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Infinite Matrices in Algebraic K-Theory and Topology

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A first step in the program of classifying finite dimensional paracompact manifolds in the spirit of surgery theory is to develop an algebraic theory of infinite simple homotopy types for finite dimensional, locally finite CW complexes. In [2] and [12] the geometric foundations of such a theory are discussed and in [12] some important progress was made on the algebraic part. In [3] a complete, à priori algebraic description of the torsion was given for the special case of a finite dimensional, locally finite CW complex with finitely many stable ends. The present paper together with [14, Chap I, § 5] and [4] extend the methods of [3] to the general case. In fact the algebraic approach to finite simple types as expounded in [9] or [11] can be developed in a completely similar way in the theory of infinite simple types using the concept of a "locally finite" matrix. Locally finite algebraic objects seem to provide the right setting for extending much of the theory of compact manifolds to open manifolds. For example, see [14] for a very comprehensive treatment of surgery theory for open manifolds. The locally finite matrix idea has also arisen in the work of Karoubi and Villamayor on K-theory from the Fredholm operator viewpoint. For example, see [6] and [7]. Other examples of its use can be found in [5], [15], and [16].

The present paper is purely algebraic. The first section discusses locally finite matrices. The second section defines the K_1 type object in which the torsion of an infinite simple type lies (more exactly, see 3.5). The third section gives the basic exact sequence that allows one to make calculations in important special cases. Finally in the fourth section we define a $K_1(f)$ for any ring homomorphism $f: R \to S$ which extends the usual definition of the relative group of a surjection.

In [14] the algebraic part of the theory of infinite simple types is developed along the lines of [9] and in [4] we complete the series by discussing the geometric part; for example, it is shown that a proper h-cobordism is a product iff its torsion vanishes.

§ 1. Locally Finite Matrices

In this paper all modules will be considered right modules unless otherwise stated.

Let R be a ring with identity. In this paper $K_0(R)$ will denote the Grothendieck group of the category of finitely generated, projective R-modules; $GL(R) = \lim_{n\to\infty} GL(n, R)$ will be the general linear group; $E(R) \subset GL(R)$ will be the group

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of elementary matrices; and $K_1(R)=GL(R) \bmod [GL(R), GL(R)]$. See [1] or [13] for example. Recall also the following matrix identities.

$$\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -X^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{1.1}$$

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix} \tag{1.2}$$

Let E and F be free R-modules based on countable sets $\{e_\alpha\}$ and $\{f_\beta\}$ respectively. An R-linear transformation $h: E \to F$ is *locally finite* provided that for each f_β there are at most finitely many e_α such that f_β appears in $h(e_\alpha)$ with a non-zero coefficient. If $h(e_\alpha) = \sum f_\beta \cdot r_{\beta\alpha}$, then h is locally finite iff the matrix $(r_{\beta\alpha})$ is locally finite in the sense that each row and each column of $(r_{\beta\alpha})$ has at most finitely many non-zero terms. The ring of all locally finite transformations (matrices) of E to itself will be denoted by $l(E; R)$ or $l_R(E)$. Note that $l(E'; R)$ and $l(E''; R)$ are isomorphic if there is a bijection between the bases $\{e'_\alpha\}$ and $\{e''_\alpha\}$. Let $m_R(E) \subset l_R(E)$ denote the two sided ideal of *finite matrices*; i.e., of those matrices which have at most finitely many non-zero entries. Finally, let $\mu_R(E)$ or $\mu(E; R)$ denote the quotient ring $l_R(E)/m_R(E)$. For economy we will let lR, mR, and μR denote $l_R(E)$, $m_R(E)$, and $\mu_R(E)$ when E is the "standard" R-module based on $\{e_1, e_2, e_3, \ldots\}$. If $A \in lR$, let $\hat{A}$ denote the corresponding element in μR.

Let the R-module M be based on $\{m_\alpha\}$. An R-submodule $N \subset M$ is a *neighborhood of infinity* iff $m_\alpha \in N$ for all but finitely many indices α. Thus $h: E \to F$ is locally finite iff for any neighborhood of infinity $L \subset F$ there is a neighborhood of infinity $A \subset E$ such that $h(A) \subset L$.

For any ring with identity R, let R^* denote the group of two sided units in R.

PROPOSITION 1.3. *There is a surjective homomorphism*

$$\varrho : (\mu R)^* \to K_0(R).$$

Remark. In (1.12) below we show that ϱ induces an isomorphism $K_1(\mu R) \cong K_0(R)$.

Proof of (1.3). Let $E^n \subset E$ be the free R-submodule based on $\{e_1, \ldots, e_n\}$; let $E^{n,m} \subset E$ be the free R-submodule based on $\{e_{n+1}, \ldots, e_m\}$; let $E_n \subset E$ be the free R-submodule based on $\{e_{n+1}, e_{n+2}, \ldots\}$.

Step 1. Let $\alpha \in lR$ be a locally finite matrix which is invertible modulo mR. Then there is an $n > 0$ such that $\alpha: E_n \to E$ is injective and $E/\alpha(E_n)$ is a finitely generated, projective R-module.

Assuming step 1 for the moment, here is how to define ϱ. If P is finitely generated, projective, let $\langle P \rangle \in K_0(R)$ be the class it determines. Now let $x \in (\mu R)^*$ and choose an $\alpha \in lR$ with $\hat{\alpha} = x$. Define

$$\varrho(x) = \langle E/\alpha(E_n)\rangle - \langle E^n\rangle \tag{1.4}$$

The argument showing that ϱ is well defined and is a homomorphism mirrors the Bass-Heller-Swan argument constructing a homomorphism $K_1(R[t, t^{-1}]) \to K_0(R)$. See [13, p. 227]. Hence we just give the proof of

Step 1 *(cont.)*. Since $\hat{\alpha} \in (\mu R)^*$ there are integers m, n, p with $m < n$ such that $\alpha \mid E_m$ is injective and

$$E \supset \alpha(E_m) \supset E_p \supset \alpha(E_n).$$

First, $E/\alpha(E_m)$ has projective dimension $=1$ because there is an exact sequence $0 \to E_m \xrightarrow{\alpha} E \to E/\alpha(E_m) \to 0$. Hence $\alpha(E_m)/E_p$ is projective because there is an exact sequence $0 \to \alpha(E_m)/E_p \to E/E_p \to E/\alpha(E_m) \to 0$ where $E/E_p \cong E^p$ is free (cf. [13, p. 102]). Thus $E_p/\alpha(E_n)$ is projective and finitely generated because there is an exact sequence

$$\begin{array}{c} 0 \to E_p/\alpha(E_n) \to \alpha(E_m)/\alpha(E_n) \to \alpha(E_m)/E_p \to 0 \\ \hphantom{0 \to E_p/\alpha(E_n) \to } \Vert \hphantom{ \to \alpha(E_m)/E_p \to 0} \\ \hphantom{0 \to E_p/\alpha(E_n) \to } E^{m,n} \hphantom{ \to \alpha(E_m)/E_p \to 0} \end{array}$$

Finally we see that $E/\alpha(E_n) \cong E^p \oplus E_p/\alpha(E_n)$ is finitely generated and projective as required. Note that if $E_q \subset \alpha(E_n)$ then $\alpha(E_n)/E_q$ is also finitely generated and projective because there is an exact sequence

$$0 \to \alpha(E_n)/E_q \to E/E_q \to E/\alpha(E_n) \to 0.$$

In fact, if $n < q$, we have,

$$\langle E/\alpha(E_n)\rangle - \langle E^n\rangle = \langle E^{n,q}\rangle - \langle \alpha(E_n)/E_q\rangle \tag{1.5}$$

