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**Autor:** Farrell, F.T. / Wagoner, J.B.

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# Algebraic Torsion for Infinite Simple Homotopy Types

F. T. FARRELL[1] and J. B. WAGONER[2]

This paper is the last of a series dealing with the problem of giving an algebraic description of the torsion invariant for proper $h$-cobordisms using the concept of a locally finite infinite matrix. The other two papers are [1] and [6]. The reader should also consult [5]. The main results of this paper are (3.1) and (4.1) which describe the group of proper simple types on a strongly locally finite CW-complex as a $K_1$-type group $\mathrm{Wh}(\pi)$. The exact sequence (3.6) of [1] allows $\mathrm{Wh}(\pi)$ to be computed in a number of cases.

## § 1. Infinite simple types

In this section we give a definition of infinite simple homotopy type equivalen to the one in [5] but in a form more convenient for our purposes.

Let $\mathscr{C}$ denote the category of strongly locally finite, countable, CW-complexes and proper homotopy classes of continuous maps. Recall from [2] that a CW-complex is strongly locally finite provided it is the union of a countable, locally finite collection of finite subcomplexes. Let $\mathscr{C}^+ \subset \mathscr{C}$ denote the full subcategory whose objects are finite dimensional. A *proper expansion* $K \nearrow L$ in the category $\mathscr{C}$ is an inclusion $K \subset L$ where $L = K \cup (\bigcup_{i=1}^{\infty} L_i)$ and each $L_i$ is a finite subcomplex such that

a) $(L_i - K) \cap (L_j - K) = \emptyset$ for $i \neq j$

b) $L_i$ collapses to $K_i = K \cap L_i$.

A *proper contraction* $L \searrow K$ is the homotopy inverse of a proper expansion $K \nearrow L$. A proper map $f: X \to Y$ is a *proper simple homotopy equivalence* (in $\mathscr{C}$) iff there is a sequence of proper expansions and contractions $X = X_0 \to X_1 \to X_2 \to \cdots X_{n-1} \to X_n = Y$ whose composition is properly homotopic to $f$. If $f$ is a morphism in $\mathscr{C}^+$, then it is a *proper simple equivalence* (in $\mathscr{C}^+$) provided each map $X_i \to X_{i+1}$ is a morphism in $\mathscr{C}^+$. In particular each proper expansion $K \nearrow L$ in $\mathscr{C}^+$ must satisfy the condition

c) there is an integer $n$ such that $\dim(L_i - K) \leqslant n$ for all $i$.

Now given the notion of proper simple equivalence in $\mathscr{C}$ we have as in [5] the group $\mathscr{S}(X)$ of proper simple homotopy types of an object $X$ of $\mathscr{C}$. An element of $\mathscr{S}(X)$ is represented by a proper homotopy equivalence $f: X \to Y$ and two such maps $f_0: X \to Y_0$ and $f_1: X \to Y_1$ are considered the same iff there is a simple equivalence $s: Y_0 \to Y_1$ in $\mathscr{C}$ such that $s \circ f_0$ is properly homotopic to $f_1$. Similarly one has the

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group of proper simple types $\mathscr{S}^+(X)$ for any object $X$ in $\mathscr{C}^+$. If $X$ is an object of $\mathscr{C}^+$ there is a natural map $\mathscr{S}^+(X)\to\mathscr{S}(X)$ which we show to be an isomorphism in (4.2).

Throughout this paper any CW-complex will always be assumed to be strongly locally finite. We need this in order to say as in [2] that any proper map can be properly deformed to a cellular map – the starting point for the algebraic theory of simple types describing $\mathscr{S}(X)$ as a functor of $\pi_1 X$ and the system of fundamental groups of neighborhoods of infinity. If one works in the category of all locally finite, countable CW-complexes, then $\mathscr{S}(X)$ may be non-zero even though $X$ is simply connected and simply connected at infinity. For example, let $K=e^0\cup e^1\cup\cdots$ where $e^n$ is attached to $e^{n-1}$ by collapsing $\partial e^n$ to a point in the interior of $e^{n-1}$. The property of being strongly locally finite is preserved under proper simple equivalence. Hence, if $K'$ is any subdivision of $K$ which is strongly locally finite, then $K$ and $K'$ are not simply equivalent although they ought to be. This minor technical point could be remedied if one knew that any locally finite CW-complex had a strongly locally finite subdivision. We get around the difficulty by working only with strongly locally finite CW-complexes. In passing we remark that (**) of [5] is false if the category of all locally finite, countable CW-complexes is used. One must stay with strongly locally finite complexes.

Suppose $L=K\cup_{f_i}\{e_i^n\}\cup_{g_j}\{e_j^{n+1}\}$. Suppose the attaching map $g_j$ of some $(n+1)$-cell $e_j^{n+1}$ misses all the $n$-cells except, say, $e_i^n$ and suppose $g_j$ takes the top hemisphere of $\partial e_j^{n+1}$ homeomorphically onto $e_i^n$ and takes the bottom hemisphere into $L$. Thus the pair $e_j^n\cup e_j^{n+1}$ forms an elementary expansion. We shall say that $e_j^{n+1}$ *cancels* $e_i^n$.

