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## Algebraic Torsion for Infinite Simple Homotopy Types

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This paper is the last of a series dealing with the problem of giving an algebraic description of the torsion invariant for proper  $h$ -cobordisms using the concept of a locally finite infinite matrix. The other two papers are [1] and [6]. The reader should also consult [5]. The main results of this paper are (3.1) and (4.1) which describe the group of proper simple types on a strongly locally finite CW-complex as a  $K_1$ -type group  $\text{Wh}(\pi)$ . The exact sequence (3.6) of [1] allows  $\text{Wh}(\pi)$  to be computed in a number of cases.

### § 1. Infinite simple types

In this section we give a definition of infinite simple homotopy type equivalence to the one in [5] but in a form more convenient for our purposes.

Let  $\mathcal{C}$  denote the category of strongly locally finite, countable, CW-complexes and proper homotopy classes of continuous maps. Recall from [2] that a CW-complex is strongly locally finite provided it is the union of a countable, locally finite collection of finite subcomplexes. Let  $\mathcal{C}^+ \subset \mathcal{C}$  denote the full subcategory whose objects are finite dimensional. A *proper expansion*  $K \nearrow L$  in the category  $\mathcal{C}$  is an inclusion  $K \subset L$  where  $L = K \cup (\bigcup_{i=1}^{\infty} L_i)$  and each  $L_i$  is a finite subcomplex such that

- a)  $(L_i - K) \cap (L_j - K) = \emptyset$  for  $i \neq j$
- b)  $L_i$  collapses to  $K_i = K \cap L_i$ .

A *proper contraction*  $L \searrow K$  is the homotopy inverse of a proper expansion  $K \nearrow L$ . A proper map  $f: X \rightarrow Y$  is a *proper simple homotopy equivalence* (in  $\mathcal{C}$ ) iff there is a sequence of proper expansions and contractions  $X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots X_{n-1} \rightarrow X_n = Y$  whose composition is properly homotopic to  $f$ . If  $f$  is a morphism in  $\mathcal{C}^+$ , then it is a *proper simple equivalence* (in  $\mathcal{C}^+$ ) provided each map  $X_i \rightarrow X_{i+1}$  is a morphism in  $\mathcal{C}^+$ . In particular each proper expansion  $K \nearrow L$  in  $\mathcal{C}^+$  must satisfy the condition

- c) there is an integer  $n$  such that  $\dim(L_i - K) \leq n$  for all  $i$ .

Now given the notion of proper simple equivalence in  $\mathcal{C}$  we have as in [5] the group  $\mathcal{S}(X)$  of proper simple homotopy types of an object  $X$  of  $\mathcal{C}$ . An element of  $\mathcal{S}(X)$  is represented by a proper homotopy equivalence  $f: X \rightarrow Y$  and two such maps  $f_0: X \rightarrow Y_0$  and  $f_1: X \rightarrow Y_1$  are considered the same iff there is a simple equivalence  $s: Y_0 \rightarrow Y_1$  in  $\mathcal{C}$  such that  $s \circ f_0$  is properly homotopic to  $f_1$ . Similarly one has the

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group of proper simple types  $\mathcal{S}^+(X)$  for any object  $X$  in  $\mathcal{C}^+$ . If  $X$  is an object of  $\mathcal{C}^+$  there is a natural map  $\mathcal{S}^+(X) \rightarrow \mathcal{S}(X)$  which we show to be an isomorphism in (4.2).

Throughout this paper any CW-complex will always be assumed to be strongly locally finite. We need this in order to say as in [2] that any proper map can be properly deformed to a cellular map – the starting point for the algebraic theory of simple types describing  $\mathcal{S}(X)$  as a functor of  $\pi_1 X$  and the system of fundamental groups of neighborhoods of infinity. If one works in the category of all locally finite, countable CW-complexes, then  $\mathcal{S}(X)$  may be non-zero even though  $X$  is simply connected and simply connected at infinity. For example, let  $K = e^0 \cup e^1 \cup \dots$  where  $e^n$  is attached to  $e^{n-1}$  by collapsing  $\partial e^n$  to a point in the interior of  $e^{n-1}$ . The property of being strongly locally finite is preserved under proper simple equivalence. Hence, if  $K'$  is any subdivision of  $K$  which is strongly locally finite, then  $K$  and  $K'$  are not simply equivalent although they ought to be. This minor technical point could be remedied if one knew that any locally finite CW-complex had a strongly locally finite subdivision. We get around the difficulty by working only with strongly locally finite CW-complexes. In passing we remark that (\*\*) of [5] is false if the category of all locally finite, countable CW-complexes is used. One must stay with strongly locally finite complexes.

Suppose  $L = K \cup_{f_i} \{e_i^n\} \cup_{g_j} \{e_j^{n+1}\}$ . Suppose the attaching map  $g_j$  of some  $(n+1)$ -cell  $e_j^{n+1}$  misses all the  $n$ -cells except, say,  $e_i^n$  and suppose  $g_j$  takes the top hemisphere of  $\partial e_j^{n+1}$  homeomorphically onto  $e_i^n$  and takes the bottom hemisphere into  $L$ . Thus the pair  $e_j^n \cup e_j^{n+1}$  forms an elementary expansion. We shall say that  $e_j^{n+1}$  *cancels*  $e_i^n$ .

