that for any $P, Q\in\mathfrak{P}$, we have*

$\bar{U}_{x,P,t}\cap\bar{U}_{y,Q,t}\neq\emptyset$ *if and only if* $\overline{e(P)}\cap\overline{e(Q)}\neq\emptyset$.

This follows from 10.4 and the finiteness of $\mathfrak{P}/\Gamma$.

10.6. PROPOSITION. *Let $P, Q\in\mathfrak{P}$, $x, y\in X$. For $t>0$, let*

$$E_t=\{\gamma\in\Gamma\,|\,\bar{U}_{x,P,t}\cdot\gamma\cap\bar{U}_{x,Q,t}\neq\emptyset\}.$$

(i) *E_t is the union of finitely many double cosets modulo $(\Gamma\cap P)$ and $(\Gamma\cap Q)$.*

(ii) *For t small enough,*

$$E_t=\{\gamma\in\Gamma\mid\overline{e(P)}\cdot\gamma\cap\overline{e(Q)}\neq\emptyset\}=\{\gamma\in\Gamma\mid P^\gamma\cap Q \text{ is parabolic}\}.$$

(i) Let D_t and D_t' be as in 10.4(1). Then $E_t=(\Gamma\cap P)\cdot F_t\cdot(\Gamma\cap Q)$, where

$$F_t=\{\gamma\in\Gamma\mid D_t\cdot\gamma\cap D_t'\neq\emptyset\},$$

and F_t is finite since D_t and D_t' are compact and Γ acts properly on $\bar{X}$ (9.3).

(ii) Since F_t is finite and decreasing as $t\to 0$, it is independent of t for t small enough. Our assertion then follows from 10.4.

10.7. Let $P\in\mathfrak{P}$. Let S' be a maximal split torus of R_dP and A' the identity component of $S'(\mathbf{R})$. Then $P(\mathbf{R})=A'\ltimes{}^0P(\mathbf{R})$, and there is a natural projection $\sigma:A'\to A_P$. We let for $t>0$

$$P(t)=\sigma^{-1}(A_{P,t})\cdot{}^0P(\mathbf{R}). \tag{1}$$

In the notation of 6.1, we can also write this $P(t)=A'_{P,t}\cdot{}^0P(\mathbf{R})$, and (1) shows that $P(t)$ does not depend on the choice of A'.

10.8. PROPOSITION. *Let $P, Q\in\mathfrak{P}$ and K, K' be maximal compact subgroups of $G(\mathbf{R})$.*

(i) *Let $g\in G(\mathbf{Q})$ and assume that $K\cdot P(t)\cdot g\cap K'\cdot Q(t)\neq\emptyset$ for all $t>0$. Then $P^g\cap Q$ is parabolic.*

(ii) *Given $t>0$, the set of $\gamma\in\Gamma$ for which $K\cdot P(t)\cdot\gamma\cap K'\cdot Q(t)\neq\emptyset$ is the union of finitely many double cosets modulo $(\Gamma\cap P)$ and $(\Gamma\cap Q)$.*

(iii) *There exists $t>0$ such that $K\cdot P(t)\cdot\Gamma\cap K'\cdot Q(t)=\emptyset$ unless $P^\gamma\cap Q$ is parabolic for some $\gamma\in\Gamma$.*

Let x (resp. y) be a point of X fixed under K (resp. K'). Then $K\cdot P(t)$ (resp. $K'\cdot Q(t)$) is the inverse image of $U_{x,P,t}$ (resp. $U_{y,Q,t}$) under the orbital map $g\mapsto x\cdot g$ (resp. $g\mapsto y\cdot g$). Therefore (i) follows from 10.4, (ii) from 10.6 and (iii) from 10.4, 10.6.

10.9. COROLLARY. *Assume G to be connected and $K\cdot P(t)\cdot g\cap K'\cdot P(t)\neq\emptyset$ for all $t>0$. Then $g\in P(\mathbf{Q})$.*

Indeed, $P^g \cap P$ is parabolic by 10.8. But in a connected algebraic group, two conjugate parabolic subgroups whose intersection is parabolic are identical. Hence g normalizes P, and then $g \in P$.

10.10. *Remark.* In § 10, the ground field k is the field of rational numbers. However, the definitions in 10.1, 10.7 and the statements 10.4, 10.5, 10.8(i), (ii) and 10.9 make sense in the context of §§ 6, 7 where k is any subfield of $\mathbf{R}$. For P, Q minimal these assertions can be proved using representative functions as in [3, §§ 14, 15], but we do not know whether they are true in general.