## § 2. Definition of torsion in $\mathscr{C}^+$

In this section we briefly recall the definition of torsion for a proper homotopy equivalence in $\mathscr{C}^+$. For details and terminology see [1] and [6, Chap. I, § 5].

Let $X$ be a non-compact, connected, strongly locally finite CW-complex and let $t{:}T\to X$ be a tree for $X$. This means that $T$ is a locally finite, contractible, one dimensional simplicial complex with a base vertex $0\in T$ such that if $v\in T$ is a vertex different from 0 then at least two 1-cells branch off from $v$. Furthermore, $t{:}T\to X$ is required to be a cellular map which is properly $\frac{1}{2}$-connected in the sense that $t^*{:}H^0(X)\to H^0(T)$ and $t^*{:}H^0_{\text{end}}(X)\to H^0_{\text{end}}(T)$ are isomorphisms.

The obstruction group $\mathrm{Wh}(X;t)$, which is defined in [6, Chap. I, § 5] to capture the torsion of a proper homotopy equivalence $f{:}\,W\to X$ in $\mathscr{C}^+$, is an abelian group which depends only on $\pi_1 X$ and the inverse system of fundamental groups of neighborhoods of infinity in $X$. Up to isomorphism $\mathrm{Wh}(X;t)$ is also independent of the choice of tree $t{:}T\to X$. The group $\mathrm{Wh}(X;t)$ can be computed as follows:

First some generalities. Let $t{:}T\to X$ be any tree for $X$. The set $J$ of vertices of $T$

can be partially ordered by letting $u \leqslant v$ iff the arc from $v$ to the base vertex 0 passes through $u$. Let $|v|$ denote the number of 1-simplices in the arc from $v$ to 0. Let $T_v \subseteq T$ denote the smallest subcomplex containing all vertices $w$ of $T$ with $v \leqslant w$. Let $J' \subset J$ be a cofinal subset (containing the vertex 0) obtained as follows: choose an increasing sequence $0 = n_0 < n_1 < \cdots$. Then let $j \in J'$ iff there is some $n_k$ with $|j| = n_k$. Associated to $J'$ is a tree $T'$ obtained by inserting a 1-simplex between vertices $u$ and $v$ of $J'$ whenever $u < v$ and there is no vertex $w$ of $J'$ with $u < w < v$. The natural map $T' \to T$ is properly $\frac{1}{2}$-connected and the composition $T' \to T \to X$ is a tree for $X$.

Now start with the original tree $t: T \to X$. Then there is a tree $t': T' \to X$ derived from $t: T \to X$ by the above process and there is a collection $\{X_u\}$ of infinite, connected subcomplexes of $X$ (one subcomplex $X_u$ for each vertex $u$ of $T'$) satisfying the following conditions (cf. [6, Chap. I, § 5]):

a) $X_0 = X$, $X_u \supset X_v$ when $u \leqslant v$, and $t'(T_u) \subset X_u$

b) $X_u \cap X_v = \emptyset$ if $|u| = |v|$ and $u \neq v$

c) for each $n \geqslant 0$, $X - \bigcup_{|v|=n} X_v$ is contained in some finite subcomplex of $X$

d) given any finite subcomplex $K$ of $X$, there is some $n \geqslant 0$ such that $K \cap (\bigcup_{|v|=n} X_v) = \emptyset$.

Now for each vertex $u$ of $T'$ let $\pi_u = \pi_1(X_u, t'(u))$. If $u \leqslant v$ define the homomorphism $\gamma_{uv}: \pi_v \to \pi_u$ to be "conjugation" by the path $t'(\alpha_{vu}) \subset X_u$ where $\alpha_{vu} \subset T'_u$ is the arc from $u$ to $v$. The collection $\pi = \{\pi_u, \gamma_{uv}\}$ is a tree of groups over the set $J'$ of vertices of $T'$. Let $Z[\pi] = \{Z[\pi_u], \gamma_{uv}\}$ denote the associated tree of group rings. Let $\mathrm{Wh}(\pi)$ be as defined in [1] and [6, Chap. I, § 5]. Then there is an isomorphism

$$\mathrm{Wh}(\pi) \cong \mathrm{Wh}(X; t)$$

See [6, Chap. I, § 5]. The point of using $\mathrm{Wh}(X; t)$ as the obstruction group rather than one of its "representatives" $\mathrm{Wh}(\pi)$ is to make the torsion well defined and independent of various choices such as the $X_u$ above. However in proving certain things one often uses a convenient choice of a $\mathrm{Wh}(\pi)$. Also, there is the basic algebraic exact sequence (see (4.3) below) which relates $\mathrm{Wh}(\pi)$ with the $\mathrm{Wh}(\pi_u)$ and $\tilde{K}_0(\pi_u)$ and allows one to compute $\mathrm{Wh}(\pi)$ in a number of cases.

