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# On the Extension of Real Places 1)

MANFRED KNEBUSCH

#### Introduction

About twenty years ago S. Lang studied places  $\varphi: K \to R \cup \infty$  on a field K with values in a fixed real closed field R ([L]). One of his main results was the theorem, that any such place  $\varphi$  can be extended to an R-valued place on a suitable real closure of K ([L], Th. 6). Now the real closures of K correspond up to K-isomorphisms uniquely to the (total) orderings of K. Thus one may ask whether it is possible to obtain a more precise version of Lang's theorem by a more thorough analysis of the relations between orderings and real places. This question is the starting point of the present paper.

We say that an ordering  $\alpha$  of K lies over the place  $\varphi: K \to R \cup \infty$  or that  $\varphi$  and  $\alpha$  are *compatible*, if any element  $\alpha$  of K which is positive with respect to  $\alpha$  has value  $\varphi(\alpha) = \infty$  or  $\varphi(\alpha) \ge 0$ . (Recall that R is ordered in a unique way). In §1 we first show that over any real place  $\varphi$  lies at least one ordering  $\alpha$ . Then we prove the following refinement of Lang's theorem:

THEOREM 1.6. Assume that L is an algebraic field extension of K, that  $\beta$  is an ordering on L and that  $\varphi$  is an R-valued place on K, compatible with the restriction of  $\beta$  to K. Then there exists a unique R-valued place  $\psi$  on L extending  $\varphi$  and compatible with  $\beta$ .

Harrison ([H]), and Leicht, Lorenz ([LL]) showed that the orderings  $\alpha$  of a field K correspond uniquely to the *signatures*  $\sigma$  of K, i.e. the ring homomorphisms  $\sigma \colon W(K) \to \mathbb{Z}$ , where W(K) denotes the Witt ring of non singular symmetric bilinear forms over K([W]). As usual we denote for any  $a \neq 0$  in K by (a) the element of W(K) represented by the form  $B \colon K \times K \to K$ , B(x, y) = axy. The signature  $\sigma$  corresponding to the ordering  $\alpha$  is characterized by  $\sigma(a) = +1$  if a > 0 with respect to  $\alpha$ , and  $\sigma(a) = -1$  if a < 0. (Recall that W(K) is generated by the elements (a).) We shall make strong use of this connection between orderings and Witt rings, and we shall always identify an ordering  $\alpha$  with the corresponding signature  $\sigma$ . The unique signature  $W(R) \to \mathbb{Z}$  will be denoted by  $\rho$ .

As will be explained in §2, any R-valued place  $\varphi$  on K yields a well defined additive map  $\varphi_*: W(K) \to \mathbb{Z}$ , whose value on an element (a) is obtained in the following way: If the square class  $aK^{*2}$  contains an element  $a' = ab^2$  such that  $\varphi(a') \neq 0$  and  $\varphi = \varphi(\varphi(a'))$  with an arbitrary choice of a'. If  $aK^{*2}$  contains no such elements,

<sup>1)</sup> The main results of this paper have been announced in [K2, part B].

then  $\varphi_*(a) = 0$ . Obviously a signature  $\sigma$  lies over  $\varphi$  if and only if  $\sigma(a) = \varphi_*(a)$  for all a in  $K^*$  such that  $\varphi_*(a) \neq 0$ .

We prove in §2 the following counterpart of the theorem above:

THEOREM 2.6. Assume that L is an arbitrary field extension of K, that  $\psi$  is an R-valued place on L and that  $\sigma$  is a signature of K lying over  $\psi \mid K$ . There exists a signature  $\tau$  of L lying over  $\psi$  and extending  $\sigma$  {i.e.  $\tau(a) = \sigma(a)$  for all a in  $K^*$ } if and only if  $\sigma(a) = \psi_*(a)$  for all a in  $K^*$  with  $\psi_*(a) \neq 0$ .

We further prove in §2 a theorem about the real places in the field composites of an algebraic extension  $L_1/K$  and an arbitrary extension  $L_2/K$ .

Our work in §3 originates from the question, how many R-valued extensions has a given R-valued place  $\varphi$  of K in a finite field extension L/K. Recall that the regular trace  $\operatorname{Tr}_{L/K}$  induces an additive map  $\operatorname{Tr}_{L/K}^*: W(L) \to W(K)$  mapping the class of a symmetric bilinear space (E, B) over L to the class of the space  $(E, \operatorname{Tr}_{L/K} \circ B)$  over K (cf [S]). We prove the following trace formula: For any x in W(L)

$$\varphi_* \big( \mathrm{Tr}_{L/K}^*(x) \big) = \sum_{\psi \mid \varphi} \psi_*(x)$$

where  $\psi$  runs through all R-valued places of L extending  $\varphi$ , with the convention that the right hand side is zero if there are no such places  $\psi$ . Applying this formula to the unit element (1) of W(L) one obtains that  $\varphi$  has exactly  $\varphi_*(\operatorname{Tr}_{L/K}^*(1))$  R-valued extensions to L.

The final section 4 gives an application of the theorem 1.6 cited above to the problem of extending an R-valued place  $\varphi$  on K to a field L which is finitely generated over K but not necessarily algebraic.

To prevent misunderstandings I remark that in this paper different places with the same valuation ring are never identified and that a place is allowed to be trivial, i.e. to avoid the value  $\infty$ .

## **§1**

We first recall some well known facts and notations (cf [L], [AS]). Assume that on a field K an ordering  $\sigma$  is given and that k is a subfield of K. An element a of K is called infinitely large over k (with respect to  $\sigma$ ) if there is no element c>0 in k such that |a| < c. Here |a| denotes the element a if  $a \ge 0$  and -a if  $a \le 0$  with respect to  $\sigma$ . The set of elements of K, which are not infinitely large over k, is a valuation ring of K ([AS], p. 95) which we call the valuation ring  $\mathfrak{o} = \mathfrak{o}(K/k, \sigma)$  associated with  $\sigma$  over k. Obviously the maximal ideal  $\mathfrak{m}$  of  $\mathfrak{o}$  is the set of all elements a in K which are infinitely small over k, i.e. |a| < c for all c > 0 in k.

Clearly  $k \subset \mathfrak{o}$ . A field k' with  $k \subset k' \subset K$  is called archimedian over k (with respect

to  $\sigma$ ) if  $k' \subset \mathfrak{o}$ . Then  $\mathfrak{o}$  is also the valuation ring associated with  $\sigma$  over k'. It is clear from general valuation theory that any algebraic extension k' of k in K is archimedian over k.

As follows from Zorn's lemma, there exists at least one intermediate field  $\tilde{k} \supset k$  which is *maximal* archimedian over k, i.e.  $\tilde{k}$  is archimedian over k, but no field  $k' \supset \tilde{k}$  different from  $\tilde{k}$  is archimedian over k ([L], p. 379). We say that k is *maximal archimedian in K*, if  $k = \tilde{k}$ . It is clear from general valuation theory that the field  $\mathfrak{o}/\mathfrak{m}$  is always algebraic over  $\tilde{k}$ .

