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Šmulian-Eberlein Spaces[1])

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Introduction

A subset of a Hausdorff topological space will be called *nearly relatively compact* (resp. *relatively countably compact*) if any filter (resp. any sequence) on this subset has a nonempty adherence in the space. Any relatively compact set (i.e. a subset of a compact set) is nearly relatively compact and any nearly relatively compact set is relatively countably compact. For regular spaces the relatively compact and nearly relatively compact sets coincide. A subset of a Hausdorff topological space will be called *sequentially dense* if every point of its closure is the limit point of a convergent sequence in this subset. The theorem of Šmulian (resp. Eberlein) gives a sufficient condition for a relatively countably compact set to be sequentially dense (resp. nearly relatively compact). Many generalizations of these important theorems were given, the most general by J. D. Pryce [4]. In order to study these problems D. H. Fremlin defined a Hausdorff topological space as *angelic* [4] if any relatively countably compact set is relatively compact and sequentially dense.

In potential theory there exists a theorem proved by A. Cornea [2], which deals with the convergence of harmonic functions, and which is closely related to the theorem of Eberlein. Because of the order relation appearing in the theorem of Cornea, neither of these theorems can be deduced from the other. In order to obtain a more general result, containing at the same time the theorems of Pryce and Cornea, we have introduced the notions of *Eberlein space* and *Šmulian space*. On an Eberlein (resp.

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[1]) This paper was elaborated during the period the author visited the ETH Zürich Forschungs institut für Mathematik and the EPF Lausanne, Département de Mathématiques.

Šmulian) space any relatively countably compact set is nearly relatively compact (Theorem 2.13) (resp. sequentially dense (Theorem 3.22)), so that a regular Šmulian-Eberlein space (i.e. a regular space which is both a Šmulian and an Eberlein space) is angelic. But there exist completely regular spaces such that any relatively countably compact set is relatively compact which are not Eberlein spaces (Example 2.15), Eberlein spaces for which any subset is sequentially dense which are not Šmulian spaces (Example 3.23) and completely regular spaces for which any relatively countably compact set is sequentially dense and which are not Šmulian spaces (Example 3.24).

Nevertheless many of the usual topological spaces are Eberlein or Šmulian spaces. For instance the paracompact (Corollary 2.4), Lindelöf (Corollary 2.5) and topological spaces underlying a complete uniform space (Theorem 2.7) are Eberlein spaces. The product of any family of Eberlein spaces is an Eberlein space (Theorem 2.8) (the category of Eberlein spaces possesses inductive and projective limits (Corollaries 2.11 and 2.9)). Any topological space on which there exists a coarser metrizable topology is a Šmulian-Eberlein space (Corollary 3.19 and Proposition 3.1; generalization of [3] Théorème 3). In particular the topological groups for which the one-point sets are of type G_δ are Šmulian-Eberlein spaces (Corollary 3.20 and Proposition 3.1). A topological space which may be injected continuously in a Šmulian (resp. Šmulian-Eberlein) space is a Šmulian space (Corollaries 3.6 and 3.7) (resp. a Šmulian-Eberlein space (Corollaries 3.17 and 3.18))

If X is a Hausdorff topological space which contains a dense σ-compact set and if Y is a regular space on which there exists a coarser metrizable topology, then the space $\mathscr{C}(X, Y)$ of continuous maps of X into Y (endowed with the topology of pointwise convergence) is a Šmulian-Eberlein space (Corollary 5.16; generalization of [4], Theorem 3.2). Suppose X is a Hausdorff topological space with the property that a real valued function f on X is continuous if for any σ-compact set A of X there exists a continuous real valued function g on X such that $f=g$ on $\bar{A}$; then $\mathscr{C}(X, Y)$ is an Eberlein space for any completely regular Eberlein space Y (Corollary 5.19; generalization of [4], Theorem 2.4). If $\mathscr{C}(X, Y)$ is an Eberlein space, Y is a separated uniform space, and $\mathfrak{S}$ is a covering of X, the $\mathscr{C}_{\mathfrak{S}}(X, Y)$ (i.e. $\mathscr{C}(X, Y)$ endowed with the topology of uniform convergence on the sets of $\mathfrak{S}$) is an Eberlein space (Corollary 5.2.). Let X be a Hausdorff topological space, $\mathfrak{S}$ be a covering of X such that any set of $\mathfrak{S}$ is contained in the closure of a σ-compact set, Y be a separated uniform space, $\mathscr{F}$ be a subset of $\mathscr{C}(X, Y)$ and let $\mathscr{F}_{\mathfrak{S}}$ denote the uniform space obtained by endowing $\mathscr{F}$ with the uniform structure of uniform convergence on the sets of $\mathfrak{S}$. If any Cauchy filter $\mathfrak{F}$ on $\mathscr{F}_{\mathfrak{S}}$ which has the property that the intersection of any countable family in $\mathfrak{F}$ belongs to $\mathfrak{F}$ is convergent, then $\mathscr{F}$ endowed with the topology of pointwise convergence is an Eberlein space (Theorem 5.23.).

Applying these results to locally convex vector spaces, we get:

a) Let E, F be locally convex vector spaces such that there exists a σ-compact dense set in E and such that the one point sets of F are of type G_δ. Then the set $\mathscr{L}(E, F)$ of continuous linear maps of E into F is a Šmulian-Eberlein space for any topology finer than the topology of pointwise convergence (Theorem 6.2.).

As a corollary we get:

b) If E is a locally convex vector space such that $\{0\}$ is a G_δ-set in the Mackey topology, then E endowed with any topology, finer than the weak topology, is a Šmulian-Eberlein space (Corollary 6.3.; generalization of [3] Proposition 6 and [4] Theorem 4.2.).

c) Let E, F be locally convex vector spaces such that F is an Eberlein space, $\mathfrak{S}$ be a covering of E with bounded sets such that any set of $\mathfrak{S}$ is contained in the closure of a σ-compact set of E and $\mathscr{F}$ be a set of continuous linear maps of E into endowed with the uniform structure of uniform convergence on the sets of $\mathfrak{S}$. If any Cauchy filter $\mathfrak{F}$ on $\mathscr{F}$ is convergent to an element of $\mathscr{F}$ if it possesses the property that the intersection of any countable family in $\mathfrak{F}$ belongs to $\mathfrak{F}$ then $\mathscr{F}$ is an Eberlein space for the topology of pointwise convergence (Proposition 6.6.).

As a corollary we get:

d) Let E be a locally convex vector space, E' be its dual and A be a subset of E with the property that any Cauchy filter $\mathfrak{F}$ on A converges to a point of A if the intersection of any countable family in $\mathfrak{F}$ belongs to $\mathfrak{F}$. Then A is an Eberlein space for any topology consistent with the duality $\langle E, E'\rangle$ (Corollary 6.7.; generalization of [3] Proposition 2 and [4] Theorems 4.3. and 4.4.).

e) Let E be a bornological locally convex vector space, $\mathfrak{S}$ be a covering of E and F be an Eberlein locally convex vector space. The $\mathscr{L}(E, F)$ endowed with the topology of uniform convergence on the sets of $\mathfrak{S}$ is an Eberlein space (Theorem 6.8.).

f) If X is a locally compact space, $\mathscr{K}(X)$ is the vector space of continuous real valued (resp. complex valued) functions on X which have compact carrier and $\mathscr{M}(X)$ is the vector space of Radon real (resp. complex) measures on X, then $\mathscr{K}(X)$ is a Šmulian-Eberlein space with respect to the weak topology $\sigma(\mathscr{K}(X), \mathscr{M}(X))$ (Theorem 6.13.) and $\mathscr{M}(X)$ is an Eberlein space with respect to the vague topology (i.e. the weak topology $\sigma(\mathscr{M}(X), \mathscr{K}(X))$ (Corollary 6.11.).

g) Let X be a Hausdorff topological space, $\mathscr{C}(X)$ (resp. $\mathscr{C}^b(X)$) be the set of continuous (resp. bounded continuous) real or complex functions on X, and $\mathscr{M}^c(X)$ (resp. $\mathscr{M}^b(X)$) be the set of real or complex measures with compact carrier (resp. bounded measures) on X. A subset of $\mathscr{C}(X)$ (resp. $\mathscr{C}^b(X)$) is compact for the $\sigma(\mathscr{C}(X), \mathscr{M}^c(X))$-(resp. $\sigma(\mathscr{C}^b(X), \mathscr{M}^b(X))$-) topology if and only if it is bounded for this topology and is compact for the topology of pointwise convergence (Corollary 7.4.; generalization of [3] Théorème 5).

h) Let E be a complete locally convex vector space which contains a weakly σ-compact dense set. Let X be a measurable space, μ be a measure on X and f be a map of X into E such taht for any $x' \in E'$ the function $x' \circ f$ is μ-integrable. If for any equicontinuous sequence $(x'_n)_{n\in N}$ in E' converging to 0 for the $\sigma(E', E)$-topology we have

$$\lim_{n\to\infty} \int x'_n \circ f \, d\mu = 0$$

then there exists $x \in E$ such that for any $x' \in E'$ we have

$$\int x' \circ f \, d\mu = \langle x, x' \rangle$$

(Theorem 7.9; generalization of [1]).

As corollaries we get:

i) Let E be a quasi-complete locally convex vector space, X be a Hausdorff topological space, μ be a bounded measure on X and f be a μ-measurable map of X into E for the weak topology of E such that $f(X)$ is bounded. Then $x' \circ f$ is μ-integrable for any $x' \in E'$ and there exists $x \in E$ such that

$$\int x' \circ f \, d\mu = \langle x, x' \rangle$$

for any $x' \in E'$ (Corollary 7.14).

j) Let X, Y be Hausdorff topological spaces, μ, ν be bounded complex measures on X, Y respectively and f be a bounded complex function on $X \times Y$ which is separatedly continuous in each variable; then the restriction of the function

$$x \mapsto \int f(x, y) \, d\nu(y) : X \to \mathbf{C},$$

to the closure of any relatively countably compact set of X is continuous (Corollary 7.6 and Corollary 4.4) and we have

$$\int \left(\int f(x, y) \, d\mu(x) \right) d\nu(y) = \int \left(\int f(x, y) \, d\nu(y) \right) d\mu(x)$$

(Corollary 7.15).

For esthetic reasons the whole theory was done only for Hausdorff topological spaces, although practically all results remain true on arbitrary topological spaces.

The used notion which were not defined in the present paper may be found in N. Bourbaki or H. H. Schaefer [5].

The notions introduced in this paper were also studied here for their own sake. For the reader who is only interested in some of the above quoted results, we gige an index and the list of the logical connections at the end of the paper.

I. Nets

A *preorder relation* $\leqslant$ on a set I is a binary relation on I such that

a) $\iota\in I\Rightarrow\iota\leqslant\iota$;

b) $\iota, \iota', \iota''\in I, \iota\leqslant\iota', \iota'\leqslant\iota''\Rightarrow\iota\leqslant\iota''$.

An *upper directed preordered* set is a set I endowed with a preorder relation $\leqslant$ such that for any $\iota', \iota''\in I$ there exists $\iota\in I$ with $\iota'\leqslant\iota, \iota''\leqslant\iota$.

The *section filter* of an upper directed preordered non-empty set I is the filter generated by the filter-base $\{\{\iota\in I \mid \iota\geqslant\kappa\} \mid \kappa\in I\}$. A *net* on a set X is a pair (I,f), where I is an upper directed preordered non-empty set and f is a function defined on I such that $f(I)\subset X$. If $\mathfrak{F}$ is a filter on I, we shall denote, abusively, by $f(\mathfrak{F})$ the filter on X generated by the filterbase $\{f(A) \mid A\in\mathfrak{F}\}$. A net (I,f) on a topological space is called *countably compact* if for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I the adherence of the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ is non-empty.

PROPOSITION 1.1. *Let $\mathfrak{V}$ be a covering of a set X, (I, f) be a net on X, $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I, and $(I_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{U}$. If $f^{-1}(V)\notin\mathfrak{U}$ for any $V\in\mathfrak{V}$, then there exist an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I and a sequence $(V_n)_{n\in\mathbf{N}}$ in $\mathfrak{V}$ such that for any $n\in\mathbf{N}$ we have*

$$\iota_n\in I_n,\quad f(\iota_n)\in V_n\setminus\bigcup_{m<n} V_m.$$

Suppose that the sequences were constructed up to $n-1$ and let us find ι_n and V_n. Since $f^{-1}(V_m)\notin\mathfrak{U}$ for any $m<n$ and since $\mathfrak{U}$ is an ultrafilter we get

$$f^{-1}\left(\bigcup_{m<n} V_m\right)\notin\mathfrak{U},\quad f^{-1}\left(X\setminus\bigcup_{m<n} V_m\right)\in\mathfrak{U}.$$

On the other hand

$$\{\iota\in I \mid \iota\geqslant\iota_{n-1}\}\in\mathfrak{U},\quad I_n\in\mathfrak{U}$$

and therefore there exists

$$\iota_n\in f^{-1}(X\setminus\textstyle\bigcup_{m<n}V_m)\cap\{\iota\in I \mid \iota\geqslant\iota_{n-1}\}\cap I_n.$$

We take an arbitary V_n in $\mathfrak{V}$ such that $f(\iota_n)\in V_n$. †

PROPOSITION 1.2. *Let X be a topological space, $\mathfrak{V}$ be a covering of X and $\mathfrak{W}$ be a countable subset of open sets of $\mathfrak{V}$ such that any point $x\in X\setminus\bigcup_{V\in\mathfrak{W}} V$ possesses a neighbourhood U with the property that the set $\{V\in\mathfrak{V} \mid V\cap U\neq\emptyset\}$ is finite. If (I, f) is a countably compact net on X and if $\mathfrak{U}$ is an ultrafilter on I, finer than the section filter of I, then there exists $V\in\mathfrak{V}$ such that $f^{-1}(V)\in\mathfrak{U}$.*

Let φ be a map of $\mathbf{N}$ onto $\mathfrak{W}$ and let us denote

$$I_n := f^{-1}\left(X \setminus \bigcup_{m \leqslant n} \varphi(m)\right)$$

for any $n \in \mathbf{N}$.

Let us suppose that the proposition is not true. Then, since $\mathfrak{U}$ is an ultrafilter,

$$f^{-1}\left(\bigcup_{m \leqslant n} \varphi(m)\right) \notin \mathfrak{U}, \quad I_n \in \mathfrak{U}$$

for any $n \in \mathbf{N}$. By the preceding proposition there exist an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I and a sequence $(V_n)_{n \in \mathbf{N}}$ in $\mathfrak{V}$ such that

$$\iota_n \in I_n, \quad f(\iota_n) \in V_n \setminus \bigcup_{m < n} V_m$$

for any $n \in \mathbf{N}$. Let x be an adherent point in X of the sequence $(f(\iota_n))_{n \in \mathbf{N}}$. Assume first that there exists $V \in \mathfrak{W}$ such that $x \in V$. Then there exists $m \in \mathbf{N}$ such that $V = \varphi(m)$ and therefore $f(\iota_n) \notin V$ for any $n > m$ and this is a contradiction since V is open. But then, by the hypothesis of the proposition, x possesses a neighbourhood U such that the set $\{V \notin \mathfrak{V} \mid V \cap U \neq \emptyset\}$ is finite. There exists then a natural number m such that $V_n \cap U = \emptyset$ for any $n \geqslant m$ and this leads also to the contradictary relation $f(\iota_n) \notin U$ for any $n > m$. Our initial assumption led us therefore to the absurd conclusion that the adherence of the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ is empty. †

PROPOSITION 1.3. *Let (I, f) be a net on a set X, $\mathfrak{F}$ be a filter on I, finer than the section filter of I, and $F \in \mathfrak{F}$. Then there exists a net (J, g) on $f(F)$ and an increasing map φ of J onto F such that:* a) *$g = f \circ \varphi$;* b) *$\mathfrak{F} = \varphi(\mathfrak{G})$ where $\mathfrak{G}$ denotes the section filter of J;* c) *for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in F there exists an increasing sequence $(\kappa_n)_{n \in \mathbf{N}}$ in J such that $\varphi(\kappa_n) = \iota_n$ for any $n \in \mathbf{N}$. If X is a topological space and if the net (I, f) is countably compact, then any such net (J, g) is countably compact.*

We set

$$J := \bigcup_{\substack{A \in \mathfrak{F} \\ A \subset F}} (A \times \{A\})$$

and endow J with the preorder relation

$$(\iota, A) \leqslant (\kappa, B) :\Leftrightarrow \iota \leqslant \kappa \quad \text{and} \quad A \supset B.$$

This preorder relation is upper directed. Indeed let $(\iota', A') \in J$, $(\iota'', A'') \in J$. There exists $\iota \in I$ such that $\iota' \leqslant \iota$, $\iota'' \subset \iota$. Since $\mathfrak{F}$ is finer than the section filter of I it follows that the set $A := \{\kappa \in I \mid \kappa \geqslant \iota\} \cap A' \cap A''$ belongs to $\mathfrak{F}$. Let $\kappa \in A$. Then $(\kappa, A) \in J$ and $(\iota', A') \leqslant (\kappa, A)$, $(\iota'', A'') \leqslant (\kappa, A)$. We denote by φ the map $(\iota, A) \mapsto \iota : J \to F$ and set

$g := f \circ \varphi$. It is obvious that φ is increasing and $\varphi(J) = F$. Let $A \in \mathfrak{F}$. Then, obviously

$$\bigcup_{\substack{B \in \mathfrak{F} \\ B \subset A \cap F}} (B \times \{B\}) \in \mathfrak{G} \quad \text{and} \quad \varphi\Big(\bigcup_{\substack{B \in \mathfrak{F} \\ B \subset A \cap F}} (B \times \{B\})\Big) = A \cap F.$$

Hence $\varphi(\mathfrak{G}) \supset \mathfrak{F}$. Conversely let $(\iota, A) \in J$. Then

$$\varphi(\{(\kappa, B) \in J \mid (\kappa, B) \geqslant (\iota, A)\}) \supset A \cap \{\kappa \in I \mid \kappa \geqslant \iota\} \in \mathfrak{F}$$

$\varphi(\mathfrak{G}) \subset \mathfrak{F}$, $\varphi(\mathfrak{G}) = \mathfrak{F}$. The property c) is obvious.

The last assertion is obvious. †

PROPOSITION 1.4. *Let* $(\mathbf{N}, f)$ *be a countably compact net on a topological space. If the sequence* $(f(n))_{n \in \mathbf{N}}$ *has a unique adherent point, then it converges to this adherent point.*

Let x be the unique adherent point of the sequence $(f(n))_{n \in \mathbf{N}}$ and assume that the sequence does not converge to x. Then there exists a neighbourhood U of x and a subsequence $(f(n_k))_{k \in \mathbf{N}}$ of the sequence $(f(n))_{n \in \mathbf{N}}$ such that $f(n_k) \notin U$ for any $k \in \mathbf{N}$. Let y be an adherent point of the sequence $(f(n_k))_{k \in \mathbf{N}}$. Then y is an adherent point of the sequence $(f(n))_{n \in \mathbf{N}}$ different from x and this is a contradiction. †

II. Eberlein Spaces

A Hausdorff topological space will be called an *Eberlein space* if for any countably compact net (I, f) on it and for any filter $\mathfrak{F}$ on I, finer than the section filter of I, the adherence of the filter $f(\mathfrak{F})$ is non-empty. This is equivalent to the assertion that for any countably compact net (I, f) on it and for any ultrafilter $\mathfrak{U}$ on I, finer than the section filter of I, the ultrafilter $f(\mathfrak{U})$ is convergent.

An *Eberlein closed* set of a Hausdorff topological space X is a subset Y of X such that for any countably compact net (I, f) on the topological subspace Y and for any ultrafilter $\mathfrak{U}$ on I, finer than the section filter of I and such that $f(\mathfrak{A})$ converges in X to a point $x \in X$, we have $x \in Y$. Any closed set of a Hausdorff topological space is Eberlein closed. The intersection of any family and, by Proposition 1.3, the union of any finite family of Eberlein closed sets are also Eberlein closed. If Y is an Eberlein closed subset of X and if Z is a subspace of X then $Y \cap Z$ is Eberlein closed in Z.

PROPOSITION 2.1. *Let X be an Eberlein space and Y be a subset of X. Y is an Eberlein subspace if and only if it is Eberlein closed. In particular any closed subspace of an Eberlein space is also an Eberlein space. If Y is Eberlein closed in X, then it is Eberlein closed in X for any finer topology.*

The proof is obvious. †

PROPOSITION 2.2. *Let X be a topological space, Y be an Eberlein subspace of X, (I, f) be a countably compact net on X and $\mathfrak{F}$ be a filter on I, finer than the section filter of I. If $f^{-1}(Y) \in \mathfrak{F}$ then the adherence of $f(\mathfrak{F})$ is non-empty.*

By Proposition 1.3 there exists a countably compact net (J, g) on Y and an increasing map φ of J into I such that $g = f \circ \varphi$ and $\mathfrak{F} = \varphi(\mathfrak{G})$, where $\mathfrak{G}$ is the section filter of J. Since Y is an Eberlein space the adherence of $g(\mathfrak{G}) = f(\mathfrak{F})$ is non-empty. †

We shall give some criteria for a topological space to be an Eberlein space. The criteria given in the corollaries 2.4, 2.5 and 2.6 having similar proofs, we prove first a somehow complicated theorem, which has the advantage of avoiding repetitions.

THEOREM 2.3. *Let X be a Hausdorff topological space and let $\mathfrak{A}$ be a set of Eberlein subspaces of X. If for every open covering $\mathfrak{B}$ of X there exists a covering $\mathfrak{V}$ of X, finer than the covering $\mathfrak{A} \cup \mathfrak{B}$ and a countable subset $\mathfrak{W}$ of open sets of $\mathfrak{V}$ such that any point $x \in X \setminus \bigcup_{V \in \mathfrak{W}} V$ possesses a neighbourhood U with the property that the set $\{V \in \mathfrak{V} \mid V \cap U \neq \emptyset\}$ is finite, then X is an Eberlein space.*

Assume that X is not an Eberlein space. Then there exists a countably compact net (I, f) on X and an ultrafilter $\mathfrak{U}$ on I, finer than the section filter of I, such that the ultrafilter $f(\mathfrak{U})$ is not convergent. By Proposition 2.2 $f^{-1}(A) \notin \mathfrak{U}$ for any $A \in \mathfrak{A}$.

Since $f(\mathfrak{U})$ is not convergent any $x \in X$ possesses an open neighbourhood U_x such that $f^{-1}(U_x) \notin \mathfrak{U}$. Let $\mathfrak{V}$ be the covering of X, finer than the covering $\mathfrak{A} \cup \{U_x \mid x \in X\}$ of X, with the proporties indicated in the proposition. By Proposition 1.2 there exists $V \in \mathfrak{V}$ such that $f^{-1}(V) \in \mathfrak{U}$. This leads to a contradiction since then there exists an $A \in \mathfrak{A}$ such that $f^{-1}(A) \in \mathfrak{U}$ or an $x \in X$ such that $f^{-1}(U_x) \in \mathfrak{U}$. †

COROLLARY 2.4. *Any paracompact (and therefore any metrisable) space in an Eberlein space.*

It is sufficient to take $\mathfrak{A}$ (and $\mathfrak{W}$) empty in the theorem. †

COROLLARY 2.5. *Any Lindelöf space*[2]) *is an Eberlein space.*

It is sufficient to take $\mathfrak{A}$ empty (and $\mathfrak{V} = \mathfrak{B}$) in the theorem. †

COROLLARY 2.6. *A topological space which possesses a locally finite covering with Eberlein subspaces is an Eberlein space. In particular a topological sum of Eberlein spaces is an Eberlein space.*

Let $\mathfrak{A}$ be a locally finite covering of a topological space with Eberlein subspaces. It is sufficient to take $\mathfrak{V} = \mathfrak{A}$ and $\mathfrak{W} = \emptyset$ in the theorem. †

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[2]) A Lindelöf space is a Hausdorff topological space such that any open covering contains a countable subcovering. Any Hausdorff topological space which is σ-compact is a Lindelöf space.

Remark. There exist locally compact spaces which are not Eberlein spaces. Indeed let ω_1 be the first uncountable ordinal number and let a subset U of ω_1 be open if for any $\xi \in U$ there exists $\eta < \xi$ such that $]\eta, \xi] \subset U$; then ω_1 endowed with this topology is a locally compact space which is not an Eberlein space.

THEOREM 2.7. *Let X be a separated uniform space, (I, f) be a countably compact net on X and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. Then $f(\mathfrak{U})$ is a Cauchy filter. Hence the topological space canonically associated to a complete separated uniform space is an Eberlein space.*

Assume that $f(\mathfrak{U})$ is not a Cauchy filter. Then there exists a uniformly continuous ecart d of X with the property that for any $A \in \mathfrak{U}$ there exist $\iota, \kappa \in A$ such that $d(f(\iota), f(\kappa)) > 1$. We shall construct by induction an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $d(f(\iota_m, f(\iota_n)) > \frac{1}{2}$ for any two different natural numbers m, n. We take an arbitrary ι_0 in I. Assume that the sequence was constructed up to $n-1$ and let us find a ι_n. We have

$$\{\iota \in I \mid \iota \geqslant \iota_{n-1}\} \in \mathfrak{U}, \quad m < n \Rightarrow \{\iota \in I \mid d(f(\iota), f(\iota_m)) \leqslant \tfrac{1}{2}\} \notin \mathfrak{U}.$$

$\mathfrak{U}$ being an ultrafilter we get

$$\{\iota \in I \mid \iota \geqslant \iota_{n-1}\} \cap \left(\bigcap_{m<n} \{\iota \in I \mid d(f(\iota), f(\iota_m)) > \tfrac{1}{2}\}\right) \in \mathfrak{U}.$$

Hence there exists $\iota_n \in I$ such that

$$\iota_n \geqslant \iota_{n-1}, \quad m < n \Rightarrow d(f(\iota_m), f(\iota_n)) > \tfrac{1}{2}.$$

Let x be an adherent point of the sequence $(f(\iota_n))_{n \in \mathbf{N}}$. There exists then a subsequence $(\iota_{n_k})_{k \in \mathbf{N}}$ of $(\iota_n)_{n \in \mathbf{N}}$ such that

$$d(x, f(\iota_{n_k})) \leqslant \tfrac{1}{4}$$

for any $k \in \mathbf{N}$ and this leads to the contradictory relation

$$d(f(\iota_{n_0}), f(\iota_{n_1})) \leqslant \tfrac{1}{2}.$$

Hence $f(\mathfrak{U})$ is a Cauchy filter. †

THEOREM 2.8. *The product of any family of Eberlein spaces is an Eberlein space.*

Let $(X_\lambda)_{\lambda \in L}$ be a family of Eberlein spaces, (I, f) be a countably compact net on $\prod_{\lambda \in L} X_\lambda$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. If we denote for any $\lambda \in L$ by π_λ the projection

$$\prod_{\lambda \in L} X_\lambda \to X_\lambda$$

then $(I, \pi_\lambda \circ f)$ is a countably compact net on X_λ. Since X_λ is an Eberlein space it follows that the ultrafilter $\pi_\lambda \circ f(\mathfrak{U})$ is convergent. Since λ is arbitrary we deduce that the ultrafilter $f(\mathfrak{U})$ is convergent. Hence $\prod_{\lambda \in L} X_\lambda$ is an Eberlein space. †

COROLLARY 2.9. *The projective limit of Eberlein spaces is also an Eberlein space.*
In fact this projective limit is homeomorphic with a closed subspace of a product of Eberlein spaces. †

PROPOSITION 2.10. *For any topological space X there exists an Eberlein space X_0 and a continuous map φ_X of X into X_0 such that $\varphi_X(X)$ is dense in X_0 and such that for any Eberlein space Y and for any continuous map φ of X into Y there exists a unique continuous map ψ of X_0 into Y such that $\varphi = \psi \circ \varphi_X$.*

Let us denote by M the set of pairs (Y, φ), where Y is an Eberlein space whose underlying set is a subset of 2^{2^X} and φ is a continuous map of X into Y. We denote by $\mathfrak{A}$ the small category whose objects are the elements of M and such that for any two objects (Y, φ), (Y', φ')

$$\mathrm{Hom}((Y, \varphi), (Y', \varphi')) := \{\psi \in \mathscr{C}(Y, Y') \mid \varphi' = \psi \circ \varphi\},$$

where $\mathscr{C}(Y, Y')$ denotes the set of continuous maps of Y into Y'; the composition of two morphisms in $\mathfrak{A}$ is the usual one. Let F be the imbedding functor of $\mathfrak{A}$ into the category of topological spaces and let $(X', (\psi_{(Y,\varphi)})_{(Y,\varphi)\in M})$ be its projective limit. By the above corollary X' is an Eberlein space.