We will briefly indicate how to define the torsion

$$\tau(L, K) \in \mathrm{Wh}(\pi) \cong \mathrm{Wh}(L; t)$$

of an inclusion $K \to L$ where $K$ is a proper deformation retract of $L$ and $\dim(L-K) < \infty$. Here $\pi = \{\pi_u, \gamma_{uv}\}$ is the tree of groups corresponding to any choice of a tree $t': T' \to L$ derived from $t: T \to L$ as above and any choice of a system $\{L_u\}$ satisfying (a) through (d) with respect to $t': T' \to L$. The definition of $\tau(L, K)$ given below is in the spirit of [4]. Using the general machinery of [6, Chap. I, § 5] and [1] it is not hard to see that this approach to torsion for infinite simple types is equivalent to the one worked out in [6, Chap. I, § 5] which follows the lines of [3]. The argument showing the equivalence is entirely similar to the one in the compact case.

For any CW-complex $X$, let $\tilde{X}$ denote the universal covering space $p:\tilde{X}\to X$. If $Y\subset X$, let $\tilde{Y}=p^{-1}(Y)$.

By condition (a) above $t'(u)\in L_u$ for every vertex $u$ of $T'$. Select a fixed lifting $\hat{u}\in\tilde{L}_u$ of $t'(u)$. If $v$ is a vertex of $T'$ and $u\leqslant v$, let $v'\in\tilde{L}_u$ be the lifting of $t'(v)\in L_u$ obtained as the end point of the lifting of the path $t(\alpha_{vu})$ to a path in $\tilde{L}_u$ starting at $\hat{u}$. Here $\alpha_{vu}$ denotes the arc from $u$ to $v$ in $T'_u$. If $v$ is a vertex of $T'$ with $u\leqslant v$ there is a unique map $\tilde{L}_v\to\tilde{L}_u$ covering the inclusion $L_v\to L_u$ such that $\hat{v}\in\tilde{L}_v$ goes to $v'\in\tilde{L}_u$. Furthermore, if $u\leqslant v\leqslant w$ the map $\tilde{L}_w\to\tilde{L}_u$ is the composition $\tilde{L}_w\to\tilde{L}_v\to\tilde{L}_u$.

The next choice we make is to select a locally finite collection $\Lambda$ of paths $\alpha(\sigma)$ from the barycenters of cells $\sigma$ of $L$ to the images $t'(\alpha(\sigma))$ of vertices $u(\sigma)$ of $T'$ such that if $\sigma\subset L_u$ then $\alpha(\sigma)\subset L_u$. If $\sigma\subset L_u$ the path $\alpha(\sigma)$ determines a path $\beta_u(\sigma)$ from $\sigma$ to $t'(u)$ in $L_u$: first follow $\alpha(\sigma)$ to $t'(u(\sigma))$ and then follow $t'(\alpha_{u,u(\sigma)})$ to $t'(u)$. Here $\alpha_{u,u(\sigma)}$ is the arc in $T'$ from $u(\sigma)$ to $u$.

If $X$ is any CW-complex, let $X^n$ denote the $n$-skeleton of $X$.

Now define the based $Z[\pi]$-module $C_n(L,K)$ as follows:

$$C_n(L,K)=\{C_n(L,K)_u\}$$

where for each vertex $u$ of $T'$

$$C_n(L,K)_u=H_n(\overline{L^n_u},\overline{L^{n-1}_u}\cup\overline{(L^n_u\cap K)})$$

The "bar" is taken with respect to the universal cover $\tilde{L}_u\to L_u$. The $Z[\pi_u]$-module $C_n(L,K)_u$ is free with one basis element for each $n$-cell of $L_u-K$. The basis element corresponding to an $n$-cell $\sigma$ of $L_n-K$ is given by the lifting of $\sigma$ to $\tilde{L}_u$ determined by the path $\beta_u(\sigma)$. If $u\leqslant v$ the map $\tilde{L}_v\to\tilde{L}_u$ determines a homomorphism $C_n(L,K)_v\to$ $\to C_n(L,K)_u$ and in fact we have an injection

$$C_n(L,K)_v\otimes_{Z[\pi_v]}Z[\pi_u]\to C_n(L,K)_u$$

whose image is the free submodule generated by the $n$-cells of $L_u-K$ which lie in $L_v-K$. The boundary operators $\partial^u_n:C_n(L,K)_u\to C_{n-1}(L,K)_u$ are compatible with the maps $C_n(L,K)_v\to C_n(L,K)_u$ and therefore define a morphism of $Z[\pi]$-modules

$$\partial_n:C_n(L,K)\to C_{n-1}(L,K)$$

which satisfies $\partial_{n-1}\circ\partial_n=0$. This gives a chain complex

$$(C_*,\partial_*)=\{C_n(L,K),\partial_n\}$$

of based $Z[\pi]$-modules. In fact, if $S^n=\{S^n_u\}$ is the tree of sets over $J'$ where $S^n_u$

consists of the $n$-cells of $L_u - K$, then $C_n(L, K)$ is the free $Z[\pi]$-module generated by $S^n$. Since $\dim(L-K)<\infty$ at most finitely many of the chain groups $C_n(L, K)$ are not zero.