§ 11. Cohomology of Arithmetic Groups

We keep the hypotheses and notations of § 9: $k = \mathbf{Q}$ and Γ is an *arithmetic subgroup* of $G(\mathbf{Q})$. Moreover, we assume that Γ is *torsion-free.*

11.1. *Qualitative results*

By 9.3 and 9.5, $\bar{X}/\Gamma$ is a compact C^∞-manifold with corners, hence is homeomorphic to a compact C^∞-manifold with boundary (cf. Appendix), and can be triangulated ([23], § 10). Moreover, $\bar{X}$ is contractible (8.3.1), hence is a universal covering of $\bar{X}/\Gamma$. These properties imply:

a) The group Γ is isomorphic to the fundamental group of $\bar{X}/\Gamma$, hence is *finitely presented.*

b) The space $\bar{X}/\Gamma$ is a $K(\Gamma, 1)$-*space.* Its cohomology (or homology) is isomorphic to the one of Γ. More precisely, if A is a Γ-module, and $\tilde{A}$ the corresponding local system on $\bar{X}/\Gamma$, there are canonical isomorphisms

$$H_q(\Gamma, A) \approx H_q(\bar{X}/\Gamma, \tilde{A}) \quad \text{and} \quad H^q(\Gamma, A) \approx H^q(\bar{X}/\Gamma, \tilde{A})$$

for any q.

c) The group Γ is *of type* (FL) in the sense of [26], p. 84. Indeed, a triangulation of $\bar{X}/\Gamma$ lifts to a Γ-invariant triangulation of $\bar{X}$ and the corresponding complex of simplicial chains

$$0 \to C_d \to C_{d-1} \to \cdots \to C_0 \to \mathbf{Z} \to 0 \quad (d = \dim X)$$

gives a $\mathbf{Z}[\Gamma]$-free resolution of finite type of the $\mathbf{Z}[\Gamma]$-module $\mathbf{Z}$.

Remark. The above results depend only on the *existence* of a compactification of X/Γ as a manifold with boundary (or even as a finite complex), and not on the structure of the compactification; in the semi-simple case, they are due to Raghunathan [25].

11.2. *Comparison between $\bar{X}/\Gamma$ and its boundary $\partial\bar{X}/\Gamma$*

Let l be the $\mathbf{Q}$-rank of G/RG.

PROPOSITION. *The inclusion map $\partial\bar{X}/\Gamma \to \bar{X}/\Gamma$ is an $(l-2)$-homotopy equivalence.*

(This means that the natural maps $\pi_i(\partial\bar{X}/\Gamma) \to \pi_i(\bar{X}/\Gamma)$ are bijective for $i \leqslant l-2$.)

Since $\pi_1(\bar{X}/\Gamma)=\Gamma$ and $\pi_i(\bar{X}/\Gamma)=0$ for $i \neq 1$, we have to prove that $\pi_1(\partial\bar{X}/\Gamma) \to \Gamma$ is an isomorphism if $l \geqslant 3$, and that $\pi_i(\partial\bar{X}/\Gamma)=0$ for $i \leqslant l-2$, $i \neq 1$. This in turn follows from the fact that $\partial\bar{X}$ has the homotopy type of a bouquet of $(l-1)$-spheres (8.5.1), hence is simply connected if $l \geqslant 3$, and $\pi_i(\partial\bar{X})=0$ for $i \leqslant l-2$.

11.3. *Euler-Poincaré characteristics*

If Y is a finite complex, we denote by $\chi(Y)$ its Euler-Poincaré characteristic; we put $\chi(\Gamma)=\chi(\bar{X}/\Gamma)$, cf. [26], p. 91, prop. 9.

PROPOSITION. (a) $\chi(\partial\bar{X}/\Gamma)=0$.

(b) *If $d=\dim X$ is odd, or if $R_uG \neq \{e\}$, we have $\chi(\Gamma)=0$.*

Assume first Γ to be "net" [3, §17], hence contained in G^0, let $V=R_uG(\mathbf{R})$. The space $\bar{X}$ has a natural structure of principal V-bundle, cf. 7.2, (iii). By [3, 17.3], $\Gamma/(\Gamma \cap V)$ is torsion-free, hence acts freely on $\bar{X}/V$. This implies that $\bar{X}/\Gamma$ has a fibering with typical fiber $N=V/(\Gamma \cap V)$. If $\dim V \geqslant 1$, it is well-known that $\chi(N)=0$, hence $\chi(\Gamma)=\chi(\bar{X}/\Gamma)=0$ which proves the second assertion of (b).

Now, if $P \in \mathfrak{P}$ is distinct from G^0, its unipotent radical is non-trivial. Hence, by the above, applied to P, we see that the image $\overline{e'(P)}=\overline{e(P)}/\Gamma_P$ of $\overline{e(P)}$ in $\bar{X}/\Gamma$ (cf. 9.4) is such that $\chi(\overline{e'(P)})=0$. But the $\overline{e'(P)}$ make up a finite cover of $\partial\bar{X}/\Gamma$, and their intersections are either empty or of the form $\overline{e'(Q)}$ for some $Q \neq G^0$, hence have zero Euler-Poincaré characteristic. By an easy combinatorial argument, this implies that $\chi(\partial\bar{X}/\Gamma)=0$.