Let (Y, φ), (Y', φ') be two objects of $\mathfrak{A}$ and $\psi \in \mathrm{Hom}((Y, \varphi), (Y', \varphi'))$. Then, by the definition, $\varphi' = \psi \circ \varphi$. Hence there exists a unique continuous map φ'_X of X into X' such that $\varphi = \psi_{(Y,\varphi)} \circ \varphi'_X$ for any object (Y, φ) of $\mathfrak{A}$. Let X_0 be the subspace $\overline{\varphi'_X(X)}$ of X'. By Proposition 2.1 X_0 is an Eberlein space. Let us denote by i the imbedding map of X_0 into X' and by φ_X the continuous map of X into X_0 defined by φ'_X. We want to show that the pair (X_0, φ_X) possesses the announced properties.

It is obvious that $\varphi_X(X)$ is dense in X_0. Let Y be an Eberlein space and φ be a continuous map of X into Y. Then the subspace $\overline{\varphi(X)}$ of Y is also an Eberlein space (Proposition 2.1). Since the cardinal number of $\varphi(X)$ is smaller than the cardinal number of X, the cardinal number of $\overline{\varphi(X)}$ is smaller than the cardinal number of 2^{2^X}. We may therefore identify $\overline{\varphi(X)}$ with a subset of 2^{2^X}. If φ_0 denotes the continuous map of X into $\overline{\varphi(X)}$ defined by φ, then $(\overline{\varphi(X)}, \varphi_0) \in M$.

We set

$$\psi := j \circ \psi_{(\overline{\varphi(X)}, \varphi_0)} \circ i,$$

where j denotes the imbedding map of $\overline{\varphi(X)}$ into Y. We get

$$\varphi = j \circ \varphi_0 = j \circ (\psi_{(\overline{\varphi(X)}, \varphi_0)} \circ \varphi'_X) = j \circ \psi_{(\overline{\varphi(X)}, \varphi_0)} \circ (i \circ \varphi_X) = \psi \circ \varphi_X.$$

The unicity of ψ follows from the fact that $\varphi_X(X)$ is dense in X_0 and Y in a Hausdorff space. †

COROLLARY 2.11. *The full subcategory of the category of topological spaces formed by the Eberlein spaces possesses inductive limits.*

Let $\mathfrak{D}$ be a small category and F be a covariant functor of $\mathfrak{D}$ into the category of Eberlein spaces. Let $(X, (\varphi_D)_{D\in \mathrm{Ob}\,\mathfrak{D}})$ be its injective limit in the category of topological spaces. Let further (X_0, φ_X) be the pair whose existence was proved in the proposition. We want to show that $(X_0, (\varphi_X \circ \varphi_D)_{D\in \mathrm{Ob}\,\mathfrak{D}})$ is the inductive limit of F.

First of all we remark that if D, D' are two objects of $\mathfrak{D}$ and if $u\in \mathrm{Hom}(D, D')$, then

$$(\varphi_X \circ \varphi_{D'}) \circ F(u) = \varphi_X \circ (\varphi_{D'} \circ F(u)) = \varphi_X \circ \varphi_D.$$

Let Y be an Eberlein space and $(\psi_D)_{D\in \mathrm{Ob}\,\mathfrak{D}}$ be a family such that for any $D\in \mathrm{Ob}\,\mathfrak{D}$, $\psi_D \in \mathscr{C}(F(D), Y)$ and such that for any two objects D, D' of $\mathfrak{D}$ and for any $u\in \mathrm{Hom}(D, D')$, we have $\psi_D = \psi_{D'} \circ F(u)$. Then there exists a unique $\varphi\in\mathscr{C}(X, Y)$ such that for any $D\in \mathrm{Ob}\,\mathfrak{D}$, we have $\psi_D = \varphi\circ\varphi_D$. By the proposition there exists a unique $\psi\in\mathscr{C}(X_0, Y)$ such that $\varphi = \psi\circ\varphi_X$. We get for any $D\in \mathrm{Ob}\,\mathfrak{D}$

$$\psi_D = (\psi\circ\varphi_X)\circ\varphi_D = \psi\circ(\varphi_X\circ\varphi_D).$$

Let $\psi'\in\mathscr{C}(X_0, Y)$ such that for any $D\in \mathrm{Ob}\,\mathfrak{D}$ we have $\psi_D = \psi'\circ(\varphi_X\circ\varphi_D)$. Then, by the unicity property of ψ we get further $\psi' = \psi$. †

PROPOSITION 2.12. *A Hausdorff topological space for which there exists a proper map into an Eberlein space is an Eberlein space.*

Let φ be a proper map of a Hausdorff topological space X into an Eberlein space Y. Let further (I, f) be a countably compact net on X and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. Then $(I, \varphi\circ f)$ is a countably compact net on Y and there fore $\varphi\circ f(\mathfrak{U})$ is convergent. But φ being proper this implies that $f(\mathfrak{U})$ converges. †

THEOREM 2.13. *In an Eberlein space any relatively countably compact set is nearly relatively compact. In particular a countably compact regular Eberlein space is compact.*

Let X be an Eberlein space, A be a relatively countably compact set of X and $\mathfrak{F}$ be a filter on A. We endow A with the trivial preorder relation $\leqslant$ (i.e. we set $x\leqslant y$ for any two elements x, y of A). Then A becomes an upper directed preordered set such that $\mathfrak{F}$ is finer than the section filter of A. If we denote by f the inclusion map of A into X then (A, f) is a countably compact net on X. Since X is an Eberlein space it follows that the adherence of $f(\mathfrak{F})$ is non-empty. Since $\mathfrak{F}$ is arbitrary it follows that A is nearly relatively compact. †

The converse assertion is not true as it can be seen from the following examples.

(To be followed.)

EXAMPLE 2.14. There exists a Hausdorff topological space such that any point possesses a countable neighbourhood and a countable fundamental system of neighbourhoods (and therefore any subset is sequentially dense) and such that any relatively countably compact set is nearly relatively compact and countable, but which is not an Eberlein space.

Let ω_1 be the first uncountable ordinal number. We endow $(\mathbf{N} \cup \{\infty\}) \times \omega_1$ with a topology taking as open sets the subsets U with the following property: if $(\infty, \xi) \in U$ then there exists $m \in \mathbf{N}$ and $\eta \in \omega_1$, $\eta < \xi$ such that

$$(\{n \in \mathbf{N} \mid n > m\}) \times \{\zeta \in \omega_1 \mid \eta < \zeta \leqslant \xi\} \subset U.$$

This space is Hausdorff and any point of this space possesses a countable neighbourhood and a countable fundamental system of neighbourhoods. Moreover any relatively countably compact set is nearly relatively compact, and countable.

We endow $\mathbf{N} \times \omega_1$ with the following upper directed order relation:

$$(m, \xi) \leqslant (n, \eta) :\Leftrightarrow (m, \xi) = (n, \eta) \quad \text{or} \quad (m < n \quad \text{and} \quad \xi < \eta).$$

If f denotes the inclusion map of $\mathbf{N} \times \omega_1$ into $(\mathbf{N} \cup \{\infty\}) \times \omega_1$ and $\mathfrak{F}$ denotes the section filter of $\mathbf{N} \times \omega_1$ then for any increasing sequence $(t_n)_{n \in \mathbf{N}}$ in $\mathbf{N} \times \omega_1$ the sequence $(f(t_n))_{n \in \mathbf{N}}$ has a non-empty adherence and the adherence of $f(\mathfrak{F})$ is empty. †

EXAMPLE 2.15. There exists a completely regular space such that any relatively countably compact set is relatively compact and which is not an Eberlein space.

We construct inductively the sequence $(\aleph_n)_{n \in \mathbf{N}}$ in the following way: $\aleph_0$ is the smallest infinite cardinal number and $\aleph_{n+1}$ is the smallest cardinal number strictly greater than $\aleph_n$. By $\aleph_\omega$ we denote the smallest cardinal number strictly greater than any $\aleph_n$ $(n \in \mathbf{N})$.

Let X be a set, whose cardinal number is $\aleph_\omega$, and Y be the set of its finite subsets endowed with the discrete topology. Let Y^* be the Stone-Čech compactification of Y and Φ be the set $(A, \mathfrak{U})$, where A is a subset of X, whose cardinal number is strictly smaller than $\aleph_\omega$, and $\mathfrak{U}$ is an ultrafilter on Y such that for any finite subset y_0 of A we have

$$\{y \in Y \mid y_0 \subset y \subset A\} \in \mathfrak{U}.$$

We denote by Z the subspace of Y^* formed by the limit points of the ultrafilters $\mathfrak{U}$ on Y for which there exists $A \subset X$ such that $(A, \mathfrak{U}) \in \Phi$. Z is obviously a completely regular space which contains Y. For any $z \in Z$ there exists a unique $(A(z), \mathfrak{U}(z)) \in \Phi$ such that $\mathfrak{U}(z)$ converges to z.

We want to show that any relatively countably compact set B of Z is relatively compact. Let $\leqslant$ be a well order relation on X such that X endowed with it is the smallest well ordered set of cardinal $\aleph_\omega$. For any $z \in Z$ let $\varphi(z)$ denote the supremum of $A(z)$

(which exists since the cardinal number of $A(z)$ is strictly smaller than $\aleph_\omega$). Assume that the set $\{\varphi(z) \mid z\in B\}$ is not bounded. Then there exists a sequence $(z_n)_{n\in\mathbf{N}}$ in B such that $(\varphi(z_n))_{n\in\mathbf{N}}$ is increasing and unbounded. Let z be an adherent point in Z of the sequence $(z_n)_{n\in\mathbf{N}}$. The closure in Z of the set $\{y\in Y \mid y\subset A(z)\}$ is a neighbourhood of z. For a sufficiently great $n\in\mathbf{N}$, $\varphi(z_n)$ is strictly greater than $\varphi(z)$ and therefore z_n does not belong to this neighbourhood of z which is a contraction. Hence the set $\{\varphi(z) \mid z\in B\}$ is bounded.

Let $\mathfrak{U}$ be an ultrafilter on Z such that $B\in\mathfrak{U}$ and let us denote for any $C\in\mathfrak{U}$

$$A(C):=\bigcup_{z\in C} A(z).$$

We assert that there exists a $D\in\mathfrak{U}$, $D\subset B$, such that $A(D)\subset A(C)$ for any $C\in\mathfrak{U}$. If this is not the case, then there exists a decreasing sequence $(C_n)_{n\in\mathbf{N}}$ in $\mathfrak{U}$ such that $C_0\subset B$ and $A(C_n)\backslash A(C_{n+1})\neq\emptyset$ for any $n\in\mathbf{N}$. This allows us to construct a sequence $(z_n)_{n\in\mathbf{N}}$ in B such that $A(z_n)\backslash\bigcup_{m>n} A(z_m)$ for any $n\in\mathbf{N}$. Let z be an adherent point in Z of the sequence $(z_n)_{n\in\mathbf{N}}$. Since the closure in Z of the set $\{y\in Y \mid y\subset A(z)\}$ is a neighbourhood of z there exists an $n\in\mathbf{N}$ such that $A(z_n)\subset A(z)$. Let $x\in A(z_n)\backslash\bigcup_{m>n} A(z_m)$. The closure in Z of the set $\{y\in Y \mid x\in y\subset A(z)\}$ is also a neigbourhood of z which does not contain any z_m for $m>n$ and this is the expected contradiction. We also remark that the cardinal number of $A(D)$ is strictly smaller than $\aleph_\omega$. Let $x\in A(D)$. The assumption

$$C:=\{z\in D \mid x\notin A(z)\}\in\mathfrak{U}$$

contradicts the relation $x\in A(D)\subset A(C)$. Hence $\{z\in C \mid x\in A(z)\}\in\mathfrak{U}$.
If y_0 is a finite subset of $A(D)$ we deduce

$$\{z\in C \mid y_0\subset A(z)\}=\bigcap_{x\in y_0}\{z\in C \mid x\in A(z)\}\in\mathfrak{U}.$$

Let us denote

$$\mathfrak{B}:=\{E\subset Y \mid \{z\in Z \mid E\in\mathfrak{U}(z)\}\in\mathfrak{U}\}.$$

Then $\mathfrak{B}$ is an ultrafilter such that for any finite subset y_0 of $A(D)$

$$\{y\in Y \mid y_0\subset y\subset A(D)\}\in\mathfrak{B}.$$

Hence $(A(D), \mathfrak{B})\in\Phi$ and $\mathfrak{B}$ converges to a $z\in Z$. The closure in Z of any set of $\mathfrak{B}$ belongs to $\mathfrak{U}$. But any neighbourhood of z contains the closure in Z of a set of $\mathfrak{B}$ and belongs therefore to $\mathfrak{U}$. Hence $\mathfrak{U}$ converges to z. We have proved that B is relatively compact.

We want to show now that Z is not an Eberlein space. Let us order Y by the inclusion relation and let f be the inclusion map of Y into Z. The (Y, f) is obviously a countably compact net on Z. Let $\mathfrak{U}$ be an ultrafilter on Y, finer than the section filter of Y. Let $z \in Z$ and $x \in X \setminus A(z)$. Then

$$\{y \in Y \mid x \in y\} \in \mathfrak{U}$$

and the closure of $\{y \in Y \mid y \subset A(z)\}$ in Z is a neighbourhood of z. Hence $\mathfrak{U}$ does not converge to z and Z is not an Eberlein space. †

Remark. It can be proved that any Hausdorff topological space whose cardinal number is strictly smaller than $\aleph_\omega$ is an Eberlein space if any relatively countably compact set is relatively compact.

III. Šmulian Spaces

A Hausdorff topological space is called a *Šmulian space* if for any countably compact net (I, f) on it, for any filter $\mathfrak{F}$ on I, finer than the section filter of I, for any adherent point x of the filter $f(\mathfrak{F})$, and for any sequence $(I_n)_{n \in \mathbf{N}}$ in $\mathfrak{F}$ there exists an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ converges to x and such that $\iota_n \in I_n$ for any $n \in \mathbf{N}$. An equivalent statement is: for any countably compact net (I, f) on it, for any ultrafilter $\mathfrak{U}$ on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point x, and for any sequence $(I_n)_{n \in \mathbf{N}}$ in $\mathfrak{U}$ there exists an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ converges to x and such that $\iota_n \in I_n$ for any $n \in \mathbf{N}$.

A Hausdorff topological space is called a *strict Šmulian space* if for any countably compact net (I, f) on it and for any ultrafilter $\mathfrak{U}$ on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point x, there exists a sequence $(A_n)_{n \in \mathbf{N}}$ in $f(\mathfrak{U})$ with the property that for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $f(\iota_n) \in A_n$ for any $n \in \mathbf{N}$, the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ converges to x.

An Eberlein space which is at the same time a Šmulian (resp. strict Šmulian) space will be called a *Šmulian-Eberlein* (resp. a *strict Šmulian-Eberlein*) space.

PROPOSITION 3.1. *Any strict Šmulian space is a Šmulian space.*

Let (I, f) be a countably compact net on a strict Šmulian space, $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point x, and $(I_n)_{n \in \mathbf{N}}$ be a sequence in $\mathfrak{U}$. There exists a sequence $(A_n)_{n \in \mathbf{N}}$ in $f(\mathfrak{U})$ with the property that for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $f(\iota_n) \in A_n$ for any $n \in \mathbf{N}$ the sequence $(f(\iota_n)_{n \in \mathbf{N}}$ converges to x. We construct inductively an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $\iota_n \in I_n \cap f^{-1}(A_n)$ for any $n \in \mathbf{N}$. Assume that this sequence was constructed up to $n-1$. Since $\mathfrak{U}$ is finer than the section filter of I $\{\iota \in I \mid \iota \geqslant \iota_{n-1}\} \in \mathfrak{U}$, and therefore $\{\iota \in I \mid \iota \geqslant \iota_{n-1}\} \cap I_n \cap f^{-1}(A_n) \in \mathfrak{U}$. We may take an arbitrary ι_n in the set $\{\iota \in I \mid \iota \geqslant \iota_{n-1}\} \cap I_n \cap f^{-1}(A_n)$. The sequence $(f(\iota_n))_{n \in \mathbf{N}}$ converges to x. †

THEOREM 3.2. *Let X be a Hausdorff topological space, Y be a Šmulian (resp. strict Šmulian) space and φ be a continuous map of X into Y. If for any $x \in X$ there exists a sequence $(U_n)_{n \in \mathbf{N}}$ of closed neighbourhoods of x such that*

$$\{x\} = \left(\bigcap_{n \in \mathbf{N}} U_n\right) \cap \varphi^{-1}(\varphi(x)),$$

then X is a Šmulian (resp. strict Šmulian) space.

We prove first the Šmulian part of the theorem. Let (I, f) be a countably compact net on X, $\mathfrak{F}$ be a filter on I, finer than the section filter of I, x be an adherent point of $f(\mathfrak{F})$ and $(I_n)_{n \in \mathbf{N}}$ be a decreasing sequence in $\mathfrak{F}$. Let further $(U_n)_{n \in \mathbf{N}}$ be a decreasing sequence of closed neighbourhoods of x such that

$$\{x\} = \left(\bigcap_{n \in \mathbf{N}} U_n\right) \cap \varphi^{-1}(\varphi(x)).$$

Let us denote by $\mathfrak{G}$ the filter on I generated by the filter-base $\{A \cap f^{-1}(U_n) \mid A \in \mathfrak{F}, n \in \mathbf{N}\}$. $(I, \varphi \circ f)$ is a countably compact not on Y, $\mathfrak{G}$ is finer than the section filter of I, $\varphi(x)$ is an adherent point of $\varphi \circ f(\mathfrak{G})$ and $(I_n \cap f^{-1}(U_n))_{n \in \mathbf{N}}$ is a sequence in $\mathfrak{G}$. Since Y is a Šmulian space, there exists an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $(\varphi(f(\iota_n)))_{n \in \mathbf{N}}$ converges to $\varphi(x)$ and such that $\iota_n \in I_n \cap f^{-1}(U_n)$ for any $n \in \mathbf{N}$. Let x' be an adherent point of the sequence $(f(\iota_n))_{n \in \mathbf{N}}$. Then $\varphi(x')$ is an adherent point of the sequence $(\varphi \circ f(\iota_n))_{n \in \mathbf{N}}$ and therefore $\varphi(x') = \varphi(x)$, $x' \in \varphi^{-1}(\varphi(x))$. On the other hand $x' \in \bigcap_{n \in \mathbf{N}} U_n$ and we deduce $x' = x$. By Proposition 1.4 the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ converges to x.

We now prove the strict Šmulian part of the theorem. Let (I, f) be a countably compact net on X, $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point x. Then $(I, \varphi \circ f)$ is a countably compact net on Y and $\varphi \circ f(\mathfrak{U})$ converges to $\varphi(x)$. Since Y is a strict Šmulian space, there exists a sequence $(A_n)_{n \in \mathbf{N}}$ in $\varphi \circ f(\mathfrak{U})$ with the property that for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $\varphi \circ f(\iota_n) \in A_n$ for any $n \in \mathbf{N}$, the sequence $(\varphi \circ f(\iota_n))_{n \in \mathbf{N}}$ converges to $\varphi(x)$. Let $(U_n)_{n \in \mathbf{N}}$ be a sequence of closed neighbourhoods of x such that

$$\{x\} = \left(\bigcap_{n \in \mathbf{N}} U_n\right) \cap \varphi^{-1}(\varphi(x)).$$

Then $(U_n \cap \varphi^{-1}(A_n))_{n \in \mathbf{N}}$ is a sequence in $f(\mathfrak{U})$. Let $(\iota_n)_{n \in \mathbf{N}}$ be an increasing sequence in I such that $f(\iota_n) \in U_n \cap \varphi^{-1}(A_n)$ for any $n \in \mathbf{N}$ and let x' be an adherent point of the sequence $(f(\iota_n))_{n \in \mathbf{N}}$. Then $\varphi(x')$ is an adherent point of the sequence $(\varphi \circ f(\iota_n))_{n \in \mathbf{N}}$ and therefore $\varphi(x') = \varphi(x)$, $x' \in \varphi^{-1}(\varphi(x))$. On the other hand $x' \in \bigcap_{n \in \mathbf{N}} U_n$ and therefore $x' = x$. By Proposition 1.4 the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ converges to x. †

COROLLARY 3.3. *Let X be a regular space, Y be a Šmulian (resp. strict Šmulian)* *space, and φ be a continuous map of X into Y such that for any $x\in X$, $\{x\}$ is of type G_δ in the space $\varphi^{-1}(\varphi(x))$. Then X is a Šmulian (resp. strict Šmulian) space.*

Let $(W_n)_{n\in\mathbf{N}}$ be a sequence of open sets of the space $\varphi^{-1}(\varphi(x))$ such that $\{x\}=\bigcap_{n\in\mathbf{N}} W_n$. Let $(V_n)_{n\in\mathbf{N}}$ be a sequence of open sets of X such that $W_n=V_n\cap\varphi^{-1}(\varphi(x))$ for any $n\in\mathbf{N}$. Since X is regular there exists for any $n\in\mathbf{N}$ a closed neighbourhood U_n of x in X such that $U_n\subset V_n$. But then

$$\{x\}\subset\left(\bigcap_{n\in\mathbf{N}} U_n\right)\cap\varphi^{-1}(\varphi(x))\subset\left(\bigcap_{n\in\mathbf{N}} V_n\right)\cap\varphi^{-1}(\varphi(x))=\bigcap_{n\in\mathbf{N}} W_n=\{x\}. \quad \dagger$$

COROLLARY 3.4. *Let X be a Hausdorff topological space, Y be a Šmulian (resp. strict Šmulian) space and φ be a continuous map of X into Y such that for any $y\in Y$, $\varphi^{-1}(y)$ is at most countable. Then X is a Šmulian (resp. strict Šmulian) space.*

Let $x\in X$. By the hypothesis there exists a sequence $(x_n)_{n\in\mathbf{N}}$ in $X\backslash\{x\}$ such that

$$\varphi^{-1}(\varphi(x))=\{x\}\cup\{x_n \mid n\in\mathbf{N}\}.$$

For any $n\in\mathbf{N}$, let U_n be a neighbourhood of x such that $x_n\notin\overline{U_n}$. Then

$$\{x\}=\left(\bigcap_{n\in\mathbf{N}}\bar{U}_n\right)\cap\varphi^{-1}(\varphi(x)). \quad \dagger$$

COROLLARY 3.5. *A Hausdorff topological space X such that for any $x\in X$, $\{x\}$ is the intersection of a countable set of closed neighbourhoods is a strict Šmulian space. In particular any metrizable space is a strict Šmulian space.* †

COROLLARY 3.6. *A subspace of a Šmulian (resp. strict Šmulian) space is a Šmulian (resp. strict Šmulian) space.* †

COROLLARY 3.7. *If X is a Šmulian (resp. strict Šmulian) space then X endowed with any finer topology is also a Šmulian (resp. strict Šmulian) space.* †

PROPOSITION 3.8. *The product of any countable family of strict Šmulian spaces is a strict Šmulian space.*

Let $(X_m)_{m\in\mathbf{N}}$ be a sequence of strict Šmulian spaces, (I, f) be a countably compact net on $\prod_{m\in\mathbf{N}} X_m$, and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to x. If for any $m\in\mathbf{N}$, π_m denotes the projection

$$\prod_{n\in\mathbf{N}} X_n\to X_m,$$

then $(I, \pi_m\circ f)$ is a countably compact net on X_m and $\pi_m\circ f(\mathfrak{U})$ converges to $\pi_m(x)$. Hence for any $m\in\mathbf{N}$ there exists a sequence $(A_{m,n})_{n\in\mathbf{N}}$ in $\pi_m\circ f(\mathfrak{U})$ with the property that for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $\pi_m\circ f(\iota_n)\in A_n$ for any $n\in\mathbf{N}$, the sequence $(\pi_m \circ f(\iota_n))_{n\in\mathbf{N}}$ converges to $\pi_m(x)$. We may even take $A_{m,n} := X_m$ for $n<m$. We set for any $n\in\mathbf{N}$

$$A_n := \prod_{m\in\mathbf{N}} A_{m,n}.$$

Then $(A_n)_{n\in\mathbf{N}}$ is a sequence in $f(\mathfrak{U})$. Let $(\iota_n)_{n\in\mathbf{N}}$ be an increasing sequence in I such that $f(\iota_n)\in A_n$ for any n. Then for any $m, n\in\mathbf{N}$ we have $\pi_m\circ f(\iota_n)\in A_{m,n}$ and therefore the sequence $(\pi_m\circ f(\iota_n))_{n\in\mathbf{N}}$ converges to $\pi_m(x)$. It follows that the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ converges to x. †

Remark. The product of an uncountable family of strict Šmulian spaces is not always a Šmulian space. Indeed let us consider the space $\{0, 1\}^{\omega_1}$, where ω_1 denotes the first uncountable ordinal number. Let f be the map of ω_1 into $\{0, 1\}^{\omega_1}$ defined by

$$[f(\xi)](\eta) := \begin{cases} 0 & \text{for} \quad \eta \leqslant \xi \\ 1 & \text{for} \quad \eta > \xi \end{cases} \qquad (\xi\in\omega_1)$$

and let $\mathfrak{F}$ be the section filter of ω_1. Then (ω_1, f) is a countably compact net on $\{0, 1\}^{\omega_1}$ and $f(\mathfrak{F})$ converges to the point

$$\eta \mapsto 0 : \omega_1 \to \{0, 1\}$$

of $\{0, 1\}^{\omega_1}$, but for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in ω_1 the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ does not converge to this point.