Let $r: L\times I \to L$ be a proper deformation retraction of $L$ down into $K$. We can assume $r$ is cellular by [2, Th. 1.7]. For each vertex $u$ of $T'$ choose a cofinite subcomplex $N_u$ of $L_u$ (i.e., $L_u - N_u$ has only finitely many cells) such that

i) $N_0 = L_0$ and $N_u \supset N_v$ whenever $u \leqslant v$

ii) $r(N_u \times I) \subset L_u$.

The map $r: N_u \times I \to L_u$ has a unique lifting $r_u: \bar{N}_u \times I \to \tilde{L}_u$ such that $r_u$ restricted to $\bar{N}_u \times 0$ is the inclusion and such that whenever $u \leqslant v$ there is a commutative diagram

$$\begin{array}{ccc} \bar{N}_v \times I & \xrightarrow[r_v]{} & \tilde{L}_v \\ \downarrow & & \downarrow \\ \bar{N}_u \times I & \xrightarrow[r_u]{} & \tilde{L}_u \end{array}$$

Let $\hat{C}_n(L, K)_u \subset C_n(L, K)_u$ be the free $Z[\pi]$-submodule generated by the $n$-cells of $L_u - K$ belonging to $N_u - K$ and let $i_u: \hat{C}_n(L, K)_u \to C_n(L, K)_u$ denote the inclusion map. The maps $r_u: \bar{N}_u \times I \to \tilde{L}_u$ induce coboundary operators

$$d_u^n: \hat{C}_n(L, K)_u \to C_{n+1}(L, K)_u$$

compatible with the morphisms $\hat{C}_n(L, K)_v \to \hat{C}_n(L, K)_u$ and $C_n(L, K)_v \to C_n(L, K)_u$ such that for each vertex $u$ of $T'$

$$\partial_{n+1}^u \circ d_u^n + d_n^{n-1} \circ \hat{\partial}_n^u = \begin{cases} id, & \text{for} \quad u = 0 \\ i_u + \text{finite matrix}, & \text{for} \quad u > 0 \end{cases}$$

Here $\hat{\partial}_n^u: \hat{C}_n(L, K)_u \to \hat{C}_{n-1}(L, K)_u$ is the restriction of $\partial_n^u$. Thus the collection $d^n = \{d_u^n\}$ defines a germ $d^n: C_n(L, K) \to C_{n+1}(L, K)$ such that on the germ level we have

$$\partial_{n+1} \circ d^n + d^{n-1} \circ \partial_n = id. \tag{$*$}$$

This shows that $(C_*, \partial_*)$ is an acyclic complex of based modules over the tree of rings $Z[\pi]$ and as in [6, Chap. I, § 5] we can define the torsion to be

$$\tau(L, K) = \tau(C_*, \partial_*) \in \mathrm{Wh}(\pi) \tag{2.1}$$

Now here is the way to define $\tau(L, K)$ in the spirit of [4]: By replacing $d^n$ with $d^n \circ \partial_{n+1} \circ d^n$ (if necessary) we can assume that $d^{n+1} \circ d^n = 0$. Let $C_{\mathrm{ev}} = \oplus_{0 \leqslant k} C_{2k}$ and $C_{\mathrm{odd}} = \oplus_{0 \leqslant k} C_{2k+1}$. The formula $(*)$ implies that $\partial_{\mathrm{ev}} + d^{\mathrm{ev}}: C_{\mathrm{ev}} \to C_{\mathrm{odd}}$ is an isomorphism on the germ level whose inverse is $\partial_{\mathrm{odd}} + d^{\mathrm{odd}}: C_{\mathrm{odd}} \to C_{\mathrm{ev}}$. Let the trees of sets $S_{\text{ev}}$ and $S_{\text{odd}}$ be defined as the disjoint unions of trees of sets

$$S_{ev} = \coprod_{0 \geqslant k} S^{2k} \quad \text{and} \quad S_{\text{odd}} = \coprod_{0 \geqslant k} S^{2k+1}$$

Then $C_{\text{ev}}$ is the free $Z[\pi]$-module generated by $S_{\text{ev}}$ and $C_{\text{odd}}$ is the free $Z[\pi]$-module generated by $S_{\text{odd}}$. Let $J'$ denote the standard tree of sets $\{J'_u\}$ determined by the partially ordered set $J'$ of vertices of $T'$; that is $J'_u = \{v \mid u \in J' \text{ and } u \leqslant v\}$. Let $F[J'; \pi]$ denote the free $Z[\pi]$-module generated by the tree of sets $J'$. As in [1, Prop. 2.2] choose proper bijections $h: S_{\text{ev}} \amalg J' \to J'$ and $g: S_{\text{odd}} \amalg J' \to J'$. Let $H: C_{\text{ev}} \oplus \oplus F[J'; \pi] \to F[J'; \pi]$ and $G: C_{\text{odd}} \oplus F[J'; \pi] \to F[J'; \pi]$ be the induced germ isomorphisms. Then $G \circ (\partial_{\text{ev}} + d^{\text{ev}}) \circ H^{-1}$ is an invertible germ taking $F[J'; \pi]$ to itself and we have

$$\tau(L, K) = \langle G \circ (\partial_{ev} + d^{ev}) \circ H^{-1} \rangle \in \mathrm{Wh}(\pi). \tag{2.2}$$

The torsion $\tau(L, K)$ is independent of the choice of the liftings $\hat{u}$ of the vertices $t'(u)$ and also of the choice of base paths $\Lambda$.

