

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 47 (1972)

Artikel: Algebraic Torsion for Infinite Simple Homotopy Types
Autor: Farrell, F.T. / Wagoner, J.B.
DOI: <https://doi.org/10.5169/seals-36381>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Algebraic Torsion for Infinite Simple Homotopy Types

F. T. FARRELL¹⁾ and J. B. WAGONER²⁾

This paper is the last of a series dealing with the problem of giving an algebraic description of the torsion invariant for proper h -cobordisms using the concept of a locally finite infinite matrix. The other two papers are [1] and [6]. The reader should also consult [5]. The main results of this paper are (3.1) and (4.1) which describe the group of proper simple types on a strongly locally finite CW-complex as a K_1 -type group $\text{Wh}(\pi)$. The exact sequence (3.6) of [1] allows $\text{Wh}(\pi)$ to be computed in a number of cases.

§ 1. Infinite simple types

In this section we give a definition of infinite simple homotopy type equivalence to the one in [5] but in a form more convenient for our purposes.

Let \mathcal{C} denote the category of strongly locally finite, countable, CW-complexes and proper homotopy classes of continuous maps. Recall from [2] that a CW-complex is strongly locally finite provided it is the union of a countable, locally finite collection of finite subcomplexes. Let $\mathcal{C}^+ \subset \mathcal{C}$ denote the full subcategory whose objects are finite dimensional. A *proper expansion* $K \nearrow L$ in the category \mathcal{C} is an inclusion $K \subset L$ where $L = K \cup (\bigcup_{i=1}^{\infty} L_i)$ and each L_i is a finite subcomplex such that

- a) $(L_i - K) \cap (L_j - K) = \emptyset$ for $i \neq j$
- b) L_i collapses to $K_i = K \cap L_i$.

A *proper contraction* $L \searrow K$ is the homotopy inverse of a proper expansion $K \nearrow L$. A proper map $f: X \rightarrow Y$ is a *proper simple homotopy equivalence* (in \mathcal{C}) iff there is a sequence of proper expansions and contractions $X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = Y$ whose composition is properly homotopic to f . If f is a morphism in \mathcal{C}^+ , then it is a *proper simple equivalence* (in \mathcal{C}^+) provided each map $X_i \rightarrow X_{i+1}$ is a morphism in \mathcal{C}^+ . In particular each proper expansion $K \nearrow L$ in \mathcal{C}^+ must satisfy the condition

- c) there is an integer n such that $\dim(L_i - K) \leq n$ for all i .

Now given the notion of proper simple equivalence in \mathcal{C} we have as in [5] the group $\mathcal{S}(X)$ of proper simple homotopy types of an object X of \mathcal{C} . An element of $\mathcal{S}(X)$ is represented by a proper homotopy equivalence $f: X \rightarrow Y$ and two such maps $f_0: X \rightarrow Y_0$ and $f_1: X \rightarrow Y_1$ are considered the same iff there is a simple equivalence $s: Y_0 \rightarrow Y_1$ in \mathcal{C} such that $s \circ f_0$ is properly homotopic to f_1 . Similarly one has the

¹⁾ Partially supported by NSF Grant GP-29697

²⁾ Partially supported by NSF Grant GP-29073

group of proper simple types $\mathcal{S}^+(X)$ for any object X in \mathcal{C}^+ . If X is an object of \mathcal{C}^+ there is a natural map $\mathcal{S}^+(X) \rightarrow \mathcal{S}(X)$ which we show to be an isomorphism in (4.2).

Throughout this paper any CW-complex will always be assumed to be strongly locally finite. We need this in order to say as in [2] that any proper map can be properly deformed to a cellular map – the starting point for the algebraic theory of simple types describing $\mathcal{S}(X)$ as a functor of $\pi_1 X$ and the system of fundamental groups of neighborhoods of infinity. If one works in the category of all locally finite, countable CW-complexes, then $\mathcal{S}(X)$ may be non-zero even though X is simply connected and simply connected at infinity. For example, let $K = e^0 \cup e^1 \cup \dots$ where e^n is attached to e^{n-1} by collapsing ∂e^n to a point in the interior of e^{n-1} . The property of being strongly locally finite is preserved under proper simple equivalence. Hence, if K' is any subdivision of K which is strongly locally finite, then K and K' are not simply equivalent although they ought to be. This minor technical point could be remedied if one knew that any locally finite CW-complex had a strongly locally finite subdivision. We get around the difficulty by working only with strongly locally finite CW-complexes. In passing we remark that (**) of [5] is false if the category of all locally finite, countable CW-complexes is used. One must stay with strongly locally finite complexes.

Suppose $L = K \cup_{f_i} \{e_i^n\} \cup_{g_j} \{e_j^{n+1}\}$. Suppose the attaching map g_j of some $(n+1)$ -cell e_j^{n+1} misses all the n -cells except, say, e_i^n and suppose g_j takes the top hemisphere of ∂e_j^{n+1} homeomorphically onto e_i^n and takes the bottom hemisphere into L . Thus the pair $e_j^n \cup e_j^{n+1}$ forms an elementary expansion. We shall say that e_j^{n+1} *cancels* e_i^n .