The ordering  $\sigma$  of K induces an ordering  $\bar{\sigma}$  of  $\mathfrak{o}/\mathfrak{m}$ , characterized in the following way ([AS], p. 95): An element  $\bar{a}$  of  $\mathfrak{o}/\mathfrak{m}$  is positive if and only if a preimage a in  $\mathfrak{o}$  is positive. (It does not matter which preimage is chosen.) If K is real closed then  $\tilde{k}$  is real closed, since  $\tilde{k}$  is algebraically closed in K. Thus in this case  $\tilde{k}$  maps bijectively onto  $\mathfrak{o}/\mathfrak{m}$  ([AS], p. 95).

All rings in this paper are commutative and have a unit element and all ring homomorphisms map 1 to 1. The unit group of a ring A is denoted by  $A^*$ . We further denote by W(A) the Witt ring of non degenerate symmetric bilinear forms over A, and for any homomorphism  $\alpha: A \to C$  into a ring C we denote by  $W(\alpha)$  the corresponding ring homomorphism from W(A) to W(C). We refer the reader to [K], [KRW, §1], or [M] for these notions. For any element a in A we denote by (a) the element of W(A) which is represented by the form  $B: A \times A \to A$ , B(x, y) = axy. These elements (a) form a subgroup Q(A) of W(A), which will be identified with the group  $A^*/A^{*2}$  of square classes. If A is local, i.e. A has only one maximal ideal, the ring W(A) is generated by Q(A) (e.g. [KRW, §1]). In this paper only the Witt rings of fields and valuation rings will play a rôle.

As explained in the introduction, the signatures  $\sigma$  of a field K, i.e. the homomorphisms  $\sigma$  from the ring W(K) to  $\mathbb{Z}$ , correspond uniquely to the orderings of K. Let L be a field extension of K and i denote the inclusion map from K into L. For any signature  $\tau$  of L we denote by  $\tau \mid K$  the signature  $\sigma = \tau \circ W(i)$  of K, and we say that  $\sigma$  is the restriction of  $\tau$  to K, or that  $\tau$  is an extension of to L. This terminology is compatible with the usual meaning of extension and restriction of orderings.

Throughout this paper R denotes a real closed field and  $\varrho$  denotes the signature of R. For a moment we forget about orderings of the field K and consider a place  $\varphi: K \to R \cup \infty$ . Let  $\mathfrak o$  denote the valuation ring of  $\varphi$ , i.e. the ring of all elements x in K with  $\varphi(x) \neq \infty$ . By composing the map  $W(\varphi \mid \mathfrak o)$  from  $W(\mathfrak o)$  to W(R) with  $\varrho: W(R) \to \mathbb Z$  we obtain a ring homomorphism from  $W(\mathfrak o)$  to  $\mathbb Z$  which we denote by  $\widehat{\varphi}$ . On a generator (a) of  $W(\mathfrak o)$ , a in  $\mathfrak o^*$ , the map takes the value 1 if  $\varphi(a) > 0$  and -1 if  $\varphi(a) < 0$ .

Since  $\mathfrak{o}$  is a Prüfer ring, the map  $W(i): W(\mathfrak{o}) \to W(K)$ , obtained from the inclusion map  $i: \mathfrak{o} \to K$  is injective ([K, Satz 11.1.1]; the reader may also consult [KRW<sub>1</sub>, Lemma 1.1] or [M, p. 93], where this fact is stated for Dedekind rings but proved for Prüfer rings). We shall always consider  $W(\mathfrak{o})$  as a subring of W(K).

We say that a signature  $\sigma$  of K lies over  $\varphi$ , or that  $\varphi$  is compatible with  $\sigma$ , if  $\sigma$  extends  $\hat{\varphi}$ . Obviously this definition coincides with the definition given in the introduction.

PROPOSITION 1.1. [KRW<sub>2</sub>, 1.13.] Over any place  $\varphi: K \to R \cup \infty$  lies at least one signature  $\sigma$  of K.

Since this fact is central for the present work, we recall the proof given in  $[KRW_2]$ : The kernel P of  $\hat{\varphi}$  is a minimal prime ideal of  $W(\mathfrak{o})$  [KRW]. Thus there exists at least one prime ideal Q of W(K) lying over P [B, Chap. II, §2, no. 6, Prop. 16]. Since  $W(\mathfrak{o})/P \cong \mathbb{Z}$  embeds into W(K)/Q we must have  $W(K)/Q \cong \mathbb{Z}$  ([LL], [H]). The only homomorphism  $\mathfrak{o}: W(K) \to \mathbb{Z}$  with kernel Q is the desired signature.

- LEMMA 1.2. (cf [L], p. 382) Let  $\varphi$  be an R-valued place on K and  $\sigma$  be a signature of K lying over  $\varphi$ . Assume further that a and b are elements of K and  $\varphi(a) \neq \infty$ . Then with respect to the orderings corresponding to  $\sigma$  and  $\varrho$  the following are true:
  - (i) b > a implies  $\varphi(b) = \infty$  or  $\varphi(b) \geqslant \varphi(a)$ .
  - (ii)  $0 < b < a \text{ implies } \varphi(b) \neq \infty \text{ and } 0 \leq \varphi(b) \leq \varphi(a)$ .
- *Proof.* (i) b-a>0 implies  $\varphi(b-a)=\infty$  or  $\varphi(b-a)\geqslant 0$  and thus  $\varphi(b)=\infty$  or  $\varphi(b)\geqslant \varphi(a)$ .
- (ii) This is clear if  $\varphi(b) = 0$ . Assume now  $\varphi(b) \neq 0$ . Then  $\varphi(ab^{-1}) = \varphi(a) \varphi(b^{-1}) \neq \infty$  and we obtain from  $ab^{-1} > 1$  and (i) that  $\varphi(a) \varphi(b^{-1}) \geqslant 1$ . Thus certainly  $\varphi(b) \neq \infty$ , and we obtain from b > 0 that  $\varphi(b) > 0$ . Thus  $\varphi(a) \geqslant \varphi(b)$ . q.e.d.