Step 2. It remains to show ϱ is onto. Again the argument is like the one in [13]. However, we will need the idea later in § 3 so it is included here for convenience. Any element of $K_0(R)$ can be represented in the form $\langle P\rangle - \langle R^n\rangle$. Where P is finitely generated projective and R^n is free on n generators. Choose an integer m so that there is a finitely generated and projective module Q with $P \oplus Q \cong R^m$. The required $\alpha \in lR$ is

$$\begin{array}{llllll} E \cong & R^n \oplus (Q \oplus & P) \oplus (Q \oplus & P) \oplus \cdots & & \\ \downarrow_\alpha & \downarrow_0 & \downarrow_{id} \quad \downarrow_{id} & \downarrow_{id} \quad \downarrow_{id} & & \\ E \cong (P \oplus & Q) \oplus (P \oplus & Q) \oplus (P \oplus Q) \oplus \cdots & & & \end{array}$$

Direct Sum Rings

Let R be an associative ring with identity. Then R is a *sum-ring* provided there are elements $\alpha_0, \alpha_1, \beta_0, \beta_1 \in R$ such that

$$\alpha_0\beta_0 = \alpha_1\beta_1 = 1$$

$$\beta_0\alpha_0 + \beta_1\alpha_1 = 1$$

Define the identity preserving ring homomorphism $\oplus : R \times R \to R$ by

$$r \oplus s = \beta_0 r \alpha_0 + \beta_1 s \alpha_1$$

for $r, s \in R$.

Strictly speaking a sum-ring is a ring with a particular choice of α_i and β_i. Let R and R' be sum-rings with respect to $\{\alpha_i, \beta_i\}$ and $\{\alpha'_i, \beta'_i\}$ respectively. A *morphism* of sum-rings $f : R \to R'$ is an identity preserving ring homomorphism f such that $f(\alpha_i) = \alpha'_i$ and $f(\beta_i) = \beta'_i$. Suppose R is a sum-ring with respect to α_i and β_i and $f : R \to R'$ is an identity preserving ring homomorphism. Then R' is a sum-ring with respect to $\alpha'_i = f(\alpha_i)$ and $\beta'_i = f(\beta_i)$ and f becomes a morphism.

A sum ring R is an *infinite sum ring* provided there is an identity preserving ring homomorphism $\infty : R \to R$ such that $r \oplus r^\infty$ for any $r \in R$ (cf. [7]).

EXAMPLE 1. lR is an infinite sum ring. To see this it will be convenient to identity lR with the ring $l_R(E)$ of locally finite R-linear transformations of the free, right R-module E with countable basis $\{e_j^k\}$ where $1 \leqslant k, j < \infty$. Partition the basis $\{e_j^k\}$ into two disjoint infinite subsets $\{e_j^k\} = A_0 \cup A_1$. Let $\beta_i : \{e_j^k\} \to A_i$ be any two bijections ($i = 0$ or 1). Let $\beta_i \in l_R(E)$ denote the corresponding locally finite matrix. Define $\alpha_i \in l_R(E)$ for $i = 0$ or 1 by

$$\alpha_i(e_j^k) = \begin{cases} \beta_i^{-1}(e_j^k), & \text{if} \quad e_j^k \in A_i \\ 0, & \text{otherwise} \end{cases}$$

This gives a sum structure on $l_R(E)$ and hence on lR. The following choice of sum structure is convenient: choose β_0 to be any bijection of $\{e_j^k\}$, $1 \leqslant k, j < \infty$, onto $\{e_1^k\}$, $1 \leqslant k < \infty$. Let $\beta_1(e_j^k) = e_{j+1}^k$. Let α_0 and α_1 be as above. To make $l_R(E)$ into an infinite sum-ring write $E = \oplus_{j=1}^\infty E_j$ where E_j is the free submodule of E spanned by the $\{e_j^k\}$, $1 \leqslant k < \infty$. Let $r \in l_R(E)$ and $e_j^k \in E$. Define

$$r^\infty(e_j^k) = \beta_1^{j-1}\beta_0 r \alpha_0 \alpha_1^{j-1}(e_j^k)$$

Then intuitively r^∞ is just the infinite direct sum of r laid out on the E_j's. We have

$r \oplus r^\infty = r^\infty$ because

$$(r \oplus r^\infty)(e_j^k) = \beta_0 r \alpha_0 (e_j^k) + \beta_1 r^\infty \alpha_1 (e_j^k)$$
$$= \begin{cases} \beta_0 r \alpha_0 (e_1^k) = r^\infty (e_1^k) \quad \text{for} \quad j = 1 \quad \text{and} \\ \beta_1 r^\infty \alpha_1 (e_j^k) = \beta_1 r^\infty (e_{j-1}^k) = \beta_1 (\beta_1^{j-1} \beta_0 r \alpha_0 \alpha_1^{j-1} (e_j^{k-1})) \\ \qquad = \beta_1^j \beta_0 r \alpha_0 \alpha_1^j (e_j^k) = r^\infty (e_j^k) \quad \text{if} \quad j > 1 \end{cases}$$

EXAMPLE 2. The homomorphic image of a sum-ring is also a sum-ring. Hence μR is a sum-ring.

Other examples will be given in § 2.

An interesting fact about an infinite sum-ring Γ is that $K_i(\Gamma)=0$ for all $i \in Z$ (cf. [16]), where for $i \geqslant 1$ the K_i is that of Quillen [10] and for $i \leqslant 1$ the K_i is that of Bass [1] or Karoubi [7]. In (1.13) below we give a simple argument showing that $K_1(lR)=0$ (cf. [7]).

LEMMA 1.6. *Let R be a sum-ring. Then*

(A) *There is a $c \in R^*$ such that for any $a, b \in R$*

$$a \oplus b = c\,(b \oplus a)\,c^{-1}$$

(B) *There is a $d \in R^*$ such that for any $a, b, c \in R$*

$$(a \oplus b) \oplus c = d\,(a \oplus (b \oplus c))\,d^{-1}.$$

Proof of 1.6. Choose $c = \beta_0\alpha_1 + \beta_1\alpha_0$ with $c^{-1} = c$ and $d = \beta_0\alpha_0^2 + \beta_1\beta_0\alpha_1\alpha_0 + \beta_1^2\alpha_1$ with $d^{-1} = \beta_0^2\alpha_0 + \beta_0\beta_1\alpha_0\alpha_1 + \beta_1\alpha_1^2$. The computation is left as an exercise.

Let $M(n, R)$ denote the ring of $n \times n$-matrices with coefficients in the ring R. Let $s_0: M(2^n, R) \to M(2^{n+1}, R)$ be the (non-identity preserving) ring homomorphism given by

$$A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $s_1: GL(2^n, R) \to GL(2^{n+1}, R)$ be the group homomorphism given by

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

PROPOSITION 1.7. *Let Γ be a sum-ring with respect to α_i, β_i. For each $n \geqslant 1$ there are ring isomorphisms $\theta_n: \Gamma \to M(2^n, \Gamma)$ and $\phi_n: M(2^n, \Gamma) \to \Gamma$ which are inverses of one another. Hence there are induced group isomorphisms $\theta_n: \Gamma^* \to GL(2^n, \Gamma)$ and*

$\phi_n: GL(2^n, \Gamma) \to \Gamma^*$. *Furthermore, for each* $r \in \Gamma$

$$\theta_{n+1}(r \oplus 0) = s_0(\theta_n(r))$$

and for each $g \in \Gamma^*$

$$\theta_{n+1}(g \oplus 1) = s_1(\theta_n(g)).$$

Proof of 1.7. Let 2^n denote the set of all function from $\{1,\ldots,n\}$. Any element $I \in 2^n$ is a sequence $I = \{i_1,\ldots,i_n\}$ where $i_\alpha = 0$ or 1. Let I' denote the sequence $\{i_n,\ldots,i_1\}$. Let $\alpha_I = \alpha_{i_1} \cdot \cdots \cdot \alpha_{i_n}$ and $\beta_I = \beta_{i_1} \cdot \cdots \cdot \beta_{i_n}$. Define $\theta_n: \Gamma \to M(2^n, \Gamma)$ by

$$\theta_n(a) = (\alpha_{I'} \cdot a \cdot \beta_J)_{I, J \in 2^n}$$

for any $a \in \Gamma$. Define $\phi_n: M(2^n, \Gamma) \to \Gamma$ by

$$\phi_n((m_{I,J})) = \sum_{I, J \in 2^n} \beta_{I'} \cdot m_{I,J} \cdot \alpha_J$$

for any $2^n \times 2^n$-matrix $(m_{I,J})$ over Γ. The verification that θ_n and ϕ_n satisfy the desired conditions is left to the reader.