## § 2. Definition of torsion in $\mathcal{C}^+$

In this section we briefly recall the definition of torsion for a proper homotopy equivalence in  $\mathcal{C}^+$ . For details and terminology see [1] and [6, Chap. I, § 5].

Let  $X$  be a non-compact, connected, strongly locally finite CW-complex and let  $t: T \rightarrow X$  be a tree for  $X$ . This means that  $T$  is a locally finite, contractible, one dimensional simplicial complex with a base vertex  $0 \in T$  such that if  $v \in T$  is a vertex different from 0 then at least two 1-cells branch off from  $v$ . Furthermore,  $t: T \rightarrow X$  is required to be a cellular map which is properly  $\frac{1}{2}$ -connected in the sense that  $t^*: H^0(X) \rightarrow H^0(T)$  and  $t^*: H_{\text{end}}^0(X) \rightarrow H_{\text{end}}^0(T)$  are isomorphisms.

The obstruction group  $\text{Wh}(X; t)$ , which is defined in [6, Chap. I, § 5] to capture the torsion of a proper homotopy equivalence  $f: W \rightarrow X$  in  $\mathcal{C}^+$ , is an abelian group which depends only on  $\pi_1 X$  and the inverse system of fundamental groups of neighborhoods of infinity in  $X$ . Up to isomorphism  $\text{Wh}(X; t)$  is also independent of the choice of tree  $t: T \rightarrow X$ . The group  $\text{Wh}(X; t)$  can be computed as follows:

First some generalities. Let  $t: T \rightarrow X$  be any tree for  $X$ . The set  $J$  of vertices of  $T$

can be partially ordered by letting  $u \leq v$  iff the arc from  $v$  to the base vertex 0 passes through  $u$ . Let  $|v|$  denote the number of 1-simplices in the arc from  $v$  to 0. Let  $T_v \subseteq T$  denote the smallest subcomplex containing all vertices  $w$  of  $T$  with  $v \leq w$ . Let  $J' \subset J$  be a cofinal subset (containing the vertex 0) obtained as follows: choose an increasing sequence  $0 = n_0 < n_1 < \dots$ . Then let  $j \in J'$  iff there is some  $n_k$  with  $|j| = n_k$ . Associated to  $J'$  is a tree  $T'$  obtained by inserting a 1-simplex between vertices  $u$  and  $v$  of  $J'$  whenever  $u < v$  and there is no vertex  $w$  of  $J'$  with  $u < w < v$ . The natural map  $T' \rightarrow T$  is properly  $\frac{1}{2}$ -connected and the composition  $T' \rightarrow T \rightarrow X$  is a tree for  $X$ .

Now start with the original tree  $t: T \rightarrow X$ . Then there is a tree  $t': T' \rightarrow X$  derived from  $t: T \rightarrow X$  by the above process and there is a collection  $\{X_u\}$  of infinite, connected subcomplexes of  $X$  (one subcomplex  $X_u$  for each vertex  $u$  of  $T'$ ) satisfying the following conditions (cf. [6, Chap. I, § 5]):

- a)  $X_0 = X$ ,  $X_u \supset X_v$  when  $u \leq v$ , and  $t'(T_u) \subset X_u$
- b)  $X_u \cap X_v = \emptyset$  if  $|u| = |v|$  and  $u \neq v$
- c) for each  $n \geq 0$ ,  $X - \bigcup_{|v|=n} X_v$  is contained in some finite subcomplex of  $X$
- d) given any finite subcomplex  $K$  of  $X$ , there is some  $n \geq 0$  such that  $K \cap (\bigcup_{|v|=n} X_v) = \emptyset$ .

Now for each vertex  $u$  of  $T'$  let  $\pi_u = \pi_1(X_u, t'(u))$ . If  $u \leq v$  define the homomorphism  $\gamma_{uv}: \pi_v \rightarrow \pi_u$  to be "conjugation" by the path  $t'(\alpha_{vu}) \subset X_u$  where  $\alpha_{vu} \subset T'_u$  is the arc from  $u$  to  $v$ . The collection  $\pi = \{\pi_u, \gamma_{uv}\}$  is a tree of groups over the set  $J'$  of vertices of  $T'$ . Let  $Z[\pi] = \{Z[\pi_u], \gamma_{uv}\}$  denote the associated tree of group rings. Let  $\text{Wh}(\pi)$  be as defined in [1] and [6, Chap. I, § 5]. Then there is an isomorphism

$$\text{Wh}(\pi) \cong \text{Wh}(X; t)$$

See [6, Chap. I, § 5]. The point of using  $\text{Wh}(X; t)$  as the obstruction group rather than one of its "representatives"  $\text{Wh}(\pi)$  is to make the torsion well defined and independent of various choices such as the  $X_u$  above. However in proving certain things one often uses a convenient choice of a  $\text{Wh}(\pi)$ . Also, there is the basic algebraic exact sequence (see (4.3) below) which relates  $\text{Wh}(\pi)$  with the  $\text{Wh}(\pi_u)$  and  $\tilde{K}_0(\pi_u)$  and allows one to compute  $\text{Wh}(\pi)$  in a number of cases.