§ 2. Definition of torsion in \mathcal{C}^+

In this section we briefly recall the definition of torsion for a proper homotopy equivalence in \mathcal{C}^+ . For details and terminology see [1] and [6, Chap. I, § 5].

Let X be a non-compact, connected, strongly locally finite CW-complex and let $t: T \rightarrow X$ be a tree for X . This means that T is a locally finite, contractible, one dimensional simplicial complex with a base vertex $0 \in T$ such that if $v \in T$ is a vertex different from 0 then at least two 1-cells branch off from v . Furthermore, $t: T \rightarrow X$ is required to be a cellular map which is properly $\frac{1}{2}$ -connected in the sense that $t^*: H^0(X) \rightarrow H^0(T)$ and $t^*: H_{\text{end}}^0(X) \rightarrow H_{\text{end}}^0(T)$ are isomorphisms.

The obstruction group $\text{Wh}(X; t)$, which is defined in [6, Chap. I, § 5] to capture the torsion of a proper homotopy equivalence $f: W \rightarrow X$ in \mathcal{C}^+ , is an abelian group which depends only on $\pi_1 X$ and the inverse system of fundamental groups of neighborhoods of infinity in X . Up to isomorphism $\text{Wh}(X; t)$ is also independent of the choice of tree $t: T \rightarrow X$. The group $\text{Wh}(X; t)$ can be computed as follows:

First some generalities. Let $t: T \rightarrow X$ be any tree for X . The set J of vertices of T

can be partially ordered by letting $u \leq v$ iff the arc from v to the base vertex 0 passes through u . Let $|v|$ denote the number of 1-simplices in the arc from v to 0 . Let $T_v \subseteq T$ denote the smallest subcomplex containing all vertices w of T with $v \leq w$. Let $J' \subset J$ be a cofinal subset (containing the vertex 0) obtained as follows: choose an increasing sequence $0 = n_0 < n_1 < \dots$. Then let $j \in J'$ iff there is some n_k with $|j| = n_k$. Associated to J' is a tree T' obtained by inserting a 1-simplex between vertices u and v of J' whenever $u < v$ and there is no vertex w of J' with $u < w < v$. The natural map $T' \rightarrow T$ is properly $\frac{1}{2}$ -connected and the composition $T' \rightarrow T \rightarrow X$ is a tree for X .

Now start with the original tree $t: T \rightarrow X$. Then there is a tree $t': T' \rightarrow X$ derived from $t: T \rightarrow X$ by the above process and there is a collection $\{X_u\}$ of infinite, connected subcomplexes of X (one subcomplex X_u for each vertex u of T') satisfying the following conditions (cf. [6, Chap. I, § 5]):

- a) $X_0 = X$, $X_u \supset X_v$ when $u \leq v$, and $t'(T_u) \subset X_u$
- b) $X_u \cap X_v = \emptyset$ if $|u| = |v|$ and $u \neq v$
- c) for each $n \geq 0$, $X - \bigcup_{|v|=n} X_v$ is contained in some finite subcomplex of X
- d) given any finite subcomplex K of X , there is some $n \geq 0$ such that $K \cap (\bigcup_{|v|=n} X_v) = \emptyset$.

Now for each vertex u of T' let $\pi_u = \pi_1(X_u, t'(u))$. If $u \leq v$ define the homomorphism $\gamma_{uv}: \pi_v \rightarrow \pi_u$ to be "conjugation" by the path $t'(\alpha_{vu}) \subset X_u$ where $\alpha_{vu} \subset T'_u$ is the arc from u to v . The collection $\pi = \{\pi_u, \gamma_{uv}\}$ is a tree of groups over the set J' of vertices of T' . Let $Z[\pi] = \{Z[\pi_u], \gamma_{uv}\}$ denote the associated tree of group rings. Let $\text{Wh}(\pi)$ be as defined in [1] and [6, Chap. I, § 5]. Then there is an isomorphism

$$\text{Wh}(\pi) \cong \text{Wh}(X; t)$$

See [6, Chap. I, § 5]. The point of using $\text{Wh}(X; t)$ as the obstruction group rather than one of its "representatives" $\text{Wh}(\pi)$ is to make the torsion well defined and independent of various choices such as the X_u above. However in proving certain things one often uses a convenient choice of a $\text{Wh}(\pi)$. Also, there is the basic algebraic exact sequence (see (4.3) below) which relates $\text{Wh}(\pi)$ with the $\text{Wh}(\pi_u)$ and $\tilde{K}_0(\pi_u)$ and allows one to compute $\text{Wh}(\pi)$ in a number of cases.