COROLLARY 1.8. *If* Γ *is a sum-ring, then*

$$K_1(\Gamma) = \varinjlim \left(\frac{\Gamma^*}{[\Gamma^*, \Gamma^*]} \xrightarrow{\oplus 1} \frac{\Gamma^*}{[\Gamma^*, \Gamma^*]} \xrightarrow{\oplus 1} \cdots \right).$$

Thus the natural map $\langle\ \rangle: \Gamma^* \to K_1(\Gamma)$ is surjective and $\langle \alpha \rangle = \langle \beta \rangle$ if there is some $n \geqslant 0$ such that

$$s^n(\alpha \cdot \beta^{-1}) \in [\Gamma^*, \Gamma^*] \tag{1.9}$$

Here $s: \Gamma^* \to \Gamma^*$ is the map $x \to x \oplus 1$. Note that since $[\Gamma^*, \Gamma^*]$ is normal and $\oplus$ is conjugate associative (cf. 1.6), condition (1.9) is equivalent to

$$(\alpha \cdot \beta^{-1}) \oplus 1 \in [\Gamma^*, \Gamma^*]. \tag{1.10}$$

Let $E_n(\Gamma) = \theta_n^{-1}(E(2^n, \Gamma))$. Then $g \in [\Gamma^*, \Gamma^*]$ implies $(g \oplus 1) \oplus 1 \in E_8(\Gamma)$ and $g \in E_n(\Gamma)$ implies $g \in [\Gamma^*, \Gamma^*]$ for $n \geqslant 2$ (cf. [9]). Hence for any two elements α and β in Γ^*, $\langle \alpha \rangle = \langle \beta \rangle$ in $K_1(\Gamma)$ iff there is an $n \geqslant 0$ and a $k \geqslant 1$ such that

$$s^n(\alpha \cdot \beta^{-1}) \in E_k(\Gamma) \tag{1.11}$$

The map $\varrho: (\mu R)^* \to K_0(R)$ satisfies $\varrho(x \oplus 1) = \varrho(x)$ so in view of (1.8) there is an induced map $\bar{\varrho}: K_1(\mu R) \to K_0(R)$.

PROPOSITION 1.12 (cf. [7]). *The homomorphism $\bar{\varrho}: K_1(\mu R) \to K_0(R)$ is isomorphism.*

Proof of 1.12. $\bar{\varrho}$ is surjective by (1.3). So suppose $\varrho(\langle\hat{\alpha}\rangle)=0$ where $\langle\hat{\alpha}\rangle \in K_1(\mu R)$ is represented by $\hat{\alpha} \in (\mu R)^*$ for some $\alpha \in lR$. This implies that for some n, coker $\alpha \cong R^n$. Hence α can be chosen to lie in $(lR)^*$. This implies $\langle\hat{\alpha}\rangle=0$ once we have

LEMMA 1.13. $K_1(lR)=0$.

Proof of 1.13. It suffices by (1.8) to show that if $\alpha \in (lR)^*$, then $\alpha \oplus 1 \in [(lR)^*, (lR)^*]$. Consider lR as $l_R(E)$ where E is as in Example 1 above. Given any sequence $A_1, A_2, \ldots$ of elements in lR we can form $A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ by letting A_i act on the submodule $E_i \subset E$. Then

$$\begin{aligned}\alpha \oplus 1 &= \alpha \oplus 1 \oplus 1 \oplus 1 \oplus \cdots \\ &= (\alpha \oplus 1 \oplus \alpha^{-1} \oplus 1 \oplus \alpha \oplus 1 \oplus \cdots) \cdot (1 \oplus 1 \oplus \alpha \oplus 1 \oplus \alpha^{-1} \oplus 1 \oplus \alpha \oplus \cdots).\end{aligned}$$

But each of the two terms in the right hand side of the equation is a product of commutators by (1.1) and (1.2).

One can actually show (cf. [5] or [16]) that $K_i(\mu R) \cong K_{i-1}(R)$ for all $i \in Z$.

§ 2. Trees and Rings

In this section we first recall for convenience the categorical description given in [14] of K_1 of a "tree of rings". Then we give an equivalent but more concrete definition which is needed for the basic exact sequence in the third section.

A *topological tree* is any connected, 1-dimensional, contractible, locally finite complex T with a base vertex v_0 such that if $v \neq v_0$ is any vertex of T then there are at least two 1-simplices branching off from v. Thus any tree has a countably infinite number of vertices and edges. If v is a vertex of T, let $|v|$ = number of edges in the arc connecting v to v_0. We call $|v|$ the "absolute value" of v. The set J of vertices of T can be partially ordered by setting $v \leqslant w$ iff there is a sequence of vertices $v=u_1, \ldots, u_n=w$ such that $|u_i|<|u_{i+1}|$ and u_i and u_{i+1} are the end points of an edge in T. The vertex v_0 is the smallest element of J. Any countably infinite, partially ordered set arising as above will be called a *tree*. If J is a tree with smallest element $0 \in J$ and $J' \subset J$ is a cofinal subset containing 0, then J' is also a tree with 0 as the smallest element. Any tree J can be considered as a category whose objects are the elements of J and whose morphisms consist of a single morphism from j to i whenever $i \leqslant j$.

A *tree of rings* (*over* J) is a covariant functor from J to the category of rings with identity and identity preserving ring homomorphisms. Thus any tree of rings R is a collection $\{R_i, \gamma_{ij}\}$ where $\gamma_{ij}: R_j \to R_i$ is a ring homomorphism for $i \leqslant j$.

A *tree of sets* is a covariant functor from J to the category of sets and inclusion maps which associates to each $j \in J$ a countable set C_j such that

a) if $|i|=|j|$ and $i\neq j$, then $C_i\cap C_j=\emptyset$
b) for each $n\geqslant 0$ the set $C_0-\bigcup_{|i|=n} C_i$ is finite
c) for each $c\in C_0$ there is an n such that $c\notin\bigcup_{|i|=n} C_i$.

If each of the sets C_j is countably infinite, then we have a *tree of infinite sets*. In this case condition (c) is not really needed.

Let $R=\{R_j, \gamma_{ij}\}$ be a tree of rings over J. A *module M over R* consists of a collection $\{M_j, h_{ij}\}$ where M_j is a R_j-module and whenever $i\leqslant j$, $h_{ij}; M_j\to M_i$ is an additive map satisfying

(i) $h_{ij}(r\cdot m)=\gamma_{ij}(r)\cdot h_{ij}(m)$
(ii) $h_{ij}\circ h_{jk}=h_{ik}$ for $i\leqslant j\leqslant k$.

If $M=\{M_j, f_{ij}\}$ and $M'=\{M_j', f_{ij}'\}$ are two modules over R a *morphism* $F: M\to M'$ is a collection $F=\{f_j\}$ of R_j-homomorphisms $f_j: M_j\to M_j'$ such that whenever $i\leqslant j$ we have $h_{ij}'\circ f_j=f_i\circ h_{ij}$. Morphisms can be added and composed by adding and composing the f_j's.