PROPOSITION 3.9. *The product of a Šmulian space with a strict Šmulian space is a Šmulian space.*

Let X be a Šmulian space and Y be a strict Šmulian space. Let (I, f) be a countably compact net on $X\times Y$, $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point (x, y), and $(I_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{U}$. If p (resp. q) denotes the projection

$$X\times Y \mapsto X \quad (\text{resp. } X\times Y \to Y)$$

then $(I, p\circ f)$ (resp. $I, q\circ f$) is a countably compact net on X (resp. Y) and $p\circ f(\mathfrak{U})$ (resp. $q\circ f(\mathfrak{U})$) converges to x (resp. y). Since Y is a strict Šmulian space there exists a sequence $(A_n)_{n\in\mathbf{N}}$ in $q\circ f(\mathfrak{U})$ with the property that for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $q\circ f(\iota_n)\in A_n$ for any $n\in\mathbf{N}$, the sequence $(q\circ f(\iota_n))_{n\in\mathbf{N}}$ converges to y. Then $(I_n\cap f^{-1}(q^{-1}(A_n)))_{n\in\mathbf{N}}$ is a sequence in $\mathfrak{U}$ and, X being a Šmulian space, there exists an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that the sequence $(p\circ f(\iota_n))_{n\in\mathbf{N}}$ converges to x and $\iota_n\in I_n\cap f^{-1}(q^{-1}(A_n))$ for any $n\in\mathbf{N}$. We deduce that the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ converges to (x, y). †

Remark. We do not know if the product of two Šmulian spaces is a Šmulian space, but this result seems to us improbably.

PROPOSITION 3.10. *Let X be a Hausdorff topological space, (I, f) be a countably compact net on X and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point x. If there exists a Šmulian subspace Y of X which is Eberlein closed and such that $f^{-1}(Y)\in\mathfrak{U}$, and if $(I_n)_{n\in\mathbf{N}}$ is a sequence in $\mathfrak{U}$, then there exists an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $(f(\iota_n))_{n\in\mathbf{N}}$ converges to x and $\iota_n\in I_n$ for any $n\in\mathbf{N}$. If there exists a strict Šmulian subspace Y of X which is Eberlein closed and such that $f^{-1}(Y)\in\mathfrak{U}$ then there exists a sequence $(A_n)_{n\in\mathbf{N}}$ in $f(\mathfrak{U})$ with the property that for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $f(\iota_n)\in A_n$ for any $n\in\mathbf{N}$ the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ converges to x.*

Let first Y be an Eberlein closed subset of X such that $f^{-1}(Y)\in\mathfrak{U}$. By Proposition 1.3 there exists a countably compact net (J, g) on Y and an increasing map φ of J into I such that $g=f\circ\varphi$, $\mathfrak{U}=\varphi(\mathfrak{G})$ where $\mathfrak{G}$ denotes the section filter of J, and such for that any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in $f^{-1}(Y)$ there exists an increasing sequence $(\kappa_n)_{n\in\mathbf{N}}$ in J such that $\varphi(\kappa_n)=\iota_n$ for any $n\in\mathbf{N}$. Since Y is Eberlein closed we deduce $x\in Y$.

Let us now prove the Šmulian part of the proposition. Since $g(\mathfrak{G})=f(\mathfrak{U})$ converges to $x\in Y$ and since $\varphi^{-1}(I_n)\in\mathfrak{G}$ for any $n\in\mathbf{N}$, there exists an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in J such that $(g(\iota_n))_{n\in\mathbf{N}}$ converges to x and such that $\iota_n\in\varphi^{-1}(I_n)$ for any $n\in\mathbf{N}$. Then $(\varphi(\iota_n))_{n\in\mathbf{N}}$ is an increasing sequence in I such that $(f(\varphi(\iota_n)))_{n\in\mathbf{N}}$ converges to x and such that $\varphi(\iota_n)\in I_n$ for any $n\in\mathbf{N}$.

Let us now prove the strict Šmulian part of the proposition. Since $g(\mathfrak{G})=f(\mathfrak{U})$ converges to $x\in Y$ and since Y is a strict Šmulian space there exists a sequence $(A_n)_{n\in\mathbf{N}}$ in $g(\mathfrak{G})$ with the property that for any increasing sequence $(\kappa_n)_{n\in\mathbf{N}}$ in J such that $g(\kappa_n)\in A_n$ for any $n\in\mathbf{N}$ the sequence $(g(\kappa_n))_{n\in\mathbf{N}}$ converges to x. Let $(\iota_n)_{n\in\mathbf{N}}$ be an increasing sequence in I such that $f(\iota_n)\in A_n$ for any $n\in\mathbf{N}$. By Proposition 1.3 there exists an increasing sequence $(\kappa_n)_{n\in\mathbf{N}}$ in J such that $\varphi(\kappa_n)=\iota_n$, for any $n\in\mathbf{N}$. Then the sequence $(f(\iota_n))_{n\in\mathbf{N}}=(g(\kappa_n))_{n\in\mathbf{N}}$ converges to x. †

PROPOSITION 3.11. *Let X be a Hausdorff topological space. If there exists a locally finite covering of X with Šmulian (resp. strict Šmulian) subspaces which are Eberlein closed, then X is a Šmulian (resp. strict Šmulian) space.*

Let (I, f) be a countably compact net on X and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to x. Let $\mathfrak{V}$ be a locally finite covering of X with Šmulian (resp. strict Šmulian) subspaces which are Eberlein closed. By Proposition 1.2 there exists $V\in\mathfrak{V}$ such that $f^{-1}(V)\in\mathfrak{U}$. The assertion now follows from the preceding proposition. †

COROLLARY 3.12. *A paracompact space which is locally a Šmulian (resp. strict Šmulian) space is a Šmulian (resp. strict Šmulian) space.*

Let X be a paracompact space such that any $x\in X$ possesses a closed neighbourhood

U_x which is a Šmulian (resp. strict Šmulian) space. Let $\mathfrak{B}$ be a locally finite covering of X which is finer than the covering $(U_x)_{x\in X}$. Then $(\bar{V})_{V\in\mathfrak{B}}$ is a locally finite covering of X with Šmulian (resp. strict Šmulian) subspaces, which are Eberlein closed. †

COROLLARY 3.13. *The topological sum of Šmulian (resp. strict Šmulian) spaces is a Šmulian (resp. strict Šmulian) space.* †

PROPOSITION 3.14. *Let X be a Šmulian space. If for any $x\in X$ there exists a filter $\mathfrak{F}$ on X with a countable base such that any convergent sequence on X converges to x if and only if its associated elementary filter on X is finer than $\mathfrak{F}$, then X is a strict Šmulian space. If for any $x\in X$ there exists a real function g on X equal to 0 at x and strictly positive on $X\backslash\{x\}$ and such that for any sequence $(y_n)_{n\in\mathbf{N}}$ in X which converges to a point $y\in X$, the sequence $(g(y_n))_{n\in\mathbf{N}}$ converges to $g(y)$, then for any $x\in X$ there exists a filter $\mathfrak{F}$ on X with the indicated properties.*

Let (I, f) be a countably compact net on X and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point $x\in X$. Let $\mathfrak{F}$ be the filter on X with the indicated properties and let $A\in\mathfrak{F}$. Assume that $A\notin f(\mathfrak{U})$. Then $I\backslash f^{-1}(A)\in\mathfrak{U}$ and, since X is a Šmulian space, there exists an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $(f(\iota_n))_{n\in\mathbf{N}}$ converges to x and such that $\iota_n\in I\backslash f^{-1}(A)$ for any $n\in\mathbf{N}$. From the hypotheses about $\mathfrak{F}$ we deduce that the elementary filter $\mathfrak{F}'$ associated with the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ is finer than $\mathfrak{F}$. This is a contradiction, since $X\backslash A$ belongs to $\mathfrak{F}'$. Hence $A\in f(\mathfrak{U})$.

Let $(A_n)_{n\in\mathbf{N}}$ be a countable base of $\mathfrak{F}$. Let $(\iota_n)_{n\in\mathbf{N}}$ be an increasing sequence in I such that $f(\iota_n)\in\bigcap_{m\leqslant n}A_m$ for any $n\in\mathbf{N}$. Since (I, f) is a countably compact net, the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ has an adherent point $y\in X$. Using again the fact that X is a Šmulian space, we deduce that there exists a subsequence $(f(\iota_{n_k}))_{k\in\mathbf{N}}$ of the sequence $(f(\iota_n))_{n\in\mathbf{N}}$, which converges to y. But the elementary filter associated to the sequence $(f(\iota_{n_k}))_{k\in\mathbf{N}}$ is finer than $\mathfrak{F}$. Hence $y=x$. Since x is the only adherent point of the sequence $(f(\iota_n))_{n\in\mathbf{N}}$, we deduce by Proposition 1.4 that $(f(\iota_n))_{n\in\mathbf{N}}$ converges to x. This shows that X is a strict Šmulian space.

In order to prove the last assertion let x be a point of X and g a real function on X with the indicated properties. We set

$$\mathfrak{F}:=\{A\subset X \mid \inf_{y\in X\backslash A} g(y)>0\}.$$

It is easy to see that $\mathfrak{F}$ is a filter on X with a countable base. Let $(x_n)_{n\in\mathbf{N}}$ be a convergent sequence on X. We want to show that $(x_n)_{n\in\mathbf{N}}$ converges to x if and only if its associated elementary filter $\mathfrak{F}'$ is finer than $\mathfrak{F}$. Assume that $(x_n)_{n\in\mathbf{N}}$ does not converge to x. Then it converges to a $y\in X$, $y\neq x$ and we get

$$\lim_{n\to\infty} g(x_n)=g(y)>0, \quad \inf_{n\in\mathbf{N}} g(x_n)>0.$$

Hence $\mathfrak{F}'$ is not finer than $\mathfrak{F}$. Assume that $(x_n)_{n\in\mathbf{N}}$ converges to x. Then $\lim_{n\to\infty} g(x_n)=g(x)=0$ and $\mathfrak{F}'$ is finer than $\mathfrak{F}$. †

PROPOSITION 3.15. *Let X be a Hausdorff topological space, Y be a Šmulian space, φ be a continuous map of X into Y, (I, f) be a countably compact net on X and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. If $\varphi\circ f(\mathfrak{U})$ converges to a point $y\in Y$ such that $\varphi^{-1}(y)$ is a Lindelöf space (with respect to the induced topology), then $f(\mathfrak{U})$ is convergent.*

Assume the contrary. Then any $x\in\varphi^{-1}(y)$ possesses an open neighbourhood V_x which does not belong to $f(\mathfrak{U})$. Since $(V_x\cap\varphi^{-1}(y))_{x\in\varphi^{-1}(y)}$ is an open covering of $\varphi^{-1}(y)$, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ in $\varphi^{-1}(y)$ such that $(V_{x_n}\cap\psi^{-1}(y))_{n\in\mathbf{N}}$ is a covering of $\varphi^{-1}(y)$. Since $\mathfrak{U}$ is an ultrafilter and since $f^{-1}(V_{x_n})\notin\mathfrak{U}$ for any $n\in\mathbf{N}$, we get

$$f^{-1}\left(X\setminus\bigcup_{m\leqslant n} V_{x_m}\right)\in\mathfrak{U}$$

for any $n\in\mathbf{N}$. Y being a Šmulian space and $(I, \varphi\circ f)$ being a countably compact net on Y, there exists an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $(\varphi\circ f(\iota_n))_{n\in\mathbf{N}}$ converges to y and such that $\iota_n\in f^{-1}(X\setminus\bigcup_{m\leqslant n} V_{x_m})$ for any $n\in\mathbf{N}$. Since the net (I, f) is countably compact we deduce that there exists an adherent point x of the sequence $(f(\iota_n))_{n\in\mathbf{N}}$. It is obvious that $x\in\varphi^{-1}(y)$. Therefore there exists $m\in\mathbf{N}$ such that $x\in V_{x_m}$. But $f(\iota_n)\notin V_{x_m}$ for $n\geqslant m$ and this is a contradiction. †

COROLLARY 3.16. *Let X be a Hausdorff topological space, Y be a Šmulian-Eberlein space and φ be a continuous map of X into Y such that for any $y\in Y$, $\varphi^{-1}(y)$ is a Lindelöf space (with respect to the induced topology). Then X is an Eberlein space.* †

COROLLARY 3.17. *A subspace of a Šmulian-Eberlein (resp. strict Šmulian-Eberlein) space is a Šmulian-Eberlein (resp. strict Šmulian-Eberlein) space.*

The corollary follows from the preceding one with the aid of Corollary 3.6. †

COROLLARY 3.18 *If X is a Šmulian-Eberlein (resp. strict Šmulian-Eberlein) space then X endowed with any finer topology is a Šmulian-Eberlein (resp. strict Šmulian-Eberlein) space.*

The Corollary follows from Corollary 3.16 with the aid of Corollary 3.7. †

COROLLARY 3.19. *A topological space for which there exists a coarser metrizable topology is a strict Šmulian-Eberlein space.*

The assertion follows from the preceding corollary with the aid of Corollaries 2.4 and 3.5. †

COROLLARY 3.20. *Any topological group (and therefore any topological vector space) for which the one-point sets are of type G_δ is a strict Šmulian-Eberlein space.*

The assertion follows immediately from the preceding corollary and the next lemma. †

LEMMA 3.21. *If the one-point sets of topological group are of type G_δ, then there exists a coarser metrizable topology on the group.*

Let $(U_n)_{n\in\mathbf{N}}$ be a sequence of open sets of the topological group X such that

$$\{1\} = \bigcap_{n\in\mathbf{N}} U_n,$$

where 1 denotes the neutral element of the group. Then there exists a sequence $(V_n)_{n\in\mathbf{N}}$ of open neighbourhoods of 1 such that for any $n\in\mathbf{N}$

$$V_n = V_n^{-1}, \quad V_{n+1}V_{n+1} \subset V_n \cap U_n.$$

We set for any $n\in\mathbf{N}$

$$\hat{V}_n := \{(x, y)\in X^2 \mid xy^{-1}\in V_n\}.$$

Then $\{\hat{V}_n \mid n\in\mathbf{N}\}$ is a fundamental system of vicinities (entourages) for a separated uniformity on X. Being countable the uniform space defined by it is metrizable. Its topology is obviously coarser than the initial topology of X. †

THEOREM 3.22. *Any relatively countably compact set of a Šmulian space is sequentially dense.*

Let A be a relatively countably compact set of a Šmulian space X. We endow A with the trivial preorder relation; i.e. we set $x\leqslant y$ for any two elements x, y of A. Then A becomes an upper directed preordered set. If f denotes the inclusion map of A into X then (A, f) is a countably compact net on X. Any adherent point of A is adherent to the filter $f(\mathfrak{F})$, where $\mathfrak{F}$ denotes the section filter of A. Hence for any adherent point of A there exists a sequence in A converging to this point. †

The converse assertion is not true as it can be seen from following examples.

EXAMPLE 3.23. There exists an Eberlein space for which any subset is sequentially dense and which is not a Šmulian space.

Let ω_1 be the first uncountable ordinal number. If the continuum hypothesis $2^{\aleph_0}=\aleph_1$ is assumed then there exists a family $(f_\xi)_{\xi\in\omega_1}$ such that: a) for any $\xi\in\omega_1$, f_ξ is an increasing map of $\mathbf{N}$ into $\mathbf{N}$; b) if $\xi, \eta\in\omega_1$ and $\xi<\eta$ then there exists $i_0\in\mathbf{N}$ such that $f_\xi(i)<f_\eta(i)$ for any $i>i_0$; c) if f is an increasing map of $\mathbf{N}$ into $\mathbf{N}$, then there exists $\xi\in\omega_1$ and $i_0\in\mathbf{N}$ such that $f(i)<f_\xi(i)$ for any $i>i_0$.

We set

$$X := \mathbf{N}^2 \cup \omega_1 \cup \{\omega_1\}.$$

A subset U of X will be called open if it possesses the following properties:

a) if $\omega_1 \in U$ then for any $i \in \mathbf{N}$ the set $\{j \in \mathbf{N} \mid (i, j) \notin U\}$ is finite;

b) if $0 \in U$ then there exists $i_0 \in \mathbf{N}$ such that

$$\{(i,j) \in \mathbf{N}^2 \mid i_0 \leqslant i, j \leqslant f_0(i)\} \subset U$$

c) if there exists $\xi \in \omega_1$, $\xi \neq 0$, such that $\xi \in U$ then there exists $\xi_0 < \xi$ and $i_0 \in \mathbf{N}$ such that

$$\{(i,j) \in \mathbf{N}^2 \mid i_0 \leqslant i, f_{\xi_0}(i) < j \leqslant f_\xi(i)\} \subset U.$$

X endowed with this topology is an Eberlein space for which any set is sequentially dense.

Let $\mathbf{N}^2$ be endowed with the following uper directed order relation

$$(i,j) \leqslant (i',j') :\Leftrightarrow (i \leqslant i' \quad \text{and} \quad j \leqslant j')$$

and let f be the inclusion map of $\mathbf{N}^2$ into X. Then $(\mathbf{N}^2, f)$ is a net on X. We want to show that it is countably compact. Let $((i_n, j_n))_{n \in \mathbf{N}}$ be an increasing sequence in $\mathbf{N}^2$. If there exist $n_0 \in \mathbf{N}$ such that $i_n = i_{n_0}$ for $n \geqslant n_0$ then the sequence $((i_n, j_n))_{n \in \mathbf{N}}$ converges to ω_1 or, if $(j_n)_{n \in \mathbf{N}}$ is also stationary from a certain $n \in \mathbf{N}$, to a point of $\mathbf{N}^2$. Assume now that $(i_n)_{n \in \mathbf{N}}$ is strictly increasing and let I be the set of $\eta \in \omega_1$ such that there exists $n_0 \in \mathbf{N}$ with the property that for any $n \geqslant n_0$ we have $j_n \leqslant f_\eta(i_n)$. Let ξ be the smallest element of I. If $\xi = 0$ then the sequence $((i_n, j_n))_{n \in \mathbf{N}}$ converges to 0. If $\xi \neq 0$ then for any $\xi_0 < \xi$ the set

$$\{n \in \mathbf{N} \mid f_{\xi_0}(i_n) < j_n \leqslant f_\xi(i_n)\}$$

is infinite and therefore ξ is an adherent point of the sequence $((i_n, j_n))_{n \in \mathbf{N}}$. If $\mathfrak{F}$ denotes the section filter of $\mathbf{N}^2$ then ω_1 is adherent to $f(\mathfrak{F})$. Set for any $n \in \mathbf{N}$

$$I_n := \{(i,j) \in I \mid i \geqslant n, j \geqslant n\}.$$

Then $(I_n)_{n \in \mathbf{N}}$ is a sequence in $\mathfrak{F}$. Let $((i_n, j_n))_{n \in \mathbf{N}}$ be an increasing sequence in $\mathbf{N}^2$ such that $(i_n, j_n) \in I_n$ for any $n \in \mathbf{N}$. It is obvious that ω_1 is not an adherent point of this sequence. Hence X is not a Šmulian space. †

EXAMPLE 3.24. There exists a completely regular space for which any relatively countably compact set is sequentially dense and which is not a Šmulian space.

Let Y be an uncountable set, X_0 be the set of subsets of Y of cardinal $\aleph_0$ and $X := X_0 \cup \{Y\}$. We endow X with a topology by taking as open sets the subsets V of X such that for any $A \in V$ there exists a finite subset B of A such that $\{C \in X \mid B \subset C \subset A\} \subset V$. It is easy to see that the above set $\{C \in X \mid B \subset C \subset A\}$ is closed and open for this topology. From that we get immediately that X is completely regular. Since for any $A \in X_0$, $\{A\}$ is of type G_δ it follows from Corollary 3.3. that X_0 is a Šmulian space with respect to the induced topology. On the other hand Y does not belong to the closure of any countable subset of X_0. Hence by Theorem 3.22. any relatively countably compact set of X is sequentially dense.

X_0 endowed with the inclusion relation is an upper directed ordered set. If f denotes the inclusion map of X_0 into X then (X_0, f) is a countably compact net on X such that, if $\mathfrak{F}$ denotes the section filter of X_0 then $f(\mathfrak{F})$ converges to Y. This and the above remark show that X is not a Šmulian space. †

PROPOSITION 3.25. *Let X be an Eberlein (resp. Šmulian) space and $\leqslant$ an order relation on X such that for any upper directed subset A of X and for any $x \in X$ the following two assertions are equivalent:*

a) *x is the supremum of A;*

b) *x is adherent to the section filter of A.*

Let A be an upper directed subset of X such that any increasing sequence in A has a supremum. Then A has a supemum (resp. if A has a suprenum x then x is the supremum of an increasing sequence in A).

If we denote by f the inclusion map of A into X, then (A, f) is a countably compact net in X. If X is an Eberlein space, then the adherence in X of the section filter of A is non-empty and therefore A has a supremum. If X is a Šmulian space and A has a supremum x, then x belongs to the adherence of the section filter of A and there exists therefore an increasing sequence in A converging to x; but then x is the supremum of this increasing sequence. †

IV. Eberlein Continuous Maps

Let X, Y be two Hausdorff topological spaces. A map g of X into Y is called *Eberlein continuous* if for any countably compact net (I, f) on X and for any ultrafilter $\mathfrak{U}$ on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point $x \in X$, the ultrafilter $g(f(\mathfrak{U}))$ converges to $g(x)$. We denote by $\mathscr{E}(X, Y)$ the set of Eberlein continuous maps of X into Y. Of course any continuous map is Eberlein continuous. If g is an Eberlein continuous map of X into Y, and (I, f) is a countably compact net on X, then $(I, g \circ f)$ is a countably compact net on Y. The composition of two Eberlein continuous maps is Eberlein continuous.

THEOREM 4.1. *Let X be a Šmulian space (I, f) be a countably compact net on X, $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a point $x \in X$, and g be a map of X into a Hausdorff topological space Y such that for any increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I with the property that $(f(\iota_n))_{n \in \mathbf{N}}$ is convergent, $(g(f(\iota_n)))_{n \in \mathbf{N}}$ converges to $g(\lim_{n \to \infty} f(\iota_n))$. Then $g(f(\mathfrak{U}))$ converges to $g(x)$. In particular a map g of a Šmulian space X into a Hausdorff topological space is*

Eberlein continuous if for any $x \in X$ *and for any sequence* $(x_n)_{n\in\mathbf{N}}$ *in* X *converging to* x, *the sequence* $(g(x_n))_{n\in\mathbf{N}}$ *converges to* $g(x)$.

Let V be a neighbourhood of $g(x)$. If $V \notin g(f(\mathfrak{U}))$ then $f^{-1}(g^{-1}(Y\backslash V)) \in \mathfrak{U}$. Since X is a Šmulian space there exists an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $(f(\iota_n))_{n\in\mathbf{N}}$ converges to x and such that $\iota_n \in f^{-1}(g^{-1}(Y\backslash V))$ for any $n \in \mathbf{N}$. We get the contradictory relations

$$\lim_{n\to\infty} g(f(\iota_n)) = g(x), \quad n \in \mathbf{N} \Rightarrow g(f(\iota_n)) \in Y\backslash V.$$

Hence $V \in g(f(\mathfrak{U}))$ and, V being arbitrary, $g(f(\mathfrak{U}))$ converges to $g(x)$. †

Remark. The example 3.24. shows that we may not replace in Theorem 4.1. the hypothesis that X is a Šmulian space with the weaker one that any relatively countably compact set is sequentially dense. Indeed if we denote by g the function on X which is equal to 0 on X_0 and equal to 1 at Y then for any convergent sequence $(x_n)_{n\in\mathbf{N}}$ on X the sequence $(g(x_n))_{n\in\mathbf{N}}$ converges to $g(\lim_{n\to\infty} x_n)$. In order to see that g is not Eberlein continuous it is sufficient to take the countably compact net (X_0, f) on $X$and an ultrafilter $\mathfrak{U}$ on X_0 finer than the section filter of X_0 and such that $f(\mathfrak{U})$ converges to Y. It is possible to construct such examples even with Eberlein spaces X.

THEOREM 4.2. *The restriction of any Eberlein continuous map to any relatively countably compact set is continuous. In particular any Eberlein continuous map is universally measurable.*

Let X, Y be two Hausdorff topological spaces, g be an Eberlein continuous map of X into Y and A be a relatively countably compact set of X. Let $x \in A$ and $\mathfrak{U}$ be an ultrafilter on A converging to x. If we endow A with the trivial preorder relation $y \leqslant z$ for any $y, z \in A$ and if we denote by f the inclusion map of A into X, then (A, f) is a countably compact net on X and $\mathfrak{U}$ is an ultrafilter on A, finer than the section filter of A and such that $f(\mathfrak{U})$ converges to x. It follows that $g(\mathfrak{U})$ converges to $g(x)$. Since $\mathfrak{U}$ and x are arbitrary we deduce that the restricton of g to A is continuous. †

Remark. The Theorems 4.1 and 4.2. have important consequences for the integration of vector valued functions. Indeed by Lebesgue theorem any integral may be considered as a map possessing the property indicated in Theorem 4.1. If the ground space is a Šmulian space then, by Theorem 4.1., it is Eberlein continuous. With the aid of Theorem 4.2. and of Grothendieck's completeness criterium it follows that the integral is even continuous. These considerations will be applied in Chapter VII.

COROLLARY 4.3. *Let X be a Hausdorff topological space such that any map defined on X is continuous if its restriction to any compact set of X is continuous. Then any Eberlein continuous map on X is continuous.* †

COROLLARY 4.4. *Let X be a Hausdorff topological space, Y be a regular space,*

f be an Eberlein continuous map of X into Y and A be a relatively countably compact set of X. Then the restriction of f to $\bar{A}$ is continuous.