We will briefly indicate how to define the torsion

$$\tau(L, K) \in \text{Wh}(\pi) \cong \text{Wh}(L; t)$$

of an inclusion  $K \rightarrow L$  where  $K$  is a proper deformation retract of  $L$  and  $\dim(L - K) < \infty$ . Here  $\pi = \{\pi_u, \gamma_{uv}\}$  is the tree of groups corresponding to any choice of a tree  $t': T' \rightarrow L$  derived from  $t: T \rightarrow L$  as above and any choice of a system  $\{L_u\}$  satisfying (a) through (d) with respect to  $t': T' \rightarrow L$ . The definition of  $\tau(L, K)$  given below is in the spirit of [4]. Using the general machinery of [6, Chap. I, § 5] and [1] it is not hard to see that this approach to torsion for infinite simple types is equivalent to the one worked out



in [6, Chap. I, § 5] which follows the lines of [3]. The argument showing the equivalence is entirely similar to the one in the compact case.

For any CW-complex  $X$ , let  $\tilde{X}$  denote the universal covering space  $p: \tilde{X} \rightarrow X$ . If  $Y \subset X$ , let  $\tilde{Y} = p^{-1}(Y)$ .

By condition (a) above  $t'(u) \in L_u$  for every vertex  $u$  of  $T'$ . Select a fixed lifting  $\hat{u} \in \tilde{L}_u$  of  $t'(u)$ . If  $v$  is a vertex of  $T'$  and  $u \leq v$ , let  $v' \in \tilde{L}_u$  be the lifting of  $t'(v) \in L_u$  obtained as the end point of the lifting of the path  $t(\alpha_{vu})$  to a path in  $\tilde{L}_u$  starting at  $\hat{u}$ . Here  $\alpha_{vu}$  denotes the arc from  $u$  to  $v$  in  $T'_u$ . If  $v$  is a vertex of  $T'$  with  $u \leq v$  there is a unique map  $\tilde{L}_v \rightarrow \tilde{L}_u$  covering the inclusion  $L_v \rightarrow L_u$  such that  $\hat{v} \in \tilde{L}_v$  goes to  $v' \in \tilde{L}_u$ . Furthermore, if  $u \leq v \leq w$  the map  $\tilde{L}_w \rightarrow \tilde{L}_u$  is the composition  $\tilde{L}_w \rightarrow \tilde{L}_v \rightarrow \tilde{L}_u$ .

The next choice we make is to select a locally finite collection  $\mathcal{A}$  of paths  $\alpha(\sigma)$  from the barycenters of cells  $\sigma$  of  $L$  to the images  $t'(\alpha(\sigma))$  of vertices  $u(\sigma)$  of  $T'$  such that if  $\sigma \subset L_u$  then  $\alpha(\sigma) \subset L_u$ . If  $\sigma \subset L_u$  the path  $\alpha(\sigma)$  determines a path  $\beta_u(\sigma)$  from  $\sigma$  to  $t'(u)$  in  $L_u$ : first follow  $\alpha(\sigma)$  to  $t'(u(\sigma))$  and then follow  $t'(\alpha_{u, u(\sigma)})$  to  $t'(u)$ . Here  $\alpha_{u, u(\sigma)}$  is the arc in  $T'$  from  $u(\sigma)$  to  $u$ .

If  $X$  is any CW-complex, let  $X^n$  denote the  $n$ -skeleton of  $X$ .

Now define the based  $Z[\pi]$ -module  $C_n(L, K)$  as follows:

$$C_n(L, K) = \{C_n(L, K)_u\}$$

where for each vertex  $u$  of  $T'$

$$C_n(L, K)_u = H_n(\overline{L_u^n}, \overline{L_u^{n-1}} \cup \overline{(L_u^n \cap K)})$$

The “bar” is taken with respect to the universal cover  $\tilde{L}_u \rightarrow L_u$ . The  $Z[\pi_u]$ -module  $C_n(L, K)_u$  is free with one basis element for each  $n$ -cell of  $L_u - K$ . The basis element corresponding to an  $n$ -cell  $\sigma$  of  $L_u - K$  is given by the lifting of  $\sigma$  to  $\tilde{L}_u$  determined by the path  $\beta_u(\sigma)$ . If  $u \leq v$  the map  $\tilde{L}_v \rightarrow \tilde{L}_u$  determines a homomorphism  $C_n(L, K)_v \rightarrow C_n(L, K)_u$  and in fact we have an injection

$$C_n(L, K)_v \otimes_{Z[\pi_v]} Z[\pi_u] \rightarrow C_n(L, K)_u$$

whose image is the free submodule generated by the  $n$ -cells of  $L_u - K$  which lie in  $L_v - K$ . The boundary operators  $\partial_n^u: C_n(L, K)_u \rightarrow C_{n-1}(L, K)_u$  are compatible with the maps  $C_n(L, K)_v \rightarrow C_n(L, K)_u$  and therefore define a morphism of  $Z[\pi]$ -modules

$$\partial_n: C_n(L, K) \rightarrow C_{n-1}(L, K)$$

which satisfies  $\partial_{n-1} \circ \partial_n = 0$ . This gives a chain complex

$$(C_*, \partial_*) = \{C_n(L, K), \partial_n\}$$

of based  $Z[\pi]$ -modules. In fact, if  $S^n = \{S_u^n\}$  is the tree of sets over  $J'$  where  $S_u^n$

consists of the  $n$ -cells of  $L_u - K$ , then  $C_n(L, K)$  is the free  $Z[\pi]$ -module generated by  $S^n$ . Since  $\dim(L - K) < \infty$  at most finitely many of the chain groups  $C_n(L, K)$  are not zero.