Let $\alpha: J\to N^+$ be a function from J to the non-negative integers such that $|i|\leqslant\alpha(i)$ and $\alpha(i)\leqslant\alpha(j)$ whenever $i\leqslant j$. This induces a "shift functor" from the category of modules over the tree of rings R to itself as follows: Form $M^\alpha=\{M_i^\alpha, h_{ij}^\alpha\}$ from $M=\{M_i, h_{ij}\}$ by letting $M_i^\alpha=\oplus_k(M_k\otimes R_i)$ where $i\leqslant k$ and $|k|=\alpha(i)$. To get h_{ij}^α: $M_j^\alpha\to M_i^\alpha$ when $i\leqslant j$, let $l\in J$ satisfy $|l|=\alpha(j)$. Then there is a unique $k\leqslant l$ with $|k|=\alpha(i)$. The map $h_{kl}: M_l\to M_k$ induces a map $h_{kl}': M_l\otimes R_j\to M_k\otimes R_i$ and h_{ij}^α: $\oplus_l(M_l\otimes R_j)\to\oplus_k(M_k\otimes R_i)$ is obtained by summing up the h_{kl}'.

If $F: M\to N$ is a morphism, let $F^\alpha: M^\alpha\to N^\alpha$ be given by $F^\alpha=\{f_i^\alpha\}$ where $f_i^\alpha=\oplus_k(f_k\otimes id):[\oplus_k(M_k\otimes R_i)]\to[\oplus_k(N_k\otimes R_i)]$. Now let $\beta: J\to N^+$ be another "shift map" such that $\alpha\leqslant\beta$; that is, $\alpha(i)\leqslant\beta(i)$ for all $i\in J$. For each l with $i\leqslant l$ and $|l|=\beta(i)$ there is a unique l' such that $l'\leqslant l$ and $|l'|=\alpha(i)$. Therefore we have a map $M_l\otimes R_i\to M_{l'}\otimes R_i$. These sum together to produce a map

$$\bigoplus_l (M_l\otimes R_i)\to\bigoplus_{l'}(M_{l'}\otimes R_i)$$

This in turn gives a morphism $\pi_{\alpha\beta}: M^\beta\to M^\alpha$ of modules over the tree of rings R.

Now if $f: M^\alpha\to N$ is a morphism and $\alpha\leqslant\beta$ there is the composition $f\circ\pi_{\alpha\beta}: M^\beta\to\to M^\alpha\to N$.

A *germ* $[f]: M\to N$ consists of an equivalence class of morphisms $f: M^\alpha\to N$, each f being defined on some M^α, where $f: M^\alpha\to N$ and $g: M^\beta\to N$ are *equivalent* iff there is a shift map $\gamma: J\to N^+$ such that $\alpha\leqslant\gamma$ and $\beta\leqslant\gamma$ and $f\circ\pi_{\alpha\gamma}=f\circ\pi_{\beta\gamma}$ as morphisms from M^γ to N. Addition and composition of morphisms induces addition and composition of germs.

The category $\mathscr{M}_R$ with modules over the tree of rings R as objects and germs as morphisms is an abelian category. See [14] for full details. The category $\mathscr{M}_R$ has a "direct sum" operation where $M\oplus N=\{M_j\oplus N_j\}$.

Let $C=\{C_j\}$ be a tree of sets and $R=\{R_j, \gamma_{ij}\}$ be a tree of rings. The *free module* $F[C; R]$ *generated by* C *over* R is given by $\{F_j, h_{ij}\}$ where F is the free R_j module generated by C_j (if C_j is empty set $F_j=0$) and $h_{ij}: F_j \to F_i$ is induced by the inclusion $C_j \subset C_i$. A module $M \in \mathcal{M}_R$ is said to be *locally finitely generated* provided there is an epimorphism $F[C; R] \to M$ for some tree of sets C. Let $\mathcal{P}_R \subset \mathcal{M}_R$ denote the full subcategory of locally finitely generated projectives. Then $\mathcal{P}_R$ is an admissible, semi-simple subcategory of $\mathcal{M}_R$ in the sense of [1, p. 388] and one can define

$$K_1(R) = K_1(\mathcal{P}_R). \tag{2.0}$$

An alternate but equivalent definition of $K_1(R)$ which we discuss in the remainder of this section goes as follows: the tree J has associated to it a natural tree of infinite sets $E=\{E_j\}$ where $E_j=\{i \in J \mid i \geqslant j\}$. Let $\Gamma(R)$ denote the ring of endomorphisms of $F[E; R]$. The addition in $\Gamma(R)$ is the addition of germs and the multiplication is composition of germs. Then we can set

$$K_1(R) = K_1(\Gamma(R)). \tag{2.0'}$$

There is a natural map $K_1(\Gamma(R)) \to K_1(\mathcal{P}_R)$ which is an isomorphism. The argument showing this is entirely similar to the argument in [1, p. 353]; one uses (1.8) together with (2.4) below or Lemma 6 of [14, Chap. I, § 5].

Now let R be a tree ring and let C be a tree of infinite sets over J. Consider a collection $A=\{A^k\}$, $k \in J$, which satisfies

1) $A^k=(a^k_{pq})$ is a locally finite matrix with $(p, q) \in C_0 \times C_0$ such that there are at most finitely many pairs $(p, q) \notin C_k \times C_k$ with $a^k_{pq} \neq 0$, and

2) for each $k \in J$, $A^k = A^{l_1} \otimes R_k + \cdots + A^{l_n} \otimes R_k +$ finite matrix where $l_1, \ldots, l_n$ are the elements of J with $|l_i| = |k|+1$ and $k \leqslant l_i$. Here $A \otimes R_k = (\gamma_{kl}(a^l_{pq}))$ for $l=l_1, \ldots, l_n$.

Two such collections $A=\{A^k\}$ and $B=\{B^k\}$ are *equivalent* iff $A^0=B^0$ and $A^k=B^k$ + finite matrix whenever $k>0$. A *germ* is an equivalence class of such collections. If $[A]$ and $[B]$ are two germs represented by $A=\{A^k\}$ and $B=\{B^k\}$, define

$$[A]+[B] = [\{A^k + B^k\}]$$

and

$$[A]\cdot[B] = [\{A^k \cdot B^k\}]$$

It is easy to check that the addition and multiplication of germs is well defined and that this definition of germs of $F[C; R]$ to itself agrees with the previous one. Let $\Gamma(C; R)$ denote this ring of germs.

Here is an alternate description of $\Gamma(C; R)$. Let $\mu(C_i; R_i)$ be the ring of locally finite matrices operating on the free R_i-module based on C_i modulo the ideal of finite matrices. Whenever $i \leqslant j$ the inclusion $C_j \subset C_i$ and the ring homomorphism $\gamma_{ij}: R_j \to R_i$ induce a ring homomorphism

$$\mu_{ij}: \mu(C_j; R_j) \to \mu(C_i; R_i)$$

via the correspondence $(a^j_{pq}) \to (\gamma_{ij}(a^j_{pq}))$. For each non-negative integer n let

$$\mu_n(C; R) = \bigoplus_{|i|=n} \mu(C_i; R_i)$$

whenever $k \leqslant l$, there is an identity preserving ring homomorphism

$$\mu_l(C; R) \to \mu_k(C; R)$$

obtained by summing up the maps μ_{ij} where $|i| = k$ and $|j| = l$. Define

$$\mu(C; R) = \lim_{\leftarrow} \mu_n(C; R).$$

Then $\Gamma(C; R)$ is the pull back of the diagram

$$\begin{array}{ccc} \Gamma(C; R) & \dashrightarrow & \mu(C; R) \\ \downarrow & & \downarrow \\ l(C_0; R_0) & \longrightarrow & \mu(C_0, R_0) \end{array}$$

Let $J' \subset J$ be a cofinal subset of J containing the smallest element 0 and suppose that there is a sequence of positive integers $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ such that $j \in J'$ iff $|j| = \alpha_k$ for some $k \geqslant 0$. Let C' and R' be the corresponding trees derived from C and R. If $A = \{A^k\}$ satisfies (1) and (2) so does $A' = \{A^k\}_{k \in J'}$.