PROPOSITION 1.3. (cf. [L], Th. 5). Let  $\varphi$  be an R-valued place on K and k be a subfield of K on which  $\varphi$  is trivial. Assume that R is archimedian over  $\varphi(k)$ . Then for a signature  $\tau$  on K the following are equivalent:

- (i)  $\tau$  lies over  $\varphi$ .
- (ii) The valuation ring  $\mathfrak{o}$  of  $\varphi$  coincides with the valuation ring  $\mathfrak{o}(K/k, \tau)$  of  $\tau$  over k. The homomorphism  $\bar{\varphi}: \mathfrak{o}/\mathfrak{m} \to R$  induced by  $\varphi$  on the residue class field of  $\sigma$  is order preserving with respect to the ordering  $\bar{\tau}$ , induced by  $\tau$  on  $\mathfrak{o}/\mathfrak{m}$ , and  $\varrho$ .
- PROOF. a) Let  $\mathfrak{o}'$  denote the ring  $\mathfrak{o}(K/k,\tau)$  and  $\mathfrak{m}'$  denote the maximal ideal of  $\mathfrak{o}'$ . We first assume  $\mathfrak{o}'=\mathfrak{o}$  and analyze the situation in this case. That  $\overline{\varphi}$  is order preserving with respect to  $\overline{\tau}$  and  $\varrho$  means in the language of quadratic forms that the map  $\varrho \circ W(\overline{\varphi})$  from  $W(\mathfrak{o}/\mathfrak{m})$  to Z coincides with  $\overline{\tau}$ . We denote the canonical map from  $\mathfrak{o}$  onto  $\mathfrak{o}/\mathfrak{m}$  by  $\alpha$ . Clearly  $\tau \mid W(\mathfrak{o}) = \overline{\tau} \circ W(\alpha)$ . Now  $W(\alpha)$  is surjective, since all generators  $(\overline{a})$ ,  $\overline{a}$  in  $(\mathfrak{o}/\mathfrak{m})^*$ , can be lifted to elements (a) in  $W(\mathfrak{o})$ . Therefore the equation  $\overline{\tau} = \varrho \circ W(\overline{\varphi})$  is equivalent to  $\tau \mid W(\mathfrak{o}) = \varrho \circ W(\varphi \mid \mathfrak{o}) = \widehat{\varphi}$ . Thus (ii)  $\Rightarrow$  (i) is clear, and to prove (i)  $\Rightarrow$  (ii) it only remains to be shown that if  $\tau$  lies over  $\varphi$  the rings  $\mathfrak{o}'$  and  $\mathfrak{o}$  coincide.
  - b) Assume that  $\tau$  lies over  $\varphi$ . We first show that  $\mathfrak{m} \subset \mathfrak{m}'$ . Let a be an element of K

which is not in m' and let b=|a| with respect to  $\tau$ . There exists some c>0 in k with b>c. By Lemma 1.2(i) we obtain  $\varphi(b)=\infty$  or  $\varphi(b)\geqslant \varphi(c)\geqslant 0$ . Since  $\varphi(c)\neq 0$ , certainly  $\varphi(b)\neq 0$  and thus  $\varphi(a)\neq 0$ , i.e. a lies not in m. This proves  $m \subset m'$ .

Now we show  $\mathfrak{m}' \subset \mathfrak{m}$ . Then  $\mathfrak{o} = \mathfrak{o}'$  will be clear. Assume a is an element of  $\mathfrak{m}'$  and without loss of generality a > 0. For any c > 0 in k we have 0 < a < c and thus by Lemma 1.2 (ii)  $0 \le \varphi(a) \le \varphi(c)$ . Since R is archimedian over  $\varphi(k)$  the value  $\varphi(a)$  must be zero i.e. a lies in  $\mathfrak{m}$ . q.e.d.

EXAMPLE 1.4. If k is a maximal subfield of K such that a given place  $\varphi: K \to R \cup \infty$  is trivial on k then for any signature  $\tau$  of K the conditions (i) and (ii) of Proposition 1.3 are equivalent. In fact, all values of  $\varphi$  lie in the algebraic closure R' of  $\varphi(k)$  in R, which is archimedian over  $\varphi(k)$ . Replace R by R'!

COROLLARY 1.5. Let  $\sigma$  be a signature on a field K and k be a subfield of K, whose algebraic closure k' in K is maximal archimedian with respect to  $\sigma$ . Further assume that  $\chi: k \to R$  is an order preserving homomorphism with respect to  $\sigma \mid k$  and  $\varrho$ . Then there exists a unique place  $\varphi: K \to R \cup \infty$  with the following properties:

- (i)  $\varphi$  is compatible with  $\sigma$ ,
- (ii)  $\varphi \mid k = \chi$ ,
- (iii)  $\varphi$  is zero dimensional over k, i.e. all values  $\neq \infty$  of  $\varphi$  are algebraic over  $\chi(k)$ .

**Proof.** We replace R by the algebraic closure of  $\varphi(k)$  in R and then forget condition (iii). According to Proposition 1.3 the valuation ring of the place  $\varphi$  to be constructed must coincide with  $\mathfrak{o} = \mathfrak{o}(K/k, \tau)$ . Now, by the assumption about k', the residue class ring  $\mathfrak{o}/\mathfrak{m}$  of  $\mathfrak{o}$  is algebraic over k. Thus it follows from a well known theorem of Artin and Schreier [AS, Satz 8] that there exists a unique order preserving homomorphism  $\beta:\mathfrak{o}/\mathfrak{m} \to R$  with respect to  $\bar{\tau}$  and  $\varrho$  which extends  $\chi$  (cf [K<sub>1</sub>, Cor. 5.1], where this is proved by similar methods as are used in the present paper). By Proposition 1.3 the place which has the valuation ring  $\mathfrak{o}$  and induces on  $\mathfrak{o}/\mathfrak{m}$  the map  $\beta$  fulfills the conditions (i) and (ii) and is the only R-valued place with these properties. q.e.d.

From Proposition 1.3 and Corollary 1.5 we obtain

THEOREM 1.6. (cf [L], Th. 6). Assume that L is an algebraic field extension of K, that  $\tau$  is a signature of L and that  $\varphi$  is an R-valued place of K, compatible with  $\tau \mid K$ . Then there exists a unique R-valued place  $\psi$  of L extending  $\varphi$  and compatible with  $\tau$ .

**Proof.** We chose a maximal subfield k of K on which  $\varphi$  is trivial. Then  $\varphi$  is zero dimensional over k. By Example 1.4 the valuation ring  $\mathfrak{o}$  of  $\varphi$  coincides with the valuation ring  $\mathfrak{o}(K/k, \sigma)$  associated with the restriction  $\sigma$  of  $\tau$  to K. Since L/K is algebraic, the residue class ring of  $\mathfrak{o}(L/k, \tau)$  is algebraic over the residue class ring of  $\mathfrak{o}(K/k, \sigma)$ . Thus the algebraic closure k' of k in L is maximal archimedian with respect to  $\tau$ . Now we can apply Cor. 1.5 to  $\sigma$  and the homomorphism  $\chi = \varphi \mid k$  from k to R and also to

 $\tau$  and  $\chi$ . Clearly  $\varphi$  is the place of K corresponding to  $\sigma$  and  $\chi$  in the sense of Cor. 1.5. We denote by  $\psi$  the place of L corresponding to  $\tau$  and  $\chi$ . Since  $\psi \mid K$  must correspond to  $\sigma$  and  $\chi$  we have  $\psi \mid K = \varphi$ . On the other hand any R-valued extension  $\psi'$  of  $\varphi$  to L is zero dimensional over k. Thus if  $\psi'$  is compatible with  $\tau$ , Cor. 1.5 yields  $\psi' = \psi$ .