Let  $r: L \times I \rightarrow L$  be a proper deformation retraction of  $L$  down into  $K$ . We can assume  $r$  is cellular by [2, Th. 1.7]. For each vertex  $u$  of  $T'$  choose a cofinite subcomplex  $N_u$  of  $L_u$  (i.e.,  $L_u - N_u$  has only finitely many cells) such that

i)  $N_0 = L_0$  and  $N_u \supset N_v$  whenever  $u \leq v$

ii)  $r(N_u \times I) \subset L_u$ .

The map  $r: N_u \times I \rightarrow L_u$  has a unique lifting  $r_u: \bar{N}_u \times I \rightarrow \tilde{L}_u$  such that  $r_u$  restricted to  $\bar{N}_u \times 0$  is the inclusion and such that whenever  $u \leq v$  there is a commutative diagram

$$\begin{array}{ccc} \bar{N}_v \times I & \xrightarrow{r_v} & \tilde{L}_v \\ \downarrow & & \downarrow \\ \bar{N}_u \times I & \xrightarrow{r_u} & \tilde{L}_u \end{array}$$

Let  $\hat{C}_n(L, K)_u \subset C_n(L, K)_u$  be the free  $Z[\pi]$ -submodule generated by the  $n$ -cells of  $L_u - K$  belonging to  $N_u - K$  and let  $i_u: \hat{C}_n(L, K)_u \rightarrow C_n(L, K)_u$  denote the inclusion map. The maps  $r_u: \bar{N}_u \times I \rightarrow \tilde{L}_u$  induce coboundary operators

$$d_u^n: \hat{C}_n(L, K)_u \rightarrow C_{n+1}(L, K)_u$$

compatible with the morphisms  $\hat{C}_n(L, K)_v \rightarrow \hat{C}_n(L, K)_u$  and  $C_n(L, K)_v \rightarrow C_n(L, K)_u$  such that for each vertex  $u$  of  $T'$

$$\partial_{n+1}^u \circ d_u^n + d_u^{n-1} \circ \hat{\partial}_n^u = \begin{cases} id, & \text{for } u = 0 \\ i_u + \text{finite matrix}, & \text{for } u > 0 \end{cases}$$

Here  $\partial_n^u: \hat{C}_n(L, K)_u \rightarrow \hat{C}_{n-1}(L, K)_u$  is the restriction of  $\partial_n^u$ . Thus the collection  $d^n = \{d_u^n\}$  defines a germ  $d^n: C_n(L, K) \rightarrow C_{n+1}(L, K)$  such that on the germ level we have

$$\partial_{n+1} \circ d^n + d^{n-1} \circ \partial_n = id. \quad (*)$$

This shows that  $(C_*, \partial_*)$  is an acyclic complex of based modules over the tree of rings  $Z[\pi]$  and as in [6, Chap. I, § 5] we can define the torsion to be

$$\tau(L, K) = \tau(C_*, \partial_*) \in \text{Wh}(\pi) \quad (2.1)$$

Now here is the way to define  $\tau(L, K)$  in the spirit of [4]: By replacing  $d^n$  with  $d^n \circ \partial_{n+1} \circ d^n$  (if necessary) we can assume that  $d^{n+1} \circ d^n = 0$ . Let  $C_{\text{ev}} = \bigoplus_{0 \leq k} C_{2k}$  and  $C_{\text{odd}} = \bigoplus_{0 \leq k} C_{2k+1}$ . The formula  $(*)$  implies that  $\partial_{\text{ev}} + d^{\text{ev}}: C_{\text{ev}} \rightarrow C_{\text{odd}}$  is an isomorphism on the germ level whose inverse is  $\partial_{\text{odd}} + d^{\text{odd}}: C_{\text{odd}} \rightarrow C_{\text{ev}}$ . Let the trees of

sets  $S_{\text{ev}}$  and  $S_{\text{odd}}$  be defined as the disjoint unions of trees of sets

$$S_{\text{ev}} = \coprod_{0 \geq k} S^{2k} \quad \text{and} \quad S_{\text{odd}} = \coprod_{0 \geq k} S^{2k+1}$$

Then  $C_{\text{ev}}$  is the free  $Z[\pi]$ -module generated by  $S_{\text{ev}}$  and  $C_{\text{odd}}$  is the free  $Z[\pi]$ -module generated by  $S_{\text{odd}}$ . Let  $J'$  denote the standard tree of sets  $\{J'_u\}$  determined by the partially ordered set  $J'$  of vertices of  $T'$ ; that is  $J'_u = \{v \mid u \in J' \text{ and } u \leq v\}$ . Let  $F[J'; \pi]$  denote the free  $Z[\pi]$ -module generated by the tree of sets  $J'$ . As in [1, Prop. 2.2] choose proper bijections  $h: S_{\text{ev}} \amalg J' \rightarrow J'$  and  $g: S_{\text{odd}} \amalg J' \rightarrow J'$ . Let  $H: C_{\text{ev}} \oplus \oplus F[J'; \pi] \rightarrow F[J'; \pi]$  and  $G: C_{\text{odd}} \oplus \oplus F[J'; \pi] \rightarrow F[J'; \pi]$  be the induced germ isomorphisms. Then  $G \circ (\partial_{\text{ev}} + d^{\text{ev}}) \circ H^{-1}$  is an invertible germ taking  $F[J'; \pi]$  to itself and we have

$$\tau(L, K) = \langle G \circ (\partial_{\text{ev}} + d^{\text{ev}}) \circ H^{-1} \rangle \in \text{Wh}(\pi). \quad (2.2)$$

The torsion  $\tau(L, K)$  is independent of the choice of the liftings  $\hat{u}$  of the vertices  $t'(u)$  and also of the choice of base paths  $\lambda$ .