The assertion follows immediately from the theorem with the aid of the following lemma. †

LEMMA 4.5. *Let X be a Hausdorff topological space and $\mathfrak{S}$ be a set of subsets of X such that if $A \in \mathfrak{S}$ and $x \in \bar{A}$ then $A \cup \{x\} \in \mathfrak{S}$. If f is a map of X into a regular space Y such that its restriction to any set of $\mathfrak{S}$ is continuous, then for any $A \in \mathfrak{S}$ the restriction of f to $\bar{A}$ is continuous.*

Let $A \in \mathfrak{S}$, $x \in \bar{A}$ and V be a closed neighbourhood of $f(x)$. Since the restriction of f to $A \cup \{x\}$ is continuous there exists an open neighbourhood U of x such that $f(U \cap A) \subset V$. Let $y \in U \cap \bar{A}$. Since the restriction of f to $A \cup \{y\}$ is continuous, we get $f(y) \in V$. Hence $f(U \cap \bar{A}) \subset V$. V being arbitrary, the restriction of f to $\bar{A}$ is continuous at x. But x being arbitrary the restriction of f to $\bar{A}$ is continuous. †

Remark. The converse of Theorem 4.2. or of Corollary 4.4. is not true since there exist real functions on Eberlein spaces whose restrictions to the closures of relatively countably compact sets are continuous (and even to countable unions of such sets) but which are not Eberlein continuous. Indeed let ω_1 be the first uncountable ordinal number. We set

$$X := (\omega_1 \times (\mathbf{N} \cup \{\infty\})) \cup \{0\}$$

and endow X with a topology by taking as open sets the subsets U of X such that

a) if $(\xi, \infty) \in U$ then there exists $\eta \in \omega_1$ such that $\eta < \xi$ and $m \in \mathbf{N}$ such that $\{(\zeta, n) \in \omega_1 \times \mathbf{N} \mid \eta < \zeta \leqslant \xi, m \leqslant n\} \subset U$;

b) If $0 \in U$ then there exists $\xi \in \omega_1$ and $m \in N$ such that $\{(\eta, n) \in \omega_1 \times \mathbf{N} \mid \xi < \eta, m < n\} \subset U$.

X endowed with this topology is an Eberlein space. Any relatively countably compact set of X is at most countable. It follows that the real function g on X equal to 0 on $\omega_1 \times (\mathbf{N} \cup \{\infty\})$ and equal to 1 at 0 has the property that for any relatively countably compact set A of X the restriction of g to $\bar{A}$ is continuous. We want to show that g is not Eberlein continuous. Let us endow $\omega_1 \times \mathbf{N}$ with the order relation

$$(\{\xi, m) \leqslant (\eta, n) :\Leftrightarrow ((\{\xi, m) = (\eta, n) \quad \text{or} \quad (\{\xi < n \quad \text{and} \quad m < n))$$

and let f be the inclusion map of $\omega_1 \times \mathbf{N}$ into X. Then $(\omega_1 \times \mathbf{N}, f)$ is a countably compact net on X. If $\mathfrak{F}$ denotes the section filter of $\omega_1 \times \mathbf{N}$ then $f(\mathfrak{F})$ converges to 0 and the filter $g(f(\mathfrak{F}))$ converges to $0 \neq g(0)$.

The following example will show that there exists a completely regular and countably compact space X and a real function g on X whose restrictions to any compact set of X is continuous and which is not continuous. Let $\mathbf{N}^*$ be the Stone-Cech compactification of $\mathbf{N}$ and let Φ be the set of non-trivial ultrafilters $\mathfrak{U}$ on $\mathbf{N}$ with the property that for any map $f:\mathbf{N}\to\mathbf{N}$ there exists $M\in\mathfrak{U}$ such that the restriction of f to M is injective. Let A be the set of points x of $\mathbf{N}^*$ for which there exists $\mathfrak{U}\in\Phi$ such that $\mathfrak{U}$ converges to x. By the continuum hypothesis $\bar{A}=\mathbf{N}^*\backslash\mathbf{N}$, $\overline{\mathbf{N}^*\backslash A}=\mathbf{N}^*$. We set $X:=\mathbf{N}^*\backslash A$. X endowed with the induced topology is a completely regular space such that for any compact set K of X the set $K\cap\mathbf{N}$ is finite. It follows that the real function on X equal to 0 on $\mathbf{N}$ and equal to 1 elsewhere, which obviously is not continuous, has the property that its restrictions to any compact set is continuous. It can be shown, using the properties of Φ, that X is a countably compact space.

We call *c-space* a Hausdorff topological space X such that any Eberlein continuous map of X into a regular space is continuous.

COROLLARY 4.6. *Let X be a Hausdorff topological space such that any subset V of X is open if for any relatively countably compact set A the set $\bar{A}\cap V$ is open in $\bar{A}$ for the induced topology. Then X is a c-space. In particular any Kelley-space is a c-space.*

The assertion follows immediately from the preceding corollary. †

PROPOSITION 4.7. *Let X be a Hausdorff topological space and $\mathfrak{T}$ be the coarsest topology on X for which any Eberlein continuous map on X into an arbitrary Hausdorff topological space is continuous. If (I,f) is a countably compact net on X with respect to the initial topology and if $\mathfrak{U}$ is an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to $x\in X$ in the initial topology of X, then (I,f) is a countably compact net for the $\mathfrak{T}$-topology and $f(\mathfrak{U})$ converges to x in $\mathfrak{T}$. In particular any Eberlein continuous map on X endowed with $\mathfrak{T}$ is an Eberlein continous map on X endowed with the initial topology. X endowed with $\mathfrak{T}$ is a c-space. The identical map of X into X endowed with $\mathfrak{T}$ is Eberlein continuous.*

Let $(Y_\lambda)_{\lambda\in L}$ be a family of Hausdorff topological spaces and $(\varphi)_{\lambda\in L}$ be a family such that for any $\lambda\in L$, φ_λ is an Eberlein continuous map of X into Y_λ. Then $\prod_{\lambda\in L}\varphi_\lambda$ is an Eberlein continuous map of X into $\prod_{\lambda\in L}Y_\lambda$. Hence the sets of the form $\varphi^{-1}(W)$, where φ is an Eberlein continuous map of X into a Hausdorff topological space Y and W is an open set of Y, form a base for the $\mathfrak{T}$-topology.

Let U be a neighbourhood of x in the $\mathfrak{T}$-topology. Then there exists an Eberlein continuous map φ of X into a Hausdorff topological space Y and a neighbourhood V of $\varphi(x)$ such that $\varphi^{-1}(V)\subset U$. Since φ is Eberlein continuous, $\varphi(f(\mathfrak{U}))$ converges to $\varphi(x)$ and therefore $\varphi^{-1}(V)\in f(\mathfrak{U})$. Hence $U\in f(\mathfrak{U})$ and $f(\mathfrak{U})$ converges to x in the $\mathfrak{T}$-topology.

Let now $(\iota_n)_{n\in\mathbf{N}}$ be an increasing sequence in I and y be an adherent point of the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ with respect to the initial topology of X. If we denote by g the map $n\mapsto f(\iota_n):\mathbf{N}\to X$ then $(\mathbf{N},g)$ is a countably compact net on X such that there exists an ultrafilter $\mathfrak{B}$ on $\mathbf{N}$ such that $g(\mathfrak{B})$ converges to y with respect to the initial topology of X. By the above considerations $g(\mathfrak{B})$ converges to y with respect to the $\mathfrak{T}$-topology.

Hence y is an adherent point of the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ in the $\mathfrak{T}$-topology. But this means that (I,f) is a countably compact net on X with respect to the $\mathfrak{T}$-topology.

The last assertions are obvious. †

COROLLARY 4.8. *Let X be a Hausdorff topological sace and $\mathscr{F}$ be a set of maps of X into arbitrary Hausdorff topological spaces. We denote by $\mathscr{F}'$ the maps of $\mathscr{F}$ which are Eberlein continuous and $\mathfrak{T}(\mathscr{F})$ the corsest topology on X which is finer than the initial one and for which any $f\in\mathscr{F}'$ is continuous. Then any $f\in\mathscr{F}$ which is Eberlein continuous with respect to $\mathfrak{T}(\mathscr{F})$ is continuous with respect to $\mathfrak{T}(\mathscr{F})$.*

Let $f\in\mathscr{F}$ be Eberlein continuos with respect to $\mathfrak{T}(\mathscr{F})$. Since the topology $\mathfrak{T}$ introduced in the proposition is finer than $\mathfrak{T}(\mathscr{F})$, f is Eberlein continuous with respect to $\mathfrak{T}$. By the proposition it follows that f is Eberlein continuous for the initial topology of X. Hence $f\in\mathscr{F}'$ and is therefore continuous with respect to $\mathfrak{T}(\mathscr{F})$. †

PROPOSITION 4.9. *Let X be a Hausdorff topological space, $\mathfrak{F}$ be a filter on X converging to a point $x\in X$, $(I_\lambda, f_\lambda)_{\lambda\in L}$ be a family of countably compact nets on X and $(\mathfrak{F}_\lambda)_{\lambda\in L}$ be a family with the following properties:*

a) *for any $\lambda\in L$, $\mathfrak{F}_\lambda$ is a filter on I_λ, finer than the section filter of I_λ;*

b) $\mathfrak{F}=\bigcap_{\lambda\in L} f_\lambda(\mathfrak{F}_\lambda)$.

Then for any Eberlein continuous map φ defined on X the filter $\varphi(\mathfrak{F})$ converges to $\varphi(x)$.

By b) $f_\lambda(\mathfrak{F}_\lambda)$ converges to x for any $\lambda\in L$. Hence $\varphi(f_\lambda(\mathfrak{F}_\lambda))$ converges to $\varphi(x)$ for any $\lambda\in L$. If V is a neighbourhood of $\varphi(x)$ then $\varphi^{-1}(V)$ belongs to $f_\lambda(\mathfrak{F}_\lambda)$ for any $\lambda\in L$ and therefore to $\mathfrak{F}$. It follows that $\varphi(\mathfrak{F})$ converges to $\varphi(x)$. †

COROLLARY 4.10. *Let X be a Hausdorff topological space, x be a point of X, $(I_\lambda, f_\lambda)_{\lambda\in L}$ be a family of countably compact nets on X and $(\mathfrak{F}_\lambda)_{\lambda\in L}$ be a family such that for any $\lambda\in L$, $\mathfrak{F}_\lambda$ is a filter on I_λ, finer than the section filter of I_λ and such that $\bigcap_{\lambda\in L} f_\lambda(\mathfrak{F}_\lambda)$ is the filter of neighbourhoods of x. Then any Eberlein continuous map defined on X is continuous at x.* †

PROPOSITION 4.11. *Let X be a Hausdorff topological space, $(X_\alpha)_{\alpha\in A}$ be a family of Šmulian spaces, $(\varphi_\alpha)_{\alpha\in A}$ be a family such that for any $\alpha\in A$, $\varphi_\alpha\in\mathscr{E}(X, X_\alpha)$ and Y be a subspace of X such that any $x\in X$ belongs to Y if for any $\alpha\in A$ there exists $y\in Y$ such that $\varphi_\alpha(x)=\varphi_\alpha(y)$. Then Y is Eberlein closed in X. Hence if X is an Eberlein space, then so is Y.*

Let (I, f) be a countably compact net on Y and $\mathfrak{U}$ be an ultrafilter on I finer than the section filter of I and such that $f(\mathfrak{U})$ converges in X to an x. We have to show that $x\in Y$. Let $\alpha\in A$. Then $(I, \varphi_\alpha\circ f)$ is a countably compact net on X_α and $\varphi_\alpha\circ f(\mathfrak{U})$ converges to $\varphi_\alpha(x)$. Since X_α is a Šmulian space, there exists an increasing sequence

$(\iota_n)_{n\in\mathbf{N}}$ in I such that $(\varphi_\alpha \circ f(\iota_n))_{n\in\mathbf{N}}$ converges to $\varphi_\alpha(x)$. Since (I, f) is a countably compact net the sequence $(f(\iota_n))_{n\in\mathbf{N}}$ possesses an adherent point $y\in Y$. It is obvious that $\varphi_\alpha(y)$ is an adherent point of the sequence $(\varphi_\alpha \circ f(\iota_n))_{n\in\mathbf{N}}$ and therefore $\varphi_\alpha(y)=\varphi_\alpha(x)$. Since α is arbitrary we deduce $x\in Y$. The last assertion follows with the aid of Proposition 2.1. †

V. Spaces of Continuous Maps

For any sets X, Y we denote by Y^X the set of maps of X into Y. If Y is a Hausdorff topological space then any subset of Y^X will be considered endowed with the topology of pointwise convergence. If Y is a separated uniform space and $\mathfrak{S}$ is a covering of X, then, for any subset $\mathscr{F}\subset Y^X$, $\mathscr{F}_{\mathfrak{S}}$ will denote the uniform space (respectively the topological space) obtained by endowing $\mathscr{F}$ with the uniform structure (respectively the topology) of uniform convergence on the sets of $\mathfrak{S}$.

If X and Y are non-empty Hausdorff topological spaces, then $\mathscr{C}(X, Y)$ will denote the subspace of Y^X formed by the continuous maps. We remark that Y is homeomorphic to the closed subspace of $\mathscr{C}(X, Y)$ formed by the constant maps. Hence if $\mathscr{C}(X, Y)$ is an Eberlein (respectively Šmulian, strict Šmulian) space, then Y is an Ebelerin (respectively Šmulian, strict Šmulian) space. If (I, f) is a net on Y^X we shall write f_ι instead of $f(\iota)$ for any $\iota\in I$.

If, in addition, Y is endowed with a preorder relation, we shall consider Y^X endowed with the preorder relation

$$f \leqslant g :\Leftrightarrow (\forall x)\,(x\in X \Rightarrow f(x)\leqslant g(x)).$$

The following proposition allows to extend the results enunciated for the spaces $\mathscr{F}\subset Y^X$ to the spaces $\mathscr{F}_{\mathfrak{S}}$.

PROPOSITION 5.1. *Let $\mathfrak{S}$ be a covering of X, Y be a separated uniform space, $\mathscr{F}$ be a subset of Y^X, (I, f) be a countably compact net on $\mathscr{F}_{\mathfrak{S}}$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to a $g\in\mathscr{F}$ in the topology of pointwise convergence. Then $f(\mathfrak{U})$ converges to g in $\mathscr{F}_{\mathfrak{S}}$.*

We may suppose that the union of any finite family of sets of $\mathfrak{S}$ belongs to $\mathfrak{S}$. Assume that the proposition is not true. Then there exists a set $A\in\mathfrak{S}$ and a uniformly continuous ecart d on Y such that

$$\{h\in\mathscr{F} \mid x\in A \Rightarrow d(g(x), h(x))\leqslant 1\}\notin f(\mathfrak{U}).$$

Since $\mathfrak{U}$ is an ultrafilter

$$\{\iota\in I \mid (\exists x)\,(x\in A,\, d(g(x), f_\iota(x))>1)\}\in\mathfrak{U}.$$

We shall construct inductively a sequence $(x_n)_{n\in\mathbf{N}}$ in A and an increasing sequence

$(\iota_n)_{n\in\mathbf{N}}$ in I such that for any $n\in\mathbf{N}$

$$d(g(x_n), f_{\iota_n}(x_n)) > 1, \quad m < n \Rightarrow d(g(x_m), f_{\iota_n}(x_m)) < \tfrac{1}{3}.$$

Assume that the sequences were constructed up to $n-1$ and let us find ι_n and x_n. Since $f(\mathfrak{U})$ converges to g in the topology of pointwise convergence, we have

$$\{\iota\in I \mid m < n \Rightarrow d(g(x_m), f_\iota(x_m)) < \tfrac{1}{3}\}\in\mathfrak{U}.$$

Hence the set

$$\{\iota\in I \mid (m < n \Rightarrow d(g(x_m), f_\iota(x_m)) < \tfrac{1}{3}) \\ \& (\iota \geqslant \iota_{n-1}) \& (\exists x)(x\in A, d(g(x), f_\iota(x)) > 1)\}$$

belongs to U and is therefore non-empty. We take an arbitrary ι_n in this set and an arbitrary $x_n\in A$ such that

$$d(g(x_n), f_{\iota_n}(x_n)) > 1.$$

Since (I, f) is a countably compact net on $\mathscr{F}_{\mathfrak{S}}$ the sequence $(f_{\iota_n})_{n\in\mathbf{N}}$ possesses an adherent point h in $\mathscr{F}_{\mathfrak{S}}$. Hence there exists a subsequence$(f_{\iota_{n_k}})_{k\in\mathbf{N}}$ such that

$$x\in A \Rightarrow d(h(x), f_{\iota_{n_k}}(x)) < \tfrac{1}{3}$$

for any $k\in\mathbf{N}$. This leads to the contradictory relation

$$\tfrac{2}{3} < d(g(x_{n_0}) f_{\iota_{n_0}}(x_{n_0})) - d(f_{\iota_{n_0}}(x_{n_0}), h(x_{n_0})) \leqslant d(g(x_{n_0}), h(x_{n_0})) \\ \leqslant d(g(x_{n_0}), f_{\iota_{n_1}}(x_{n_0})) + d(f_{\iota_{n_1}}(x_{n_0}), h(x_{n_0})) < \tfrac{2}{3}. \quad \dagger$$

COROLLARY 5.2. *Let $\mathfrak{S}$ be a covering of X, Y be a separated uniform space and $\mathscr{F}$ be a subset of Y^X. If $\mathscr{F}$ is an Eberlein (resp. Šmulian, resp. strict Šmulian) space for a topology $\mathfrak{T}$, which is finer than the topology of pointwise convergence and coarser than the topology of uniform convergence on the sets of $\mathfrak{S}$, then $\mathscr{F}_{\mathfrak{S}}$ is an Eberlein (resp. Šmulian, resp. strict Šmulian) space. In particular if Y is an Eberlein space $Y^X_{\mathfrak{S}}$ is an Eberlein space.*

The Šmulian and strict Šmulian parts of the corollary follow immediately from Corollary 3.7.

Let (I, f) be a countably compact net on $\mathscr{F}_{\mathfrak{S}}$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. Then (I, f) is a countably compact net on $\mathscr{F}$ endowed with $\mathfrak{T}$. Hence $f(\mathfrak{U})$ converges to a function $g\in\mathscr{F}$ in the $\mathfrak{T}$-topology. But then $f(\mathfrak{U})$ converges to g in the topology of pointwise convergence. By the proposition $f(\mathfrak{U})$ converges to g in $\mathscr{F}_{\mathfrak{S}}$ .The last assertion follows from Theorem 2.8. †

Remark. In the sequel we shall state the results only for the topology of pointwise convergence, this corollary extending them automatically to the topologies of uniform convergence on the sets of a covering $\mathfrak{S}$.

THEOREM 5.3. *Let X be a set, $(X_\alpha)_{\alpha\in A}$ be a family of sets, Y be a Hausdorff topological space $(Y_\beta)_{\beta\in B}$ be a family of Hausdorff topological spaces, $\mathscr{F}$ be a subset of Y^X, and $(\varphi_\alpha)_{\alpha\in A}$, $(\psi_\beta)_{\beta\in B}$, $(\mathscr{F}_{\alpha\beta})_{(\alpha,\beta)\in A\times B}$ be families such that for any $(\alpha,\beta)\in A\times B$, $\varphi_\alpha\in X^{X_\alpha}$, $\psi_\beta\in\mathscr{E}(Y, Y_\beta)$ and $\mathscr{F}_{\alpha\beta}$ is an Eberlein closed set of $Y_\beta^{X_\alpha}$. If for any $f\in Y^X$ we have*

$$f\in\mathscr{F} \Leftrightarrow ((\alpha,\beta)\in A\times B \Rightarrow \psi_\beta\circ f\circ\varphi_\alpha\in\mathscr{F}_{\alpha\beta})$$

then $\mathscr{F}$ is Eberlein closed in Y^X. Hence if Y is an Eberlein space, then so is $\mathscr{F}$.

Let (I, f) be a countably compact net on $\mathscr{F}$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges in Y^X to a g. For any $(\alpha,\beta)\in A\times B$ we denote by $f_{\alpha\beta}$ the map

$$\iota\mapsto\psi_\beta\circ f_\iota\circ\varphi_\alpha : I\to\mathscr{F}_{\alpha\beta}\,.$$

Then $(I, f_{\alpha\beta})$ is a countably compact net on $\mathscr{F}_{\alpha\beta}$ and $f_{\alpha\beta}(\mathfrak{U})$ converges in $Y_\beta^{X_\alpha}$ to $\psi_\beta\circ g\circ\varphi_\alpha$. Since $\mathscr{F}_{\alpha\beta}$ is Eberlein closed in $Y_\beta^{X_\alpha}$ it follows that $\varphi_\beta\circ g\circ\varphi_\alpha\in\mathscr{F}_{\alpha\beta}$. By the hypothesis $g\in F$.

The last assertion follows from Theorem 2.8. and Proposition 2.1. †

COROLLARY 5.4. *Let X, Y be Hausdorff topological spaces. If $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X then $\mathscr{C}(X, Z)$ is Eberlein closed in Z^X for any subspace Z of Y. In particular if Z is an Eberlein space then $\mathscr{C}(X, Z)$ is an Eberlein space.* †

COROLLARY 5.5. *If X is a Hausdorff topological space such that $\mathscr{C}(X, \mathbf{R})$ is an Eberlein space, then $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X (resp. is an Ebelerin space) for any completely regular space (resp. for any completely regular Eberlein space) Y.*

It is sufficient to take in the theorem

$$A:=\{0\},\ B:=\mathscr{C}(Y,\mathbf{R}),\ X_0:=X,\ \varphi_0:=\text{identity map},$$

and for any $\beta\in B$

$$Y_\beta:=\mathbf{R},\ \psi_\beta:=(y\mapsto\beta(y):Y\to Y_\beta),\ \mathscr{F}_{0,\beta}:=\mathscr{C}(X_0, Y_\beta).\quad †$$

THEOREM 5.6. *Let $(X_n)_{n\in\mathbf{N}}$, $(Y_{n\in\mathbf{N}})$ be two sequences of Hausdorff topological spaces such that for any $(m,n)\in\mathbf{N}^2$, $\mathscr{C}(X_m, Y_n)$ is a strict Šmulian space. Let X, Y be two Hausdorff topological spaces and $(\varphi_n)_{n\in\mathbf{N}}$, $(\psi_n)_{n\in\mathbf{N}}$ be two sequences such that:*

a) *for any $n\in\mathbf{N}$ we have $\varphi_n\in\mathscr{C}(X_n, X)$, $\psi_n\in\mathscr{C}(Y, Y_n)$;*

b) *$X=\bigcup_{n\in\mathbf{N}}\varphi_n(X_n)$;*

c) *any two points y', y'' of Y coincide if $\psi_n(y')=\psi_n(y'')$ for any $n\in\mathbf{N}$.*

Then $\mathscr{C}(X, Y)$ is a strict Šmulian space.

We denote by Φ the map of $\mathscr{C}(X, Y)$ into $\prod_{(m,n)\in\mathbf{N}^2}\mathscr{C}(X_m, Y_n)$ defined by

$$[\Phi(f)]\,(m,n):=\psi_n\circ f\circ\varphi_m\qquad((m,n)\in\mathbf{N}^2,\ f\in\mathscr{C}(X,Y)).$$

It is obvious that Φ is a continuous injection. The theorem now follows from Proposition 3.8 and Corollary 3.4. †

COROLLARY 5.7. *If X is a Hausdorff topological space such that $\mathscr{C}(X, \mathbf{R})$ is a strict Šmulian space, then $\mathscr{C}(X, Y)$ is a strict Šmulian space for any topological space Y which may be injected continuously in $\mathbf{R}^{\mathbf{N}}$ (this is equivalent that Y possesses a coarser metrizable topology which has a countable base).*

Let for any $n \in \mathbf{N}$, π_n be the n-th projection of $\mathbf{R}^{\mathbf{N}}$ and let ψ be a continuous injection of Y into $\mathbf{R}^{\mathbf{N}}$. We may take in the theorem

$$X_n := X,\ Y_n := \mathbf{R},\ \varphi_n := \text{identity map},\ \psi_n := \pi_n \circ \psi,$$

for any $n \in \mathbf{N}$. †

PROPOSITION 5.8. *Let X, Y be Hausdorff topological spaces such that $\mathscr{C}(X, Y)$ is a Šmulian space. Then for any quotient space X' of X and for any subspace Y' of Y, $\mathscr{C}(X', Y')$ is a Šmulian space.*

Let (I, f') be a countably compact net on $\mathscr{C}(X', Y')$, $\mathfrak{F}$ be a filter on I, finer than the section filter of I, g be an adherent point of $f'(\mathfrak{F})$ and $(I_n)_{n \in \mathbf{N}}$ be a sequence in $\mathfrak{F}$. If φ (resp. ψ) denotes the canonical map of X into X' (resp. of Y' into Y) and if f denotes the map

$$\iota \mapsto \psi \circ f'_\iota \circ \varphi : I \to \mathscr{C}(X, Y),$$

then (I, f) is a countably compact net on $\mathscr{C}(X, Y)$ such that $\psi \circ g \circ \varphi$ is an adherent point of $f(\mathfrak{F})$. Since $\mathscr{C}(X, Y)$ is a Šmulian space there exists an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that $(f(\iota_n))_{n \in \mathbf{N}}$ converges to $\psi \circ g \circ \varphi$ and such that $\iota_n \in I_n$ for any $n \in \mathbf{N}$. But then $(f'(\iota_n))_{n \in \mathbf{N}}$ converges to g. Hence $\mathscr{C}(X', Y')$ is a Šmulian space. †

THEOREM 5.9. *Let X be a Hausdorff topological space, (I, f) be a countably compact net on X and Y be a regular space. Let $\mathscr{G}(I, f)$ be the set of those $g \in Y^X$ which have the following property: if I' is an upper directed subset of I and $\mathfrak{U}$ is an ultrafilter on I', finer than the section filter of I' and such that $f(\mathfrak{U})$ converges to a point $x \in X$, then $g(f(\mathfrak{U}))$ converges to $g(x)$. Then $\mathscr{G}(I, f)$ is Eberlein closed in Y^X.*

Let (J, g) be a countably compact net on $\mathscr{G}(I, f)$ and $\mathfrak{B}$ be an ultrafilter on J, finer than the section filter of J and such that $g(\mathfrak{B})$ converges in Y^X to a map h. We have to show that $h \in \mathscr{G}(I, f)$.

Assume the contrary. Then there exists an upper directed subset I' of I and an ultrafilter $\mathfrak{U}$ on I', finer than the section filter of I' and such that $f(\mathfrak{U})$ converges to a point x, with the property that $h(f(\mathfrak{U}))$ does not converge to $h(x)$. There exists therefore a closed neighbourhood V of $h(x)$, which does not belong to $h(f(\mathfrak{U}))$.

Since $\mathfrak{U}$ is an ultrafilter it follows that $f^{-1}(h^{-1}(Y\backslash V))\in\mathfrak{U}$. Let W be an open neighbourhood fo $h(x)$, whose closure lies in the interior of V.

We shall construct inductively an increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I', an increasing sequence $(\kappa_n)_{n\in\mathbf{N}}$ in J and a decreasing sequence $(I_n)_{n\in\mathbf{N}}$ in $\mathfrak{U}$ such that for any $n\in N$ we have:

a) $\iota_n\in I_{n-1}\cap f^{-1}(h^{-1}(Y\backslash V))$;
b) $g_{\kappa_n}(f(I_n))\subset W$;
c) $m\leqslant n\Rightarrow g_{\kappa_n}(f(\iota_m))\in Y\backslash V$.