In [6, Chap I, § 5] the torsion is shown to be invariant under subdivision and to be additive in the following sense: Let $M \subset L \subset K$ where $M$ is a proper deformation retract of $L$ and $L$ is a proper deformation retract of $K$. Let $t: T \to L$ be a tree for $L$. Then

$$\tau(K, M) = \tau(K, L) + i_* \tau(L, M) \tag{2.3}$$

where $i_*: \mathrm{Wh}(L; t) \to \mathrm{Wh}(K; i \circ t)$ is the isomorphism induced by the inclusion $i: L \subsetneq K$

Now let $f: X \to Y$ be a proper homotopy equivalence in the category $\mathscr{C}^+$ and let $t: T \to Y$ be a tree. Deform $f$ properly to a proper cellular map $\hat{f}$ and as in [6, Chap I, § 5] define

$$\tau(f) = r_* \tau(M_{\hat{f}}, X) \in \mathrm{Wh}(Y; t) \tag{2.4}$$

where $r: M_f \to Y$ is the standard deformation retraction. If $i: K \subsetneq L$ is an inclusion and $K$ is a deformation retraction of $L$ then $\tau(i) = \tau(L, K)$. This is Lemma 20 of Chap I. of [6]. By Lemma 21 of Chap I of [6] the torsion $\tau(f)$ doesn't depend on the choice of cellular "approximation" $\hat{f}$. Furthermore the following additivity property holds (Lemma 22 of Chap I of [6]): Let $f: X \to Y$ and $g: Y \to X$ be proper homotopy equivalences. Let $t: T \to Y$ be a tree. Then

$$\tau(g \circ f) = \tau(g) + g_* \tau(f) \tag{2.5}$$

where $g$ induces the isomorphism $g_*: \mathrm{Wh}(Y; t) \to \mathrm{Wh}(Z; g \circ t)$.

LEMMA 2.5. *Suppose $f:X\to Y$ is a simple equivalence in the category $\mathscr{C}^+$. Then $\tau(f)=0$.*

*Proof.* The additivity property of torsion reduces the argument to showing that $\tau(L, K)=0$ where $K\nearrow L$ is an expansion in $\mathscr{C}^+$. Write $L=K\cup(\bigcup_{i+1}^{\infty} L_i)$ where $L_i$ is a finite subcomplex which collapses to $K_i=K\cap L_i$ and $\dim(L_i-K)\leqslant n$ for all $i$. Each $L_i$ can be collapsed to $K_i$ by performing elementary collapses in order of decreasing dimension. The additivity property again reduces the problem to showing that $\tau(L, K)=0$ whenever each $K_i\nearrow L_i$ is a sequence of elementary expansions of dimension $k$. However the torsion certainly vanishes in this case because $\partial_{\mathrm{ev}}+d^{\mathrm{ev}}=\partial_k: C_k(L, K)\to C_{k-1}(L, K)$ is a blocked germ [1, §2] with each block being a product of elementary matrices.

Now let $X$ be an object of $\mathscr{C}^+$ and let $t:T\to X$ be a tree for $X$. Let $[f]\in\mathscr{S}^+(X)$ be represented by a proper homotopy equivalence $f:X\to Y$. Choose a proper homotopy inverse $g:Y\to X$ of $f$ and as in [6, Chap. I, § 5] let

$$\tau^+(f)=\tau(g)\in \mathrm{Wh}(X;t). \tag{2.6}$$

Then (2.4) and (2.5) imply that (2.6) gives a well defined homomorphism

$$\tau^+:\mathscr{S}^+(X)\to \mathrm{Wh}(X;t) \quad .$$

and we show in the next section that this is an isomorphism.

## § 3. $\tau^+$ is an isomorphism

In this section we prove

THEOREM 3.1. *Let $X$ be an object of $\mathscr{C}^+$ and let $t:T\to Y$ be a tree for $X$. Then*

$$\tau^+:\mathscr{S}^+(X)\to \mathrm{Wh}(X;t)$$

*is an isomorphism.*

First we prove that $\tau^+$ is injective.

Let $f:X\to Y$ be a cellular proper homotopy equivalence and let $M_f$ be the mapping cylinder of $f$.