§2

We want to study the orderings of a field K which lie over a given R-valued place of K. For this purpose we first consider more generally an arbitrary valuation ring  $\mathfrak o$  with maximal ideal  $\mathfrak m$ , residue class field  $k=\mathfrak o/\mathfrak m$ , and quotient field K. For any a in  $\mathfrak o$  we denote by  $\bar a$  the image in k. The following proposition has been proved in this generality in  $[K_3, \S 3]$  (cf [Sp], [M, Chap. V] if  $\mathfrak o$  is discrete, and  $[K, \S 12]$  if  $\mathfrak o$  has rank one).

PROPOSITION 2.1. There exists a unique additive map  $\partial: W(K) \to W(k)$  such that  $\partial(a) = (\bar{a})$  for every a in  $\mathfrak{D}^*$  and  $\partial(a) = 0$  for every (a) in Q(K) which lies not in  $Q(\mathfrak{D})$ , i.e. with  $aK^{*2} \cap \mathfrak{D}^*$  empty.<sup>2</sup>

Let  $v: K^* \to \Gamma$  denote a valuation corresponding to  $\mathfrak p$  with value group  $\Gamma$ . This valuation induces a map  $\tilde v$  from  $Q(K) = K^*/K^{*2}$  to  $\Gamma/2\Gamma$ . We chose a subgroup M of Q(K) such that  $\tilde v$  gives a bijection from M to  $\Gamma/2\Gamma$ . Such a subgroup M clearly exists, since Q(K) and  $\Gamma/2\Gamma$  are vector spaces over the field of two elements. We call M a group of representatives for  $\Gamma/2\Gamma$ . Any element z of W(K) can be written – possibly in different ways – in the form  $z = \sum_{m \in M} x_m m$  with  $x_m$  in  $W(\mathfrak p)$  and only finitely many  $x_m \neq 0$ , since this is true for the generators (a) of W(K). We denote for any x in  $W(\mathfrak p)$  by  $\bar x$  the image under the natural map from  $W(\mathfrak p)$  to W(K). Then one immediately computes for m in M:

$$\partial (mz) = \bar{x}_m. \tag{2.2}$$

In particular  $\partial(mz)$  is zero for nearly all m in M. Thus we have a map

$$\Delta: W(K) \to W(k)[M]$$

into the group ring of M over W(k), defined by

$$\Delta(z) = \sum_{m \in M} \partial(mz) m. \tag{2.3}$$

It is clear from (2.2) that  $\Delta$  is a ring homomorphism which is surjective, since the

<sup>&</sup>lt;sup>2</sup>) In [K<sub>3</sub>] this map  $\partial$  is denoted by  $\varphi_*$  with  $\varphi$  the place  $K \to k \cup \infty$  corresponding to  $\emptyset$ . In the present paper  $\varphi_*$  will have a slightly different meaning.

natural map  $W(\mathfrak{o}) \to W(k)$  is surjective. It is further clear from (2.2) that  $\Delta(z) = 0$  if and only if all  $x_m$  lie in the kernel of  $W(\mathfrak{o}) \to W(k)$ , which we denote by  $W(\mathfrak{o}, \mathfrak{m})$ . Summarizing we obtain

**PROPOSITION** 2.4.  $\Delta: W(K) \to W(k)$  [M] is a ring-epimorphism whose kernel is the ideal of W(K) generated by  $W(\mathfrak{o}, \mathfrak{m})$ .

*Remarks*. i) It has been shown in [K, §12] that the set  $W(\mathfrak{o}, \mathfrak{m})$  itself is an ideal of W(K) if  $\mathfrak{o}$  has rank one.

ii) It is not difficult to prove for any local ring  $\mathfrak o$  with maximal ideal  $\mathfrak m$  that  $W(\mathfrak o, \mathfrak m)$  is generated as an ideal by the elements 1-(1+d) with d in  $\mathfrak m$ . We shall not need this fact.

We now consider a real place  $\varphi: K \to R \cup \infty$  and denote by  $\mathfrak o$  the valuation ring of  $\varphi$ . We continue to use the notations  $\mathfrak m$ , k, v,  $\Gamma$ , M with respect to  $\mathfrak o$  as above.  $\varphi$  induces a homomorphism  $\overline{\varphi}$  from k into R. The composite map

$$\varphi_* = \varrho \circ W(\bar{\varphi}) \circ \partial : W(K) \to W(k) \to W(R) \to \mathbb{Z}$$

has the description given in the introduction, and the restriction of  $\varphi_*$  to  $W(\mathfrak{o})$  is the ring homomorphism  $\hat{\varphi}$  considered in §1. As an easy consequence of Proposition 2.4 we obtain

THEOREM 2.5. The signatures  $\sigma: W(K) \to \mathbb{Z}$  lying over a given place  $\varphi: K \to R \cup \infty$  correspond uniquely to the characters  $\chi: M \to \{\pm 1\}$  by the following formulas:

$$\chi(m) = \sigma(m)$$

for m in M, and

$$\sigma(z) = \sum_{m \in M} \chi(m) \varphi_*(mz)$$

for z in W(K).

Remark. Except the last formula this has already been proved by Krull in a different way [Kr, p. 189].

**Proof.** Clearly  $\hat{\varphi}$  vanishes on  $W(\mathfrak{o}, \mathfrak{m})$ . Thus any signature  $\sigma$  lying over  $\varphi$  must vanish on the ideal generated by  $W(\mathfrak{o}, \mathfrak{m})$  in W(K). By Prop. 2.4 such a signature must have the form  $\sigma = \alpha \circ \Delta$  with a uniquely determined ring homomorphism  $\alpha$  from W(k)[M] to  $\mathbb{Z}$ . These homomorphisms  $\alpha$  correspond uniquely to the pairs  $(\alpha_0, \chi)$  consisting of a homomorphism  $\alpha_0 \colon W(k) \to \mathbb{Z}$  and a character  $\chi \colon M \to \{\pm 1\}$ , the correspondence being given by  $\alpha_0 = \alpha \mid W(k)$  and  $\chi = \alpha \mid M$ . From the definition (2.3) of  $\Delta$  we obtain for the signature  $\sigma = \alpha \circ \Delta$  and z in W(K):

$$\sigma(z) = \sum_{m \in M} \chi(m) (\alpha_0 \circ \partial) (mz).$$

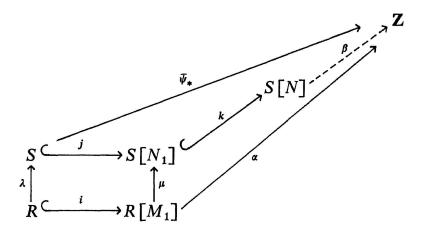
Such a signature  $\sigma$  coincides with  $\hat{\varphi}$  on  $W(\mathfrak{o})$  if and only if  $\alpha_0(\bar{z}) = \hat{\varphi}(z)$  for all z in  $W(\mathfrak{o})$ . This means  $\alpha_0 \circ \partial = \varphi_*$ . Theorem 2.5 is now obvious. q.e.d.