Assume that the sequences were constructed up to $n-1$. Since I_{n-1}, $f^{-1}(h^{-1}(Y\backslash V))$, and $\{\iota\in I' \mid \iota\geqslant\iota_{n-1}\}$ belong to $\mathfrak{U}$, their intersection is non-empty. We take an arbitrary ι_n in this intersection, so that condition a) is fulfilled. The sets $Y\backslash V$ and W are open and

$$h(x)\in W, \quad m\leqslant n\Rightarrow h(f(\iota_m))\in Y\backslash V.$$

Since $g(\mathfrak{V})$ converges to h, we get

$$\{\kappa\in J \mid g_\kappa(x)\in W\}\cap\{\kappa\in J \mid m\leqslant n\Rightarrow g_\kappa(f(\iota_m))\in Y\backslash V\}\in\mathfrak{V}.$$

The set $\{\kappa\in J \mid \kappa\geqslant\kappa_{n-1}\}$ also belongs to $\mathfrak{V}$ and therefore there exists a $\kappa_n\in J$, $\kappa_n\geqslant\kappa_{n-1}$, satisfying the above condition c) and such that $g_{\kappa_n}(x)\in W$. Since $g_{\kappa_n}\in\mathscr{G}(I,f)$, $g_{\kappa_n}(f(\mathfrak{U}))$ converges to $g_{\kappa_n}(x)$. Hence $f^{-1}(g_{\kappa_n}^{-1}(W))\in\mathfrak{U}$. We set

$$I_n := I_{n-1}\cap f^{-1}(g_{\kappa_n}^{-1}(W)).$$

I_n satisfies the above condition b).

The net (J, g) being countably compact in $\mathscr{G}(I,f)$, there exists a $h_0\in\mathscr{G}(I,f)$ which is adherent to the sequence $(g_{\kappa_n})_{n\in\mathbf{N}}$. The net (I,f) being countably compact in X, there exists an $x_0\in X$ which is adherent to the sequence $(f(\iota_n))_{n\in\mathbf{N}}$. Let us denote

$$I'_0 := \{\iota_n \mid n\in\mathbf{N}\}.$$

Then I' is an upper directed set of I and there exists and ultrafilter $\mathfrak{U}_0$ on I', finer than the section filter of I' and such that $f(\mathfrak{U}_0)$ converges to x_0. Since $h_0\in\mathscr{G}(I,f)$ and $g_{\kappa_n}\in\mathscr{G}(I,f)$ for any $n\in\mathbf{N}$ it follows that $h_0(f(\mathfrak{U}_0))$ converges to $h_0(x_0)$ and $g_{\kappa_n}(f(\mathfrak{U}_0))$ converges to $g_{\kappa_n}(x_0)$ for any $n\in N$. We deduce (by a), b)) $g_{\kappa_n}(x_0)\in\bar{W}$ for any $n\in\mathbf{N}$ and therefore $h_0(x_0)\in\bar{W}$. By c) we get $h_0(f(\iota_m))\in\overline{Y\backslash V}$ for any $m\in\mathbf{N}$ and therefore $h_0(x_0)\in\overline{Y\backslash V}$ and this is a contradiction since

$$\bar{W}\cap\overline{Y\backslash V}=\emptyset. \quad \dagger$$

COROLLARY 5.10. *If X is a Hausdorff topological space and Y is a regular space, then $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X. Hence if Y is an Eberlein space, then $\mathscr{C}(X, Y)$ is an Eberlein space.*

The first assertion follows immediately from the theorem and from the relation $\mathscr{E}(X, Y)=\bigcap\{\mathscr{G}(I, f)\,|(I, f)$ is a countably compact net on $X\}$.

The last assertion follows from the first one, Theorem 2.8 and Proposition 2.1. †

COROLLARY 5.11. *For any c-space X and for any regular space Y, $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X. Hence if Y is a regular Eberlein space, $\mathscr{C}(X, Y)$ is an Eberlein space.* †

COROLLARY 5.12. *Let X be a c-space, Y be a regular space and $\leqslant$ be an order relation on Y such that any upper directed set A of Y has a supremum which is adherent to A. If $\mathscr{F}$ is an upper directed set of $\mathscr{C}(X, Y)$ such that the supremum of any increasing sequence in $\mathscr{F}$ is continuous, then the supremum of $\mathscr{F}$ is continuous.*

If we denote by f the inclusion map of $\mathscr{F}$ into $\mathscr{C}(X, Y)$ the $(\mathscr{F}, f)$ is a countably compact net on $\mathscr{C}(X, Y)$ such that the supremum of $\mathscr{F}$ is adherent to $f(\mathfrak{F})$, where $\mathfrak{F}$ denotes the section filter of $\mathscr{F}$. Since by the preceding corollary $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X, this supremum is continuous. †

This corollary contains Proposition 1.1 of [2].

PROPOSITION 5.13. *Let X, Y be Hausdorff topological spaces, $(K_n)_{n\in\mathbf{N}}$ be an increasing sequence of compact sets of X whose union is X, $(U_n)_{n\in\mathbf{N}}$ be a decreasing sequence of open sets of Y^2 such that*

$$\bigcap_{n\in\mathbf{N}} U_n=\bigcap_{n\in\mathbf{N}} \bar{U}_n=\{(y, y)\,|\,y\in Y\},$$

(I, f) be a countably compact net on Y^X, $\mathfrak{F}$ be a filter on I, finer than the section filter of I and such that $f(\mathfrak{F})$ converges to a $g\in Y^X$ and $(I_n)_{n\in\mathbf{N}}$ be a decreasing sequence in $\mathfrak{F}$. If for any $\iota\in I$ and any $n\in\mathbf{N}$ the restrictions of the functions f_ι and g to K_n are continuous, then there exists an upper directed countable subset J of I such that $\{I_n\cap J\,|\,n\in\mathbf{N}\}$ generates the section filter $\mathfrak{G}$ of J and such that g is adherent to the filter $f(\mathfrak{G})$

We shall construct inductively a sequence $(J_n)_{n\in\mathbf{N}}$ of finite subsets of I such that we have for any $n\in\mathbf{N}$:

a) $J_n\subset I_n$;

b) $\iota\in J_{n-1},\ \kappa\in J_n\Rightarrow\iota\leqslant\kappa$;

c) for any family $(x_m)_{m\leqslant n}$ in K_n there exists $\iota\in J_n$ such that for any $m\leqslant n$ we have $(g(x_m), f_\iota(x_m))\in U_n$.

Assume that the sequence was constructed up to $n-1$ and let $\kappa\in I$ be an upper bound for $\bigcup_{m<n} J_m$. Let $x:=(x_m)_{m\leqslant n}$ be a point of $(K_n)^{n+1}$. Since g is adherent to the set $\{f_\iota\,|\,\iota\in I_n,\ \iota\geqslant\kappa\}$ there exists $\iota_x\in I_n$, $\iota_x\geqslant\kappa$, such that for any $m\leqslant n$

$$(g(x_m), f_{\iota_x}(x_m))\in U_n.$$

The restrictions of g and f_{ι_x} to K_n being continuous, the set

$$U_x := \{x' \in (K_n)^{n+1} \mid m \leqslant n \Rightarrow (g(x'_m), f_{\iota_x}(x'_m)) \in U_n\}$$

is open in $(K_n)^{n+1}$ and contains x. Since $(K_n)^{n+1}$ is compact, there exists a finite subset A of $(K_n)^{n+1}$ such that

$$(K_n)^{n+1} = \bigcup_{x \in A} U_x.$$

We set

$$J_n := \{\iota_x \mid x \in A\}.$$

It is obvious that J_n is a finite subset of I satisfying the above conditions a) and b). In order to show that it also satisfies condition c) let $(x_m)_{m \leqslant n}$ be a family in K_n. Then there exists $x' \in A$ such that $(x_m)_{m \leqslant n} \in U_{x'}$ and therefore for any $m \leqslant n$ we have

$$(g(x_m), f_{\iota_{x'}}(x_m)) \in U_n.$$

We set now

$$J := \bigcup_{n \in \mathbf{N}} J_n.$$

J is obviously an upper directed countable subset of I and $\{I_n \cap J \mid n \in N\}$ generates the section filter of J. In order to prove the last assertion let $\kappa \in J$, $(x_\lambda)_{\lambda \in L}$ be a finite family in X and for any $\lambda \in L$ let V_λ be a neighbourhood of $g(x_\lambda)$. There exists $n_0 \in \mathbf{N}$ such that:

α) $\operatorname{card} L \leqslant n_0 + 1$, β) $\lambda \in L \Rightarrow x_\lambda \in K_{n_0}$, γ) $\kappa \in \bigcup_{m < n_0} J_m$.

There exists for any $n \geqslant n_0$, $\iota_n \in J_n$ such that for any $\lambda \in L$

$$(g(x_\lambda), f_{\iota_n}(x_\lambda)) \in U_n.$$

We want to show that there exists $n \geqslant n_0$ such that $f_{\iota_n}(x_\lambda) \in V_\lambda$ for any $\lambda \in L$. Assume the contrary. Then there exist $\lambda \in L$ and a strictly increasing sequence $(n(k))_{k \in \mathbf{N}}$ in $\mathbf{N}$ such that $f_{\iota_{n(k)}}(x_\lambda) \notin V_\lambda$ for any $k \in \mathbf{N}$. By b) the sequence $(\iota_{n(k)})_{k \in \mathbf{N}}$ is increasing in I. Since the net (I, f) is countably compact there exists $h \in Y^X$ which is adherent to the sequence $(f_{\iota_{n(k)}})_{k \in \mathbf{N}}$. Then $h(x_\lambda)$ is adherent to the sequence $(f_{\iota_{n(k)}}(x_\lambda))_{k \in \mathbf{N}}$. From $f_{\iota_{n(k)}}(x_\lambda) \notin V_\lambda$ we deduce $h(x_\lambda) \neq g(x_\lambda)$. From $(g(x_\lambda), f_{\iota_{n(k)}}(x_\lambda)) \in U_{n(k)}$ for any $k \in \mathbf{N}$ we get

$$(g(x_\lambda), h(x_\lambda)) \in \bigcap_{k \in \mathbf{N}} \overline{U_{n(k)}}$$

and therefore $g(x_\lambda) = h(x_\lambda)$. This is the expected contradiction. Hence g is adherent to $f(\mathfrak{G})$. †

PROPOSITION 5.14. *Let X be a σ-compact Hausdorff topological space and Y be an Eberlein space whose diagonal is the intersection of a countable set of closed neigh-*

bourhoods (i.e. there exists a countable set of closed neighbourhoods of $\{(y, y) \mid y \in Y\}$ *in* Y^2 *whose intersection is* $\{(y, y) \mid y \in Y\}$*). Then* $\mathscr{E}(X, Y)$ *is a Šmulian Eberlein space. The above conditions for* Y *are satisfied by any topological space on which there exists a coarser metrizable topology.*

Since Y is an Eberlein space, $\mathscr{E}(X, Y)$ is an Eberlein space too (Corollary 5.10). Let (I, f) be a countably compact net on $\mathscr{E}(X, Y)$, $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to $g \in \mathscr{E}(X, Y)$, and $(I_n)_{n \in \mathbf{N}}$ be a decreasing sequence in $\mathfrak{U}$. Let $(K_n)_{n \in \mathbf{N}}$ be an increasing sequence of compact sets of X whose union is X, $(U_n)_{n \in \mathbf{N}}$ be a decreasing sequence of open sets of Y^2 such that

$$\bigcap_{n \in \mathbf{N}} U_n = \bigcap_{n \in \mathbf{N}} \bar{U}_n = \{(y, y) \mid y \in Y\}.$$

By Theorem 4.2 for any $\iota \in I$ and any $n \in \mathbf{N}$ the restrictions of f_ι and g to K_n are continuous. Hence by the preceding proposition there exists an upper directed countable subset J of I such that $\{I_n \cap J \mid n \in \mathbf{N}\}$ generates the section filter $\mathfrak{G}$ of J and such that g is adherent to the filter $f(\mathfrak{G})$.

For any $n \in \mathbf{N}$ let φ_n be the map of K_n into Y^J defined by

$$(\varphi_n(x))(\iota) := f_\iota(x) \qquad (x \in K_n, \iota \in J).$$

Since φ_n is continuous, $\varphi_n(K_n)$ is compact and is the quotient space of K_n with respect to the equivalence relation

$$\iota \in J \Rightarrow f_\iota(x) = f_\iota(y) \qquad (x, y \in K_n).$$

The diagonal of Y^J is obviously a G_δ-set in $(Y^J)^2$ and therefore the diagonal of $\varphi_n(K_n)$ is a G_δ-set in $(\varphi_n(K_n))^2$. It follows that $\varphi_n(K_n)$ is metrizable. Hence there exists a countable subset A_n of K_n such that $\varphi_n(A_n)$ is dense in $\varphi_n(K_n)$.

By Corollary 3.5 Y is a strict Šmulian space. Hence $Y^{\bigcup_{n \in \mathbf{N}} A_n}$ is a strict Šmulian space (Proposition 3.8) and therefore a Šmulian space (Proposition 3.1). We deduce that there exists an increasing sequence $(\iota_m)_{m \in \mathbf{N}}$ in J such that $\iota_m \in I_m$ for any $m \in \mathbf{N}$ and such that $(f_{\iota_m}(x))_{m \in \mathbf{N}}$ converges to $g(x)$ for any $x \in \bigcup_{n \in \mathbf{N}} A_n$. Let h be an adherent point in $\mathscr{E}(X, Y)$ of the sequence $(f_{\iota_m})_{m \in \mathbf{N}}$. Since g and h are adherent points in $\mathscr{E}(X, Y)$ of $\{f_\iota \mid \iota \in J\}$, there exists for any $n \in N$ two map $\bar{g}_n$, $\bar{h}_n$ in $\mathscr{C}(\varphi(K_n), Y)$ such that

$$g = \bar{g}_n \circ \varphi_n, \qquad h = \bar{h}_n \circ \varphi_n$$

on K_n. But for any $n \in \mathbf{N}$ and any $x \in A_n$ we have $g(x) = h(x)$ and therefore for any $n \in \mathbf{N}$ the functions $\bar{g}_n$, $\bar{h}_n$ coincide on $\varphi(A_n)$. This set being dense in $\varphi(K_n)$ it follows $\bar{g}_n = \bar{h}_n$ for any $n \in \mathbf{N}$ and therefore $g = h$. g being the only adherent point of the sequence $(f_{\iota_m})_{m \in \mathbf{N}}$ this sequence converges to g (Proposition 1.4.)

Let now Y be a topological space on which there exists a coarser metrizable topology $\mathfrak{T}$. By Corollary 2.4. Y is an Eberlein space. Let d be a metric on Y consistent with $\mathfrak{T}$.

We set for any $n \in \mathbf{N}$

$$V_n := \left\{ (x, y) \in Y^2 \mid d(x, y) \leqslant \frac{1}{n+1} \right\}.$$

Then $\{V_n \mid n \in \mathbf{N}\}$ is a countable set of closed neighbourhoods of the diagonal of Y whose intersection is the diagonal of Y. †

Remark. It was shown (Corollary 4.4) that if X is a Hausdorff topological space and Y a regular space, then the restriction to the closure of any relatively countably compact set of X of any Eberlein continuous map of X into Y is continuous. But the restriction of an Eberlein continuous map to a σ-compact set is not always continuous even if Y is $\mathbf{R}$ and X is σ-compact. Indeed set

$$X := (\mathbf{R} \times \mathbf{N}) \cup \{0\}$$

and let a subset U of X be open if it fulfills the following two conditions:

a) $U \cap (\mathbf{R} \times \mathbf{N})$ is open with respect to the product topology of $\mathbf{R} \times \mathbf{N}$;

b) if $0 \in U$ then there exists $n \in \mathbf{N}$ such that $\mathbf{R} \times \{m \in \mathbf{N} \mid m \geqslant n\} \backslash U$ is countable.

Then X is a Hausdorff space and is σ-compact. The function on X equal to 0 on $\mathbf{R} \times \mathbf{N}$ and equal 1 at 0 is Eberlein continuous and not continuous.

THEOREM 5.15. *Let X be a Hausdorff topological space, Y be a regular space on which there exists a coarser metrizable topology, (I, f) be a countably compact net on $\mathscr{C}(X, Y)$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. Then* $\lim_{\iota, \mathfrak{U}} f_\iota(x)$ *exists for any $x \in X$ and for any σ-compact set A of X and for any sequence $(I_n)_{n \in \mathbf{N}}$ in $\mathfrak{U}$ there exists an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I, such that $(f_{\iota_n}(x))_{n \in \mathbf{N}}$ converges to* $\lim_{\iota, \mathfrak{U}} f_\iota(x)$ *for any $x \in \bar{A}$. In particular the map*

$$x \mapsto \lim_{\iota, \mathfrak{U}} f_\iota(x) : \bar{A} \to Y$$

is continuous and equal to the restriction to $\bar{A}$ of a continuous map of X into Y.

By Corollaries 3.19 and 5.10, $\mathscr{E}(X, Y)$ is an Eberlein space and this shows that $\lim_{\iota, \mathfrak{U}} f_\iota(x)$ exists for any $x \in X$ and that the map g

$$x \mapsto \lim_{\iota, \mathfrak{U}} f_\iota(x) : X \to Y$$

is Eberlein continuous. By the preceding proposition $\mathscr{E}(A, Y)$ is a Šmulian space. Hence there exists an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I, such that $(f_{\iota_n}(x))$ converges to $g(x)$ for any $x \in A$ and such that $\iota_n \in I_n$ for any $n \in \mathbf{N}$. The net (I, f) being countably compact there exists an adherent point h of the sequence $(f_{\iota_n})_{n \in \mathbf{N}}$ in $\mathscr{C}(X, Y)$. We get $g = h$ on A. Hence the restriction of g to any σ-compact set of X is continuous. By Lemma 4.5 the restriction of g to $\bar{A}$ is continuous. It follows that h and g coincide on $\bar{A}$. The map h being arbitrary it follows that for any $x \in \bar{A}$, $g(x)$ is the unique adherent point of the sequence $(f_{\iota_n}(x))_{n\in\mathbf{N}}$. By Proposition 1.4 it follows that $(f_{\iota_n}(x))_{n\in\mathbf{N}}$ converges to $g(x)$ for any $x\in\bar{A}$. †

COROLLARY 5.16. *If a Hausdorff topological space X possesses a dense σ-compact set, the $\mathscr{C}(X, Y)$ is a Šmulian-Eberlein space for any regular space Y on which there exists a coarser metrizable topology.* †

Remark. There exist compact sets X such that $\mathscr{C}(X, \mathbf{R})$ is not a strict Šmulian space. Indeed let α be an ordinal number and $\mathfrak{U}$ be an ultrafilter on α, finer than the section filter of α and such that the intersection of any countable family in $\mathfrak{U}$ is non-empty. We endow $X:=\alpha\cup\{\alpha\}$ with the usual topology, i.e. a subset U of X is open if for any $\xi\in U$ there exists $\eta<\xi$ such that

$$\{\zeta\in X \mid \eta<\zeta\leqslant\xi\}\subset U.$$

X is then a compact space. For any $\xi\in\alpha$ we denote by f_ξ the function on X equal to 1 at $\xi+1$ and equal to 0 elsewhere. Then (α, f) is a countably compact net on $\mathscr{C}(X, \mathbf{R})$ such that $f(\mathfrak{U})$ converges to the identically 0 function. Let $(A_n)_{n\in\mathbf{N}}$ be a sequence in $f(\mathfrak{U})$ and $\xi\in\bigcap_{n\in\mathbf{N}} f^{-1}(A_n)$. If we set

$$\iota_n:=\xi$$

for any $n\in\mathbf{N}$, then $(\iota_n)_{n\in\mathbf{N}}$ is increasing sequence in α such that $f_{\iota_n}\in A_n$ for any $n\in\mathbf{N}$, but the sequence $(f_{\iota_n})_{n\in\mathbf{N}}$ does not converge to the identically 0 function on X. Hence $\mathscr{C}(X, \mathbf{R})$ is not a strict Šmulian space.

COROLLARY 5.17. *Let X be a Hausdorff topological space and $(\mathscr{F}_n)_{n\in\mathbf{N}}$ be a sequence of compact subsets of $\mathscr{C}(X, \mathbf{R})$. If for any $x\in X$, $\{y\in X \mid f\in\bigcup_{n\in\mathbf{N}}\mathscr{F}_n \Rightarrow f(x)= = f(y)\}$ is a Lindelöf subspace of X (resp. is equal to $\{x\}$), then X is an Eberlein space (resp. is a Šmulian-Eberlein space).*

Let φ be the continuous map of X into $\mathscr{C}(\bigcup_{n\in\mathbf{N}}\mathscr{F}_n, \mathbf{R})$ defined by

$$[\varphi(x)](f):=f(x)$$

for any $x\in X$ and any $f\in\bigcup_{n\in\mathbf{N}}\mathscr{F}_n$. By the preceding corollary $\mathscr{C}(\bigcup_{n\in\mathbf{N}}\mathscr{F}_n, \mathbf{R})$ is a Šmulian-Eberlein space. The assertions follow now from Corollary 3.16 and Corollary 3.18. †

COROLLARY 5.18. *For any completely regular space X and for any topological group G for which the one point sets are of type G_δ the space $\mathscr{K}(X, G)$ of continuous maps of X into G with compact carrier enowed with the topology of pointwise convergence is a Šmulian-Eberlein space.*

Let X_0 be a compactification of X. For any $f\in\mathscr{K}(X, G)$ we denote by $\varphi(f)$ the map of X_0 into G equal to f on X and equal to 1 (the neutral element of G) elsewhere. Then φ is an injective map of $\mathscr{K}(X, G)$ into $\mathscr{C}(X_0, G)$ which induces a homeomorphism of $\mathscr{K}(X, G)$ onto $\varphi(\mathscr{K}(X, G))$. By the Corollary 5.16 and Lemma 3.21, $\mathscr{C}(X_0, G)$ is a Šmulian-Eberlein space. Hence by Corollary 3.17 $\varphi(\mathscr{K}(X, G))$ and therefore $\mathscr{K}(X, G)$ is a Šmulian-Eberlein space. †

COROLLARY 5.19. *Let X be a Hausdorff topological space such that any real function g on X is continuous if for any σ-compact set A of X there exists a real continuous function on X which coincides with g on $\bar{A}$. Then $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X (resp. is an Eberlein space) for any completely regular (resp. for any completely regular Eberlein) space Y.*

By Corollary 5.5 it is sufficient to show that $\mathscr{C}(X, \mathbf{R})$ is an Eberlein space and this follows immediately from the theorem. †

COROLLARY 5.20. *Let X be a Hausdorff topological space such that any real continuous function g on X is continuous if for any σ-compact set A of X there exists a real continuous function on X which coincides with g on $\bar{A}$ (resp. let X be a Hausdorff topological space which possesses a dense σ-compact set). Let Y be a completely regular space (resp. let Y be a regular space which possesses a coarser metrizable topology) and let $\leqslant$ be an order relation on Y such that any upper directed set B of Y has a supremum, which is adherent to B. If $\mathscr{F}$ is an upper directed set of $\mathscr{C}(X, Y)$ such that the supremum of any increasing sequence in $\mathscr{F}$ is continuous, then the supremum of $\mathscr{F}$ is continuous (resp. and there exists an increasing sequence in $\mathscr{F}$ converging to this supremum).*

By the preceding corollary $\mathscr{C}(X, Y)$ is Eberlein closed in Y^X (resp. by Corollary 5.16 $\mathscr{C}(X, Y)$ is a Šmulian-Eberlein space). If f denotes the inclusion map of $\mathscr{F}$ into $\mathscr{C}(X, Y)$ and if $\mathfrak{F}$ denotes the section filter of $\mathscr{F}$, then $(\mathscr{F}, f)$ is a countably compact net on $\mathscr{C}(X, Y)$ and $f(\mathfrak{F})$ converges in Y^X to the supremum of $\mathscr{F}$. Hence this supremum is continuous (resp. and there exists an increasing sequence in $\mathscr{F}$ converging to this supremum). †

THEOREM 5.21. *Let X be a Hausdorff topological space on which there exists a bounded measure (i.e. mesure borneé in Bourbalei, Intégration, Ch. IX) whose carrier coincides with X. Then for any regular space Y on which there exists a coarser metrizable topology with a countable base, $\mathscr{C}(X, Y)$ is a strict Šmulian-Eberlein space.*

The existence of a bounded measure μ on X whose carrier coincides with X implies that X possesses a dense σ-compact set and so, by Corollary 5.16, $\mathscr{C}(X, Y)$ is a Šmulian-Eberlein space. In order to show that $\mathscr{C}(X, Y)$ is a strict Šmulian space we only have to show that $\mathscr{C}(X, \mathbf{R})$ is a strict Šmulian space (Corollary 5.7). We denote by d the map

$$(f, g) \mapsto \int \frac{|f-g|}{1+|f-g|}\, d|\mu| : \mathscr{C}(X, \mathbf{R})^2 \to \mathbf{R}_+ .$$

It is easy to see that d is a metric on $\mathscr{C}(X, \mathbf{R})$. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence on $\mathscr{C}(X, \mathbf{R})$ which converges to an $f\in\mathscr{C}(X, \mathbf{R})$. By Lebesgue theorem

$$\lim_{n\to\infty} d(f, f_n) = 0.$$

The assertion now follows from Proposition 3.14. †

We shall say that a filter $\mathfrak{F}$ is *δ-stable* if the intersection of any countable family in $\mathfrak{F}$ belongs to $\mathfrak{F}$.