LEMMA 3.2. *There is an inclusion $X\subsetneq M$ with $X$ a proper deformation retract of $M$ such that the pair $(M, X)$ is simply equivalent* rel $X$ *in $\mathscr{C}^+$ to the pair $(M_f, X)$ and such that $M-X$ has cells in only two dimensions.*

The proof of this is a straight forward generalization to the proper category of the argument for Lemma 3 of [7]. In fact, $M$ can be chosen to have cells only in dimensions $n+1$ and $n$ where $n \geqslant \max(\dim X, \dim Y)$. Thus $M$ can be constructed to have cells only in dimensions $2k$ and $2k-1$ where $2k-1 \geqslant \max(\dim X, \dim Y)$.

Now suppose $f: X \to Y$ represents an element of $\mathscr{S}^+(X)$ on which $\tau^+$ vanishes. Replace $M_f$ by $M$ as above. Choose $t': T' \to X$ and $\{X_u\}$ as in § 2. Choose a collection $\{M_u\}$ satisfying (a) through (d) of §2 as follows: Let

$M'_u = X_u \cup \{(2k-1)$-cells of $M$ whose attaching maps lie in $X_u\}$. Then set

$M_u = M'_u \cup \{2k$-cells of $M$ whose attaching maps lie in $M'_u\}$. Assume that $2k \geqslant 4$ and let $\tau = \{\tau_u, \gamma_{uv}\}$ be the tree of groups where $\pi_u = \pi_1(M_u, t'_{(u)}) \simeq \pi_1(X_u, t'(u))$. Since $\tau^+(f) = 0$ we know that $\tau = \tau(M, X) \in \mathrm{Wh}(\pi) \cong \mathrm{Wh}(M; t)$ also vanishes. The torsion $\tau$ is represented by the germ

$$\partial = \partial_{2k}: C_{2k}(M, X) \to C_{2k-1}(M, X).$$

Since $\tau = 0$ we know by Lemma 2.7 of [1] that after stabilization of $\partial$ to $(\partial \oplus 1) \oplus \cdots \oplus 1$ it is possible to find blocked germs $A = \sum_{0 \leqslant u} A^u$ and $B = \sum_{0 \leqslant u} B_u$ of $C_{2k}(M, X)$ to itself such that

$$[(\partial \oplus 1) \oplus \cdots \oplus 1] \cdot A \cdot B = P$$

where $P: C_{2k}(M, X) \to C_{2k-1}(M, X)$ is a $\pi$-permutation germ. We also know that each of the square matrices $A^u$ and $B^u$ is a product of elementary matrices over $Z[\pi_u]$. Since $P$ is a $\pi$-permutation germ it has a matrix representative $\{P_u\}$ where $P_u: C_{2k} \times (M, X)_u \to C_{2k-1}(M, X)_u$ satisfies $P_u$(basis element) $= \pm g \cdot$(basis element) where $g \in \pi_u$. The stabilization of $\partial$ to $(\partial \oplus 1) \oplus \cdots \oplus 1$ is achieved geometrically by stabilizing $M$; that is, we replace $M$ by $M \cup \{e_u^{2k-1} \cup e_u^{2k}\}$ where each pair $e_u^{2k-1} \cup e_u^{2k}$ is an elementary expansion attached to the vertex $t'(u) \in X$. To simplify notation we shall still denote the stabilized $\partial$ and the stabilized $M$ by $\partial$ and $M$. Let $S = \{S_u\}$ denote the tree of sets over $J'$ where $S_u = 2k$-cells of $M_u - X$. Since $A$ is blocked we can (as in §2 of [1]) replace the tree $S = \{S_u\}$ by an equivalent tree of sets $D = \{D_u\}$ with $D_u \subset S_u$ and we can amalgamate $A$ so that $\hat{D}_u = D_u - \bigcup_{u<v} D_v$ is a finite set which is the support of $A^u$; that is, $A^u$ is an invertible $Z[\pi_u]$-homomorphism from $F_u = F[\hat{D}_u; \pi_u]$ to itself.

Recall the following: Suppose $L = K \cup_f e^n$ where $f: S^{n-1} \to K$ is the attaching map. If $f$ is deformed by a homotopy $H: S^{n-1} \times I \to K$ to a map $g: S^{n-1} \to L$ then $L' = K \cup_g e^n$ has the same simple type as $L$. Let $W = K \cup_H (e^n \times I)$ where $H: S^{n-1} \times I \to K$ is the attaching map. Then $L \nearrow W \searrow L'$ is the simple equivalence from $L$ to $L'$. Also recall that if $K_0 \to K_1 \to \cdots \to K_{n-1} \to K_n$ is a sequence of elementary expansions and/or contractions then there is a complex $W$ containing $K_0$ and $K_n$ such that $K_0 \nearrow W \searrow K_n$ and $\dim W \leqslant \max(\dim K_i)$.