If $\dim X$ is odd, the duality of manifolds with boundary implies that $\chi(\bar{X}/\Gamma) = \frac{1}{2}\chi(\partial\bar{X}/\Gamma)$, which is 0 by (a). This concludes the proof when Γ is net; the general case follows by a covering argument, using [3, 17.4].

Remark. The fact that $\chi(\Gamma)=0$ when d is odd can also be proved by the method of Harder [18].

11.4. *Duality theorem*

We keep the above notation. In particular $d=\dim X$ and l is the $\mathbf{Q}$-rank of G/RG.

11.4.1. THEOREM. *We have $H^i(\Gamma, \mathbf{Z}[\Gamma])=0$ for $i \neq d-l$ and the group $I=H^{d-l}(\Gamma, \mathbf{Z}[\Gamma])$ is free abelian of rank 1 if $l=0$ and of infinite rank if $l \geqslant 1$.*

This follows from theorem 8.6.5 together with the elementary fact that $H^i(\Gamma, \mathbf{Z}[\Gamma]) = H^i_c(\bar{X}, \mathbf{Z})$ for all i (cf. for instance [1], n° 6.3).

Note that the right action of Γ on $\mathbf{Z}[\Gamma]$ defines on $H^{d-l}(\Gamma, \mathbf{Z}[\Gamma])$ a structure of Γ-module. This Γ-module is the *dualizing module* of Γ:

11.4.2. THEOREM. *There is a homology class $e \in H_{d-l}(\Gamma, I)$ such that, for every*

Γ-module A, and for every integer q, the cap-product by e defines an isomorphism

$$H^q(\Gamma, A) \to H_{d-l-q}(\Gamma, I \otimes A).$$

This follows from theorem 4.5 of [1], combined with theorem 11.4.1 and 11.1.

Remark. In the language of [1], Γ is a *duality group.* It is a "Poincaré duality group" if I is isomorphic to $\mathbf{Z}$, i.e. if $l=0$, or, equivalently, if $\partial\bar{X}=\emptyset$.

11.4.3. COROLLARY. *We have $cd(\Gamma)=d-l$.*

(Recall that $cd(\Gamma)$ is the *cohomological dimension* of Γ, cf. [26], p. 84.)

This is clear from theorem 11.4.2.

11.4.4. THEOREM. *Let Δ be any arithmetic subgroup of $G(\mathbf{Q})$ (which may have torsion). Then Δ is of type (WFL), $vcd(\Delta)=d-l$, $H^i(\Delta, \mathbf{Z}[\Delta])=0$ for $i\neq d-l$ and $H^{d-l}(\Delta, \mathbf{Z}[\Delta])$ is isomorphic to I.*

(For the definitions of "type (*WFL*)" and "*vcd*," see [26], n° 1.8.)

This follows from 11.1, 11.4.1, 11.4.3 applied to the torsion-free subgroups of finite index of Δ. (Notice that $H^i(\Delta, \mathbf{Z}[\Delta])$ is isomorphic to $H^i(\Gamma, \mathbf{Z}[\Gamma])$ if Γ is of finite index in Δ, cf. [1], prop. 3.1.)

EXAMPLES. $vcd(\mathbf{SL}_3(\mathbf{Z}))=5-2=3$; $vcd(\mathbf{Sp}_4(\mathbf{Z}))=6-2=4$.

11.5. *Duality in cohomology*

Let R be a commutative ring and Ω an injective R-module. If V is any R-module, we define V' as $\mathrm{Hom}_R(V, \Omega)$. We also define I' as $\mathrm{Hom}_{\mathbf{Z}}(I, \Omega)$.

11.5.1. THEOREM. *For every $R[\Gamma]$-module V and every integer q, $H^q(\Gamma, V)'$ is naturally isomorphic to $H^{d-l-q}(\Gamma, \mathrm{Hom}_R(V, I'))$.*

This follows from the isomorphisms

$$\begin{array}{ll} H^q(\Gamma, V)' \cong H_{d-l-q}(\Gamma, V\otimes I)' & \text{(theorem 11.4.2)} \\ H_{d-l-q}(\Gamma, V\otimes I)' \cong H^{d-l-q}(\Gamma, (V\otimes I)') & \text{(elementary)} \\ (V\otimes I)' \cong \mathrm{Hom}_k(V, I') & \text{(linear algebra)}. \end{array}$$

EXAMPLES. a) R is a field, $\Omega=R$, and V' is the dual vector space of V.

b) $R=\mathbf{Z}$ and $\Omega=\mathbf{Q}/\mathbf{Z}$.

11.6. *Remark.* If K is a number field, L an affine K-group, and Δ an arithmetic subgroup of $L(K)$, the above results can be applied to Δ, viewed as an arithmetic subgroup of the $\mathbf{Q}$-group $G=R_{K/\mathbf{Q}}L$, cf. [3], n° 7.16. We leave the details to the reader; he will notice, in particular, that the $\mathbf{Q}$-rank l of G/RG is equal to the K-rank of L/RL [8; 6.21].