PROPOSITION 5.22. *Let X be a set, $\mathfrak{S}'$ and $\mathfrak{S}''$ be two coverings of X, Y be a separated uniform space, $\mathscr{F}$ be a subset of Y^X and $f\in Y^X$. Consider the following two assertions:*

a) *for any sequence $(A'_n)_{n\in\mathbf{N}}$ in $\mathfrak{S}'$ there exists $g\in\mathscr{F}$ such that $f=g$ on $\bigcup_{n\in\mathbf{N}} A'_n$;*

b) *there exists a δ-stable filter on $\mathscr{F}$ converging to f in $Y^X_{\mathfrak{S}''}$.*

If any set of $\mathfrak{S}''$ is contained in the union of a sequence in $\mathfrak{S}'$, then a ⇒ b. *If any set of $\mathfrak{S}'$ is contained in the union of a sequence in $\mathfrak{S}''$ and if there exists a coarser metrizable uniform structure on Y, then* b ⇒ a.

a ⇒ b. Let $\mathfrak{F}$ be the set of subsets $\mathscr{G}$ of $\mathscr{F}$ with the following property: there exists a sequence $(A'_n)_{n\in\mathbf{N}}$ in $\mathfrak{S}'$ such that

$$\{g\in\mathscr{F} \mid g = f \quad \text{on} \quad \bigcup_{n\in\mathbf{N}} A'_n\} \subset \mathscr{G}.$$

Then $\mathfrak{F}$ is a δ-stable filter on $\mathscr{F}$ converging to f in $Y^X_{\mathfrak{S}''}$.

b ⇒ a. Let d be a metric on Y whose associated uniform structure is coarser than the initial uniform structure of Y and let $\mathfrak{F}$ be a δ-stable filter on $\mathscr{F}$ converging to f in $Y^X_{\mathfrak{S}}$. Let $(A'_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{S}'$ and for any $n\in\mathbf{N}$ let $(A''_{n,m})_{m\in\mathbf{N}}$ be a sequence in $\mathfrak{S}''$ such that

$$A'_n \subset \bigcup_{m\in\mathbf{N}} A''_{n,m}.$$

For any $p\in\mathbf{N}$

$$\mathscr{F}_p := \left\{f'\in\mathscr{F} \mid x\in \bigcup_{\substack{n\leqslant p\\ m\leqslant p}} A''_{n,m} \Rightarrow d(f(x), f'(x)) < \frac{1}{p+1}\right\}$$

belongs to $\mathfrak{F}$. Let g be an element of $\bigcap_{p\in\mathbf{N}} \mathscr{F}_p$. It is obvious that $g=f$ on $\bigcup_{n\in\mathbf{N}} A'_n$. †

THEOREM 5.23. *Let X be a Hausdorff topological space and $\mathfrak{S}$ be a covering of X such that any set of $\mathfrak{S}$ is contained in the closure of a σ-compact set. Let Y be a separated uniform space and $\mathscr{F}\subset\mathscr{C}(X, Y)$. If any δ-stable Cauchy filter on $\mathscr{F}_{\mathfrak{S}}$ is convergent then $\mathscr{F}$ is Eberlein closed in Y^X. Hence if Y is an Eberlein space then so is $\mathscr{F}$.*

Let (I, f) be a countably compact net on $\mathscr{F}$ and $\mathfrak{U}$ be an ultrafilter on I finer than the section filter of I and such that $f(\mathfrak{U})$ converges in Y^X to g. Let $\mathscr{H}$ be the set of uniformly continuous maps of Y into metrizable uniform spaces and for any $h\in\mathscr{H}$ let us denote

$$\mathscr{F}(h):=\{h\circ\varphi \mid \varphi\in\mathscr{F}\},\qquad h^*:=(\iota\mapsto h\circ f_\iota : I\to\mathscr{C}(X, Z_h)),$$

where Z_h denotes the range of h. Then (I, h^*) is a countably compact net on $\mathscr{F}(h)$ and $h^*(\mathfrak{U})$ converges to $h\circ g$.

By Theorem 5.15 there exists for any σ-compact set A of X and any $h\in\mathscr{H}$ a function of $\mathscr{F}(h)$ equal to $h\circ g$ on $\bar{A}$. Let us denote by $\mathfrak{S}'$ the set of the countable unions of sets of $\mathfrak{S}$ and for any $B\in\mathfrak{S}'$ and any $h\in\mathscr{H}$

$$\mathscr{F}(h, B):=\{\varphi\in\mathscr{F} \mid h\circ\varphi=h\circ g \quad \text{on} \quad B\}.$$

Since any set of $\mathfrak{S}'$ is contained in the closure of a σ-compact set of X it follows from the above remark that $\mathscr{F}(h, B)$ is not empty. It is easy to see that $\{\mathscr{F}(h, B) \mid h\in\mathscr{H}, B\in\mathfrak{S}'\}$ generates a δ-stable Cauchy filter on $\mathscr{F}_{\mathfrak{S}}$ converging to g. We deduce $g\in\mathscr{F}$.

The last assertion follows from Theorem 2.8 and Proposition 2.1. †

PROPOSITION 5.24. *Let X be a Hausdorff topological space, $\mathfrak{S}$ be a covering of X such that there exists a sequence in $\mathfrak{S}$ whose union is dense in X and Y be a uniform space which possesses a coarser metrizable uniform structure. Then $\mathscr{C}_{\mathfrak{S}}(X, Y)$ is a strict Šmulian-Eberlein space.*

It follows immediately from the hypotheses that $\mathscr{C}_{\mathfrak{S}}(X, Y)$ possesses a coarser metrizable topology and the assertion follows from Corollary 3.19. †

PROPOSITION 5.25. *Let X be a Hausdorff topological space which contains a countable dense set and Y be a strict Šmulian space. Then $\mathscr{C}(X, Y)$ is a strict Šmulian space.*

Let A be a countable dense set of X and φ be the map of $\mathscr{C}(X, Y)$ into Y^A defined by

$$(\varphi(f))(x):=f(x) \qquad (f\in\mathscr{C}(X, Y), x\in A).$$

It is obvious that φ is a continuous injection of $\mathscr{C}(X, Y)$ into Y^A. Since Y^A is a strict Šmulian space (Proposition 3.8) it follows that $\mathscr{C}(X, Y)$ is also a strict Šmulian space (Corollary 3.4.) †

THEOREM 5.26. *Let X be a set, $(X_\alpha)_{\alpha\in A}$ be a family of sets, Y be a Hausdorff topological space, $(Y_\beta)_{\beta\in B}$ be a family of Hausdorff topological spaces, $\mathscr{F}$, $\mathscr{G}$ be subsets of Y^X such that $\mathscr{F}\subset\mathscr{G}$, and $(\varphi_\alpha)_{\alpha\in A}$, $(\psi_\beta)_{\beta\in B}$ be families such that for any $\alpha\in A$ and any $\beta\in B$, $\varphi_\alpha\in X^{X_\alpha}$, $\psi_\beta\in\mathscr{C}(Y, Y_\beta)$. We assume:*

a) *for any $(\alpha, \beta)\in A\times B$, $\{\psi_\beta\circ g\circ\varphi_\alpha \mid g\in\mathscr{G}\}$ is a Šmulian space;*

b) *any $g\in\mathscr{G}$ belongs to $\mathscr{F}$ if for any $(\alpha, \beta)\in A\times B$ there exists $f\in\mathscr{F}$ such that $\psi_\beta\circ f\circ\varphi_\alpha=\psi_\beta\circ g\circ\varphi_\alpha$.*

Then $\mathcal{F}$ is Eberlein closed in $\mathcal{G}$. Hence if $\mathcal{G}$ is an Eberlein space, then so is $\mathcal{F}$.

For any $(\alpha, \beta)\in A\times B$ the map

$$g\mapsto \psi_\beta\circ g\circ\varphi_\alpha : \mathcal{G}\to\{\psi_\beta\circ g\circ\varphi_\alpha \mid g\in\mathcal{G}\}.$$

is Eberlein continuous. The theorem follows from Proposition 4.11. †

COROLLARY 5.27. *Let X, Y be Hausdorff topological spaces, $(Y_\lambda)_{\lambda\in L}$ be a family of metrizable topological spaces, $(\varphi_\lambda)_{\lambda\in L}$ be a family such that for any $\lambda\in L$, $\varphi_\lambda\in\mathscr{C}(Y, Y_\lambda)$ and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}\subset\mathcal{G}$, be subsets of $\mathscr{C}(X, Y)$ such that any $g\in\mathcal{G}$ belongs to $\mathcal{F}$ if for any σ-compact set A of X and for any $\lambda\in L$ there exists $f\in\mathcal{F}$ such that $\varphi_\lambda\circ f = \varphi_\lambda\circ g$ on A. Then $\mathcal{F}$ is Eberlein closed in $\mathcal{G}$. Hence if $\mathcal{G}$ is an Eberlein space, then so is $\mathcal{F}$.*

By Proposition 5.14. for any σ-compact set A of X and for any $\lambda\in L$, $\mathscr{C}(A, Y_\lambda)$ is a Šmulian space. By Corollary 3.6. $\{\varphi_\lambda\circ g\circ i_A \mid g\in\mathcal{G}\}$ is a Šmulian space, where i_A denotes the inclusion map $A\to X$. The assertions follow now immediately from the theorem. †

COROLLARY 5.28. *Let X, Y be Hausdorff topological spaces such that on Y there exists a coarser completely regular topology and $\mathcal{F}$, $\mathcal{G}$ be subsets of $\mathscr{C}(X, Y)$ such that $\mathcal{F}\subset\mathcal{G}$ and any $g\in\mathcal{G}$ belongs to $\mathcal{F}$ if for any σ-compact set A of X there exists an $f\in\mathcal{F}$ equal to g on A. Then $\mathcal{F}$ is Eberlein closed in $\mathcal{G}$. Hence if $\mathcal{G}$ is an Eberlein space, then so is $\mathcal{F}$.*

Since Y possesses a coarser completely regular topology, it may be injected continuously into a product of metrizable spaces. Hence we may apply the preceding corollary. †

COROLLARY 5.29. *Let X be a set, Y be a Hausdorff topological space, $(Y_\lambda)_{\lambda\in L}$ be a family of strict Šmulian spaces, $(\varphi_\lambda)_{\lambda\in L}$ be a family such that for any $\lambda\in L$, $\varphi_\lambda\in\mathscr{C}(Y, Y_\lambda)$ and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}\subset\mathcal{G}$, be subsets of X such that any $g\in\mathcal{G}$ belongs to $\mathcal{F}$ if for any countable subset A of X and for any $\lambda\in L$ there exists $f\in\mathcal{F}$ such that $\varphi_\lambda\circ f=\varphi_\lambda\circ g$ on A. Then $\mathcal{F}$ is Eberlein closed in $\mathcal{G}$. Hence if $\mathcal{G}$ is an Eberlein space, then so is $\mathcal{F}$.*

By Theorem 3.8. and Proposition 3.1. for any countable subset A of X and for any $\lambda\in L$, Y_λ^A is a Šmulian space. By Corollary 3.6. $\{\varphi_\lambda\circ g\circ i_A \mid g\in\mathcal{G}\}$ is a Šmulian space, where i_A denotes the inclusion map $A\to X$. The assertions follow now from the theorem. †

PROPOSITION 5.30. *Let X be a set, Y be a Hausdorff topological space, Z be a Šmulian space, φ be a continuous map of Y into Z such that for any $z\in Z$, $\varphi^{-1}(z)$ is a Lindelöf space (with respect to the induced topology) and $\mathcal{F}$ be a subset of Y^X such that any $h\in Y^X$ belongs to $\mathcal{F}$ if there exists $g\in\mathcal{F}$ such that $\varphi\circ g=\varphi\circ h$. If $\{\varphi\circ g \mid g\in\mathcal{F}\}$ is an Eberlein space, then $\mathcal{F}$ is an Eberlein space.*

Let (I, f) be a countably compact net on $\mathscr{F}$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I. Let us denote by $\varphi \circ f$ the map

$$\iota \mapsto \varphi \circ f_\iota : I \to Z^X .$$

Then $(I, \varphi \circ f)$ is a countably compact net on $\{\varphi \circ g \mid g \in \mathscr{F}\}$ and therefore there exists $g \in \mathscr{F}$ such that $\varphi \circ f(\mathfrak{U})$ converges to $\varphi \circ g$.

Let $x \in X$ and let g_x be the map,

$$\iota \mapsto f_\iota(x) : I \to Y .$$

Then (I, g_x) is a countably compact net on Y and $\varphi \circ g_x(\mathfrak{U})$ converges to $\varphi(g(x))$. By Proposition 3.15, $g_x(\mathfrak{U})$ is convergent.

Let us denote by h the map

$$x \mapsto \lim g_x(\mathfrak{U}) : X \to Y .$$

Then $\varphi(h(x)) = \varphi(g(x))$ for any $x \in X$ and therefore $\varphi \circ h = \varphi \circ g$. We deduce $h \in \mathscr{F}$. Since it is clear that $f(\mathfrak{U})$ converges to h it follows that $\mathscr{F}$ is an Eberlein space. †

COROLLARY 5.31. *Let X be a set, Y be a Hausdorff topological space and Z be a Šmulian space obtained by endowing Y with a coarser topology and $\mathscr{F}$ be an Eberlein subspace of Z^X. Then $\mathscr{F}$ is an Eberlein subspace of Y^X.*

It is sufficient to take φ the identical map of Y into Z and to apply the proposition. †

PROPOSITION 5.32. *Let X be a set, $\mathfrak{A}$ be a set of subsets of X, Y be a Hausdorff topological space and $\mathscr{F}$ be the set of those $f \in Y^X$ for which $f(A)$ is relatively countably compact for any $A \in \mathfrak{A}$. If Y is a locally compact, paracompact space or if Y is a strict Šmulian space then $\mathscr{F}$ is Eberlein closed in Y^X.*

Let (I, f) be a countably compact net on $\mathscr{F}$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges in Y^X to a map g. We want to show that $g \in \mathscr{F}$.

a) The case when Y is locally compact and paracompact. Then there exists a family $(Y_\lambda)_{\lambda \in \Lambda}$ of pairwise disjoint open, σ-compact sets of such Y that

$$Y = \bigcup_{\lambda \in \Lambda} Y_\lambda .$$

Assume first that there exist $A \in \mathfrak{A}$, a sequence $(x_n)_{n \in \mathbf{N}}$ in A and a sequence $(\lambda_n)_{n \in \mathbf{N}}$ in Λ such that $g(x_n) \in Y_{\lambda_n}$ for any $n \in N$ and such that $\lambda_m \neq \lambda_n$ for any $m \neq n$. Then we may construct inductively an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that for any m, $n \in \mathbf{N}$, $m \leqslant n$, we have $f_{\iota_n}(x_m) \in Y_{\lambda_m}$. Let h be an adherent point in $\mathscr{F}$ of the sequence $(f_{\iota_n})_{n \in \mathbf{N}}$. Then

$$h(x_m) \in Y_{\lambda_m}$$

for any $m \in \mathbf{N}$ and this contradicts the fact that $h(A)$ is relatively countably compact.

Assume now that there exist $A \in \mathfrak{A}$, a sequence $(x_n)_{n \in \mathbf{N}}$ in A and a $\lambda \in \Lambda$ such that $g(x_n) \in Y_\lambda$ for any $n \in N$ and such that the adherence of the sequence $(g(x_n))_{n \in \mathbf{N}}$ is empty. Let $(U_m)_{m \in \mathbf{N}}$ be a sequence of open relatively compact sets of Y_λ such that for any $n \in \mathbf{N}$: a) $\bar{U}_n \subset U_{n+1}$, b) $Y_\lambda = \bigcup_{m \in \mathbf{N}} U_m$, c) $g(x_n) \notin \bar{U}_n$. Then we may construct inductively an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that for any $m, n \in \mathbf{N}, m \leqslant n$, we have $f_{\iota_n}(x_m) \notin \bar{U}_m$. Let h be an adherent point in $\mathscr{F}$ of the sequence $(f_{\iota_n})_{n \in \mathbf{N}}$. Then $h(x_m) \notin U_m$ for any $m \in \mathbf{N}$ and this contradicts the fact that $h(A)$ is relatively countably compact.

We deduce by the above results that $g(A)$ is relatively countably compact for any $A \in \mathfrak{A}$ and therefore $g \in \mathscr{F}$. Hence $\mathscr{F}$ is Eberlein closed in Y^X.

b) The case when Y is a strict Šmulian space. Assume that $g \notin \mathscr{F}$. Then there exists $A \in \mathfrak{A}$ such that $g(A)$ is not relatively countably compact. Hence there exists a sequence $(x_n)_{n \in \mathbf{N}}$ in A such that the sequence $(g(x_n))_{n \in \mathbf{N}}$ has no adherent point in Y. Let B the set $\{x_n \mid n \in \mathbf{N}\}$. Since Y^B is a strict Šmulian space (Proposition 3.8) there exists an increasing sequence $(\iota_m)_{m \in \mathbf{N}}$ in I such that $(f_{\iota_m}(x_n))_{m \in \mathbf{N}}$ converges to $g(x_n)$ for any $n \in \mathbf{N}$. If h denotes an adherent function in $\mathscr{F}$ of the sequence $(f_{\iota_m})_{m \in \mathbf{N}}$ then $h(x_n) = g(x_n)$ for any $n \in \mathbf{N}$. But then $h(A)$ is not relatively countably compact and this is a contradiction. Hence $g \in \mathscr{F}$ and $\mathscr{F}$ is Eberlein closed in Y^X. †

COROLLARY 5.33. *Let X be a Hausdorff topological space such that any real function g on X is continuous if for any σ-compact set A of X there exists a real continuous function on X which coincides with g on $\bar{A}$. Let further $\mathfrak{A}$ be a set of subsets of X, Y be a Hausdorff topological space and $\mathscr{C}^{\mathfrak{A}}(X, Y)$ be the set of those $f \in \mathscr{C}(X, Y)$ for which $f(A)$ is relatively compact for any $A \in \mathfrak{A}$. If Y is a locally compact, paracompact space or if Y is a completely regular, strict Šmulian-Eberlein space then $\mathscr{C}^{\mathfrak{A}}(X, Y)$ is an Eberlein space.*

Since Y is an Eberlein space (Corollary 2.4), $\mathscr{C}(X, Y)$ is an Eberlein space (Corollary 5.19). By the proposition $\mathscr{C}^{\mathfrak{A}}(X, Y)$ is Eberlein closed in $\mathscr{C}(X, Y)$ and therefore an Eberlein space (Proposition 2.1). †

VI. Locally Convex Vector Spaces

A locally convex vector space (over $\mathbf{R}$ or $\mathbf{C}$) will always be Hausdorff. If E, F are locally convex vector spaces then $\mathscr{L}(E, F)$ denotes the set of continuous linear maps of E into F. E' denotes the dual (i.e. $\mathscr{L}(E, \mathbf{R})$ or $\mathscr{L}(E, \mathbf{C})$) of E. If $\langle E, F \rangle$ is a duality separated in E, then E may be identified with a subset of $\mathbf{R}^F$ or $\mathbf{C}^F$; the induced topology is called the *weak topology on E* (associated to the duality $\langle E, F \rangle$) and is denoted by $\sigma(E, F)$. If E is a locally convex vector space, then by *weak topology on E* we understand the weak topology with respect to the duality $\langle E, E' \rangle$.

PROPOSITION 6.0. *Let E, F be locally convex vector spaces such that E possesses a countable dense set and F is a strict Šmulian space. Then $\mathscr{L}(E,F)$ is a strict Šmulian space for any topology finer than the topology of pointwise convergence.*

By Proposition 5.25. $\mathscr{C}(E, F)$ is a strict Šmulian space. The assertion follows now from Corollary 3.4. †

PROPOSITION 6.1. *Let E be a locally convex vector space such that for any Banach space G, $\mathscr{L}(E, G)$ is an Eberlein space. Then $\mathscr{L}(E, F)$ is an Eberlein space for any locally convex vector space F which is an Eberlein space.*

Let $\mathscr{P}$ be the set of all continuous semi-norms on F and for any $p \in \mathscr{P}$ let F_p be the Banach space obtained by completion of the quotient space of F over $p^{-1}(0)$ and u_p be the canonical map $F \to F_p$. We apply Theorem 5.3. (and Proposition 2.1.) by taking $A := \{0\}$, $B := \mathscr{P}$, $X_0 := E$, $Y_p := F_p$ for any $p \in \mathscr{P}$, $X := E$, $Y := F$, $\varphi_0 :=$ the identical map of E, $\psi_p := u_p$ for any $p \in \mathscr{P}$, $\mathscr{F} := \mathscr{L}(E, F)$, $\mathscr{F}_{0p} := \mathscr{L}(E, F_p)$. †

THEOREM 6.2. *Let E, F, be locally convex vector spaces. If there exists a σ-compact (resp. countable) dense set in E and if the one point sets of F are of type G_δ, then $\mathscr{L}(E, F)$ is a Šmulian-Eberlein (resp. strict Šmulian-Eberlein) space for any topology finer than the typology of pointwise convergence. If there exists a σ-compact dense set in E and if F is an Eberlein space, then $\mathscr{L}(E, F)$ is an Eberlein space.*

Assume first that E possesses a σ-compact dense set and that the one point sets of F are of type G_δ. By Corollary 5.16. and Lemma 3.21. $\mathscr{C}(E, F)$ is a Šmulian-Eberlein space and the assertion follows from Corollaries 3.17. and 3.18.

Assume now that E possesses a countable dense set and that the one point sets of F are of type G_δ. Since any countable set is σ-compact it follows from the above proof that $\mathscr{L}(E, F)$ is an Eberlein space and so we have to prove only that it is a strict Šmulian space and this follows from the Proposition 6.0. and Corollary 3.5.

The last assertion follows from the first one and from the preceding proposition. †

COROLLARY 6.3. *If E is a locally convex vector space such that the one point sets are of type G_δ, then E endowed with any topology finer than the weak topology is a Šmulian-Eberlein space. If moreover E possesses a countable dense set, then E endowed with any topology finer than the weak topology is a strict Šmulian-Eberlein space.*

If we endow E' with the $\sigma(E', E)$-topology, then $E = \mathscr{L}(E', \mathbf{R} \text{ or } \mathbf{C})$ and the assertion follows from the theorem with the aid of the following lemma. †

LEMMA 6.4. *Let E be a locally convex vector space and let us endow E' with the weak topology associated to the duality $\langle E', E \rangle$. If the one point sets of E are of type G_δ, then E' possesses a σ-compact dense set. If moreover E possesses a countable dense set, then E' possesses a countable dense set.*

Let $(U_n)_{n\in\mathbf{N}}$ be a sequence of convex closed neighbourhoods of the origin of E whose intersection is equal to $\{0\}$. If for any $n\in\mathbf{N}$, U_n° denotes the polar set of U_n with respect to the duality $\langle E, E'\rangle$, then $(U_n^\circ)_{n\in\mathbf{N}}$ is a sequence of compact sets of E'. Let $x\in E$ be equal to 0 on $\bigcup_{n\in\mathbf{N}} U_n^\circ$. Then $x\in U_n$ for any $n\in\mathbf{N}$ and therefore $x=0$. This shows that $\bigcup_{n\in\mathbf{N}} U_n^\circ$ is dense in E'.

If moreover E possesses a countable dense set, then any U_n° is metrizable and possesses therefore a countable dense set. We deduce that E' possesses a countable dense set. †

Remark. If E is metrizable or one of the continuous seminorms of E is a norm then the one point sets of E are of type G_δ.

PROPOSITION 6.5. *Let E, F be locally convex vector spaces such that:*

a) *there exists a finite measure on E whose carrier is equal to E (this happens e.g. if E possesses a countable dense set);*

b) *there exists a countable dense set in F;*

c) *the one point sets of F are of type G_δ.*

If F_s denotes the locally convex vector space obtained by enodwing F with its weak topology, then $\mathscr{L}(E, F_s)$ is a strict Šmulian-Eberlein space for any topology finer than the topology of pointwise convergence.

By the preceding lemma, F' possesses a countable dense set. Hence there exists on F a metrizable topology which is coarser than the weak topology. Since F possesses a countable dense set this topology has a countable base. By Theorem 5.21. $\mathscr{C}(E, F_s)$ is a strict Šmulian-Eberlein space. The assertion follows now from Corollaries 3.17. and 3.18. †

PROPOSITION 6.6. *Let E, F be locally convex vector spaces such that F is an Eberlein space, $\mathfrak{S}$ be a covering of E with bounded sets such that any set of $\mathfrak{S}$ is contained in the closure of a σ-compact set of E, and $\mathscr{F}$ be a set of continuous linear maps of E into F. If any δ-stable Cauchy filter on $\mathscr{F}_\mathfrak{S}$ is convergent, then $\mathscr{F}$ is an Eberlein space (with respect to the topology of pointwise convergence).*

The proposition is an immediate consequence of Theorem 5.23. †

Proposition 6.6. contains Theorem 6. of [6].

COROLLARY 6.7. *A subset A of a locally convex vector space is an Eberlein space with respect to the weak topology if any δ-stable Cauchy filter on A converges to a point of A (this happens if particular in A is complete).*

Let us denote by E the locally convex vector space, by E' its dual endowed with the $\sigma(E', E)$-topology and by $\mathfrak{S}$ the set of equicontinuous subsets of E'. The assertion follows now from the proposition by replacing E, F and $\mathscr{F}$ with E' the scalar field and A respectively. †

THEOREM 6.8. *Let E be a bornological locally convex vector space, $\mathfrak{S}$ be a covering of E, and F be an Eberlein locally convex vector space. Then $\mathscr{L}(E, F)$ endowed with the topology of uniform convergence on the sets of $\mathfrak{S}$ is an Eberlein space.*

By Theorem 5.9. the set of linear maps g of E into F such that for any sequence $(x_n)_{n\in\mathbf{N}}$ in E which converges to 0 the sequence $(g(x_n))_{n\in\mathbf{N}}$ converges to 0 is Eberlein closed in F^E and therefore an Eberlein space (Propositions 2.1. and 2.8.). But E being bornological this set is exactly $\mathscr{L}(E, F)$. The theorem follows now from Corollary 5.2. †

This theorem contains Corollary 2.3. of [6].

PROPOSITION 6.9. *Let X be a set, Φ be a set of sequences in X, Y be a regular space and $\mathscr{G}$ be the set of those $g\in Y^X$ for which $(g(x_n))_{n\in\mathbf{N}}$ converges to $g(x_0)$ for any $(x_n)_{n\in\mathbf{N}}\in\Phi$. Then $\mathscr{G}$ is Eberlein closed in Y^X. In particular if Y is an Eberlein space then so is $\mathscr{G}$.*

Let $(x_n)_{n\in\mathbf{N}}\in\Phi$ and let us endow X with the following topology: a subset U of X is open if either $x_0\notin U$ or $x_0\in U$ and there exists $m\in\mathbf{N}$ such that $x_n\in U$ for any $n\geqslant m$. X endowed with this topology is a Hausdorff topological space and $(x_n)_{n\in\mathbf{N}}$ converges to x_0. Hence if we denote by f the map

$$n\mapsto x_n:\mathbf{N}\to X,$$

then $(\mathbf{N}, f)$ is a countably compact net on X. By Theorem 5.9., $\mathscr{G}(\mathbf{N}, f)$ is Eberlein closed in Y^X. $\mathscr{G}$ is Eberlein closed in Y^X as the intersection of the sets $\mathscr{G}(\mathbf{N}, f)$, where $(x_n)_{n\in\mathbf{N}}$ runs through Φ. The last assertion follows from Proposition 2.1. and Theorem 2.8. †

COROLLARY 6.10. *Let E be an ordered vector space and F be the set of linear forms x' on E with the property that for any decreasing sequence $(x_n)_{n\in\mathbf{N}}$ in E whose infimum is 0 we have*

$$\lim_{n\to\infty} x'(x_n)=0.$$

Then F is an Eberlein space with respect to the weak topology associated to the duality $\langle F, E\rangle$.