The correspondence $A \to A'$ induces a ring homomorphism

$$\phi: \Gamma(C; R) \to \Gamma(C'; R')$$

PROPOSITION 2.1. *ϕ is an isomorphism.*

Proof. Exercise. Similar to proof that inverse limits are not changed by taking cofinal subsets.

Actually, the ring $\Gamma(C; R)$ only depends up to isomorphism only on the tree of rings R. Corollary 2.3 below shows that for any two trees of infinite sets C and D over J the rings $\Gamma(C; R)$ and $\Gamma(D; R)$ are isomorphic in a very natural way.

A function $f: C_0 \to D_0$ is *proper* provided that for each $j \in J$ there are at most finitely many elements $v \in C_j$ such that $f(v) \notin D_j$.

PROPOSITION 2.2. *Let $C = \{C_j\}$ and $D = \{D_j\}$ be trees of infinite sets over J. There is a bijection $h: C_0 \to D_0$ such that both h and h^{-1} are proper.*

Proof of 2.2. Associated to the tree J is the "standard" tree of sets $E=\{E_j\}$ where $E_j=\{i \mid i\in J \text{ and } j\leqslant i\}$. To prove (2.2) it is sufficient to show that for any tree of infinite sets C there is a bijection $h: C_0 \to E_0$ such that h and h^{-1} are proper.

For any tree of sets C define, for each non-negative integer n and each $j\in J$ with $|j|=n$, the set $\hat{C}_j$ as

$$\hat{C}_j = C_j - \bigcup_i C_i$$

where $|i|=n+1$ and $j\leqslant i$.

By condition (b) above $\hat{C}_j$ is finite. Let $C^n = \bigcup_{|j|\leqslant n} \hat{C}_j$. Note that

$$C^{n+1} = C^n \cup \hat{C}_{j_1} \cup \cdots \cup \hat{C}_{j_k}$$

where $j_1, \ldots, j_k \in J$ are the vertices with absolute value $n+1$. Also $C_0 = \bigcup_n C^n$.

We shall construct the required bijection $h: C_0 \to E_0$ by first defining a sequence of injections $h_n: C^n \to E_0$ such that

$$(\alpha)\ h_{n+1} \mid C^n = h_n;\ (\beta)\ h(C^n) \supset E^n;\ (\gamma)\ h(\hat{C}_j) \subset E_j.$$

Step 1. Construction of an h satisfying (α) through (γ).

We can assume that each $\hat{C}_j$ is non-empty by the following argument: Using the fact that each C_j is countably infinite choose a collection $\{\gamma_j\}_{j\in J}$ of distinct elements γ_j of C_u such that $\gamma_j \in C_j$ for each $j\in J$. Define for $j\in J$

$$D_j = C_j - \{\gamma_i\}_{|i|<|j|}$$

Then D_k contains D_l properly whenever $k<l$ and $D=\{D_j\}$ is a tree of infinite sets with each $\hat{D}_j$ non-empty. Furthermore the identity maps $C_0 \to D_0$ and $D_0 \to C_0$ are proper; so to prove (2.2) we can, if necessary, replace C by D to insure that $\hat{D}_j \neq \emptyset$. Now let $S \subset E_0$ be any finite subset. We say S is *full* in E_j provided there is some $n>|j|$ so that $S\cap E_j = \{i \mid i\in E_j \text{ and } |i|<n\}$. If S is full in E_j, then S is full in E_l whenever $j<l$ and $S\cap E_l \neq 0$. Note that $E_j=\{j\}$ for each $j\in J$.

Now let $h_0: C^0 \to E_0$ be any injection with $h_0(C^0)$ full in E_0. Suppose that $h_n: C^n \to E_0$ is defined for $n>0$ and satisfies (α), (β), and (γ). Recall that $C^{n+1} = C^n \cup \hat{C}_{j_1} \cup \cdots \cup \hat{C}_{j_k}$ as above. Define $h_{n+1}: C^{n+1} \to E_0$ by letting $h_{n+1} \mid C^n = h_n$ and $h_{n+1} \mid \hat{C}_p$ for $p=j_1, \ldots, j_k$ be any injection of $\hat{C}_p$ into $E_p - h_n(C^n)$ such that $h_{n+1}(C^n \cup \hat{C}_p)$ is full in E_p. This sequence of h_n's satisfies (α) through (γ).

Now define the injection $h: C_0 \to E_0$ by letting $h \mid C^n = h_n$. Condition (β) implies h is a surjection and condition (γ) implies that $h(C_j) \subset E_j$ for each $j\in J$.

Step 2. It remains to show that $h^{-1}: E_0 \to C_0$ is proper. We show that for any $j\in J$ there are at most finitely many elements $v\in C_0$ such that $v\notin C_j$ but $h(v)\in E_j$.

Suppose to the contrary that there infinitely many such v's; say, $v_1, v_2, v_3, \ldots$. Condition (b) above implies there is some v_n and a vertex $p \in J$ with $v_n \in C_p$ and $|p| \geqslant |j|$. Since $v_n \notin C_j$ there is a vertex $i \in J$ with $|i| = |j|$, $i \neq j$, and $p \geqslant i$. Condition (a) above says that $E_i \cap E_j = \emptyset$; bat $h(C_i) \subset E_i$ so $h(v_n) \notin E_j$. This is a contradiction. Hence h^{-1} is proper. This completes the proof of (2.2).

Now let C and D be trees of infinite sets over J and let $h: C_0 \to D_0$ be as in (2.2). The bijections h and h^{-1} induce isomorphisms on the germ level $[h]: F[C; R] \to F[D; R]$ and $[h^{-1}]: F[D; R] \to F[C; R]$. The germs $[h]$ and $[h^{-1}]$ are inverses of one another. Thus we have

COROLLARY 2.3. *If C and D are trees of infinite sets over J, then $F[C; R]$ and $F[D; R]$ are isomorphic. Hence $\Gamma(C; R)$ and $\Gamma(D; R)$ are isomorphic.*

A useful useful special case of (2.3) is the following: we say two trees of infinite sets $C = \{C_\alpha\}$ and $D = \{D_\alpha\}$ are *equivalent* iff $C_0 = D_0$ and, for $\alpha > 0$, both $C_\alpha - C_\alpha \cap D_\alpha$ and $D_\alpha - C_\alpha \cap D_\alpha$ are finite sets. If C and D are equivalent, then the identity map $id: C_0 \to D_0$ and its inverse are proper bijections; so (2.3) applies and we have $\Gamma(C; R) = \Gamma(D; R)$.

If $[A]$ is a germ in $\Gamma(C; R)$ there is a representative $A = \{A^k\}$ of $[A]$ and a tree of infinite sets $D = \{D_k\}$ equivalent to C with $C_k \supseteq D_k$ for all $k \in J$ and such that for any $k \in J$

$$a^k_{pq} = 0 \quad \text{for} \quad (p, q) \notin D_k \times D_k \tag{1'}$$

and for any k and l in J with $k \leqslant l$

$$a^k_{pq} = \delta_{kl}(a^l_{pq}) \quad \text{for} \quad (p, q) \in D_l \times D_l. \tag{2'}$$

Any such representative A of the germ $[A]$ will be called a *matrix*. Note that the matrix $A = \{A^k\}$ is a morphism (not just a germ) of $F[D; R]$ to itself.