We will briefly indicate how to define the torsion

$$\tau(L, K) \in \text{Wh}(\pi) \cong \text{Wh}(L; t)$$

of an inclusion $K \rightarrow L$ where K is a proper deformation retract of L and $\dim(L - K) < \infty$. Here $\pi = \{\pi_u, \gamma_{uv}\}$ is the tree of groups corresponding to any choice of a tree $t': T' \rightarrow L$ derived from $t: T \rightarrow L$ as above and any choice of a system $\{L_u\}$ satisfying (a) through (d) with respect to $t': T' \rightarrow L$. The definition of $\tau(L, K)$ given below is in the spirit of [4]. Using the general machinery of [6, Chap. I, § 5] and [1] it is not hard to see that this approach to torsion for infinite simple types is equivalent to the one worked out

in [6, Chap. I, § 5] which follows the lines of [3]. The argument showing the equivalence is entirely similar to the one in the compact case.

For any CW-complex X , let \tilde{X} denote the universal covering space $p: \tilde{X} \rightarrow X$. If $Y \subset X$, let $\tilde{Y} = p^{-1}(Y)$.

By condition (a) above $t'(u) \in L_u$ for every vertex u of T' . Select a fixed lifting $\hat{u} \in \tilde{L}_u$ of $t'(u)$. If v is a vertex of T' and $u \leq v$, let $v' \in \tilde{L}_u$ be the lifting of $t'(v) \in L_u$ obtained as the end point of the lifting of the path $t(\alpha_{vu})$ to a path in \tilde{L}_u starting at \hat{u} . Here α_{vu} denotes the arc from u to v in T'_u . If v is a vertex of T' with $u \leq v$ there is a unique map $\tilde{L}_v \rightarrow \tilde{L}_u$ covering the inclusion $L_v \rightarrow L_u$ such that $\hat{v} \in \tilde{L}_v$ goes to $v' \in \tilde{L}_u$. Furthermore, if $u \leq v \leq w$ the map $\tilde{L}_w \rightarrow \tilde{L}_u$ is the composition $\tilde{L}_w \rightarrow \tilde{L}_v \rightarrow \tilde{L}_u$.

The next choice we make is to select a locally finite collection \mathcal{A} of paths $\alpha(\sigma)$ from the barycenters of cells σ of L to the images $t'(\alpha(\sigma))$ of vertices $u(\sigma)$ of T' such that if $\sigma \subset L_u$ then $\alpha(\sigma) \subset L_u$. If $\sigma \subset L_u$ the path $\alpha(\sigma)$ determines a path $\beta_u(\sigma)$ from σ to $t'(u)$ in L_u : first follow $\alpha(\sigma)$ to $t'(u(\sigma))$ and then follow $t'(\alpha_{u,u(\sigma)})$ to $t'(u)$. Here $\alpha_{u,u(\sigma)}$ is the arc in T' from $u(\sigma)$ to u .

If X is any CW-complex, let X^n denote the n -skeleton of X .

Now define the based $Z[\pi]$ -module $C_n(L, K)$ as follows:

$$C_n(L, K) = \{C_n(L, K)_u\}$$

where for each vertex u of T'

$$C_n(L, K)_u = H_n(\overline{L_u^n}, \overline{L_u^{n-1}} \cup \overline{L_u^n \cap K})$$

The “bar” is taken with respect to the universal cover $\tilde{L}_u \rightarrow L_u$. The $Z[\pi_u]$ -module $C_n(L, K)_u$ is free with one basis element for each n -cell of $L_u - K$. The basis element corresponding to an n -cell σ of $L_n - K$ is given by the lifting of σ to \tilde{L}_u determined by the path $\beta_u(\sigma)$. If $u \leq v$ the map $\tilde{L}_v \rightarrow \tilde{L}_u$ determines a homomorphism $C_n(L, K)_v \rightarrow C_n(L, K)_u$ and in fact we have an injection

$$C_n(L, K)_v \otimes_{Z[\pi_v]} Z[\pi_u] \rightarrow C_n(L, K)_u$$

whose image is the free submodule generated by the n -cells of $L_u - K$ which lie in $L_v - K$. The boundary operators $\partial_n^u: C_n(L, K)_u \rightarrow C_{n-1}(L, K)_u$ are compatible with the maps $C_n(L, K)_v \rightarrow C_n(L, K)_u$ and therefore define a morphism of $Z[\pi]$ -modules

$$\partial_n: C_n(L, K) \rightarrow C_{n-1}(L, K)$$

which satisfies $\partial_{n-1} \circ \partial_n = 0$. This gives a chain complex

$$(C_*, \partial_*) = \{C_n(L, K), \partial_n\}$$

of based $Z[\pi]$ -modules. In fact, if $S^n = \{S_u^n\}$ is the tree of sets over J' where S_u^n

consists of the n -cells of $L_u - K$, then $C_n(L, K)$ is the free $Z[\pi]$ -module generated by S^n . Since $\dim(L - K) < \infty$ at most finitely many of the chain groups $C_n(L, K)$ are not zero.