THEOREM 2.6. Let L be an arbitrary field extension of K and  $\psi: L \to R \cup \infty$  a real place. Further let  $\sigma$  be a signature of K. Then the following are equivalent:

- (i) There exists a signature  $\tau$  of L which lies over  $\psi$  and extends  $\sigma$ .
- (ii)  $\sigma(a) = \psi_*(a)$  for all a in  $K^*$  such that  $\psi_*(a) \neq 0$ .

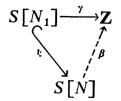
Proof. (i)  $\Rightarrow$  (ii) is evident. We now assume (ii), which in particular implies that  $\sigma$  lies over the restriction  $\varphi = \psi \mid K$ . We denote by  $\mathfrak o$  and  $\mathfrak O$  the valuation rings of  $\varphi$  resp.  $\psi$ , and by  $\bar{\mathfrak o}$  and  $\bar{\mathfrak O}$  their residue class fields, further by  $w: L^* \to \Gamma'$  a valuation corresponding to  $\mathfrak O$  with value group  $\Gamma'$  and by  $\Gamma$  the value group  $w(K^*)$  of  $w \mid K^*$ . We chose a group M of representatives for  $\Gamma/2\Gamma$  in Q(K). Let  $M_0$  denote the subgroup of all m in M with w(m) in  $2\Gamma'$ , and let  $M_1$  denote an arbitrary chosen subgroup of M such that  $M = M_0 \times M_1$ . The map from  $M_1$  into  $\Gamma'/2\Gamma'$  induced by w is injective. Thus also the natural map from  $M_1$  into Q(L) is injective, and we can chose a group N of representatives of  $\Gamma'/2\Gamma'$  in Q(L) which contains the image  $N_1$  of  $M_1$ . We further chose a subgroup  $N_0$  of N such that  $N = N_0 \times N_1$ . Finally let  $\Delta$  denote the map (2.3) from W(K) onto  $W(\bar{\mathfrak o})$  M and  $\Delta'$  the analogous map from W(L) onto  $W(\bar{\mathfrak o})$  M.

By the proof of Theorem 2.5 we have  $\sigma = \alpha \circ \Delta$  with a homomorphism  $\alpha$  from  $W(\bar{\mathfrak{o}})$  [M] to Z which extends the homomorphism  $\bar{\varphi}_* = \varrho \circ W(\bar{\varphi})$  from  $W(\bar{\mathfrak{o}})$  to Z. Similarly  $\tau$  must have the form  $\tau = \beta \circ \Delta'$  with a homomorphism  $\beta$  from  $W(\bar{\mathfrak{o}})$  [N] to Z which we have to construct. Since the natural map from Q(K) to Q(L) maps  $M_0$  into  $Q(\mathfrak{D}) \subset W(\mathfrak{D})$ , there is an obvious map  $\lambda$  from the ring  $R := W(\bar{\mathfrak{o}})$  [ $M_0$ ] to the ring  $S := W(\bar{\mathfrak{o}})$ . Combining  $\lambda$  with the bijection  $M_1 \cong N_1$  induced by  $Q(K) \to Q(L)$  we obtain a map  $\mu$  from the ring  $R[M_1] = W(\bar{\mathfrak{o}})$  [M] to  $S[N_1]$ . Consider now the diagram



with inclusion maps i, j, k. That  $\tau$  lies over  $\psi$  means that the upper triangle is commutative, and that  $\tau$  extends  $\sigma$  means that the lower triangle is commutative. Our hypothesis (ii) means  $\overline{\psi}_* \circ \lambda = \alpha \circ i$ . Now the square with the arrows  $i, \mu, \lambda, j$  is a pushout

{i.e. gives a description of  $S[N_1]$  as the tensor product of S and  $R[M_1]$  over R}. Thus there is a unique map  $\gamma: S[N_1] \to \mathbb{Z}$  with  $\gamma \circ j = \overline{\psi}_*$  and  $\gamma \circ \mu = \alpha$ . The only condition which  $\beta$  has to fulfill is the commutativity of the following triangle:



Since  $S[N] = S[N_1][N_0]$  clearly such a homomorphism  $\beta$  exists. q.e.d.

Remark 2.7. More precisely the extensions of  $\gamma: S[N_1] \to \mathbb{Z}$  to  $\mathbb{Z}$ -valued homomorphisms  $\beta$  of S[N] correspond bijectively to the characters of  $N_0$ . The map from  $N_0$  to  $\Gamma'/(2\Gamma' + \Gamma)$  induced by w is bijective. Thus the signatures  $\tau$  of L extending  $\sigma$  and lying over  $\psi$  correspond bijectively to the characters of  $\Gamma'/(2\Gamma' + \Gamma)$  (in a non-canonical way), if there are any such signatures  $\tau$ .

We close this section with an application of the last two theorems.

THEOREM 2.8. Assume that  $L_1$  and  $L_2$  are field extensions of a field K and that  $L_1$  is algebraic over K. Further assume that on each  $L_i$  an R-valued place  $\varphi_i$  is given such that  $\varphi_1$  and  $\varphi_2$  coincide on K. Then the following are equivalent:

