Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 47 (1972)

Artikel: Homological Methods and the Third Dimension Subgroup
Autor: Bachmann, F. / Grünenfelder, L.
DOI: https://doi.org/10.5169/seals-36383

Homological Methods and the Third Dimension Subgroup

by F. BACHMANN and L. GRÜNENFELDER[1])

In this paper we use a certain commutative diagram to derive some well-known and some new results in group theory. Our approach is homological in the sense that we make use of various exact sequences arising in homology of groups.

Let us recall some notation (see [1]): Associated to the lower central series $\{G_n \mid G_1 = G\}$ of a group G is the Lie-ring LG whose underlying abelian group is given by

$$\bigoplus_{i \geqslant 1} G_i/G_{i+1}.$$

The bracket operation on LG is induced by the commutator in G. There is a natural surjection of graded rings

$$ULG \xrightarrow{\varphi_G} \mathrm{gr}\mathbf{Z}G,$$

where ULG denotes the universal envelope of LG and $gr\mathbf{Z}G$ is given as follows: Let JG be the augmentation ideal in $\mathbf{Z}G$; then $gr\mathbf{Z}G$ is the graded ring associated to the JG-adic filtration of $\mathbf{Z}G$.

Our map φ_G is induced by the Lie-algebra map

$$LG \xrightarrow{i_G} gr\mathbf{Z}G$$

which in degreen is defined by the inclusion of G_n into the nth dimension subgroup $D_n(G) = \{x \in G \mid x - 1 \in JG^n\}$.

In the present investigation we consider φ_G in degree 2. Let U_nLG and $Q_n(G) = JG^n/JG^{n+1}$ be the nth components in ULG and $gr\mathbf{Z}G$ respectively. Our main results are:

THEOREM I. *For any group G there is a commutative diagram (of abelian*

1) We thank Prof. B. Eckmann for having given us the opportunity to work at the Forschungsinstitut für Mathemathik, ETH.

groups) with exact rows

$$\begin{array}{ccccccccc} 0 \to & G_2/G_3 & \xrightarrow{i_G} & Q_2(G) & \xrightarrow{p} & Q_2(G_{ab}) & \to 0 \\ & \| & & \uparrow \varphi_G & & \uparrow \varphi_{G_{ab}} & \\ 0 \to & G_2/G_3 & \xrightarrow[j_G]{} & U_2LG & \to & U_2(G_{ab}) & \to 0 . \end{array}$$

Here, j_G is the canonical map from LG to ULG, p is the obvious projection, and $G_{ab}=G/G_2$.

COROLLARY 1. *$D_3(G)=G_3$ for any group G.*

COROLLARY 2. *$U_2LG \xrightarrow{\varphi_G} Q_2(G)$ is bijective for any group G.*

THEOREM II. *If G is finitely generated the sequence*

$$0 \to G_2/G_3 \xrightarrow{i_G} Q_2(G) \xrightarrow{p} Q_2(G_{ab}) \to 0$$

is split exact.

§ 1. Diagrams

1. LEMMA. *For any group G and all $n \geqslant 2$ there are exact sequences of abelian groups*

$$(A_n)\colon\ H_2(G, \mathbf{Z}) \to H_2(G/G_n, \mathbf{Z}) \xrightarrow{\beta} G_n/G_{n+1} \to 0$$

$$(B_n)\colon\ H_2(G, \mathbf{Z}) \to H_1(G, JG/JG^n) \to Q_n(G) \to 0$$

$$(C_n)\colon\ 0 \to G_n/G_{n+1} \xrightarrow{j_G} U_n(LG) \to U_n(L(G/G_n)) \to 0$$

Moreover, (C_n) is split exact in the following two cases:
(a) *$n=2$ and G finitely generated*
(b) *$n>2$ and G arbitrary.*

Proof. (A_n) forms a part of the "5-term-sequence" associated to the group epimorphism $G \twoheadrightarrow G/G_n$ (see e.g. [6]).

(B_n) forms a part of the homology sequence corresponding to the G-module sequence

$$0 \to JG^n \to JG \to JG/JG^n \to 0 .$$

We used the identification $H_1(G, JG) \simeq H_2(G, \mathbf{Z})$.