Let $r : L \times I \rightarrow L$ be a proper deformation retraction of L down into K . We can assume r is cellular by [2, Th. 1.7]. For each vertex u of T' choose a cofinite subcomplex N_u of L_u (i.e., $L_u - N_u$ has only finitely many cells) such that

- i) $N_0 = L_0$ and $N_u \supset N_v$ whenever $u \leq v$
- ii) $r(N_u \times I) \subset L_u$.

The map $r : N_u \times I \rightarrow L_u$ has a unique lifting $r_u : \tilde{N}_u \times I \rightarrow \tilde{L}_u$ such that r_u restricted to $\tilde{N}_u \times 0$ is the inclusion and such that whenever $u \leq v$ there is a commutative diagram

$$\begin{array}{ccc} \tilde{N}_v \times I & \xrightarrow{r_v} & \tilde{L}_v \\ \downarrow & & \downarrow \\ \tilde{N}_u \times I & \xrightarrow{r_u} & \tilde{L}_u \end{array}$$

Let $\hat{C}_n(L, K)_u \subset C_n(L, K)_u$ be the free $Z[\pi]$ -submodule generated by the n -cells of $L_u - K$ belonging to $N_u - K$ and let $i_u : \hat{C}_n(L, K)_u \rightarrow C_n(L, K)_u$ denote the inclusion map. The maps $r_u : \tilde{N}_u \times I \rightarrow \tilde{L}_u$ induce coboundary operators

$$d_u^n : \hat{C}_n(L, K)_u \rightarrow C_{n+1}(L, K)_u$$

compatible with the morphisms $\hat{C}_n(L, K)_v \rightarrow \hat{C}_n(L, K)_u$ and $C_n(L, K)_v \rightarrow C_n(L, K)_u$ such that for each vertex u of T'

$$\partial_{n+1}^u \circ d_u^n + d_n^{n-1} \circ \hat{\partial}_n^u = \begin{cases} id, & \text{for } u = 0 \\ i_u + \text{finite matrix}, & \text{for } u > 0 \end{cases}$$

Here $\partial_n^u : \hat{C}_n(L, K)_u \rightarrow \hat{C}_{n-1}(L, K)_u$ is the restriction of ∂_n^u . Thus the collection $d^n = \{d_u^n\}$ defines a germ $d^n : C_n(L, K) \rightarrow C_{n+1}(L, K)$ such that on the germ level we have

$$\partial_{n+1} \circ d^n + d^{n-1} \circ \partial_n = id. \tag{*}$$

This shows that (C_*, ∂_*) is an acyclic complex of based modules over the tree of rings $Z[\pi]$ and as in [6, Chap. I, § 5] we can define the torsion to be

$$\tau(L, K) = \tau(C_*, \partial_*) \in \text{Wh}(\pi) \tag{2.1}$$

Now here is the way to define $\tau(L, K)$ in the spirit of [4]: By replacing d^n with $d^n \circ \partial_{n+1} \circ d^n$ (if necessary) we can assume that $d^{n+1} \circ d^n = 0$. Let $C_{\text{ev}} = \bigoplus_{0 \leq k} C_{2k}$ and $C_{\text{odd}} = \bigoplus_{0 \leq k} C_{2k+1}$. The formula (*) implies that $\partial_{\text{ev}} + d^{\text{ev}} : C_{\text{ev}} \rightarrow C_{\text{odd}}$ is an isomorphism on the germ level whose inverse is $\partial_{\text{odd}} + d^{\text{odd}} : C_{\text{odd}} \rightarrow C_{\text{ev}}$. Let the trees of

sets S_{ev} and S_{odd} be defined as the disjoint unions of trees of sets

$$S_{\text{ev}} = \coprod_{0 \geq k} S^{2k} \quad \text{and} \quad S_{\text{odd}} = \coprod_{0 \geq k} S^{2k+1}$$

Then C_{ev} is the free $Z[\pi]$ -module generated by S_{ev} and C_{odd} is the free $Z[\pi]$ -module generated by S_{odd} . Let J' denote the standard tree of sets $\{J'_u\}$ determined by the partially ordered set J' of vertices of T' ; that is $J'_u = \{v \mid u \in J' \text{ and } u \leq v\}$. Let $F[J'; \pi]$ denote the free $Z[\pi]$ -module generated by the tree of sets J' . As in [1, Prop. 2.2] choose proper bijections $h: S_{\text{ev}} \amalg J' \rightarrow J'$ and $g: S_{\text{odd}} \amalg J' \rightarrow J'$. Let $H: C_{\text{ev}} \oplus \oplus F[J'; \pi] \rightarrow F[J'; \pi]$ and $G: C_{\text{odd}} \oplus \oplus F[J'; \pi] \rightarrow F[J'; \pi]$ be the induced germ isomorphisms. Then $G \circ (\partial_{\text{ev}} + d^{\text{ev}}) \circ H^{-1}$ is an invertible germ taking $F[J'; \pi]$ to itself and we have

$$\tau(L, K) = \langle G \circ (\partial_{\text{ev}} + d^{\text{ev}}) \circ H^{-1} \rangle \in \text{Wh}(\pi). \quad (2.2)$$

The torsion $\tau(L, K)$ is independent of the choice of the liftings \hat{u} of the vertices $t'(u)$ and also of the choice of base paths λ .