In [6, Chap I, § 5] the torsion is shown to be invariant under subdivision and to be additive in the following sense: Let  $M \subset L \subset K$  where  $M$  is a proper deformation retract of  $L$  and  $L$  is a proper deformation retract of  $K$ . Let  $t: T \rightarrow L$  be a tree for  $L$ . Then

$$\tau(K, M) = \tau(K, L) + i_* \tau(L, M) \quad (2.3)$$

where  $i_*: \text{Wh}(L; t) \rightarrow \text{Wh}(K; i \circ t)$  is the isomorphism induced by the inclusion  $i: L \hookrightarrow K$ .

Now let  $f: X \rightarrow Y$  be a proper homotopy equivalence in the category  $\mathcal{C}^+$  and let  $t: T \rightarrow Y$  be a tree. Deform  $f$  properly to a proper cellular map  $\hat{f}$  and as in [6, Chap I, § 5] define

$$\tau(f) = r_* \tau(M_{\hat{f}}, X) \in \text{Wh}(Y; t) \quad (2.4)$$

where  $r: M_{\hat{f}} \rightarrow Y$  is the standard deformation retraction. If  $i: K \hookrightarrow L$  is an inclusion and  $K$  is a deformation retract of  $L$  then  $\tau(i) = \tau(L, K)$ . This is Lemma 20 of Chap I. of [6]. By Lemma 21 of Chap I of [6] the torsion  $\tau(f)$  doesn't depend on the choice of cellular "approximation"  $\hat{f}$ . Furthermore the following additivity property holds (Lemma 22 of Chap I of [6]): Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  be proper homotopy equivalences. Let  $t: T \rightarrow Y$  be a tree. Then

$$\tau(g \circ f) = \tau(g) + g_* \tau(f) \quad (2.5)$$

where  $g$  induces the isomorphism  $g_*: \text{Wh}(Y; t) \rightarrow \text{Wh}(X; g \circ t)$ .

LEMMA 2.5. *Suppose  $f: X \rightarrow Y$  is a simple equivalence in the category  $\mathcal{C}^+$ . Then  $\tau(f) = 0$ .*

*Proof.* The additivity property of torsion reduces the argument to showing that  $\tau(L, K) = 0$  where  $K \nearrow L$  is an expansion in  $\mathcal{C}^+$ . Write  $L = K \cup (\bigcup_{i=1}^{\infty} L_i)$  where  $L_i$  is a finite subcomplex which collapses to  $K_i = K \cap L_i$  and  $\dim(L_i - K) \leq n$  for all  $i$ . Each  $L_i$  can be collapsed to  $K_i$  by performing elementary collapses in order of decreasing dimension. The additivity property again reduces the problem to showing that  $\tau(L, K) = 0$  whenever each  $K_i \nearrow L_i$  is a sequence of elementary expansions of dimension  $k$ . However the torsion certainly vanishes in this case because  $\partial_{ev} + d^{ev} = \partial_k: C_k(L, K) \rightarrow C_{k-1}(L, K)$  is a blocked germ [1, §2] with each block being a product of elementary matrices.

Now let  $X$  be an object of  $\mathcal{C}^+$  and let  $t: T \rightarrow X$  be a tree for  $X$ . Let  $[f] \in \mathcal{S}^+(X)$  be represented by a proper homotopy equivalence  $f: X \rightarrow Y$ . Choose a proper homotopy inverse  $g: Y \rightarrow X$  of  $f$  and as in [6, Chap. I, § 5] let

$$\tau^+(f) = \tau(g) \in \text{Wh}(X; t). \quad (2.6)$$

Then (2.4) and (2.5) imply that (2.6) gives a well defined homomorphism

$$\tau^+: \mathcal{S}^+(X) \rightarrow \text{Wh}(X; t) \quad .$$

and we show in the next section that this is an isomorphism.

### § 3. $\tau^+$ is an isomorphism

In this section we prove

THEOREM 3.1. *Let  $X$  be an object of  $\mathcal{C}^+$  and let  $t: T \rightarrow X$  be a tree for  $X$ . Then*

$$\tau^+: \mathcal{S}^+(X) \rightarrow \text{Wh}(X; t)$$

*is an isomorphism.*

First we prove that  $\tau^+$  is injective.

Let  $f: X \rightarrow Y$  be a cellular proper homotopy equivalence and let  $M_f$  be the mapping cylinder of  $f$ .