If J is a tree and R is a tree of rings over J we let

$$\Gamma(R) = \Gamma(E; R)$$

where E is the standard tree of sets obtained from J. The need for considering various "presentations" $\Gamma(C; R)$ of $\Gamma(R)$ for different trees of infinite sets C arises in topological applications.

EXAMPLE 1. R is the inverse sequence $R_0 \leftarrow R_1 \leftarrow R_{i-1} \xleftarrow{f_i} R_i \leftarrow \cdots$. Any element of $\Gamma(R)$ is represented by a sequence $(M^0, M^1, M^2, \ldots)$ of locally finite matrices $M^\alpha = (m^\alpha_{ij})$ where $m^\alpha_{ij} \in R_\alpha$, $0 \leqslant i, j < \infty$, such that

$$M^\alpha = M^{\alpha+1} \otimes R_\alpha + \text{finite matrix}.$$

EXAMPLE 2. Same as above but where the f_α are isomorphisms for, say, $\alpha \geqslant k$. Any element of $\Gamma(R)$ is represented by a pair of locally finite matrices (M, N) such that $M = N \otimes R_0 +$ finite matrix. Here M has entries in R_0 and N has entries in R_k. In particular if $f: A \to B$ is a ring homomorphism we shall denote the ring Γ of the system $B \xleftarrow{f} A \xleftarrow{id} A \xleftarrow{id} \cdots$ by γf. In this case γf can be described as the pullback of the diagram

$$\begin{array}{ccc} \gamma f & \dashrightarrow & \mu A \\ \downarrow & & \downarrow \\ lB & \longrightarrow & \mu B \end{array}$$

EXAMPLE 3. The tree R has two ends:

$$\to R_{-\alpha} \xrightarrow[f_{-\alpha}]{} R_{-\alpha+1} \to \cdots \to R_{-1} \to R_0 \leftarrow R_1 \leftarrow \cdots \leftarrow R_{\alpha-1} \xleftarrow[f_\alpha]{} R_\alpha \leftarrow \cdots.$$

Suppose f_α and $f_{-\alpha}$ are isomorphisms for $\alpha \geqslant k$. Then any element of $\Gamma(R)$ is represented by a triple (M^{-k}, M^0, M^k) of locally finite matrices $M^\alpha = (m_{ij}^\alpha)$ over R^α where $\alpha = -k$, 0, k and $-\infty < i, j < \infty$ such that

(a) $m_{ij}^{-k} = 0$ if $i > 0$ or $j > 0$

(b) $m_{ij}^k = 0$ if $i < 0$ or $j < 0$

(c) $M^0 = M^{-k} \otimes R_0 + M^k \otimes R_0 +$ finite matrix.

EXAMPLE 4. All the ring maps $\delta_{ij}: R_j \to R_i$ are isomorphisms. $\Gamma(R)$ is the ring of all locally finite matrices $M = (m_{ij})$ over R_0 (where $i, j \in J$) such that for each $i \in J$ there are at most finitely many $j \notin E_i$ such that $m_{ji} \neq 0$.

LEMMA 2.4. *The ring $\Gamma(R)$ is a sum-ring.*

Proof. If $C = \{C_j\}$ and $D = \{D_j\}$ are trees of infinite sets over J, define the *sum of C and D*, written $C \amalg D$, to be the collection $\{C_j \amalg D_j\}$ where $C_j \amalg D_j$ denotes the disjoint union of C_j and D_j. In view of (2.3) it suffices to show that $\Gamma(E^0 \amalg E^1, R)$ is a sum-ring where E^0 and E^1 are two copies of the standard tree E. If X is any set and R a ring let $F(X, R)$ be the free R-module generated by X. Now for $i = 0$ or 1 let $\beta_i: E^0 \amalg E^1 \to E^i$ be a proper bijection as in (2.2). For each $k \in J$ let β_i^k denote the corresponding locally finite R_k-transformation from $F(E_k^0 \amalg E_k^1, R_k)$ to itself; β_i^k is only determined up to a finite matrix for $k > 0$. For $i = 0$ or 1 and $k \in J$, define the locally finite R_k-transformation α_i^k of $F(E_k^0 \amalg E_k^1, R_k)$ to itself by

$$\alpha_i^k(e) = \begin{cases} (\beta_i^k)^{-1}(e), & \text{for } e \in E_k^i \\ 0 \quad , & \text{for } e \notin E_k^i. \end{cases}$$

Each α_i^k is well determined modulo a finite matrix. The germs $\alpha_i = \{\alpha_i^k\}$ and $\beta_i = \{\beta_i^k\}$ make $\Gamma(E^0 \amalg E^1; R)$ into a sum ring.

Now let C be a tree of infinite sets. The collection of finite sets $\{\hat{C}_j\}$ defined in the proof of (2.2) will be called the *block decomposition* of C. Each $\hat{C}_j$ is a *block*.

A germ $[A]$ in $\Gamma(C;\, R)$ is *blocked* provided there is a tree of infinite sets D equivalent to C and a matrix representative $A=\{A^\alpha\}$ of $[A]$ which is "blocked" with respect to D; that is, for each $\alpha \in J$, $A^\alpha = (a^\alpha_{pq})$ where

$$a^\alpha_{pq} = 0 \quad \text{if} \quad (p,q) \notin \bigcup_{\alpha \leqslant \beta} \hat{D}_\beta \times \hat{D}_\beta .$$

A germ may be blocked in many different ways. To illustrate this definition consider a germ $[A]$ in $\Gamma(E;\, R)$ where R is the tree of rings in Example 1. A blocking of the germ $[A]$ consists essentially of a sequence $0=n_0<n_1<\cdots$ of integers and a sequence $M^\alpha=(m^\alpha_{pq})$ of $(n_{\alpha+1}-n_\alpha)\times(n_{\alpha+1}-n_\alpha)$ square matrices where $n_\alpha \leqslant p,\, q < n_{\alpha+1}$ and $m^\alpha_{pq} \in R_\alpha$. The blocked representative $A=\{A^\alpha\}$ is defined by

$$A^\alpha = M^\alpha + M^{\alpha+1} \otimes R_\alpha + M^{\alpha+2} \otimes R_\alpha + \cdots$$

schematically we have

$$A = \begin{pmatrix} \boxed{M^0} & & & 0 \\ & \boxed{M^1} & & \\ & & \boxed{M^2} & \\ & 0 & & \ddots \end{pmatrix}$$

In general suppose $A=\{A^\alpha\}$ is blocked with respect to D. For $\alpha \in J$ let $M^\alpha(A)=(a^\alpha_{pq})$ where $(p,q) \in \hat{D}_\alpha \times \hat{D}_\alpha$. We shall write

$$A = \sum_\alpha M^\alpha(A) \tag{2.5}$$

to indicate that $A^\alpha = \sum_{\alpha \leqslant \beta} M^\alpha(A) \otimes R_\beta$ for each $\alpha \in J$. We call $\hat{D}_\alpha$ the *support* of $M^\alpha(A)$.

If $A=\sum_\alpha M^\alpha(A)$ is blocked with respect to the tree of infinite sets D we can block A in a new way by the operation of *amalgamation*: Choose a sequence of integers $0=n_0<n_1<n_2<\cdots$ with $i \leqslant n_i$. Form the tree of sets $D'=\{D'_\alpha\}$ by letting

$$D'_\alpha = \bigcup_{\alpha \leqslant \beta} D_\beta \quad \text{where} \quad n_{|\alpha|} \leqslant |\beta|$$

Note that D' and D are equivalent. For $\alpha \in J$ let $M^\alpha_1(A)=\sum_{\alpha \leqslant \beta} M^\beta(A) \otimes R_\alpha$ where

$n_{|\alpha|} \leqslant |\beta| < n_{|\alpha|+1}$. Then $A = \sum_\alpha M_1^\alpha(A)$ with respect to D' and we say the representation $\sum_\alpha M_1^\alpha(A)$ is obtained by "amalgamation" from $\sum_\alpha M^\alpha(A)$.