In [6, Chap I, § 5] the torsion is shown to be invariant under subdivision and to be additive in the following sense: Let $M \subset L \subset K$ where M is a proper deformation retract of L and L is a proper deformation retract of K . Let $t: T \rightarrow L$ be a tree for L . Then

$$\tau(K, M) = \tau(K, L) + i_* \tau(L, M) \quad (2.3)$$

where $i_*: \text{Wh}(L; t) \rightarrow \text{Wh}(K; i \circ t)$ is the isomorphism induced by the inclusion $i: L \hookrightarrow K$

Now let $f: X \rightarrow Y$ be a proper homotopy equivalence in the category \mathcal{C}^+ and let $t: T \rightarrow Y$ be a tree. Deform f properly to a proper cellular map \hat{f} and as in [6, Chap I, § 5] define

$$\tau(f) = r_* \tau(M_{\hat{f}}, X) \in \text{Wh}(Y; t) \quad (2.4)$$

where $r: M_{\hat{f}} \rightarrow Y$ is the standard deformation retraction. If $i: K \hookrightarrow L$ is an inclusion and K is a deformation retraction of L then $\tau(i) = \tau(L, K)$. This is Lemma 20 of Chap I. of [6]. By Lemma 21 of Chap I of [6] the torsion $\tau(f)$ doesn't depend on the choice of cellular "approximation" \hat{f} . Furthermore the following additivity property holds (Lemma 22 of Chap I of [6]): Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be proper homotopy equivalences. Let $t: T \rightarrow Y$ be a tree. Then

$$\tau(g \circ f) = \tau(g) + g_* \tau(f) \quad (2.5)$$

where g induces the isomorphism $g_*: \text{Wh}(Y; t) \rightarrow \text{Wh}(X; g \circ t)$.

LEMMA 2.5. *Suppose $f: X \rightarrow Y$ is a simple equivalence in the category \mathcal{C}^+ . Then $\tau(f) = 0$.*

Proof. The additivity property of torsion reduces the argument to showing that $\tau(L, K) = 0$ where $K \nearrow L$ is an expansion in \mathcal{C}^+ . Write $L = K \cup (\bigcup_{i+1}^\infty L_i)$ where L_i is a finite subcomplex which collapses to $K_i = K \cap L_i$ and $\dim(L_i - K) \leq n$ for all i . Each L_i can be collapsed to K_i by performing elementary collapses in order of decreasing dimension. The additivity property again reduces the problem to showing that $\tau(L, K) = 0$ whenever each $K_i \nearrow L_i$ is a sequence of elementary expansions of dimension k . However the torsion certainly vanishes in this case because $\partial_{ev} + d^{ev} = \partial_k: C_k(L, K) \rightarrow C_{k-1}(L, K)$ is a blocked germ [1, §2] with each block being a product of elementary matrices.

Now let X be an object of \mathcal{C}^+ and let $t: T \rightarrow X$ be a tree for X . Let $[f] \in \mathcal{S}^+(X)$ be represented by a proper homotopy equivalence $f: X \rightarrow Y$. Choose a proper homotopy inverse $g: Y \rightarrow X$ of f and as in [6, Chap. I, § 5] let

$$\tau^+(f) = \tau(g) \in \text{Wh}(X; t). \tag{2.6}$$

Then (2.4) and (2.5) imply that (2.6) gives a well defined homomorphism

$$\tau^+ : \mathcal{S}^+(X) \rightarrow \text{Wh}(X; t) \quad .$$

and we show in the next section that this is an isomorphism.

§ 3. τ^+ is an isomorphism

In this section we prove

THEOREM 3.1. *Let X be an object of \mathcal{C}^+ and let $t: T \rightarrow X$ be a tree for X . Then*

$$\tau^+ : \mathcal{S}^+(X) \rightarrow \text{Wh}(X; t)$$

is an isomorphism.

First we prove that τ^+ is injective.

Let $f: X \rightarrow Y$ be a cellular proper homotopy equivalence and let M_f be the mapping cylinder of f .

LEMMA 3.2. *There is an inclusion $X \hookrightarrow M$ with X a proper deformation retract of M such that the pair (M, X) is simply equivalent rel X in \mathcal{C}^+ to the pair (M_f, X) and such that $M - X$ has cells in only two dimensions.*

The proof of this is a straight forward generalization to the proper category of the argument for Lemma 3 of [7]. In fact, M can be chosen to have cells only in dimen-

sions $n+1$ and n where $n \geq \max(\dim X, \dim Y)$. Thus M can be constructed to have cells only in dimensions $2k$ and $2k-1$ where $2k-1 \geq \max(\dim X, \dim Y)$.