LEMMA 3.2. *There is an inclusion  $X \hookrightarrow M$  with  $X$  a proper deformation retract of  $M$  such that the pair  $(M, X)$  is simply equivalent rel  $X$  in  $\mathcal{C}^+$  to the pair  $(M_f, X)$  and such that  $M - X$  has cells in only two dimensions.*

The proof of this is a straight forward generalization to the proper category of the argument for Lemma 3 of [7]. In fact,  $M$  can be chosen to have cells only in dimen-

sions  $n+1$  and  $n$  where  $n \geq \max(\dim X, \dim Y)$ . Thus  $M$  can be constructed to have cells only in dimensions  $2k$  and  $2k-1$  where  $2k-1 \geq \max(\dim X, \dim Y)$ .

Now suppose  $f: X \rightarrow Y$  represents an element of  $\mathcal{S}^+(X)$  on which  $\tau^+$  vanishes. Replace  $M_f$  by  $M$  as above. Choose  $t': T' \rightarrow X$  and  $\{X_u\}$  as in § 2. Choose a collection  $\{M_u\}$  satisfying (a) through (d) of § 2 as follows: Let

$M'_u = X_u \cup \{(2k-1)\text{-cells of } M \text{ whose attaching maps lie in } X_u\}$ . Then set

$M_u = M'_u \cup \{2k\text{-cells of } M \text{ whose attaching maps lie in } M'_u\}$ . Assume that  $2k \geq 4$  and let  $\tau = \{\tau_u, \gamma_{uv}\}$  be the tree of groups where  $\pi_u = \pi_1(M_u, t'_u) \simeq \pi_1(X_u, t'(u))$ . Since  $\tau^+(f) = 0$  we know that  $\tau = \tau(M, X) \in \text{Wh}(\pi) \cong \text{Wh}(M; t)$  also vanishes. The torsion  $\tau$  is represented by the germ

$$\partial = \partial_{2k}: C_{2k}(M, X) \rightarrow C_{2k-1}(M, X).$$

Since  $\tau = 0$  we know by Lemma 2.7 of [1] that after stabilization of  $\partial$  to  $(\partial \oplus 1) \oplus \cdots \oplus 1$  it is possible to find blocked germs  $A = \sum_{0 \leq u} A^u$  and  $B = \sum_{0 \leq u} B_u$  of  $C_{2k}(M, X)$  to itself such that

$$[(\partial \oplus 1) \oplus \cdots \oplus 1] \cdot A \cdot B = P$$

where  $P: C_{2k}(M, X) \rightarrow C_{2k-1}(M, X)$  is a  $\pi$ -permutation germ. We also know that each of the square matrices  $A^u$  and  $B^u$  is a product of elementary matrices over  $Z[\pi_u]$ . Since  $P$  is a  $\pi$ -permutation germ it has a matrix representative  $\{P_u\}$  where  $P_u: C_{2k} \times (M, X)_u \rightarrow C_{2k-1}(M, X)_u$  satisfies  $P_u(\text{basis element}) = \pm g \cdot (\text{basis element})$  where  $g \in \pi_u$ . The stabilization of  $\partial$  to  $(\partial \oplus 1) \oplus \cdots \oplus 1$  is achieved geometrically by stabilizing  $M$ ; that is, we replace  $M$  by  $M \cup \{e_u^{2k-1} \cup e_u^{2k}\}$  where each pair  $e_u^{2k-1} \cup e_u^{2k}$  is an elementary expansion attached to the vertex  $t'(u) \in X$ . To simplify notation we shall still denote the stabilized  $\partial$  and the stabilized  $M$  by  $\partial$  and  $M$ . Let  $S = \{S_u\}$  denote the tree of sets over  $J'$  where  $S_u = 2k\text{-cells of } M_u - X$ . Since  $A$  is blocked we can (as in § 2 of [1]) replace the tree  $S = \{S_u\}$  by an equivalent tree of sets  $D = \{D_u\}$  with  $D_u \subset S_u$  and we can amalgamate  $A$  so that  $\hat{D}_u = D_u - \bigcup_{u < v} D_v$  is a finite set which is the support of  $A^u$ ; that is,  $A^u$  is an invertible  $Z[\pi_u]$ -homomorphism from  $F_u = F[\hat{D}_u; \pi_u]$  to itself.

Recall the following: Suppose  $L = K \cup_f e^n$  where  $f: S^{n-1} \rightarrow K$  is the attaching map. If  $f$  is deformed by a homotopy  $H: S^{n-1} \times I \rightarrow K$  to a map  $g: S^{n-1} \rightarrow L$  then  $L' = K \cup_g e^n$  has the same simple type as  $L$ . Let  $W = K \cup_H (e^n \times I)$  where  $H: S^{n-1} \times I \rightarrow K$  is the attaching map. Then  $L \nearrow W \searrow L'$  is the simple equivalence from  $L$  to  $L'$ . Also recall that if  $K_0 \rightarrow K_1 \rightarrow \cdots \rightarrow K_{n-1} \rightarrow K_n$  is a sequence of elementary expansions and/or contractions then there is a complex  $W$  containing  $K_0$  and  $K_n$  such that  $K_0 \nearrow W \searrow K_n$  and  $\dim W \leq \max(\dim K_i)$ .