Now write each matrix $A^u$ as a product of elementary matrices of the form $e_{ij}(\lambda)$: $F_u \to F_u$ where $\lambda = \pm g$ for $g \in \pi_u$. For each $u$ use this product to perform a sequence of deformations of the attaching maps of the $2k$-cells in $\hat{D}_u$ over one another as in the "handle addition" lemma of [7, Lemma 4]. At any step in the process the attaching map of a cell $e^{2k}$ in $\hat{D}_u$ is deformed with support contained in $M_u' \cup$ (other $2k$-cells in $\hat{D}_u$). This procedure changes $M$ by a proper simple equivalence in $\mathscr{C}^+$ to a complex $M'$ such the boundary map $\partial' : C_{2k}(M', X) \to C_{2k-1}(M', X)$ is just the germ $\partial \cdot A$. Repeat the process using a block decomposition of the germ $B$ to get a complex $M''$ such that $\partial'' : C_{2k}(M'', X) \to C_{2k-1}(M'', X)$ is just $\partial \cdot A \cdot B = P$. Since $P$ is a $\pi$-permutation germ the attaching maps of the $2k$-cells can be deformed in a locally finite way so that each $2k$-cell cancels just one $(2k-1)$-cell and misses all the others. This says $M''$ is properly simply equivalent in $\mathscr{C}^+$ to a complex which collapses to $X$. We conclude that $\tau^+ : \mathscr{S}^+(X) \to \mathrm{Wh}(X; t)$ is injective.

It is easy to show that $\tau^+$ is surjective: Let $t' : T' \to X$ and $\{X_u\}$ be as in §2 and let $A : F[J'; \pi] \to F[J'; \pi]$ be an invertible germ. For each vertex $u$ of $T'$ attach a 4-cell $e_u^4$ to $X$ by collapsing $\partial e_u^4$ to the point $t'(u)$. Now attach 5-cells $e_u^5$ in a locally finite way using the germ $A$. This gives a complex $M$ which has $X$ as a proper deformation retract by [2, Th. 3.1] or [5, Prop IV]. Also $\tau(X \to M) = [A] \in \mathrm{Wh}(\pi) \cong \mathrm{Wh}(X; t)$. This completes the proof that $\tau^+$ is an isomorphism.

## § 4. Torsion in the category $\mathscr{C}$

Although the methods of §2 don't directly define the torsion of a proper homotopy equivalence $f : X \to Y$ in the category $\mathscr{C}$ it is possible to prove

THEOREM 4.1. *Let $X$ be an object of $\mathscr{C}$ and let $t : T \to X$ be a tree. There is an isomorphism*

$$\mathscr{S}(X) \cong \mathrm{Wh}(X; t).$$

A consequence of (3.1) and (4.1) is

COROLLARY 4.2. *If $X$ is an object of $\mathscr{C}^+$, then $\mathscr{S}^+(X) \to \mathscr{S}(X)$ is an isomorphism.*

*Proof of* (4.1). Let $t' : T' \to X$ and $\{X_u\}$ be as in §2. Let $\pi = \{\pi_u, \gamma_{uv}\}$ be the associated tree of groups. Recall the exact sequence (3.6) of [1]:

$$\prod_{0<u} \mathrm{Wh}(\pi_u) \xrightarrow{I-S} \prod_{0\leqslant u} \mathrm{Wh}(\pi_u) \xrightarrow{\Delta} \mathrm{Wh}(\pi) \xrightarrow{\partial} \prod_{0<u} \tilde{K}_0(\pi_u) \xrightarrow{I-S} \prod_{0\leqslant u} \tilde{K}_0(\pi_u) \tag{4.3}$$

Let

$$\mathrm{Wh}(\pi)' = \mathrm{Coker}\left[\prod_{0<u} \mathrm{Wh}(\pi_u) \xrightarrow{I-S} \prod_{0\leqslant u} \mathrm{Wh}(\pi_u)\right]$$

Let

$$\tilde{K}_0(\pi)' = \mathrm{Ker}\,[\prod_{0<u} \tilde{K}_0(\pi_u) \xrightarrow{I-S} \prod_{0\leqslant u} \tilde{K}_0(\pi_u)]$$

Then there is the exact sequence

$$0 \to \mathrm{Wh}(\pi)' \to \mathrm{Wh}(\pi) \to \tilde{K}_0(\pi)' \to 0 \tag{4.4}$$

In [5] the following exact sequence is constructed:

$$0 \to \mathrm{Wh}(\pi)' \to \mathcal{S}(X) \to \tilde{K}_0(\pi)' \to 0 \tag{4.5}$$

Hence to prove (4.1) it suffices by the "5-lemma" to construct a homomorphism $\mathrm{Wh}(\pi) \to \mathcal{S}(X)$ which induces a map from the sequence (4.4) to the sequence (4.5). This was essentially done in the proof of (3.1): Take an invertible germ $A: F[J'; \pi] \to \to F[J'; \pi]$ and construct a complex $M(A)$ containing $X$ as a proper deformation retract by attaching one 4-cell $e_u^4$ to the vertex $t'(u) \in X$ and then attaching the 5-cells $e_u^5$ in a locally finite way using the germ $A$. The argument of §3 proving the injectivity of $\tau^+$ shows that $M(A)$ is simply equivalent to $M(A \cdot E)$ whenever $E$ is a blocked germ $E = \sum_u E^u$ such that each $E^u$ is a product of elementary matrices over $Z[\pi_u]$. Stabilization of $A$ to $A \oplus 1$ only changes $M(A)$ by adding elementary expansions. Hence the proper simple type of $M(A)$ doesn't change when $A$ is varied by the defining relations of $\mathrm{Wh}(\pi)$ and we get the required homomorphism $\mathrm{Wh}(\pi) \to \mathcal{S}(X)$.