Now suppose $f: X \rightarrow Y$ represents an element of $\mathcal{S}^+(X)$ on which τ^+ vanishes. Replace M_f by M as above. Choose $t': T' \rightarrow X$ and $\{X_u\}$ as in § 2. Choose a collection $\{M_u\}$ satisfying (a) through (d) of §2 as follows: Let

$M'_u = X_u \cup \{(2k-1)\text{-cells of } M \text{ whose attaching maps lie in } X_u\}$. Then set

$M_u = M'_u \cup \{2k\text{-cells of } M \text{ whose attaching maps lie in } M'_u\}$. Assume that $2k \geq 4$ and let $\tau = \{\tau_u, \gamma_{uv}\}$ be the tree of groups where $\pi_u = \pi_1(M_u, t'(u)) \simeq \pi_1(X_u, t'(u))$. Since $\tau^+(f) = 0$ we know that $\tau = \tau(M, X) \in \text{Wh}(\pi) \cong \text{Wh}(M; t)$ also vanishes. The torsion τ is represented by the germ

$$\partial = \partial_{2k}: C_{2k}(M, X) \rightarrow C_{2k-1}(M, X).$$

Since $\tau = 0$ we know by Lemma 2.7 of [1] that after stabilization of ∂ to $(\partial \oplus 1) \oplus \dots \oplus 1$ it is possible to find blocked germs $A = \sum_{0 \leq u} A^u$ and $B = \sum_{0 \leq u} B_u$ of $C_{2k}(M, X)$ to itself such that

$$[(\partial \oplus 1) \oplus \dots \oplus 1] \cdot A \cdot B = P$$

where $P: C_{2k}(M, X) \rightarrow C_{2k-1}(M, X)$ is a π -permutation germ. We also know that each of the square matrices A^u and B^u is a product of elementary matrices over $Z[\pi_u]$. Since P is a π -permutation germ it has a matrix representative $\{P_u\}$ where $P_u: C_{2k} \times (M, X)_u \rightarrow C_{2k-1}(M, X)_u$ satisfies $P_u(\text{basis element}) = \pm g \cdot (\text{basis element})$ where $g \in \pi_u$. The stabilization of ∂ to $(\partial \oplus 1) \oplus \dots \oplus 1$ is achieved geometrically by stabilizing M ; that is, we replace M by $M \cup \{e_u^{2k-1} \cup e_u^{2k}\}$ where each pair $e_u^{2k-1} \cup e_u^{2k}$ is an elementary expansion attached to the vertex $t'(u) \in X$. To simplify notation we shall still denote the stabilized ∂ and the stabilized M by ∂ and M . Let $S = \{S_u\}$ denote the tree of sets over J' where $S_u = 2k\text{-cells of } M_u - X$. Since A is blocked we can (as in §2 of [1]) replace the tree $S = \{S_u\}$ by an equivalent tree of sets $D = \{D_u\}$ with $D_u \subset S_u$ and we can amalgamate A so that $\hat{D}_u = D_u - \bigcup_{u < v} D_v$ is a finite set which is the support of A^u ; that is, A^u is an invertible $Z[\pi_u]$ -homomorphism from $F_u = F[\hat{D}_u; \pi_u]$ to itself.

Recall the following: Suppose $L = K \cup_f e^n$ where $f: S^{n-1} \rightarrow K$ is the attaching map. If f is deformed by a homotopy $H: S^{n-1} \times I \rightarrow K$ to a map $g: S^{n-1} \rightarrow L$ then $L' = K \cup_g e^n$ has the same simple type as L . Let $W = K \cup_H (e^n \times I)$ where $H: S^{n-1} \times I \rightarrow K$ is the attaching map. Then $L \nearrow W \searrow L'$ is the simple equivalence from L to L' . Also recall that if $K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_{n-1} \rightarrow K_n$ is a sequence of elementary expansions and/or contractions then there is a complex W containing K_0 and K_n such that $K_0 \nearrow W \searrow K_n$ and $\dim W \leq \max(\dim K_i)$.

Now write each matrix A^u as a product of elementary matrices of the form $e_{ij}(\lambda)$: $F_u \rightarrow F_u$ where $\lambda = \pm g$ for $g \in \pi_u$. For each u use this product to perform a sequence

of deformations of the attaching maps of the $2k$ -cells in \hat{D}_u over one another as in the “handle addition” lemma of [7, Lemma 4]. At any step in the process the attaching map of a cell e^{2k} in \hat{D}_u is deformed with support contained in $M'_u \cup$ (other $2k$ -cells in \hat{D}_u). This procedure changes M by a proper simple equivalence in \mathcal{C}^+ to a complex M' such the boundary map $\partial': C_{2k}(M', X) \rightarrow C_{2k-1}(M', X)$ is just the germ $\partial \cdot A$. Repeat the process using a block decomposition of the germ B to get a complex M'' such that $\partial'': C_{2k}(M'', X) \rightarrow C_{2k-1}(M'', X)$ is just $\partial \cdot A \cdot B = P$. Since P is a π -permutation germ the attaching maps of the $2k$ -cells can be deformed in a locally finite way so that each $2k$ -cell cancels just one $(2k - 1)$ -cell and misses all the others. This says M'' is properly simply equivalent in \mathcal{C}^+ to a complex which collapses to X . We conclude that $\tau^+ : \mathcal{S}^+(X) \rightarrow \text{Wh}(X; t)$ is injective.