LEMMA 2.6. *Let $[A] \in \Gamma(C; R)$ be any germ. Then $[A]$ has a representative of the form $X+Y$ where $X=\{X^\alpha\}$ and $Y=\{Y^\alpha\}$ are blocked (with respect to possibly different trees of infinite sets equivalent to C).*

Proof of 2.6. We discuss the special case where $J=\{1, 2. \ldots\}$ is the tree with one end. The general case when J has many ends is left to the reader. Choose a sequence $1=n_1=n_2<n_3<n_4<\cdots$ of integers and define the square $B_i \subset J \times J$ by $B_i=\{(p, q) \mid n_i \leqslant p, q < n_{i+2}\}$. This choice can be made so that $[A]$ has a representative $A=\{A^\alpha\}$ satisfying

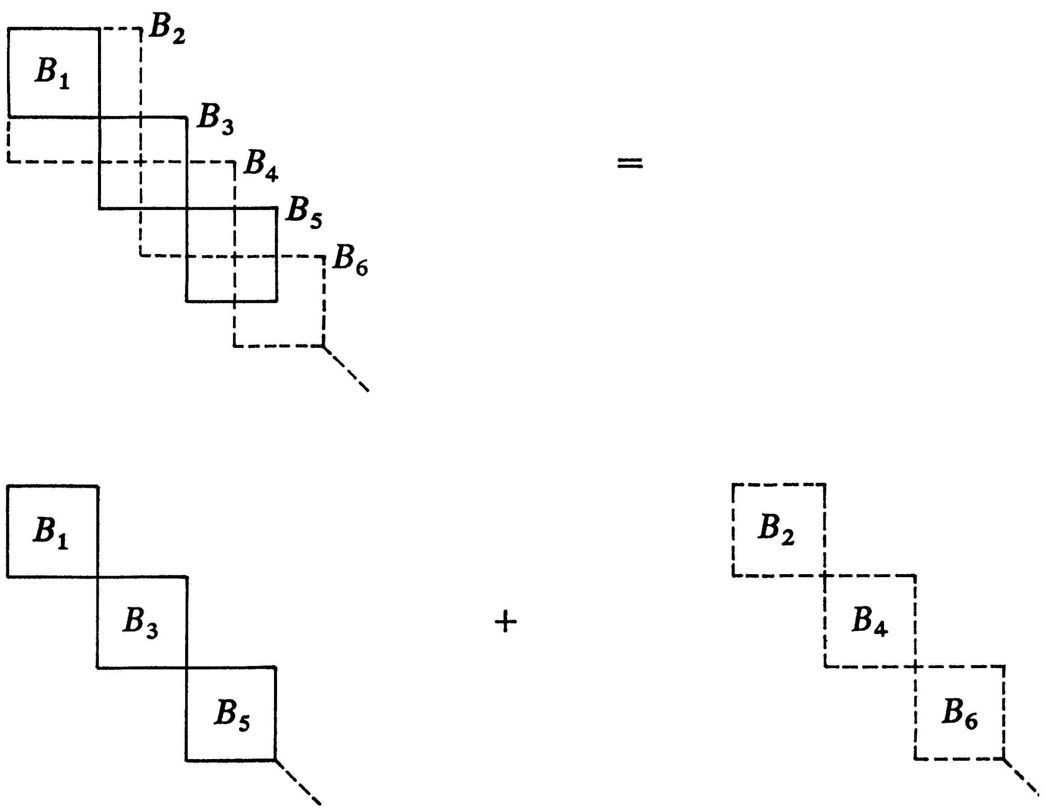
(a) $a_{pq}^\alpha = 0 \quad \text{if} \quad (p, q) \notin \bigcup_{\alpha \leqslant i} B_i$

(b) if $\alpha \leqslant \beta$, then $a_{pq}^\alpha = \delta_{\alpha\beta}(a_{pq}^\beta)$ whenever $(p, q) \in \bigcup_{\beta \leqslant i} B_i$.

Let $M^\alpha = (a_{pq}^\alpha)$ for $(p, q) \in B_\alpha$. The required $X=\{X^\alpha\}_{\alpha \geqslant 1}$ and $Y=\{Y^\alpha\}_{\alpha \geqslant 1}$ are defined as

$$X^\alpha = M^{2\alpha-1} + M^{2\alpha+1} \otimes R_{2\alpha-1} + \cdots \quad \text{and} \quad Y^\alpha = M^{2\alpha} + M^{2\alpha+2} \otimes R_{2\alpha} + \cdots.$$

Thus $[A]$ is represented by $X+Y=(\sum_\alpha M^{2\alpha-1})+(\sum_\alpha M^{2\alpha})$. Schematically we have



LEMMA 2.7. *Let $\Gamma=\Gamma(C;R)$ and let G be a germ in $E_n(\Gamma)\subset\Gamma^*$ for $n\geqslant 1$. Then G has a representative of the form $X\cdot Y$ where X and Y are blocked matrices and can be written in the form $X=\sum_\alpha M^\alpha(X)$ and $Y=\sum_\alpha M^\alpha(Y)$ where for each $\alpha\in J$ both $M^\alpha(X)$ and $M^\alpha(Y)$ are products of elementary matrices over R_α.*

Proof. We shall only deal with the following special case which contains the essential idea in (2.7): $J=\{1,2,3\ldots\}$ ordered by increasing magnitude and $C=E=$ standard tree of sets for J. We shall show that any element in $E_1(\Gamma)\subset\Gamma^*$ satisfies (2.7). The notation $e_{ij}(\gamma)$ will refer to an elementary matrix in $E(2,\Gamma)$ and also to the element in Γ^* it corresponds to under $\phi_1: GL(2,\Gamma)\to\Gamma^*$.

Let $K=\{k_i\}$ and $L=\{l_i\}$ be two copies of the tree J. Then J is equivalent to $K\amalg L$ where $k_i\in K$ is identified with $2i-1\in J$ and $l_i\in L$ is identified with $2i\in J$. Hence $\Gamma^*=\Gamma(K\amalg L;R)$.

Step 1. Consider the element $e_{12}(\lambda)\in E(2,\Gamma)$. Use (2.6) to write $\lambda=X+Y$ where $X=\sum_i M^{2i-1}$ and $Y=\sum_i M^{2i}$. Then $e_{12}(\lambda)=e_{12}(X)\cdot e_{12}(Y)$ and both $e_{12}(X)$ and $e_{12}(Y)$ are blocked as elements of Γ^*: To see this let B_i and n_i be as in (2.6). As an element of Γ^*, $e_{12}(X)$ is represented by the collection of matrices $\{A^\alpha\}$ where

$$A^\alpha=\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & \\ 0 & I_\alpha & 0 & \boxed{M^{2\alpha-1}} & \\ & & & & \boxed{M^{2\alpha+1}\otimes R_\alpha} \\ \hline & & 0 & 0 & \\ & 0 & 0 & I_\alpha & \end{array}\right)$$

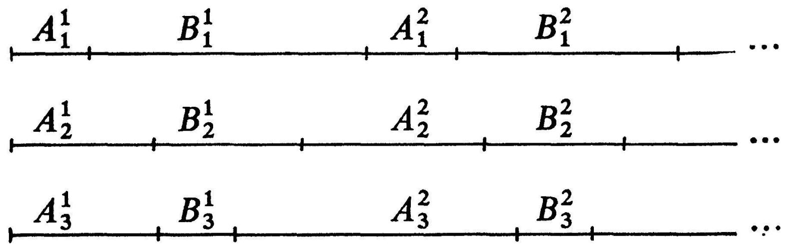
Here I denotes the inclusions $F(K_\alpha;R_\alpha)\subset F(K;R_\alpha)$ and $F(L_\alpha;R_\alpha)\subset F(L;R_\alpha)$. Hence $e_{12}(X)=\sum_{1\leqslant\alpha}N^\alpha$ where N^α is the $2.(n_{2\alpha+1}-n_{2\alpha-1})\times 2.(n_{2\alpha+1}-n_{2\alpha-1})$ elementary matrix which looks like

$$N^\alpha=\left(\begin{array}{c|c} I & M^{2\alpha-1} \\ \hline 0 & I \end{array}\right)$$

The proof that $e_{12}(Y)$ is blocked is similar. Also the above argument can be copied to show $e_{21}(\lambda)$ is the product of two blocked matrices.