By Corollary 2.4. $\mathbf{R}$ is an Eberlein space and by the proposition F is an Eberlein space. †

COROLLARY 6.11. *The set of real or complex measures on a locally compact space endowed with the vague topology is an Eberlein space.*

By Theorem 2.8. it is sufficient to prove the theoremo nly for the real measures. This follows from the proposition and the following lemma. †

LEMMA. 6.12. *Let X be a locally compact space and let $\mathscr{K}(X)$ be the real vector space of real continuous functions with compact carrier on X. A linear form u on $\mathscr{K}(X)$ is a measure if and only if for any $f \in \mathscr{K}(X)$ and for any sequence $(f_n)_{n \in \mathbf{N}}$ in $\mathscr{K}(X)$ converging to 0 and such that $|f_n| \leqslant |f|$ for any $n \in \mathbf{N}$, $(u(f_n))_{n \in \mathbf{N}}$ converges to 0.*

By Lebesgue theorem this condition is fulfilled by any measure on X. Assume that u possesses this property and let $f \in \mathscr{K}(X)$. We want to show that

$$\sup\{u(g) \mid g \in \mathscr{K}(X), |g| \leqslant |f|\} < \infty .$$

Assume the contrary. Then there exists a sequence $(f_n)_{n \in \mathbf{N}}$ in such $\mathscr{K}(X)$ such that

$$|f_n| \leqslant |f|, \quad |u(f_n)| \geqslant n+1$$

for any $n \in \mathbf{N}$. Then $((1/n+1)f_n)_{n \in \mathbf{N}}$ is a sequence in $\mathscr{K}(X)$ which converges to 0 and such that $|(1/n+1)f_n| \leqslant |f|$ for any $n \in N$. We get the contradictory relation

$$0 = \lim_{n \to \infty} \left| u\left(\frac{1}{n+1} f_n\right) \right| \geqslant 1 .$$

Hence u may be written as a difference of two positive linear forms on $\mathscr{K}(X)$ and is therefore a measure. †

THEOREM 6.13. *Let X be a locally compact space, $\mathscr{K}(X)$ be the vector space of continuous real (resp. complex) valued functions on X with compact carrier and $\mathscr{M}(X)$ be the vector space of real (resp. complex) measures on X. Then $\mathscr{K}(X)$ endowed with the weak topology $\sigma(\mathscr{K}(X), \mathscr{M}(X))$ is a Šmulian-Eberlein space.*

The $\sigma(\mathscr{K}(X), \mathscr{M}(X))$-topology being finer than the topology of pointwise convergence on $\mathscr{K}(X)$ the assertion follows from the Corollaries 5.18. and 3.18. †

PROPOSITION 6.14. *Let E be a locally convex vector space and F be the set of Eberlein continuous linear forms on E. Then F is a subspace of the algebraic dual of E and any $\sigma(E, F)$-Eberlein continous linear form on E belongs to F.*

From the definition on Eberlein continuous functions it follows immediately that F is a subspace of the algebraic dual E^* of E. Let u be a $\sigma(E, F)$-Eberlein continuous linear form on E and let $\mathfrak{T}(E^*)$ be the coarsest topology on E finer than the initial one and for which any function of F is continuous. $\mathfrak{T}(E^*)$ is finer than $\sigma(E, F)$ and so u is Eberlein continuous for $\mathfrak{T}(E^*)$. Hence by Corollary 4.8. u belongs to F. †

PROPOSITION 6.15. *Let $\langle F, G \rangle$ be a separated duality, $\mathfrak{S}$ be a saturated family, covering G, such that the sets of $\mathfrak{S}$ are $\sigma(G, F)$-relatively countably compact and let us endow F with the topology of uniform convergence on the sets of $\mathfrak{S}$. Then any $\sigma(G, F)$-Eberlein continuous linear form on G belongs to the completion of F.*

Let u be a $\sigma(G, F)$-Eberlein continuous linear form on G. By Theorem 4.2. its restriction to any set of $\mathfrak{S}$ is continuous. By Grothendieck's dual characterization of completeness u belongs to the completion of F ([5], ch. IV, 6.2., Corollary 1.). †

We shall say that a locally convex vector space E is *Eberlein complete* if any linear form on E' which is $\sigma(E', E)$-Eberlein continuous belongs to E. The notion of Eberlein complete depends only on the duality $\langle E, E' \rangle$.

COROLLARY 6.16. *Let E be a locally convex vector space, $\tilde{E}$ be its completion and $\hat{E}$ be the set of $\sigma(E', E)$-Eberlein continuous linear form on E'. Then $\hat{E} \subset \tilde{E}$ and $\hat{E}$ endowed with the induced topology of $\tilde{E}$ is Eberlein complete. In particular a complete locally convex vector space is Eberlein complete.*

The inclusion $\hat{E} \subset \tilde{E}$ follows from the proposition. E' is obviously the dual of $\hat{E}$. By Proposition 6.14. any $\sigma(E', \hat{E})$-Eberlein continuous linear form on E' belongs to $\hat{E}$. Hence $\hat{E}$ is Eberlein complete. †

Remark. The inclusion $\hat{E} \subset \tilde{E}$ may be strict. Let X be a set and

$$l^1(X) := \{f \in R^X \mid \sum_{x \in X} |f(x)| < \infty\}.$$

Then $l^1(X)$ is a real vector space,

$$f \mapsto \sum_{x \in X} |f(x)| : l^1(X) \to R_+$$

is a norm on $l^1(X)$ and $l^1(X)$ endowed with this norm is a Banach space. If we set

$$E := \{f \in l^1(X) \mid \{x \in X \mid f(x) \neq 0\} \text{ is finite}\}$$

and endow E with the induced topology, then E is a normable locally convex vector space which is Eberlein complete but not complete.

$\hat{E}$ endowed with the induced topology of $\tilde{E}$ will be called the *Eberlein completion of E*.

COROLLARY 6.17. *A locally convex vector space is Eberlein complete if and only if it coincides with its Eberlein completion.* †

PROPOSITION 6.18. *Let E, F be locally convex vector spaces, $\tilde{E}$, $\tilde{F}$ be their completions, $\hat{E}$, $\hat{F}$ be their Eberlein completions, u be a continuous linear map of E into F and $\tilde{u}$ be its unique linear continuous extension from $\tilde{E}$ into $\tilde{F}$. Then $\tilde{u}(\hat{E}) \subset \hat{F}$.*

Let $x \in \hat{E}$. Let (I, f) be a $\sigma(F', F)$-countably compact net on F' and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ is $\sigma(F', F)$-convergent to an $y' \in F'$. If $u' : F' \to E'$ denotes the adjoint map of u then u' is continuous for the $\sigma(F', F)$- and $\sigma(E', E)$-topologies. Hence $(I, u' \circ f)$ is a $\sigma(E', E)$-countably compact net on E' and $u' \circ f\,(\mathfrak{U})$ $\sigma(E', E)$-converges to $u'(y')$. We get

$$\langle \tilde{u}(x), y' \rangle = \langle x, u'(y') \rangle = \lim_{\iota, \mathfrak{U}} \langle x, u'(f(\iota)) \rangle = \lim_{\iota, \mathfrak{U}} \langle \tilde{u}(x), f(\iota) \rangle.$$

Hence $\tilde{u}(x)$ is $\sigma(F', F)$-Eberlein continuous and belongs therefore to $\hat{F}$. †

COROLLARY 6.19. *Let F be an Eberlein complete locally convex vector space and E be a subspace of F. Then the Eberlein completion of E is the smallest Eberlein complete subspace of F containing E.*

Let $\tilde{F}$ be the completion of F. Then the closure of E in $\tilde{F}$ is the completion of E. By the proposition the Eberlein completion of E lies in F. If G is an Eberlein complete subspace of F containing E the above result shows that the Eberlein completion of E is contained in G. Since by Corollary 6.16. the Eberlein completion of E is Eberlein complete, it is the smallest Eberlein complete subspace of F. †

PROPOSITION 6.20. *Let E be an Eberlein complete locally convex vector space and $\mathfrak{S}$ be a covering of E'. Then E endowed with the topology of uniform convergence on the sets of $\mathfrak{S}$ is an Eberlein space. In particular E is an Eberlein space.*

E is a closed subset of $\mathscr{E}(E', \mathbf{R}$ or $\mathbf{C})$ which is an Eberlein space (Corollaries 5.10. and 2.4.). Hence E is an Eberlein space for the weak topology (Proposition 2.1.). The assertion follows now from Corollary 5.2. †

PROPOSITION 6.21. *If E is an Eberlein complete locally convex vector space, then any closed subspace of E is Eberlein complete.*

Let F be a closed subspace of E. It is known that F' endowed with the $\sigma(F', F)$-topology is isomorphic with the quotient space E'/F° of E' endowed with the $\sigma(E', E)$-topology, where F° denotes the polar set of F in E' ([5], ch. IV, 4.1., Corollary 1.). Let u be a linear from on F' Eberlein continuous for the $\sigma(F', F)$-topology and let φ be the canonical map $E' \to E'/F^\circ = F'$. Then $u \circ \varphi$ is a linear form on E', Eberlein continuous for the $\sigma(E', E)$-topology. Since E is Eberlein complete it follows that $u \circ \varphi$ and therefore u are continuous. †

PROPOSITION 6.22. *The product of an arbitrary family of Eberlein complete locally convex vector spaces is Eberlein complete.*

Let $(E_\iota)_{\iota \in I}$ be a family of Eberlein complete locally convex vector spaces and let E be its product. Its dual may be identified algebraically with the direct sum of the family $(E_\iota')_{\iota \in I}$ ([5], ch. IV, Theorem 4.3.). Let for any $\iota \in I$, ψ_ι be the canonical map

$$\psi_\iota : E_\iota' \to E'.$$

Let u be a linear form on E' which is Eberlein continuous for the $\sigma(E', E)$-topology.

It is easy to see that $u \circ \psi_\iota$ is an Eberlein continuous on E'_ι for the $\sigma(E'_\iota, E_\iota)$-topology. Since E_ι is Eberlein complete it follows that $u \circ \varphi_\iota$ is continuous for this topology. Since this happens for any $\iota \in I$ it follows that u is continuous for the $\sigma(E', E)$-topology. Hence E is Eberlein complete. †

COROLLARY 6.23. *The projective limit in the category of locally convex vector spaces of Eberlein complete locally vector spaces is Eberlein complete.*

In fact a projective limit in the category of locally convex vector spaces is a closed subspace of a product. †

PROPOSITION 6.24. *Let E be an Eberlein complete (resp. complete) locally convex vector space which contains a weakly σ-compact dense set. If u is a linear form on E' such that $(u(x'_n))_{n \in \mathbf{N}}$ converges to 0 for any sequence (resp. equicontinuous sequence) $(x'_n)_{n \in \mathbf{N}}$ on E' which converges to 0 for the $\sigma(E', E)$-topology, then $u \in E$.*

Let us denote by E_s the topological space obtained by endowing E with the weak topology. By Corollary 5.16. and Corollary 3.6., any subset of E' endowed with the induced $\sigma(E', E)$-topology is a Šmulian space. By Theorem 4.1. u (resp. the restriction of u to any equicontinuous set of E') is Eberlein continuous (resp. is continuous (Theorem 4.2)) and therefore $u \in E$ (resp. [5], ch. IV, 6.2., Corollary 2.). †

Remark. This result contains [5], ch. IV, 6.2., Corollary 3. where it is assumed that E is complete and separable.

PROPOSITION 6.25. *Let E be a locally convex vector space which is complete for the Mackey topology, $\mathfrak{T}'$ be the topology on E' of uniform convergence on the convex compact sets of E and $\mathfrak{S}$ be the set of convex sets of E' precompact for $\mathfrak{T}'$. Then the identical map of E into E endowed with the topology $\mathfrak{T}$ of uniform convergence on the sets of $\mathfrak{S}$ is Eberlein continuous.*

Let (I, f) be a countably compact net on E and let $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges to an $x \in E$. We want to show first that (I, f) is a countably compact net on E for the topology $\mathfrak{T}$. Let $(\iota_n)_{n \in \mathbf{N}}$ be an increasing sequence in I. Then $\{f(\iota_n) \mid n \in \mathbf{N}\}$ is a relatively countably compact set of E. But E being an Eberlein space (Proposition 6.20. and Corollary 6.16.) it follows that this set is relatively compact (Theorem 2.13.). By Krein's theorem ([5], ch. IV, 11.5.) its closed convex hull A is compact hence equicontinuous for E' endowed with $\mathfrak{T}'$. We deduce that A is compact for $\mathfrak{T}$ ([5], ch. III, 4.5.). It follows that the sequence $(f(\iota_n))_{n \in \mathbf{N}}$ has a non-empty adherence for $\mathfrak{T}$. Hence (I, f) is a countably compact net for $\mathfrak{T}$. But E endowed with $\mathfrak{T}$ is an Eberlein space (Proposition 6.20. and Corollary 6.16.). We deduce that $f(\mathfrak{U})$ is convergent for $\mathfrak{T}$. Since $\mathfrak{T}$ is finer than the initial topology, $f(\mathfrak{U})$ converges to x for $\mathfrak{T}$. †

PROPOSITION 6.26. *Let X be a set, $\mathfrak{A}$ be a set of subsets of X, E be a Hausdorff topological vector space and $\mathscr{F}$ be the set of those $f \in E^X$ for which $f(A)$ is bounded for any $A \in \mathfrak{A}$. Then $\mathscr{F}$ is Eberlein closed in E^X.*

Let (I, f) be a countably compact net on $\mathscr{F}$ and $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $f(\mathfrak{U})$ converges in E^X to a map g. We want to show that $g \in \mathscr{F}$. Assume the contrary. Then there exists $A \in \mathfrak{A}$ such that $g(A)$ is not bounded. Hence there exists a sequence $(x_n)_{n \in \mathbf{N}}$ in A such that $\{g(x_n) \mid n \in \mathbf{N}\}$ is not bounded ([5], ch. I, 5.3.). There exists therefore a circled neighbourhood U of the origin of E such that $\{g(x_n) \mid n \in \mathbf{N}\} \setminus \alpha U \neq \emptyset$ for any scalar number α. Let W be a closed circled neighbourhood of the origin of E such that $W + W \subset U$. We may construct an increasing sequence $(\iota_n)_{n \in \mathbf{N}}$ in I such that for any $m, n \in \mathbf{N}$, $m \leqslant n$, we have

$$f_{\iota_n}(x_m) \in g(x_m) + W.$$

let h be an adherent point in $\mathscr{F}$ of the sequence $(f_{\iota_n})_{n \in \mathbf{N}}$. Then

$$h(x_m) \in g(x_m) + W$$

for any $m \in \mathbf{N}$. Since $h(A)$ is bounded there exists $\alpha \in \mathbf{R}$ such that $h(A) \subset \alpha W$. We get

$$g(x_m) \in \alpha W + W \subset \sup(\alpha, 1)\, U$$

for any $m \in \mathbf{N}$, and this is the expected contradiction. †

COROLLARY 6.27. *Let X be a Hausdorff topological space such that any real function g on X is continuous if for any σ-compact set B of X there exists a real continuous function on X which coincides with g on $\bar{B}$. Let further $\mathfrak{A}$ be a set of subsets of X, E be a locally convex vector space which is an Eberlein space and $\mathscr{C}^{\mathfrak{A}}(X, E)$* be the set *of those $f \in \mathscr{C}(X, E)$ for which $f(A)$ is bounded for any $A \in \mathfrak{A}$. Then $\mathscr{C}^{\mathfrak{A}}(X, E)$ is an Eberlein space.*

Since $\mathscr{C}(X, E)$ is an Eberlein space (Corollary 5.19.) the assertion follows from the proposition and Proposition 2.1. †

VII. Application to Integration Theory

A *measurable space* is a set X endowed with a σ-algebra of subsets of X, called *measurable sets*. By *measure* on a measurable space we will understand a σ-additive function on the σ-algebra of measurable sets with values in $[0, \infty]$ equal to 0 on the empty set. The measure is called *bounded* if it does not take the value ∞. Let $x \in X$; we call *Dirac measure at the points* x, and denote it by ε_x, the measure on X equal to 1 (resp. 0) on any measurable set which contains (resp. does not contain x). A measurable set A *carries* μ if $X \setminus A$ is μ-negligible (i.e. $\mu(X \setminus A) = 0$). If X is a Hausdorff topological space, then by positive measure on X we understand a measure μ on a σ-algebra $\mathfrak{B}$ of X such that:

a) the compact sets belong to $\mathfrak{B}$ and are of finite μ-measure;
b) a subset of X belongs to $\mathfrak{B}$ if its intersection with any compact set belongs to $\mathfrak{B}$;
c) for any $A \in \mathfrak{B}$ we have

$$\mu(A) = \sup\{\mu(K) \mid K \text{ compact} \subset A\};$$

d) $A \subset B$, $B \in \mathfrak{B}$, $\mu(\mathfrak{B}) = 0 \Rightarrow A \in \mathfrak{B}$.

These are exactly the "prémesures positives" introduced in Bourbaki. It is easy to show that a measure on a Hausdorff topological space is determined by its values on the compact sets. This allows to define the sum of two measures and the multiplication of a measure with a positive real number. A *complex* (resp. *real*) *measure on a Hausdorff topological space* will be a linear combination of positive measures with complex (resp. real) coefficients. This notion coincides with Bourbaki's notion of "prémesure" (resp. "prémesure réelle"). A complex (resp. real) measure on a Hausdorff topological space will be called *bounded* if it is a linear combination of bounded measures. If $\mathscr{F}$ is a vector space of complex (resp. real) functions on a Hausdorff topological space X and $\mathscr{M}$ is a vector space of complex (resp. real) measures on X such that:

a) any $f \in \mathscr{F}$ is μ-integrable for any $\mu \in \mathscr{M}$,
b) any Dirac measure belongs to $\mathscr{M}$,

then the map

$$(f, \mu) \mapsto \int f \, d\mu : \mathscr{F} \times \mathscr{M} \to \mathbf{C} \,(\text{resp. } \mathbf{R})$$

is a duality on $\langle \mathscr{F}, \mathscr{M} \rangle$ separated in $\mathscr{F}$.

Let μ be a measure on a measurable space. A set $\mathscr{F}$ of μ-integrable function will be called *equi-integrable for* μ if for any sequence $(f_n)_{n \in \mathbf{N}}$ in $\mathscr{F}$ which converges to a function f μ-allmost everywhere (i.e. at any point of X with the exception of a μ-negligible set) we have

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

Let X be a Hausdorff topological space and μ be a complex measure on X; a map φ of X into a topological space is called *μ-measurable* if for any compact set K of X there exists an increasing sequence $(K_n)_{n \in \mathbf{N}}$ of compact subsets of K such that $K \setminus \bigcup_{n \in \mathbf{N}} K_n$ is μ-negligible and for any $n \in \mathbf{N}$ the restriction of φ to K_n is continuous.

PROPOSITION 7.1. *Let X be a Hausdorff topological space, $\mathscr{F}$ be a vector space of complex (resp. real) functions on X and $\mathscr{M}$ be a vector space of complex (resp. real) measures on X such that:*

a) *any $f \in \mathscr{F}$ is μ-integrable for any $\mu \in \mathscr{M}$;*
b) *any Dirac measure belongs to $\mathscr{M}$;*
c) *for any $\mu \in \mathscr{M}$ there exists a subset A_μ of X which carries μ such that the set of the*

restrictions of the functions of $\mathscr{F}$ to A_μ is a Šmulian space with respect to the topology of pointwise convergence.
Let (I, f) be a net on $\mathscr{F}$ which is countably compact for the topology of pointwise convergence on $\mathscr{F}$ and with the following property: for any $\mu \in \mathscr{M}$ and for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $(f_{\iota_n})_{n\in\mathbf{N}}$ converges to a $g\in\mathscr{F}$ at any point of A_μ we have

$$\lim_{n\to\infty} \int f_{\iota_n}\, d\mu = \int g\, d\mu .$$

If $\mathfrak{F}$ is a filter on I finer than the section filter of I, then the adherence of $f(\mathfrak{F})$ in the topology of pointwise convergence on $\mathscr{F}$ coincides with its adherence in the $\sigma(\mathscr{F}, \mathscr{M})$-topology.

We may assume that $\mathfrak{F}$ is an ultrafilter. The $\sigma(\mathscr{F}, \mathscr{M})$-topology being finer than the topology of pointwise convergence we have only to show that if $f(\mathfrak{F})$ converges to a $g\in\mathscr{F}$ for the topology of pointwise convergence it converges to g for the $\sigma(\mathscr{F}, \mathscr{M})$-topology. Let $\mu\in\mathscr{M}$ and φ be the map

$$f \mapsto f \mid A_\mu : \to \mathscr{F} \mid A_\mu{}^{3)} .$$

Then $(I, \varphi\circ f)$ is a countably compact net on $\mathscr{F} \mid A_\mu$ (for the topology of pointwise convergence) and for any increasing sequence $(\iota_n)_{n\in\mathbf{N}}$ in I such that $\varphi\circ f_{\iota_n}$ converges to an $h\in\mathscr{F} \mid A_\mu$ we have

$$\lim_{n\to\infty} \int \varphi\circ f_{\iota_n}\, d\mu = \int h\, d\mu .$$

Since $\mathscr{F} \mid A_\mu$ is a Šmulian space we get by Theorem 4.1.

$$\int g\, d\mu = \int \varphi\circ g\, d\mu = \lim_{\iota,\mathfrak{F}} \int \varphi\circ f_\iota\, d\mu = \lim_{\iota,\mathfrak{F}} \int f_\iota\, d\mu .$$

The measure μ being arbitrary we deduce that $f(\mathfrak{F})$ converges to g for the $\sigma(\mathscr{F}, \mathscr{M})$-topology. †

COROLLARY 7.2. *Let $X, \mathscr{F}, \mathscr{M}$ be like in the proposition and $\mathscr{G}$ be a relatively countably compact set of $\mathscr{F}$ for the topology of pointwise convergence such that for any $\mu\in\mathscr{M}$, $\mathscr{G}$ is equi-integrable for μ. Then on the closure of $\mathscr{G}$ in $\mathscr{F}$ for the topology of pointwise convergence, this topology and the $\sigma(\mathscr{F}, \mathscr{M})$-topology coincide.*

We prove first a lemma.

LEMMA 7.3. *Let X be a set, $\mathfrak{T}, \mathfrak{T}'$ be two topologies on X such that $\mathfrak{T}'$ is finer than $\mathfrak{T}$ and such that X endowed with $\mathfrak{T}'$ is a regular space, and let A be a subset of X dense for $\mathfrak{T}$. If the $\mathfrak{T}$-closures and the $\mathfrak{T}'$-closures of the subsets of A coincide, then $\mathfrak{T}=\mathfrak{T}'$.*

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3) $\mathscr{F} \mid A_\mu := \{f \mid A_\mu \mid f\in\mathscr{F}\}$ where $f \mid A_\mu$ denotes the restriction of f to A_μ.

Let F be a $\mathfrak{T}'$-closed set of X and let $x \in X \backslash F$. Let U be a $\mathfrak{T}'$-closed neighbourhood of x contained in $X \backslash F$. Let $y \in F$ and V be a $\mathfrak{T}'$-neighbourhood of y. Then $V \cap (X \backslash U)$ is a $\mathfrak{T}'$-neighbourhood of y. Since y belongs to the $\mathfrak{T}$-closure of A and therefore, by the hypothesis, to the $\mathfrak{T}'$-closure of A we get

$$V \cap (A \backslash U) = V \cap (X \backslash U) \cap A \neq \emptyset .$$

Hence y belongs to the $\mathfrak{T}'$-closure of $A \backslash U$. Since y is arbitrary F is contained in the $\mathfrak{T}'$-closure of $A \backslash U$. But U being a $\mathfrak{T}'$-neighbourhood of x, x does not belong to the $\mathfrak{T}'$-closure of $A \backslash U$ which coincides with the $\mathfrak{T}$-closure of $A \backslash U$. Its complementary set is therefore a $\mathfrak{T}$-neighbourhood of x which does not intersect F. Hence F is $\mathfrak{T}$-closed. †

Let $\overline{\mathscr{G}}$ be the closure of $\mathscr{G}$ in $\mathscr{F}$ for the topology of pointwise convergence. By the above lemma it is sufficient to show that the closures of any subset of $\mathscr{G}$ in $\overline{\mathscr{G}}$ for the topology of pointwise convergence and for the $\sigma(\mathscr{F}, \mathscr{M})$-topology coincide. Let $\mathscr{H}$ be a subset of $\mathscr{G}$ and $f \in \mathscr{F}$ be an adherent point of $\mathscr{H}$ in the topology of pointwise convergence. Let $\leqslant$ be the trivial preorder relation on $\mathscr{H}$ (i.e. $g \leqslant h$ for any $g, h \in \mathscr{H}$) and φ be the identical map of $\mathscr{H}$ into $\mathscr{F}$ .Then $(\mathscr{H}, \varphi)$ is a countably compact net on $\mathscr{F}$ for the topology of pointwise convergence and possesses the following property: for any $\mu \in \mathscr{M}$ and for any sequence $(h_n)_{n \in \mathbf{N}}$ in $\mathscr{H}$ which converges to a $h \in \mathscr{F}$ at any point of A_μ we have

$$\lim_{n \to \infty} \int h_n \, d\mu = \int h \, d\mu .$$

Let $\mathfrak{U}$ be an ultrafilter on $\mathscr{H}$ which converges to f in the topology of pointwise convergence. By the proposition $\mathfrak{U}$ converges to f in the $\sigma(\mathscr{F}, \mathscr{M})$-topology. †

COROLLARY 7.4. *Let X be a Hausdorff topological space, $\mathscr{C}(X)$ be the vector space of continuous complex (resp. real) functions on X, $\mathscr{M}^b(X)$ be the vector space of bounded complex (resp. real) measures on X and*

$$\mathscr{C}^b(X) := \{ f \in \mathscr{C}(X) \mid f \text{ is bounded} \},$$
$$\mathscr{M}^c(X) := \{ \mu \in \mathscr{M}^b(X) \mid \mu \text{ has a compact carrier} \}.$$

If $\mathscr{G}$ is a relatively countably compact set of $\mathscr{C}(X)$ (resp. $\mathscr{C}^b(X)$), for the topology of pointwise convergence and a bounded set for the $\sigma(\mathscr{C}(X), \mathscr{M}^c(X))$- (resp. $\sigma(\mathscr{C}^b(X), \mathscr{M}^b(X))$-) topology then on its closure in $\mathscr{C}(X)$ (resp. $\mathscr{C}^b(X)$) for the topology of pointwise convergence, this topology and the $\sigma(\mathscr{C}(X), \mathscr{M}^c(X))$- (resp. $\sigma(\mathscr{C}^b(X), \mathscr{M}^b(X))$-) topology coincide. In particular a subset of $\mathscr{C}(X)$ (resp. $\mathscr{C}^b(X)$) is relatively countably compact, countably compact, relatively compact or compact for the $\sigma(\mathscr{C}(X), \mathscr{M}^c(X))$- (resp. $\sigma(\mathscr{C}^b(X), \mathscr{M}^b(X))$-) topology if and only if it is bounded for this topology and possesses the corresponding property for the topology of pointwise convergence.

Let us prove first that the couples $(\mathscr{C}(X), \mathscr{M}^c(X))$, $(\mathscr{C}^b(X), \mathscr{M}^b(X))$ satisfy the conditions a), b), c) of the proposition for the couple $(\mathscr{F}, \mathscr{M})$. This is obvious for a) and b). For the couple $(\mathscr{C}(X), \mathscr{M}^c(X))$ the condition c) follows from Corollary 5.16. Let us consider the couple $(\mathscr{C}^b(X), \mathscr{M}^b(X))$ and $\mu\in\mathscr{M}^b(X)$. Since μ is bounded there exists a σ-compact set A of X such that $\mu(X\backslash A)=0$. The condition c) follows from Corollary 5.16.