It is easy to show that τ^+ is surjective: Let $t' : T' \rightarrow X$ and $\{X_u\}$ be as in §2 and let $A : F[J'; \pi] \rightarrow F[J'; \pi]$ be an invertible germ. For each vertex u of T' attach a 4-cell e_u^4 to X by collapsing ∂e_u^4 to the point $t'(u)$. Now attach 5-cells e_u^5 in a locally finite way using the germ A . This gives a complex M which has X as a proper deformation retract by [2, Th. 3.1] or [5, Prop IV]. Also $\tau(X \rightarrow M) = [A] \in \text{Wh}(\pi) \cong \text{Wh}(X; t)$. This completes the proof that τ^+ is an isomorphism.

§ 4. Torsion in the category \mathcal{C}

Although the methods of §2 don't directly define the torsion of a proper homotopy equivalence $f : X \rightarrow Y$ in the category \mathcal{C} it is possible to prove

THEOREM 4.1. *Let X be an object of \mathcal{C} and let $t : T \rightarrow X$ be a tree. There is an isomorphism*

$$\mathcal{S}(X) \cong \text{Wh}(X; t).$$

A consequence of (3.1) and (4.1) is

COROLLARY 4.2. *If X is an object of \mathcal{C}^+ , then $\mathcal{S}^+(X) \rightarrow \mathcal{S}(X)$ is an isomorphism.*

Proof of (4.1). Let $t' : T' \rightarrow X$ and $\{X_u\}$ be as in §2. Let $\pi = \{\pi_u, \gamma_{uv}\}$ be the associated tree of groups. Recall the exact sequence (3.6) of [1]:

$$\prod_{0 < u} \text{Wh}(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \text{Wh}(\pi_u) \xrightarrow{A} \text{Wh}(\pi) \xrightarrow{\partial} \prod_{0 < u} \tilde{K}_0(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \tilde{K}_0(\pi_u) \quad (4.3)$$

Let

$$\text{Wh}(\pi)' = \text{Coker} \left[\prod_{0 < u} \text{Wh}(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \text{Wh}(\pi_u) \right]$$

Let

$$\tilde{K}_0(\pi)' = \text{Ker} \left[\prod_{0 < u} \tilde{K}_0(\pi_u) \xrightarrow{I-S} \prod_{0 \leq u} \tilde{K}_0(\pi_u) \right]$$

Then there is the exact sequence

$$0 \rightarrow \text{Wh}(\pi)' \rightarrow \text{Wh}(\pi) \rightarrow \tilde{K}_0(\pi)' \rightarrow 0 \tag{4.4}$$

In [5] the following exact sequence is constructed:

$$0 \rightarrow \text{Wh}(\pi)' \rightarrow \mathcal{S}(X) \rightarrow \tilde{K}_0(\pi)' \rightarrow 0 \tag{4.5}$$

Hence to prove (4.1) it suffices by the “5-lemma” to construct a homomorphism $\text{Wh}(\pi) \rightarrow \mathcal{S}(X)$ which induces a map from the sequence (4.4) to the sequence (4.5). This was essentially done in the proof of (3.1): Take an invertible germ $A: F[J'; \pi] \rightarrow F[J'; \pi]$ and construct a complex $M(A)$ containing X as a proper deformation retract by attaching one 4-cell e_u^4 to the vertex $t'(u) \in X$ and then attaching the 5-cells e_u^5 in a locally finite way using the germ A . The argument of §3 proving the injectivity of τ^+ shows that $M(A)$ is simply equivalent to $M(A \cdot E)$ whenever E is a blocked germ $E = \sum_u E^u$ such that each E^u is a product of elementary matrices over $Z[\pi_u]$. Stabilization of A to $A \oplus 1$ only changes $M(A)$ by adding elementary expansions. Hence the proper simple type of $M(A)$ doesn't change when A is varied by the defining relations of $\text{Wh}(\pi)$ and we get the required homomorphism $\text{Wh}(\pi) \rightarrow \mathcal{S}(X)$.

§ 5. The proper s -cobordism theorem

Now that $\mathcal{S}^+(X)$ and $\mathcal{S}(X)$ have been described in algebraic terms the proper s -cobordism theorem of [5] can be reformulated.

Recall that a smooth, piecewise linear or topological cobordism W^n from M_-^n to M_+^n is a proper h -cobordism provided the inclusions $M_- \hookrightarrow W$ and $M_+ \hookrightarrow W$ are proper homotopy equivalences. Suppose M_-, M_+, W are all non-compact and let $t: T \rightarrow M_-$ be a tree.