- (i) There exists a field composite F of  $L_1$  and  $L_2$  over K and an R-valued place  $\psi$  on F extending both  $\varphi_1$  and  $\varphi_2$ .
- (ii)  $\varphi_{1*}(a) = \varphi_{2*}(a)$  for all a in  $K^*$  such that both  $\varphi_{1*}(a)$  and  $\varphi_{2*}(a)$  are not zero. Proof. (i)  $\Rightarrow$  (ii) is trivial. We now assume that (ii) holds. We first construct on each  $L_i$  a signature  $\sigma_i$  lying over  $\varphi_i$  such that  $\sigma_1 \mid K = \sigma_2 \mid K$ : Let  $\varphi$  denote the restriction  $\varphi_1 \mid K = \varphi_2 \mid K$ , let  $\Gamma$  denote the value group of a valuation of K corresponding to  $\varphi$ , and M denote a group of representatives of  $\Gamma/2\Gamma$  in the group Q(K). Further let  $A_i$  (i=1,2) denote the subgroup of all m in M such that  $\varphi_{i*}(m) \neq 0$  and let  $\chi_i$  denote the character  $\varphi_{i*} \mid A_i$  of  $A_i$ . By hypothesis  $\chi_1$  and  $\chi_2$  coincide on  $A_1 \cap A_2$ . Thus it is possible to choose a character  $\chi$  of M with  $\chi \mid A_i = \chi_i$  for i=1,2. Let  $\sigma$  denote the signature of K lying over  $\varphi$  and corresponding to the character  $\chi$  as explained in Theorem 2.5. Clearly  $\sigma(m) = \varphi_{1*}(m)$  for any m in M with  $\varphi_{1*}(m) \neq 0$  and thus  $\sigma(a) = \varphi_{1*}(a)$  for any a in  $K^*$  with  $\varphi_{1*}(a) \neq 0$ . The same holds with  $\varphi_2$  instead of  $\varphi_1$ . By Theorem 2.6 there exists a signature  $\sigma_i$  on each  $L_i$  which lies over  $\varphi_i$  and extends  $\sigma$ .

We now obtain a field composite F of  $L_1$  and  $L_2$  over K and a signature  $\tau$  on F extending both  $\sigma_1$  and  $\sigma_2$  in the following way: Let S be a real closure of  $L_2$  with respect to  $\sigma_2$  and  $\gamma$  denote the signature of S. There exists a (unique) homomorphism  $f: L_1 \to S$  over K which is compatible with  $\sigma_1$  and  $\gamma$ , i.e.  $\sigma_1 = \gamma \circ W(f)$ . {Apply [AS], Satz 8. This is also a special case of our Theorem 1.6: Extend the trivial place  $K \hookrightarrow S$ 

to an S-valued place on  $L_1$  compatible with  $\sigma_1$ . The field composite  $F := f(L_1)$   $L_2 \subset S$  and the signature  $\tau := \gamma \mid F$  have the desired properties.

By Theorem 1.6 there is a unique R-valued place  $\psi$  on F which extends  $\varphi_2$  and is compatible with  $\tau$ . Clearly  $\psi \mid L_1$  is compatible with  $\sigma_1$  and extends  $\varphi$ . Thus by the same theorem  $\psi \mid L_1 = \varphi_1$ . q.e.d.

Remark 2.9. In the proof just completed we used the fact that for given signatures  $\sigma_1$ ,  $\sigma_2$  on  $L_1$  and  $L_2$  with  $\sigma_1 \mid K = \sigma_2 \mid K$  there exists a field composite F of  $L_1$  and  $L_2$  over K and a signature  $\tau$  on F with  $\tau \mid L_1 = \sigma_1$  and  $\tau \mid L_2 = \sigma_2$ . This remains true if both  $L_1$  and  $L_2$  are arbitrary field extensions of K with F a free field composite. Further it can be shown that for given  $\sigma_1$  and  $\sigma_2$  up to equivalence only one such free composite F and only one such  $\tau$  exists. These facts are closely related to the following theorem (see also  $[K_2, Th. 3]$ ): Denote by A the total quotient ring of  $L_1 \otimes_K L_2$ . The kernel and the cokernel of the obvious map from  $W(L_1) \otimes_{W(K)} W(L_2)$  to W(A) are 2-primary torsion groups. I omit the proofs since we do not need these results in the present paper.

In the situation of Theorem 3.9 it may happen that there exists more than one R-valued place  $\psi$  on a field composite F of  $L_1$  and  $L_2$  which extends both  $\varphi_1$ ,  $\varphi_2$ , as shows the following

EXAMPLE 2.10. Let  $\varphi$  be an R-valued place on a field K and let a and c be elements of K such that  $\varphi_*(a) = 0$ ,  $\varphi_*(c) = 1$ , c not a square. For example let K = R(t) with one indeterminate t, let  $\varphi$  be the place over R with  $\varphi(t) = 0$  and a = t,  $c = 1 + t^2$ . Using the trace formula proved in the next section (see also Introduction) one easily checks that there is exactly one R-valued place  $\varphi_1$  on  $L_1 := K(\sqrt{a})$  and one R-valued place  $\varphi_2$  on  $L_2 := K(\sqrt{ac})$  which extend  $\varphi$  (Of course this also follows from general valuation theory). The field  $F = K(\sqrt{a}, \sqrt{ac})$  is the only composite of  $L_1$  and  $L_2$  over K. But  $\varphi_2$  has – by the same trace formula – exactly two extensions  $\psi$ ,  $\psi'$  to F with values in R, which both must also extend  $\varphi_1$ .

§3

Assume that L is a finite extension of degree n of a field K and that an K-valued place  $\varphi$  is given on K. Let  $Tr^*: W(L) \to W(K)$  denote the transfer map from W(L) to W(K) with respect to the regular trace  $Tr = Tr_{L/K}$  ([S], cf. Introduction). The goal of this section is to prove the following trace formula:

THEOREM 3.1. For every 
$$x$$
 in  $W(L)$ 

$$\varphi_*(\operatorname{Tr}^*(x)) = \sum_{\psi \mid \varphi} \psi_*(x),$$

with the sum taken over all R-valued places  $\psi$  on L which extend  $\varphi$ .

N.B. The sum is finite, since  $\varphi$  has at most n extensions  $\psi$  to L [B, Chap VI, §8 no. 3, Th. 1].

We shall deduce this theorem from our results about the connection between signatures and real places in the previous sections and from the following trace formula for signatures, which is a consequence of Artin-Schreier's theory of real closures (see  $[K_1, \S 5]$ ):

**PROPOSITION** 3.2. For every signature  $\sigma$  of K and every element x of W(L)

$$\sigma(\operatorname{Tr}^*(x)) = \sum_{\tau \mid \sigma} \tau(x),$$

where  $\tau$  runs through the finite set of all signatures of L which extend  $\sigma$ 

Let  $\psi_i$ ,  $1 \le i \le r$ , denote the *R*-valued places of *L* which extend K(r=0), if there are no such places). We chose a valuation  $v: K^* \to \Gamma$  with value group  $\Gamma$  corresponding to the place  $\varphi$ , and for each  $\psi_i$ ,  $1 \le i \le r$ , a corresponding valuation  $w_i: L^* \to \Gamma_i$  extending v with value group  $\Gamma_i \supset \Gamma$ . Since  $\sum_i (\Gamma_i: \Gamma) \le n$  [B, loc. cit.], all  $(\Gamma_i: \Gamma)$  are finite.

To prove Theorem 3.1 we have to surmount some technical difficulties, which arise from the fact that  $\Gamma$  may not be finitely generated. To get an idea of the proof the reader is advised to follow first the proof under the additional assumption that  $\Gamma$  is finitely generated. Then it is clear (and follows from Lemma 3.3 below), that  $\Gamma/2\Gamma$  and all  $\Gamma_i/2\Gamma_i$  have the same finite cardinality. The proof goes through with the choice  $M_0 = M$ ,  $M_1 = \{1\}$ ,  $N_{i0} = N_i$  below, and Lemma 3.3 and 3.4 may be skipped.