## § 5. The proper $s$-cobordism theorem

Now that $\mathcal{S}^+(X)$ and $\mathcal{S}(X)$ have been described in algebraic terms the proper $s$-cobordism theorem of [5] can be reformulated.

Recall that a smooth, piecewise linear or topological cobordism $W^n$ from $M_-^n$ to $M_+^n$ is a proper $h$-cobordism provided the inclusions $M_- \subsetneq W$ and $M_+ \subsetneq W$ are proper homotopy equivalences. Suppose $M_-$, $M_+$, $W$ are all non-compact and let $t: T \to M_-$ be a tree.

THEOREM 5.1. *Let $n \geqslant 6$. There is a well defined torsion element $\tau(W; M_-, M_+) \in \mathrm{Wh}(M_-; t)$ which vanishes iff $(W; M_-, M_+)$ is isomorphic to $(M_- \times [0, 1]; M_- \times 0, M \times 1)$. Every element of* $\mathrm{Wh}(M_-; t)$ *can be realized as the torsion of some proper $h$-cobordism on $M_-$.*

This is just the statement of the combined theorems (3.1) and (4.2) above together with Theorem III of [5]. Alternatively, for a direct proof that elements of $\mathrm{Wh}(M_-; t)$ classify proper $h$-cobordisms on $M_-$ one can mimic the argument in the compact case using the methods of §3 in the setting of handlebody theory.

Here are some examples. Compare with [5].

a) Suppose $M_-$ is simply connected and simply connected at infinity. Then it is possible to choose a tree $t':T'\to M_-$ and a collection $\{(M_-)_u\}$ such that each $(M_-)_u$ is simply connected. Thus $\pi=\{\pi_u, \gamma_{uv}\}$ is a tree of trivial groups and (4.3) shows that $\mathrm{Wh}(\pi)\cong\mathrm{Wh}(M_-;t)$ vanishes. Hence any proper $h$-cobordism on such an $M$ is trivial.

b) Suppose $M_-$ has just one stable end $\varepsilon$ with fundamental group $\pi_1\varepsilon$ such that $\pi_1\varepsilon\to\pi_1 M_-$ is an isomorphism. Then (3.10) of [1] implies that $\mathrm{Wh}(M_-;t)=0$ and hence any proper $h$-cobordism on $M_-$ is trivial. In particular, for any non-compact $M_-$, any proper $h$-cobordism on $M_-\times R^2$ is trivial.

There are algebraic product and duality formulae similar to the ones in the compact case. Compare with [5].

Let $(W; M_-, M_+)$ be a proper $h$-cobordism and let $N$ be a compact manifold. Let $t:T\to M_-$ be a tree.

*Product formula* (see Lemma 23 of [6]).

$$\tau(W\times N; M_-\times N, M_+\times N)=\chi(N)\cdot i_*\tau(W; M_-, M_+)$$

where $\chi(N)$ is the Euler class of $N$ and $i_*:\mathrm{Wh}(M_-;t)\to\mathrm{Wh}(M_-\times N;t)$ is the induced homomorphism.

*Remark.* By constrast to the above suppose $(W^n; M_-, M_+)$ is a proper $h$-cobordism (compact or non-compact) and let $N$ be a non-compact manifold. If $n\geqslant 6$ then the proper $h$-cobordism $(W\times N; M_-\times N, M_+\times N)$ is trivial.

The torsion of a proper $h$-cobordism $(W; M_-, M_+)$ can be computed in $\mathrm{Wh}(W;t)$ where there is a conjugation $-:\mathrm{Wh}(W;t)\to\mathrm{Wh}(W;t)$ defined as follows: choose $t':T'\to W$ and $\{W_u\}$ as in §2. For each vertex $u$ of $T'$ there is the orientation homomorphism $w_u:\pi_u\to Z_2=\{+1, -1\}$. If $u\leqslant v$, then $w_v=w_u\circ\gamma_{uv}$. Define the conjugation $-:\pi\to\pi$ to be the collection of compatible conjugations $-:\pi_u\to\pi_u$ where $\bar g=w_u(g)\,g^{-1}$ for $g\in\pi_u$. The conjugation on $\pi$ induces one on $\mathrm{Wh}(\pi)\cong\mathrm{Wh}(W;t)$ by taking any invertible germ $A:F[J';\pi]\to F[J';\pi]$ to $\bar A=$conjugate transpose of $A$.

*Duality formula* (see [6, Chap I, §5])

$$\tau(W; M_+, M_-)=(-1)^{n-1}\,\bar\tau(W; M_-, M_+).$$

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