To prove (C_n) we remark that we have an exact sequence of abelian groups

$$0 \to R \to LG \to L(G/G_n) \to 0,$$

where $R = \oplus_{i \geqslant n} G_i/G_{i+1}$. The kernel V of the algebra morphism

$$ULG \to UL(G/G_n)$$

is then given by the two-sided ideal in ULG generated by $j_G(R)$. As we are working over $\mathbf{Z}$, j_G is an injective map ([2]). Therefore the nth component of V coincides with G_n/G_{n+1}. The splitting of (C_n) will be proved in section 6.

2. By naturalıty the sequence (B_2) yields the following commutative diagram (with α_2 bijective and α_3 surjective):

$$\begin{array}{ccccccc} H_2(G,\mathbf{Z}) & \to & H_1(G, Q_1(G)) & \to & Q_2(G) & \to & 0 \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & \\ H_2(G_{ab},\mathbf{Z}) & \xrightarrow{\gamma} & H_1(G_{ab}, Q_1(G_{ab})) & \to & Q_2(G_{ab}) & \to & 0 \end{array}$$

By a result of Stammbach ([6]), γ is a monomorphism. Hence, by applying the ker-coker-sequence and (A_2), we obtain an isomorphism

$$\partial: \operatorname{Ker}\alpha_3 \xrightarrow{\sim} G_2/G_3.$$

3. Consider now the square

$$\begin{array}{ccc} \operatorname{Ker}\alpha_3 = K & \overset{\kappa}{\rightarrowtail} & Q_2(G) \\ {\scriptstyle\partial^{-1}}\downarrow & * & \downarrow{\scriptstyle\varphi_G} \\ G_2/G_3 & \underset{j_G}{\to} & U_2LG \end{array},$$

where κ is the kernel map. We will prove (see section 5) that $(*)$ commutes. It follows

$$\kappa\cdot\partial^{-1} = \varphi_G\cdot j_G = i_G: G_2/G_3 \to Q_2(G).$$

Therefore we have

THEOREM I. *For any group G there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & G_2/G_3 & \overset{i_G}{\to} & Q_2(G) & \to & Q_2(G_{ab}) & \to & 0 \\ & & \| & & {\scriptstyle\varphi_G}\uparrow & & \uparrow{\scriptstyle\varphi_{G_{ab}}} & & \\ 0 & \to & G_2/G_3 & \underset{j_G}{\to} & U_2LG & \to & U_2(G_{ab}) & \to & 0 \end{array}.$$

COROLLARY 1. $D_3(G) = G_3$ *for any group G.*

COROLLARY 2. *$U_2LG \overset{\varphi_G}{\to} Q_2(G)$ is bijective for any group G.*

Indeed, Corollary 1 is an immediate consequence of the injectivity of i_G. (See [3], [5] for other proofs of this fact). For the proof of Corollary 2 we observe that $\varphi_{G_{ab}}$ is bijective ([4]) and apply the five lemma.

4. We turn now to the splitting property of the exact sequences in Theorem I. For finite groups, Sandling ([5]) established this fact using the Jennings-Hall-basis of the abelian group JG^2. The isomorphism of Corollary 2 shows that now it is enough to have a splitting map

$$s: U_2LG \to G_2/G_3 .$$

Such a map exists by lemma 1. Hence we have shown

THEOREM II. *If G is finitely generated the sequence*

$$0 \to G_2/G_3 \overset{i_G}{\to} Q_2(G) \to Q_2(G_{ab}) \to 0$$

is split exact.

§ 2. Proofs

5. To establish the commutativity of the diagram (*) we have to compute the homomorphism $\partial: K \simeq G_2/G_3$ explicitely. If S is the kernel of the canonical surjection $\mathbf{Z}G \to \mathbf{Z}G_{ab}$, i.e. S is the ideal $\langle xy - yx \mid x, y \in G \rangle$ in $\mathbf{Z}G$, then

$$K = S + JG^3/JG^3 .$$

We use the inhomogeneous bar solution $B(G_{ab})$ to compute ∂. It is clear from he ker-coker-sequence, that

$$\partial\,\overline{(xy - yx)} = \beta \bar{z} ,$$

where $\bar{z}$ is the element of $H_2(G_{ab}, \mathbf{Z})$ represented by the 2-cycle

$$z = 1 \otimes ([\bar{x} \mid \bar{y}] - [\bar{y} \mid \bar{x}])$$

in $\mathbf{Z} \otimes_{G_{ab}} B_2$. The morphism $\beta: H_2(G_{ab}, \mathbf{Z}) \to G_2/G_3$ in the sequence (A_2) is the composition