LEMMA 3.3. For each i with  $1 \le i \le r$  the kernel and the cokernel of the natural map  $\alpha_i: \Gamma/2\Gamma \to \Gamma_i/2\Gamma_i$  are finite and have the same cardinality.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} 0 \rightarrow \Gamma \rightarrow \Gamma_i \rightarrow \Gamma_i/\Gamma \rightarrow 0 \\ & \downarrow^2 & \downarrow^2 & \downarrow^2 \\ 0 \rightarrow \Gamma \rightarrow \Gamma_i \rightarrow \Gamma_i/\Gamma \rightarrow 0 \end{array}$$

of exact sequences, where the vertical arrows denote the homotheties  $x \mapsto 2x$ . Since  $\Gamma_i/\Gamma$  is finite, the kernel  $A_i$  and the cokernel  $B_i$  of  $\Gamma_i/\Gamma \xrightarrow{2} \Gamma_i/\Gamma$  have the same finite cardinality. Now the snake lemma gives an exact sequence

$$0 \to A_i \to \Gamma/2\Gamma \xrightarrow{\alpha_i} \Gamma_i/2\Gamma_i \to B_i \to 0$$

which makes the assertion obvious. q.e.d.

In  $Q(K) \cong K^*/K^{*2}$  we chose a group M of representatives of  $\Gamma/2\Gamma$ . Then after fixing an element x of W(L) we chose a decomposition  $M = M_0 \times M_1$  with the following properties:

- a)  $M_0$  is finite,
- b)  $M_0$  contains the representatives of the elements of all finite subsets  $\text{Ker }\alpha_i$ ,  $1 \le i \le r$ , of  $\Gamma/2\Gamma$ ,
  - c)  $\varphi_*(m \cdot \text{Tr}^*(x)) = 0$  for m in M but not in  $M_0$ ,
  - d)  $\psi_{i*}(m \cdot x) = 0$  for  $1 \le i \le r$  and all m in  $M_1$  with  $m \ne 1$ .

Clearly such a decomposition of M does exist. By property b) the map  $Q(K) \to Q(L)$  is injective on  $M_1$ . Thus we regard  $M_1$  also as a subgroup of Q(L). By property b) further all maps  $w_i: Q(L) \to \Gamma_i/2\Gamma_i$  are injective on  $M_1$ . For each i with  $1 \le i \le r$  we chose a group  $N_i$  of representatives of  $\Gamma_i/2\Gamma_i$  in Q(L) which contains  $M_1$ . Then we chose a decomposition  $N_i = N_{i0} \times M_1$ .

LEMMA 3.4. All  $N_{i0}$ ,  $1 \le i \le r$ , have the same cardinality as  $M_0$ . Proof. For each i,  $1 \le i \le r$ , we have a commutative diagram of exact sequences

$$\begin{array}{cccc} 0 \rightarrow \ \tilde{v}\left(M_1\right) \rightarrow \ \Gamma/2\Gamma \rightarrow C \rightarrow 0 \\ & \downarrow & \downarrow^{\alpha_i} & \downarrow \\ 0 \rightarrow \tilde{w}_i\left(M_1\right) \rightarrow \Gamma_i/2\Gamma_i \rightarrow D_i \rightarrow 0 \end{array}$$

with groups C and  $D_i$  which have the cardinalities  $|C| = |M_0|$  and  $|D_i| = |N_{i0}|$ . Let  $X_i$  denote the kernel and  $Y_i$  the cokernel of the map  $C \to D_i$ . Clearly the map  $\tilde{v}(M_1) \to \tilde{w}_i(M_1)$  is bijective. Thus  $X_i \cong \operatorname{Ker} \alpha_i$  and  $Y_i \cong \operatorname{Coker} \alpha_i$ . Lemma 3.4 now follows from Lemma 3.3 and the exact sequence

$$0 \to X_i \to C \to D \to Y_i \to 0$$
. q.e.d.

After these preparations we consider sets S and  $T_i$ ,  $1 \le i \le r$ , of signatures, defined as follows: S = set of all signatures  $\sigma$  of K lying over  $\varphi$  with  $\sigma(m) = 1$  for all m in  $M_1$ ; further  $T_i = \text{set}$  of all signatures  $\tau$  of L lying over  $\psi_i$  with  $\tau(m) = 1$  for all m in  $M_1$ . We regard the group  $\hat{M}_0$  of characters of  $M_0$  as the subgroup of characters of M which are trivial on  $M_1$ , and  $\hat{N}_{i0}$  as the group of characters of  $N_i$  which are trivial on  $M_1$ . Theorem 2.5 gives a bijection from S to  $\hat{M}_0$  mapping each  $\sigma$  in S to the corresponding character of M, and in the same way a bijection from  $T_i$  to  $\hat{N}_{i0}$ . Thus we know from Lemma 3.4 that the sets S,  $T_i$ ,  $1 \le i \le r$ , are all finite and have the same cardinality  $|M_0|$ .

Now Theorem 1.6 says, that for each signature  $\tau$  of L, whose restriction  $\tau \mid K$  lies over  $\varphi$ , there exists a unique R-valued place  $\psi$  of L which extends  $\varphi$  and is compatible with  $\tau$ . Thus the union  $T = \bigcup_i T_i$  of all  $T_i \{ T = \emptyset \text{ if } r = 0 \}$  is the set of all signatures  $\tau$  of L whose restrictions  $\tau \mid K$  belong to S, and furthermore this union is disjoint,  $T_i \cap T_j = \emptyset$  for  $i \neq j$ .

Theorem 3.1 will now come out by computing the finite sum  $\sum_{\tau \in T} \tau(x)$  with our

fixed element x in two different ways. On the one hand,

$$\sum_{\tau \in T} \tau(x) = \sum_{\sigma \in s} \sum_{\tau \mid \sigma} \tau(x)$$

Thus by Prop. 3.2 with  $z = Tr^*(x)$ :

$$\sum_{\tau \in T} \tau(x) = \sum_{\sigma \in S} \sigma(z).$$

By the last formula in Theorem 2.5 this sum equals

$$\sum_{\chi \in \widehat{M}_0} \left( \sum_{m \in M} \chi(m) \, \varphi_*(mz) \right).$$