Now write each matrix  $A^u$  as a product of elementary matrices of the form  $e_{ij}(\lambda)$ :  $F_u \rightarrow F_u$  where  $\lambda = \pm g$  for  $g \in \pi_u$ . For each  $u$  use this product to perform a sequence

of deformations of the attaching maps of the  $2k$ -cells in  $\hat{D}_u$  over one another as in the “handle addition” lemma of [7, Lemma 4]. At any step in the process the attaching map of a cell  $e^{2k}$  in  $\hat{D}_u$  is deformed with support contained in  $M'_u \cup$  (other  $2k$ -cells in  $\hat{D}_u$ ). This procedure changes  $M$  by a proper simple equivalence in  $\mathcal{C}^+$  to a complex  $M'$  such the boundary map  $\partial': C_{2k}(M', X) \rightarrow C_{2k-1}(M', X)$  is just the germ  $\partial \cdot A$ . Repeat the process using a block decomposition of the germ  $B$  to get a complex  $M''$  such that  $\partial'': C_{2k}(M'', X) \rightarrow C_{2k-1}(M'', X)$  is just  $\partial \cdot A \cdot B = P$ . Since  $P$  is a  $\pi$ -permutation germ the attaching maps of the  $2k$ -cells can be deformed in a locally finite way so that each  $2k$ -cell cancels just one  $(2k-1)$ -cell and misses all the others. This says  $M''$  is properly simply equivalent in  $\mathcal{C}^+$  to a complex which collapses to  $X$ . We conclude that  $\tau^+: \mathcal{S}^+(X) \rightarrow \text{Wh}(X; t)$  is injective.

It is easy to show that  $\tau^+$  is surjective: Let  $t': T' \rightarrow X$  and  $\{X_u\}$  be as in §2 and let  $A: F[J'; \pi] \rightarrow F[J'; \pi]$  be an invertible germ. For each vertex  $u$  of  $T'$  attach a 4-cell  $e_u^4$  to  $X$  by collapsing  $\partial e_u^4$  to the point  $t'(u)$ . Now attach 5-cells  $e_u^5$  in a locally finite way using the germ  $A$ . This gives a complex  $M$  which has  $X$  as a proper deformation retract by [2, Th. 3.1] or [5, Prop IV]. Also  $\tau(X \rightarrow M) = [A] \in \text{Wh}(\pi) \cong \text{Wh}(X; t)$ . This completes the proof that  $\tau^+$  is an isomorphism.

#### § 4. Torsion in the category $\mathcal{C}$

Although the methods of §2 don't directly define the torsion of a proper homotopy equivalence  $f: X \rightarrow Y$  in the category  $\mathcal{C}$  it is possible to prove

**THEOREM 4.1.** *Let  $X$  be an object of  $\mathcal{C}$  and let  $t: T \rightarrow X$  be a tree. There is an isomorphism*

$$\mathcal{S}(X) \cong \text{Wh}(X; t).$$

A consequence of (3.1) and (4.1) is

**COROLLARY 4.2.** *If  $X$  is an object of  $\mathcal{C}^+$ , then  $\mathcal{S}^+(X) \rightarrow \mathcal{S}(X)$  is an isomorphism.*

*Proof of (4.1).* Let  $t': T' \rightarrow X$  and  $\{X_u\}$  be as in §2. Let  $\pi = \{\pi_u, \gamma_{uv}\}$  be the associated tree of groups. Recall the exact sequence (3.6) of [1]:

$$\prod_{0 < u} \text{Wh}(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \text{Wh}(\pi_u) \xrightarrow{A} \text{Wh}(\pi) \xrightarrow{\partial} \prod_{0 < u} \tilde{K}_0(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \tilde{K}_0(\pi_u) \quad (4.3)$$

Let

$$\text{Wh}(\pi)' = \text{Coker} \left[ \prod_{0 < u} \text{Wh}(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \text{Wh}(\pi_u) \right]$$

Let

$$\tilde{K}_0(\pi)' = \text{Ker} \left[ \prod_{0 < u} \tilde{K}_0(\pi_u) \xrightarrow{I^{-S}} \prod_{0 \leq u} \tilde{K}_0(\pi_u) \right]$$

Then there is the exact sequence

$$0 \rightarrow \text{Wh}(\pi)' \rightarrow \text{Wh}(\pi) \rightarrow \tilde{K}_0(\pi)' \rightarrow 0 \quad (4.4)$$

In [5] the following exact sequence is constructed:

$$0 \rightarrow \text{Wh}(\pi)' \rightarrow \mathcal{S}(X) \rightarrow \tilde{K}_0(\pi)' \rightarrow 0 \quad (4.5)$$

Hence to prove (4.1) it suffices by the “5-lemma” to construct a homomorphism  $\text{Wh}(\pi) \rightarrow \mathcal{S}(X)$  which induces a map from the sequence (4.4) to the sequence (4.5). This was essentially done in the proof of (3.1): Take an invertible germ  $A: F[J'; \pi] \rightarrow F[J'; \pi]$  and construct a complex  $M(A)$  containing  $X$  as a proper deformation retract by attaching one 4-cell  $e_u^4$  to the vertex  $t'(u) \in X$  and then attaching the 5-cells  $e_u^5$  in a locally finite way using the germ  $A$ . The argument of §3 proving the injectivity of  $\tau^+$  shows that  $M(A)$  is simply equivalent to  $M(A \cdot E)$  whenever  $E$  is a blocked germ  $E = \sum_u E^u$  such that each  $E^u$  is a product of elementary matrices over  $Z[\pi_u]$ . Stabilization of  $A$  to  $A \oplus 1$  only changes  $M(A)$  by adding elementary expansions. Hence the proper simple type of  $M(A)$  doesn't change when  $A$  is varied by the defining relations of  $\text{Wh}(\pi)$  and we get the required homomorphism  $\text{Wh}(\pi) \rightarrow \mathcal{S}(X)$ .