THEOREM 5.1. *Let $n \geq 6$. There is a well defined torsion element $\tau(W; M_-, M_+) \in \text{Wh}(M_-; t)$ which vanishes iff $(W; M_-, M_+)$ is isomorphic to $(M_- \times [0, 1]; M_- \times 0, M \times 1)$. Every element of $\text{Wh}(M_-; t)$ can be realized as the torsion of some proper h -cobordism on M_- .*

This is just the statement of the combined theorems (3.1) and (4.2) above together with Theorem III of [5]. Alternatively, for a direct proof that elements of $\text{Wh}(M_-; t)$ classify proper h -cobordisms on M_- one can mimic the argument in the compact case using the methods of §3 in the setting of handlebody theory.

Here are some examples. Compare with [5].

a) Suppose M_- is simply connected and simply connected at infinity. Then it is possible to choose a tree $t': T' \rightarrow M_-$ and a collection $\{(M_-)_u\}$ such that each $(M_-)_u$ is simply connected. Thus $\pi = \{\pi_u, \gamma_{uv}\}$ is a tree of trivial groups and (4.3) shows that $\text{Wh}(\pi) \cong \text{Wh}(M_-; t)$ vanishes. Hence any proper h -cobordism on such an M is trivial.

b) Suppose M_- has just one stable end ε with fundamental group $\pi_1\varepsilon$ such that $\pi_1\varepsilon \rightarrow \pi_1M_-$ is an isomorphism. Then (3.10) of [1] implies that $\text{Wh}(M_-; t) = 0$ and hence any proper h -cobordism on M_- is trivial. In particular, for any non-compact M_- , any proper h -cobordism on $M_- \times R^2$ is trivial.

There are algebraic product and duality formulae similar to the ones in the compact case. Compare with [5].

Let $(W; M_-, M_+)$ be a proper h -cobordism and let N be a compact manifold. Let $t: T \rightarrow M_-$ be a tree.

Product formula (see Lemma 23 of [6]).

$$\tau(W \times N; M_- \times N, M_+ \times N) = \chi(N) \cdot i_* \tau(W; M_-, M_+)$$

where $\chi(N)$ is the Euler class of N and $i_*: \text{Wh}(M_-; t) \rightarrow \text{Wh}(M_- \times N; t)$ is the induced homomorphism.

Remark. By contrast to the above suppose $(W^n; M_-, M_+)$ is a proper h -cobordism (compact or non-compact) and let N be a non-compact manifold. If $n \geq 6$ then the proper h -cobordism $(W \times N; M_- \times N, M_+ \times N)$ is trivial.

The torsion of a proper h -cobordism $(W; M_-, M_+)$ can be computed in $\text{Wh}(W; t)$ where there is a conjugation $-: \text{Wh}(W; t) \rightarrow \text{Wh}(W; t)$ defined as follows: choose $t': T' \rightarrow W$ and $\{W_u\}$ as in §2. For each vertex u of T' there is the orientation homomorphism $w_u: \pi_u \rightarrow Z_2 = \{+1, -1\}$. If $u \leq v$, then $w_v = w_u \circ \gamma_{uv}$. Define the conjugation $-: \pi \rightarrow \pi$ to be the collection of compatible conjugations $-: \pi_u \rightarrow \pi_u$ where $\bar{g} = w_u(g) g^{-1}$ for $g \in \pi_u$. The conjugation on π induces one on $\text{Wh}(\pi) \cong \text{Wh}(W; t)$ by taking any invertible germ $A: F[J'; \pi] \rightarrow F[J'; \pi]$ to $\bar{A} = \text{conjugate transpose of } A$.

Duality formula (see [6, Chap I, §5])

$$\tau(W; M_+, M_-) = (-1)^{n-1} \bar{\tau}(W; M_-, M_+).$$

REFERENCES

- [1] FARRELL, F. T. and WAGONER, J. B., *Infinite Matrices in Algebraic K-theory and Topology*, preprint, U. C. Berkeley
- [2] FARRELL, F. T., TAYLOR, L. R. and WAGONER, J. B., *The Whitehead Theorem in the Proper Category*, preprint, U. C. at Berkeley.
- [3] MILNOR, J., *Whitehead Torsion*, Bull. Amer. Math. Soc. 72 (1966) 358–426.

- [4] DERHAM, G., KERVAIRE, M. and MAUMARY, S., *Torsion et Type Simple d'Homotopy*, Lecture Notes in Mathematics No. 48, Springer-Verlag.
- [5] SIEBENMANN, L. C., *Infinite Simple Homotopy Types*, Indag. Math., 32, (1970), No. 5.
- [6] TAYLOR, L. R., *Surgery on Paracompact Manifolds*, Thesis, U. C. Berkeley (1971).
- [7] WALL, C. T. C., *Formal Deformations*, Proc. London Math. Soc. XVI (1966), 342–352.

Received January 22, 1972