By the property c) of the decomposition  $M = M_0 \times M_1$  we may replace M by  $M_0$  in the interior sum. Then interchanging the summations we obtain

$$\sum_{z \in T} \tau(x) = |M_0| \ \varphi_*(z). \tag{3.5}$$

On the other hand

$$\sum_{\tau \in T} \tau(x) = \sum_{i=1}^{r} \sum_{\tau \in T_i} \tau(x).$$

{Read zero for the right hand side if r=0}. Again by Theorem 2.5

$$\sum_{\tau \in T_i} \tau(x) = \sum_{\chi \in \hat{N}_{i0}} \left( \sum_{m \in N_i} \chi(m_0) \psi_{i*}(mx) \right),$$

where  $m_0$  denotes the component of m in  $N_{i0}$ . Since  $N_{i0}$  is finite, we may interchange the summations and obtain for the right hand side

$$\sum_{m \in N_i} \psi_{i*}(mx) \left( \sum_{\chi \in N_{i0}} \chi(m_0) \right).$$

The interior sum is zero if  $m_0 \neq 1$ . But if  $m_0 = 1$ , then  $m \in M_1$  and by property d) of our decomposition of M the factor  $\psi_{i*}(mx)$  vanishes except for m=1. Thus this sum reduces to  $|N_{i0}| \psi_{i*}(x)$ , and we obtain

$$\sum_{t \in T} \tau(x) = \sum_{i=1}^{r} |N_{i0}| \, \psi_{i*}(x). \tag{3.6}$$

Theorem 3.1 now follows from (3.5), (3.6), and Lemma (3.4).

If  $\varphi$  is a discrete place, a more direct and more geometric proof of Theorem 3.1 without resorting to signatures follows easily from the lemma on p. 322 in [G]. It would be desirable to have a similar proof in the general case. The main difficulty

seems to be that one has to work with quadratic forms on modules, which in general are not finitely generated.

## §4

This section does not use the content of §2 and §3. Assume that  $\varphi: K \to R \cup \infty$  is a place from a field K to a real closed field R and that L is a *finitely generated* field extension of K. We ask for a criterion that  $\varphi$  is extendable to an R-valued place of L. Let  $\mathfrak{p}$  denote the valuation ring of  $\varphi$  and as in §1 let  $\hat{\varphi}$  denote the homomorphism from  $W(\mathfrak{p})$  to Z induced by  $\varphi$ .

PROPOSITION 4.1.  $\varphi$  is extendable to an R-valued place of L if and only if  $\hat{\varphi}$  vanishes on the kernel of the natural map  $W(\mathfrak{o}) \to W(L)$ .

*Proof.* We denote this kernel by  $W(\mathfrak{o}, L)$ .

- i) Assume that  $\psi: L \to R \cup \infty$  is an extension of  $\varphi$ . Then for every Z in  $W(\mathfrak{o})$  clearly  $\hat{\varphi}(Z) = \hat{\psi}(Z_L)$  with  $Z_L$  the image of Z in W(L) under the natural map  $W(K) \to W(L)$ . Thus  $\hat{\varphi}(Z) = 0$  if Z is in  $W(\mathfrak{o}, L)$ .
- ii) Now assume  $\hat{\varphi}(W(\mathfrak{o}, L) = 0$ . We first construct a signature  $\tau$  of L such that  $\tau \mid K$  lies over  $\varphi$ . We proceed as in the proof of Prop. 1.1. The kernel P of the ring-homomorphism  $\hat{\varphi}$  from  $W(\mathfrak{o})$  onto  $\mathbf{Z}$  contains  $W(\mathfrak{o}, L)$  and thus yields a prime ideal  $\bar{P}$  of  $W(\mathfrak{o})/W(\mathfrak{o}, L)$ , which is a subring of W(L) in a natural way.  $\bar{P}$  must be minimal [KRW]. Thus we can find a minimal prime ideal Q of W(L) lying over  $\bar{P}$ . Since  $W(\mathfrak{o})/P \cong \mathbf{Z}$  embeds into W(L)/Q we have  $W(L)/Q \cong \mathbf{Z}$ . The only homomorphism  $\tau: W(L) \to \mathbf{Z}$  with kernel Q is a signature whose restriction  $\sigma = \tau \mid K$  lies over  $\varphi$ .

Let S be a real closure of L with respect to  $\tau$ . Then the algebraic closure K' of K in S is a real closure of K with respect to  $\sigma$ . By Theorem 1.6 there exists a unique R-valued place  $\gamma$  of K' which extends  $\varphi$ . On the other hand the composite field L' = LK' in S is a finitely generated formally real field extension of K'. Thus by a well known theorem of Artin and Lang ([L, Theorem 7], see also  $[K_1, \S 6]$ ) there exists a place  $\chi: L' \to K' \cup \infty$  which is the identity on K'. The R-valued place  $\gamma \circ \chi$  on L' extends  $\gamma$  and thus its restriction to L extends  $\varphi$ . q.e.d.

For any field extension L/K we denote by W(K, L) the kernel of the natural map from W(K) to W(L).

THEOREM 4.2. Let  $L\supset K\supset k$  be three fields such that L/k is finitely generated. Then the following are equivalent:

- i) Any place  $\varphi: K \to R \cup \infty$  into a real closed field R which is trivial on k can be extended to an R-valued place of L.
- ii) The statement i) with the additional condition inserted, that R is algebraic over  $\varphi(k)$ .

iii) W(K, L) is a torsion group.

Notice that in statement (iii) the field k does not occur.

**Proof.** (i)  $\Rightarrow$  (ii) is trivial, and (iii)  $\Rightarrow$  (i) follows from the previous proposition 4.1. To prove (ii)  $\Rightarrow$  (iii) we assume that there exists an element Z in W(K, L) which is not torsion. We have to show that there exists a real closure R of k and a place  $\varphi: K \rightarrow R \cup \infty$  which is the identity on k, such that  $\varphi$  cannot be extended to an R-valued place of L. We write

$$Z = \sum_{i=1}^{n} \left( a_i \right)$$

with elements  $(a_i)$  in Q(K). Since Z is not a torsion element there exists a signature  $\sigma$  of K such that

$$\sigma(Z) = \sum_{i=1}^{n} \sigma(a_i) \neq 0$$

([P, Satz 22], see also [LL], [KRW]). Let S denote a real closure of K with respect to  $\sigma$  and R denote the algebraic closure of k in S. Finally let K' denote the field composite KR in S, which is finitely generated over R. According to Artin and Lang [L, Th. 8, p. 387] there exists a place  $\gamma: K' \to R \cup \infty$ , which is the identity on R, such that all  $\gamma(a_i)$ ,  $1 \le i \le n$  are finite and not zero and  $\hat{\gamma}(a_i) = \sigma(a_i)$ . Let  $\varphi$  denote the restriction of  $\gamma$  to K and  $\mathfrak{o}$  the valuation ring of  $\varphi$ . Clearly  $Z \in W(\mathfrak{o})$  and  $\hat{\varphi}(Z) = \sigma(Z) \neq 0$ . By Proposition 4.1 this place  $\varphi$  can not be extended to L. q.e.d.

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