The corollary follows now from the preceding corollary, from the next lemma and from Lebesgue theorem. †

LEMMA 7.5. *Let X be a Hausdorff topological space, $\mathscr{C}^b(X)$ be the vector space of bounded continuous real (resp. complex) functions on X, and $\mathscr{M}^b(X)$ be the vector space of bounded real (resp. complex) measures on X. If $\mathscr{F}$ is a subset of $\mathscr{C}^b(X)$ bounded for the $(\mathscr{C}^b(X), \mathscr{M}^b(X))$-topology, then*

$$\sup_{\substack{f\in\mathscr{F}\\ x\in X}} |f(x)| < \infty .$$

Assume the contrary. We shall construct inductively a sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathscr{F}$ and a sequence $(x_n)_{n\in\mathbf{N}}$ in X such that

a) $|f_n(x_n)| > \frac{1}{2}\sup_{x\in X}|f_n(x)|$,

b) $|f_n(x_n)| > 6^n\left(1+3\sum_{m<n}\sup_{f\in\mathscr{F}}|f(x_m)|\right)$

for any $n\in\mathbf{N}$. Assume that the sequences were constructed up to $n-1$. Since $\mathscr{F}$ is bounded for the $\sigma(\mathscr{C}^b(X), \mathscr{M}^b(X))$-topology we get

$$\sup_{f\in\mathscr{F}}|f(x_m)| < \infty$$

for any $m\leqslant n-1$. The existence of f_n and x_n follows from the hypothesis of the proof and from the fact that any $f\in\mathscr{F}$ is bounded.

We set

$$\mu := \sum_{n\in\mathbf{N}} \frac{1}{5^n}\varepsilon_{x_n} \in \mathscr{M}^b(X),$$

where ε_{x_n} denotes the Dirac measure at the point x_n. Then

$$\left|\int f_n\, d\mu\right| \geqslant \frac{1}{5^n}|f_n(x_n)| - \sum_{m<n}\frac{1}{5^m}|f_n(x_m)| - \sum_{m>n}\frac{1}{5^m}|f_n(x_m)|$$

$$\geqslant \left(\frac{1}{3}\frac{1}{5^n}|f_n(x_n)| - \sum_{m<n}\sup_{f\in\mathscr{F}}|f(x_m)|\right)$$

$$+\left(\frac{1}{3}\frac{1}{5^n}|f_n(x_n)|-\left(\sum_{m>n}\frac{1}{5^m}\right)\sup_{x\in X}|f_n(x)|\right)+\frac{1}{3}\frac{1}{5^n}|f_n(x_n)|\geqslant\frac{1}{3}\frac{1}{5^n}6^n$$

for any $n\in\mathbf{N}$, and we get the contradictory relation

$$\infty>\sup_{f\in\mathscr{F}}\left|\int f\,d\mu\right|\geqslant\sup_{n\in\mathbf{N}}\left|\int f_n\,d\mu\right|=\infty\ .\quad\dagger$$

Remark. If $\mathscr{F}$ denotes the vector space of bounded complex (resp. real) functions f on X which have the property that for any $\mu\in\mathscr{M}^b(X)$ the restriction of f to the carrier of μ is continuous, then we may replace in the above corollary $\mathscr{C}^b(X)$ by $\mathscr{F}$. Moreover $\mathscr{F}$ being an Eberlein space for the topology of pointwise convergence (Corollary 5.33) a subset of $\mathscr{F}$ is relatively compact for the $\sigma(\mathscr{F},\mathscr{M}^b(X))$-topology if and only if it is bounded for this topology and is relatively countably compact for the topology of pointwise convergence.

COROLLARY 7.6. *Let X, Y be Hausdorff topological spaces μ be a bounded complex measure on Y and f be a bounded complex function on $X\times Y$ such that for any $x\in X$ and any compact set K of Y the function*

$$y\mapsto f(x,y):K\to\mathbf{C}$$

is continuous and for any $y\in Y$ the function

$$x\mapsto f(x,y):X\to\mathbf{C}$$

is Eberlein continuous. Then the function

$$x\mapsto\int f(x,y)\,d\mu(y):X\to\mathbf{C}$$

is Eberlein continuous. In particular if X is a c-space (and a fortiori a compact space) this function is continuous.

By endowing Y with the finest topology which induces on the compact sets of Y the same topology as the initial one (which does not change the complex measures on X) we may assume that

$$y\mapsto f(x,y):Y\to\mathbf{C}$$

for continuous for any $x\in X$. Let $\mathscr{C}^b(Y)$ be the vector space of bounded continuous complex functions on Y and $\mathscr{M}^b(Y)$ be the vector space of bounded complex measures on Y. As it was shown in Corollary 7.4. the couple $(\mathscr{C}^b(Y),\mathscr{M}^b(Y))$ satisfies the conditions a), b), c) from the proposition. Let (I,g) be a countably compact net on X and let $\mathfrak{U}$ be an ultrafilter on I, finer than the section filter of I and such that $g(\mathfrak{U})$

converges to an $x_0 \in X$. Let $y \in Y$. By the hypothesis we get

$$\lim_{\iota, \mathfrak{U}} f(g(\iota), y) = f(x_0, y).$$

Let φ be the map of X into $\mathscr{C}^b(Y)$ which maps any $x \in X$ into the function

$$y \mapsto f(x, y): Y \to \mathbf{C}.$$

We deduce from the above relation that $(I, \varphi \circ g)$ is a countably compact net on $\mathscr{C}^b(Y)$ for the topology of pointwise convergence and that $\varphi \circ g(\mathfrak{U})$ converges to $\varphi(x_0)$ in this topology. By Lebesgue theorem and by the proposition we deduce that $\varphi \circ g(\mathfrak{U})$ converges to $\varphi(x_0)$ in the $\sigma(\mathscr{C}^b(Y), \mathscr{M}^b(Y))$-topology. Hence

$$\lim_{\iota, \mathfrak{U}} \int f(g(\iota), y)\, d\mu(y) = \lim_{\iota, \mathfrak{U}} \int \varphi \circ g(\iota)\, d\mu = \int \varphi(x_0)\, d\mu = \int f(x_0, y)\, d\mu(y). \quad \dagger$$

PROPOSITION 7.7. *Let X be a completely regular space, $\mathscr{K}(X)$ be the vector space of continuous complex (resp. real) functions with compact carriers on X and $\tilde{\mathscr{M}}(X)$ be the vector space of complex (resp. real) measures on X such that any σ-compact and relatively countably compact set of X is of finite measure. If $\mathscr{G}$ is a relatively countably compact set of $\mathscr{K}(X)$ for the topology of pointwise convergence and a bounded set for the $\sigma(\mathscr{K}(X), \mathscr{M}(X))$-topology then on its closure with respect to the topology of pointwise convergence, this topology and the $\sigma(\mathscr{K}(X), \tilde{\mathscr{M}}(X))$-topology coincide. A subset of $\mathscr{K}(X)$ is relatively compact for the $\sigma(\mathscr{K}(X), \mathscr{M}(X))$-topology if and only if it is bounded for this topology and it is relatively countably compact for the topology of pointwise convergence.*

Let $\bar{\mathscr{G}}$ be the closure of $\mathscr{G}$ in $\mathscr{K}(X)$ for the topology of pointwise convergence. By Lemma 7.3. it is sufficient to show that the closure of any subset of $\mathscr{G}$ in $\bar{\mathscr{G}}$ for the topology of pointwise convergence and for the $\sigma(\mathscr{K}(X), \tilde{\mathscr{M}}(X))$-topology coincide. Let $\mathscr{H}$ be a subset of $\mathscr{G}$ and $f \in \mathscr{K}(X)$ be an adherent point of $\mathscr{H}$ in the topology of pointwise convergence. Since $\mathscr{K}(X)$ is a Šmulian space in the topology of pointwise convergence (Corollary 5.18.) $\mathscr{H}$ is sequentially dense (Theorem 3.22.). Hence there exists a sequence $(f_n)_{n \in \mathbf{N}}$ in $\mathscr{H}$ which converges to f in the topology of pointwise convergence. We set

$$A := \bigcup_{n \in \mathbf{N}} \{x \in X \mid f_n(x) \neq 0\}.$$

A is a σ-compact set of X. We want to show that A is relatively countably compact. Assume the contrary. Then there exists a sequence $(x_m)_{m \in \mathbf{N}}$ in A with no adherent points. We shall construct inductively three sequences $(n(k))_{k \in \mathbf{N}}$, $(m(k))_{k \in \mathbf{N}}$, $(\alpha_k)_{k \in \mathbf{N}}$

of natural numbers such that for any $k \in \mathbf{N}$ we have

a) $j < k \Rightarrow f_{n(j)}(x_{m(k)}) = 0$;

b) $\alpha_k |f_{n(k)}(x_{m(k)})| - \sum_{j<k} \alpha_j |f_{n(k)}(x_{m(j)})| > k$.

Assume that the sequences were constructed up to $k-1$. We take $m(k)$ such that

$$x_{m(k)} \notin \bigcup_{j<k} \operatorname{Supp} f_{n(j)},$$

then $n(k)$ such that $f_{n(k)}(x_{m(k)}) \neq 0$ and α_k such that b) is fulfilled. If we set

$$\mu := \sum_{k \in \mathbf{N}} \alpha_k \, \varepsilon_{x_{m(k)}},$$

where ε_x denotes the Dirac measures at x, then $\mu \in \tilde{\mathscr{M}}(X)$ and we get the contradictory relation

$$\infty > \sup_{h \in \mathscr{H}} \left| \int h \, d\mu \right| \geqslant \lim_{k \to \infty} \left| \int f_{n(k)} \, d\mu \right| = \infty .$$

Hence A is relatively countably compact. By Lemma 7.5.

$$\sup_{\substack{g \in \mathscr{H} \\ x \in X}} |g(x)| < \infty .$$

By Lebesgue theorem it follows

$$\int f \, d\mu = \lim_{k \to \infty} \int f_{n(k)} \, d\mu$$

for any $\mu \in \mathscr{M}(X)$. Hence f is an adherent point of $\mathscr{H}$ in the $\sigma(\mathscr{K}(X), \tilde{\mathscr{M}}(X))$-topology.

The last assertion follows now from the fact that $\mathscr{K}(X)$ is an Eberlein space for the topology of pointwise convergence (Corollary 5.18.). †

PROPOSITION 7.8. *Let X be a locally compact space, $\mathscr{K}(X)$ be the vector space of continuous complex (resp. real) functions on X with compact carrier endowed with the topology of pointwise convergence and $\mathscr{F}$ be a relatively countably compact set of $\mathscr{K}(X)$ such that the function*

$$x \mapsto \sup_{f \in \mathscr{F}} |f(x)| : X \to R$$

is bounded and with compact carrier. If $\mathscr{M}(X)$ denotes the vector space of complex (resp. real) measures on X then the topology $\sigma(\mathscr{K}(X), \mathscr{M}(X))$ induces on $\mathscr{F}$ the same topology as the topology of pointwise convergence. In particular $\mathscr{F}$ is relatively compact for the $\sigma(\mathscr{K}(X), \mathscr{M}(X))$-topology.

Let us denote by $\mathfrak{T}$ the topology of pointwise convergence on $\mathscr{K}(X)$. By Lebesgue theorem if $(f_n)_{n\in\mathbf{N}}$ is a sequence in $\mathscr{F}$ which converges to an $f\in\mathscr{K}(X)$ in the $\mathfrak{T}$-topology then

$$\mu(f) = \lim_{n\to\infty} \mu(f_n)$$

for any $\mu\in\mathscr{M}(X)$. Since $\mathscr{F}$ is a Šmulian space for the $\mathfrak{T}$-topology (Corollaries 5.18. and 3.17.) it follows that the restriction of any $\mu\in\mathscr{M}(X)$ to $\overline{\mathscr{F}}$ is Eberlein continuous for this topology (Theorem 4.1.). Since $\mathscr{K}(X)$ is an Eberlein space for the $\mathfrak{T}$-topology (Corollary 5.18.) it follows that $\overline{\mathscr{F}}$ is compact. Hence the restriction of any $\mu\in\mathscr{M}(X)$ to $\overline{\mathscr{F}}$ is continuous (Theorem 4.2.). Hence the $\sigma(\mathscr{K}(X), \mathscr{M}(X))$-topology and the $\mathfrak{T}$-topology coincide on $\overline{\mathscr{F}}$, which is therefore compact for the $\sigma(\mathscr{K}(X), \mathscr{M}(X))$-topology. †

THEOREM 7.9. *Let E be an Eberlein complete (resp. complete) locally convex vector space which contains a weakly σ-compact dense set. Let X be a measurable space, μ be a measure on X and f be a map of X into E such that for any $x'\in E'$ the function $x'\circ f$ is μ-integrable. Then the following assertions are equivalent:*

a) *there exists $x\in E$ such that for any $x'\in E'$ we have*

$$\int x'\circ f\, d\mu = \langle x, x'\rangle\,;$$

b) *for any sequence (resp. equicontinuous sequence) $(x_n)_{n\in\mathbf{N}}$ in E' converging to* 0 *for the $\sigma(E', E)$- topology we have*

$$\lim_{n\to\infty} \int x'_n\circ f\, d\mu = 0\,.$$

a$\Rightarrow$b is trivial.

$b\Rightarrow$a follows from Proposition 6.24. †

COROLLARY 7.10. *Let E be a reflexive Banach space, X be a measurable space, μ be a meaure on X and f be a map of X into E such that for any $x'\in E'$ the function $x'\circ f$ is μ-integrable. Then he following assertions are equivalent:*

a) *there exists $x\in E$ such that*

$$\int x'\circ f\, d\mu = \langle x, x'\rangle$$

for any $x'\in E'$;

b) *for any sequence $(x'_n)_{n\in\mathbf{N}}$ in E' converging to* 0 *for the $\sigma(E', E)$-topology we have*

$$\lim_{n\to\infty} \int x'_n\circ f\, d\mu = 0\,. \quad †$$

COROLARY 7.11. *Let E be an Eberlein complete (resp. complete) locally convex vector space and F be a closed subspace of E which contains a weakly σ-compact dense set. Let X be a measurable space, μ be a measure on X and f be a map of X into F such that for any $x' \in E'$ the function $x' \circ f$ is μ-integrable. Then the following assertions are equivalent:*

a) *there exists $x \in F$ such that for any $x' \in E'$ we have*

$$\int x' \circ f \, d\mu = \langle x, x' \rangle ;$$

b) *for any sequence (resp. equicontinuous sequence) $(x'_n)_{n \in \mathbf{N}}$ in E' with the property that*

$$\lim_{n \to \infty} \langle y, x'_n \rangle = 0$$

for any $y \in F$ we have

$$\lim_{n \to \infty} \int x'_n \circ f \, d\mu = 0 .$$

a$\Rightarrow$b is trivial

b$\Rightarrow$a. By Proposition 6.21. F is Eberlein complete (resp. F is obviously complete). Since the weak topology of F coincides with the induced $\sigma(E, E')$-topolgy, F possesses a weakly σ-compact dense set. From b) and the theorem we deduce that there exists $x \in F$ such that

$$\int y' \circ f \, d\mu = \langle x, y' \rangle$$

for any $y' \in F'$. Let $x' \in E'$ and φ be the canonical map $E' \to F' = E'/F^{\circ}$, where F° denotes the polar set of F in E'. Then

$$\int x' \circ f \, d\mu = \int \varphi(x') \circ f \, d\mu = \langle x, \varphi(x') \rangle = \langle x, x' \rangle . \quad \dagger$$

COROLLARY 7.12. *Let E be a quasi-complete (even for the Mackey topology) locally convex vector space, X be a measurable space, μ be a bounded measure on X and f be a map of X into E such that $f(X)$ is bounded and contained in the closure of a weakly σ-compact set of E and such that $x' \circ f$ is μ-integrable for any $x' \in E'$. Then there exists $x \in E$ such that*

$$\int x' \circ f \, d\mu = \langle x, x' \rangle$$

for any $x' \in E'$.

Let $\tilde{E}$ be the completion of E. By the next lemma there exists a closed subspace F of $\tilde{E}$ which contains $f(X)$ and which possesses a weakly σ-compact dense set. Let $(x_n')_{n\in\mathbf{N}}$ be a sequence in E' such that

$$\lim_{n\to\infty} \langle x, x_n' \rangle = 0$$

for any $x\in F$. We set

$$A := \{x\in F \mid n\in\mathbf{N} \Rightarrow |\langle x, x_n'\rangle| \leqslant 1\}.$$

A is obviously a barrel of F. Since F is complete A absorbs any bounded set of F ([5], ch. II, 8.5.). Hence

$$\sup_{\substack{t\in X\\ n\in\mathbf{N}}} |\langle f(t), x_n'\rangle| < \infty .$$

By Lebesgue theorem we get

$$\lim_{n\to\infty} \int x_n' \circ f \, d\mu = 0 .$$

By Corollary 6.16. $\tilde{E}$ is Eberlein complete. Hence by the preceding corollary there exists $x\in F$ such that

$$\int x' \circ f \, d\mu = \langle x, x' \rangle$$

for any $x'\in E'$. But it is obvious that x belongs to the convex closed hull of $\mu(X)\, f(X)$ Since E is quasi-complete and $f(X)$ bounded this convex closed hull is contained in E. Hence $x\in E$. †

LEMMA 7.13. *If a subset of a Hausdorff topological vector space E is contained in the closure of a σ-compact set then it is contained in the closure of a σ-compact subspace of E.*

Let E be a Hausdorff topological vector space and A be a subset of E which is contained in the closure of a σ-compact set. Then there exists an increasing sequence $(K_n)_{n\in\mathbf{N}}$ of compact sets of E such that

$$A \subset \overline{\bigcup_{n\in\mathbf{N}} K_n} .$$

For any $n\in N$ we set

$$L_n := \left\{ \sum_{i=1}^{n} \alpha_i x_i \;\middle|\; |\alpha_i| \leqslant n ,\ x_i \in K_n \right\}.$$

Then L_n is compact as the continuous image of a compact set. It is obvious that $\bigcup_{n\in\mathbf{N}} L_n$ is a σ-compact subspace of E whose closure contains A †

COROLLARY 7.14. *Let E be a quasi-complete (even for the Mackey topology) locally convex vector space, X be a Hausdorff topological space, μ be a bounded measure on X and f be a μ-measurable map of X into E endowed with the weak topology such that $f(X)$ is bounded. Then $x' \circ f$ is μ-integrable for any $x' \in E'$ and there exists $x \in E$ such that*

$$\langle x, x' \rangle = \int x' \circ f \, d\mu$$

for any $x' \in E'$.

For any $x' \in E'$ the function $x' \circ f$ is bounded and μ-measurable hence μ-integrable. Let $(K_n)_{n \in \mathbf{N}}$ be a sequence of compact sets such that $X \setminus \bigcup_{n \in \mathbf{N}} K_n$ is μ-negligible and such that for any $n \in N$ the restriction of f to K_n is continuous for the weak topology of E. We may assume without changing the problem that f vanishes on $X \setminus \bigcup_{n \in \mathbf{N}} K_n$. Then $f(X)$ is contained in a weakly σ-compact set of E and the assertion follows from the preceding corollary. †

Remark. It follows from this corollary that if E is a quasi-complete locally convex vector space then any bounded measure on E which is carried by a bounded set possesses a barycenter.

COROLLARY 7.15. *Let X, Y be Hausdorff topological spaces μ, v be bounded complex measures on X, Y respectively and f be a bounded complex function on $X \times Y$ such that for any $x \in X$ (resp. $y \in Y$) and any compact set K of Y (resp. of X) the function*

$$y \mapsto f(x, y) : K \to \mathbf{C} \text{ (resp. } x \mapsto f(x, y) : K \to \mathbf{C})$$

is continuous. Then for any compact set K of Y (resp. X) the function

$$y \mapsto \int f(x, y) \, d\mu(x) : K \to \mathbf{C} \text{ (resp. } x \mapsto \int f(x, y) \, dv(y) : K \to \mathbf{C})$$

is continuous and we have

$$\int \left(\int f(x, y) \, d\mu(x) \right) dv(y) = \int \left(\int f(x, y) \, dv(y) \right) d\mu(x).$$

Let $\mathfrak{T}$ (resp. $\mathfrak{T}'$) be the finest topology on X (resp. Y) which induces on the compact sets the same topology as the initial one. Then the function f is continuous separatedly in each variable for the topologies $\mathfrak{T}$ and $\mathfrak{T}'$. By Corollary 7.6. the functions

$$y \mapsto \int f(x, y) \, d\mu(x) : Y \to \mathbf{C}, \quad x \mapsto \int f(x, y) \, dv(y) : X \to \mathbf{C}$$

are continuous for $\mathfrak{T}'$ and $\mathfrak{T}$ respectively (since X endowed with $\mathfrak{T}$ and Y endowed with $\mathfrak{T}'$ are c-spaces (Corollary 4.6.)) and this proves the first assertion.

Let us denote by $\tilde{\mathscr{C}}^b(Y)$ the vector space of bounded $\mathfrak{T}'$-continuous complex functions on Y. It is obvious that $\widetilde{\mathscr{C}}^b(Y)$ is complete for the Mackey topology associated to the duality $\langle\widetilde{\mathscr{C}}^b(Y), \mathscr{M}^b(Y)\rangle$ ($\mathscr{M}^b(Y)$ is the vector space of bounded complex measures on Y, and is equal to the vector space of bounded complex measures on Y endowed with $\mathfrak{T}'$). Let φ be the map of X into $\widetilde{\mathscr{C}}^b(Y)$ which maps any $x \in X$ into the function

$$y \mapsto f(x, y): Y \to \mathbf{C}.$$

φ is a $\mathfrak{T}$-continuous map of X into $\mathscr{C}^b(Y)$ endowed with the $(\widetilde{\mathscr{C}}^b(Y), \mathscr{M}^b(Y))$-topology (Corollary 7.4) and $\varphi(X)$ is bounded. By the preceding corollary there exists $g \in \mathscr{C}^b(Y)$ such that

$$\int g \, d\lambda = \int \left(\int \varphi \, d\lambda\right) d\mu$$

for any $\lambda \in \mathscr{M}^b(Y)$. If we take as λ the Dirac measure at y then we get

$$g(y) = \int f(x, y) \, d\mu(x).$$

Hence

$$\int \left(\int f(x, y) \, d\mu(x)\right) d\nu(y) = \int \left(\int f(x, y) \, d\nu(y)\right) d\mu(x). \quad \dagger$$

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INDEX

LIST OF LOGICAL CONNECTIONS

Explanation

6.2. (3.5, 3.17, 3.18, 3.21, 5.16, 6.0, 6.1) means that in the proof of Theorem 6.2 there were used:

Corollary 3.5, Corollary 3.17, Corollary 3.18, Lemma 3.21, Corollary 5.16, Proposition 6.0 and Proposition 6.1.

1.1, 1.2 (1.1), 1.3, 1.4;

2.1, 2.2 (1.3), 2.3 (1.2, 2.2), 2.4 (2.3), 2.5 (2.3), 2.6 (2.3), 2.7, 2.8, 2.9 (2.8), 2.10 (2.1, 2.9), 2.11 (2.10), 2.12, 2.13, 2.14, 2.15;

3.1, 3.2 (1.4), 3.3 (3.2), 3.4 (3.2), 3.5 (3.2), 3.6 (3.4), 3.7 (3.4), 3.8, 3.9, 3.10 (1.3), 3.11 (1.2, 3.10), 3.12 (3.6, 3.11), 3.13, (3.11), 3.14 (1.4), 3.15, 3.16 (3.15), 3.17 (3.6, 3.16), 3.18 (3.7, 3.16), 3.19 (2.4, 3.5, 3.18), 3.20 (3.19, 3.21), 3.21, 3.22, 3.23, 3.24 (3.3, 3.22), 3.25;

4.1, 4.2, 4.3 (4.2), 4.4 (4.2, 4.5), 4.5, 4.6 (4.4), 4.7, 4.8 (4.7), 4.9, 4.10 (4.9), 4.11 (2.1);

5.1, 5.2 (2.8, 3.7, 5.1), 5.3 (2.1, 2.8), 5.4 (5.3), 5.5 (2.1, 5.3), 5.6 (3.4, 3.8), 5.7 (5.6), 5.8, 5.9, 5.10, (2.1, 2.8, 5.9), 5.11 (5.10), 5.12 (5.11), 5.13, 5.14 (1.4, 2.4, 3.1, 3.5, 3.8, 4.2, 5.10, 5.13), 5.15 (1.4, 3.19, 4.5, 5.10, 5.14), 5.16 (5.15), 5.17 (3.16, 3.18, 5.16), 5.18 (3.17, 3.21, 5.16), 5.19 (5.5, 5.15), 5.20 (5.16, 5.19), 5.21 (3.14, 5.7, 5.16), 5.22, 5.23 (2.1, 2.8, 5.15), 5.24 (3.19), 5.25,(3.4, 3.8), 5.26 (4.11), 5.27 (3.6, 5.14, 5.26), 5.28 (5.27), 5.29 (3.1, 3.6, 3.8, 5.26), 5.30 (3.15), 5.31 (5.30), 5.32 (3.8), 5.33 (2.1, 2.4, 5.19, 5.32);

6.0 (3.4, 5.25), 6.1 (2.1, 5.3), 6.2 (3.5, 3.17, 3.18, 3.21, 5.16, 6.0, 6.1), 6.3 (6.2, 6.4), 6.4, 6.5 (3.17, 3.18, 5.21, 6.4), 6.6 (5.23,) 6.7 (6.6), 6.8 (2.1, 2.8, 5.2, 5.9), 6.9 (2.1, 2.8, 5.9), 6.10 (2.1, 2.4, 6.9), 6.11 (2.8, 6.9, 6.12), 6.12, 6.13 (3.18, 5.18), 6.14 (4.8), 6.15 (4.2), 6.16 (6.14, 6.15), 6.17 (6.16), 6.18, 6.19 (6.16, 6.18), 6.20 (2.1, 2.4, 5.2, 5.10), 6.21, 6.22, 6.23 (6.21, 6.22), 6.24 (3.6, 4.1, 5.16), 6.25 (2.13, 6.16, 6.20), 6.26, 6.27 (2.1, 5.19, 6.26);

7.1 (4.1), 7.2 (7.1, 7.3), 7.3, 7.4 (5.16, 7.2, 7.5), 7.5, 7.6 (5.16, 7.1), 7.7 (3.22, 5.18, 7.3, 7.5), 7.8 (3.17, 4.1, 4.2, 5.18), 7.9 (6.24), 7.10 (7.9), 7.11 (6.21, 7.9), 7.12 (6.16, 7.11, 7.13), 7.13, 7.14 (7.12), 7.15 (4.6, 7.4, 7.6, 7.14).

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