$$H_2(G_{ab}, \mathbf{Z}) \overset{\partial 1}{\to} H_1(G_{ab}, JG_{ab}) \overset{\partial 2}{\to} \mathbf{Z} \otimes_{G_{ab}} (G_2)_{ab} \simeq G_2/G_3 ,$$

where ∂_1 and ∂_2 are the connecting morphisms in the homology sequences of the short exact sequences

$$JG_{ab} \rightarrowtail \mathbf{Z}G_{ab} \twoheadrightarrow \mathbf{Z} \quad \text{and} \quad (G_2)_{ab} \rightarrowtail \mathbf{Z}G_{ab} \otimes_G JG \twoheadrightarrow JG_{ab}$$

respectively. For the computation of $\beta\bar{z}$ we remark that $\partial_1\bar{z}$ is represented by the 1-cycle

$$z' = (\bar{x} - 1) \otimes [\bar{y}] - (\bar{y} - 1) \otimes [\bar{x}]$$

in $JG_{ab} \otimes_{G_{ab}} B_1$ and $\partial_2\partial_1\bar{z}$ is represented by the 0-cycle

$$z'' = \overline{[x, y]}$$

in $(G_2)_{ab}$. For the typical element $xy - yx$ of K we therefore get

$$\partial\,\overline{(xy - yx)} = \beta\bar{z} = \overline{[x, y]}\,.$$

It is now immediate that the diagram (*) commutes.

6. It remains to show the splitting of the sequence (C_2). Assume G finitely generated. $G/G_2 = G_{ab}$, being finitely generated, is a finite direct sum of cyclic groups. Let $\bar{x}_1, \ldots, \bar{x}_n$ be the generators of its cyclic components. The commutator identity

$$[x^n, y] = [x, y]\,[[x, y], x^{n-1}]\,[x^{n-1}, y]$$

in G shows that there is a homomorphism

$$\sigma: (G_{ab} \otimes G_{ab}) \oplus G_2/G_3 \to G_2/G_3\,,$$

well defined by

$$\sigma(\bar{x}_i \otimes \bar{x}_j) = \begin{cases} \overline{[x_i, x_j]}\,, & \text{if } i \leqslant j \\ 0\,, & \text{if } i > j \end{cases}$$

on $G_{ab} \otimes G_{ab}$ and by the identity on G_2/G_3. As

$$\sigma(\bar{x}_i \otimes \bar{x}_j - \bar{x}_j \otimes \bar{x}_i - \overline{[x_i, x_j]}) = 0$$

it factors through U_2LG. The induced homomorphism

$$s: U_2LG \to G_2/G_3$$

is clearly left inverse to $j_G: G_2/G_3 \to U_2LG$.

A different proof of Theorem II has been given in a forthcoming paper by G. Losey.

To complete the proof of the lemma in section 1 we assume now $\bar{G}$ arbitrary and

$n>2$. It is easy to see that

$$L(G/G_n) = \bigoplus_{i=1}^{n-1} G_i/G_{i+1}\,.$$

Consider the following commutative square (where T is the tensor algebra functor and all arrows are the obvious projections):

$$\begin{array}{ccc} T(LG) & \xrightarrow{\alpha} & T(L(G/G_n)) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U(LG) & \xrightarrow{\beta} & U(L(G/G_n)) \end{array}$$

The above presentation of $L(G/G_n)$ shows that α has a splitting σ. Clearly we have

$$\sigma(\operatorname{Ker} q) \subset \operatorname{Ker} p\,.$$

Hence σ lifts to a splitting of β.

REFERENCES

[1] BACHMANN, F. and GRÜNENFELDER, L., *Ueber Lie-Ringe von Gruppen und ihre universellen Enveloppen*, Comment. Math. Helv. *47* (1972), fasc. 3.

[2] LAZARD, M., *Sur les algèbres enveloppantes universelles de certaines algèbres de Lie*, Publ. Sci. Univ. Alger (A), I (1954).

[3] PASSI, I. B. S., *Dimension subgroups*, J. Algebra *9* (1968).

[4] ——, *Polynomial functors*, Proc. Camb. Phil. Soc. *66* (1969).

[5] SANDLING, R., *Dimension subgroups over arbitrary coefficient rings*, J. Algebra *21* (1972).

[6] STAMMBACH, U., *Anwendungen der Homologietheorie der Gruppen auf Zentralreihen und auf Invarianten von Präsentierungen*, Math. Z. *94* (1966).

Forschungsinstitut für Mathematik
ETH Zürich

Received August 1972