## § 5. The proper $s$ -cobordism theorem

Now that  $\mathcal{S}^+(X)$  and  $\mathcal{S}(X)$  have been described in algebraic terms the proper  $s$ -cobordism theorem of [5] can be reformulated.

Recall that a smooth, piecewise linear or topological cobordism  $W^n$  from  $M_-^n$  to  $M_+^n$  is a proper  $h$ -cobordism provided the inclusions  $M_- \hookrightarrow W$  and  $M_+ \hookrightarrow W$  are proper homotopy equivalences. Suppose  $M_-$ ,  $M_+$ ,  $W$  are all non-compact and let  $t: T \rightarrow M_-$  be a tree.

**THEOREM 5.1.** *Let  $n \geq 6$ . There is a well defined torsion element  $\tau(W; M_-, M_+) \in \text{Wh}(M_-; t)$  which vanishes iff  $(W; M_-, M_+)$  is isomorphic to  $(M_- \times [0, 1]; M_- \times 0, M \times 1)$ . Every element of  $\text{Wh}(M_-; t)$  can be realized as the torsion of some proper  $h$ -cobordism on  $M_-$ .*

This is just the statement of the combined theorems (3.1) and (4.2) above together with Theorem III of [5]. Alternatively, for a direct proof that elements of  $\text{Wh}(M_-; t)$  classify proper  $h$ -cobordisms on  $M_-$  one can mimic the argument in the compact case using the methods of §3 in the setting of handlebody theory.



Here are some examples. Compare with [5].

a) Suppose  $M_-$  is simply connected and simply connected at infinity. Then it is possible to choose a tree  $t': T' \rightarrow M_-$  and a collection  $\{(M_-)_u\}$  such that each  $(M_-)_u$  is simply connected. Thus  $\pi = \{\pi_u, \gamma_{uv}\}$  is a tree of trivial groups and (4.3) shows that  $\text{Wh}(\pi) \cong \text{Wh}(M_-; t)$  vanishes. Hence any proper  $h$ -cobordism on such an  $M$  is trivial.

b) Suppose  $M_-$  has just one stable end  $\varepsilon$  with fundamental group  $\pi_1 \varepsilon$  such that  $\pi_1 \varepsilon \rightarrow \pi_1 M_-$  is an isomorphism. Then (3.10) of [1] implies that  $\text{Wh}(M_-; t) = 0$  and hence any proper  $h$ -cobordism on  $M_-$  is trivial. In particular, for any non-compact  $M_-$ , any proper  $h$ -cobordism on  $M_- \times R^2$  is trivial.

There are algebraic product and duality formulae similar to the ones in the compact case. Compare with [5].

Let  $(W; M_-, M_+)$  be a proper  $h$ -cobordism and let  $N$  be a compact manifold. Let  $t: T \rightarrow M_-$  be a tree.

*Product formula* (see Lemma 23 of [6]).

$$\tau(W \times N; M_- \times N, M_+ \times N) = \chi(N) \cdot i_* \tau(W; M_-, M_+)$$

where  $\chi(N)$  is the Euler class of  $N$  and  $i_*: \text{Wh}(M_-; t) \rightarrow \text{Wh}(M_- \times N; t)$  is the induced homomorphism.

*Remark.* By contrast to the above suppose  $(W^n; M_-, M_+)$  is a proper  $h$ -cobordism (compact or non-compact) and let  $N$  be a non-compact manifold. If  $n \geq 6$  then the proper  $h$ -cobordism  $(W \times N; M_- \times N, M_+ \times N)$  is trivial.

The torsion of a proper  $h$ -cobordism  $(W; M_-, M_+)$  can be computed in  $\text{Wh}(W; t)$  where there is a conjugation  $-: \text{Wh}(W; t) \rightarrow \text{Wh}(W; t)$  defined as follows: choose  $t': T' \rightarrow W$  and  $\{W_u\}$  as in §2. For each vertex  $u$  of  $T'$  there is the orientation homomorphism  $w_u: \pi_u \rightarrow Z_2 = \{+1, -1\}$ . If  $u \leq v$ , then  $w_v = w_u \circ \gamma_{uv}$ . Define the conjugation  $-: \pi \rightarrow \pi$  to be the collection of compatible conjugations  $-: \pi_u \rightarrow \pi_u$  where  $\bar{g} = w_u(g) g^{-1}$  for  $g \in \pi_u$ . The conjugation on  $\pi$  induces one on  $\text{Wh}(\pi) \cong \text{Wh}(W; t)$  by taking any invertible germ  $A: F[J'; \pi] \rightarrow F[J'; \pi]$  to  $\bar{A} = \text{conjugate transpose of } A$ .

*Duality formula* (see [6, Chap I, §5])

$$\tau(W; M_+, M_-) = (-1)^{n-1} \bar{\tau}(W; M_-, M_+).$$

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