Step 2. The first step shows that any element $Q\in E_1(\Gamma)$ is the product $Q=P_1\cdot P_2\cdot\ldots\cdot P_n$ of blocked germs where $P_i=\sum_{1\leqslant\alpha}M^\alpha(P_i)$.

To prove (2.7) it remains to show this product can be reduced to one of length two. This is done by amalgamation. Choose a sequence $1=r_1<r_2<\cdots<r_n$ so that the support of A_i^1 is contained in the support of A_{i+1}^1 for $1\leqslant i\leqslant n-1$ where $A_i^1=M^1(P_i)+\cdots+M^{r_i}(P_i)\otimes R_1$. Then choose a sequence $s_1>s_2>\cdots>s_n\geqslant 1$ such that the support of B_i^1 contains the support of B_{i+1}^1 for $1\leqslant i\leqslant n-1$ where $B_i^1=M^{r_i+1}(P_i)+\cdots+M^{r_i+s_i}(P_i)\otimes R_1$ continue in this way to get two collections of matrices A_i^α and B_i^α $1\leqslant\alpha<\infty$ and $1\leqslant i\leqslant n$, where each A_i^α and B_i^α has entries in R_α such that the supports of the A_i^α and B_i^α fit together in the following way:



(2.8)

$n=3$

Let $X=\sum_{1\leqslant\alpha}M^\alpha(X)$ and $Y=\sum_{1\leqslant\alpha}M^\alpha(Y)$ where $M^\alpha(X)=A_1^\alpha\cdot A_2^\alpha\cdot\ldots\cdot A_n^\alpha$ and $M^\alpha(Y)=B_1^\alpha\cdot B_2^\alpha\cdot\cdots\cdot B_n^\alpha$. Then $Q=X\cdot Y$ as required.

§ 3. The Basic Exact Sequence

Let $\{A_j,\gamma_{ij}\}$ be a tree of abelian groups over J and let $0\in J$ denote the smallest element. Define the *shift homomorphism*

$$S\colon\prod_{j>0}A_j\to\prod_{j\geqslant 0}A_j$$

by $S(\{a_j\})=\{b_j\}$ where for $j\geqslant 0$

$$b_j=\sum_{\substack{l\geqslant j\\ |l|=|j|+1}}\gamma_{jl}(a_l)$$

Let $I\colon\prod_{j>0}A_j\to\prod_{j\geqslant 0}A_j$ be the inclusion map.

Now let $R=\{R_j\}$ be a tree of rings over J.

THEOREM 3.1. *There is a five term exact sequence*

$$\prod_{j>0}K_1(R_j)\xrightarrow{I-S}\prod_{j\geqslant 0}K_1(R_j)\xrightarrow{\Delta}K_1(R)\xrightarrow{\partial}\prod_{j>0}K_0(R_j)\xrightarrow{I-S}\prod_{j\geqslant 0}K_0(R_j).$$

The existence of such a sequence was suggested to us by Theorems I, II and II′ of [12]. The purpose of this section is to define Δ and ∂ and to show exactness of

(3.1). There is a similar sequence (see 3.6) involving the functors "Wh" and "$\tilde{K}_0$" when the tree R is a tree of group rings.

First we define ∂: Let $\Gamma=\Gamma(S;R)$ where S is the standard tree of sets $\{S_j\}$ associated to the tree J. Let $[A]\in\Gamma^*$ be an invertible germ represented by the collection $A=\{A^j\}$ where $A^j\in\mu(S_j;R_j)^*$ for $j>0$. Each A^j determines by (1.3) an element $\varrho(A^j)\in K_0(R_j)$ and we set

$$\partial([A])=\{\varrho(A^j)\}_{j>0} \tag{3.2}$$

Then $\partial([A]\cdot[B])=\partial([A])\cdot\partial([B])$ and $\partial([A]\oplus 1)=\partial([A])$ so by (1.8) there is an induced homomorphism

$$\partial\colon K_1(R)\to\prod_{j>0}K_0(R_j)$$

Next we define Δ: This is done by defining a homomorphism

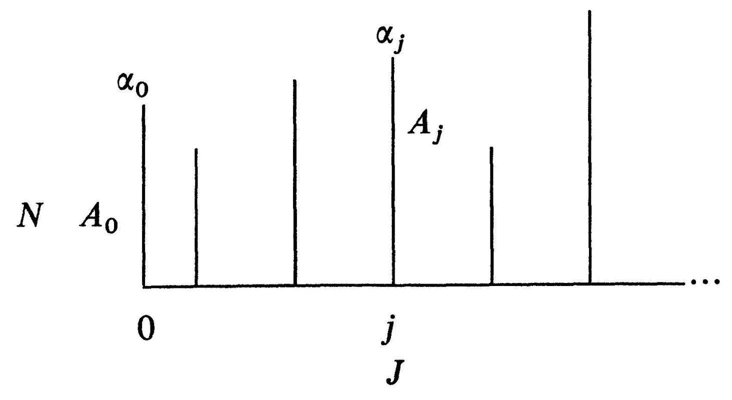
$$\Delta\colon\prod_{j\geqslant 0}GL(R_j)\to K_1(R)$$

which by [1, Cor. 1.10, p. 353] vanishes on $\prod_{j\geqslant 0}[GL(R_j), GL(R_j)]$.

Let $A=\{A_\alpha\}\in\prod_{j\geqslant 0}GL(R_j)$. Choose a function $\alpha\colon J\to N$, where $N=\{1,2,3,\ldots\}$, such that $A_j\in GL(\alpha_j,R_j)$. Form the tree of infinite sets $S(\alpha)=\{S(\alpha)_j\}$ where $S(\alpha)_j\subset J\times N$ consists of those pairs (i,k) where $J\leqslant i$ and $1\leqslant k\leqslant\alpha_i$. Consider $A_i\in GL\times(\alpha_j,R_j)$ as having support $\hat{S}(\alpha)_j=\{(j,1),\ldots,(j,\alpha_j)\}$. Then $\sum_{0\leqslant j}A_j$ is an invertible germ in $\Gamma(S(\alpha);R)^*$ blocked by $S(\alpha)$. Choose any proper bijection $h\colon S(\alpha)_0\to S_0$ as in (2.2) and set

$$\Delta_\alpha(A)=\left\langle h\cdot\left(\sum_{0\leqslant j}A_j\right)\cdot h^{-1}\oplus 1\right\rangle\in K_1(R) \tag{3.3}$$

where "$\langle\ \rangle$" denotes the class in $K_1(R)$ determined by the invertible germ $h\cdot(\sum_{0\leqslant j}A_j)\cdot h^{-1}\oplus 1$ in $\Gamma(S;R)^*$. See the following diagram:



Note that $\Delta_\alpha(A)$ is independent of the choice of bijection $h: S(\alpha)_0 \to S_0$; because any two such choices determine elements of $\Gamma(S;\, R)^*$ which are conjugate. If $A=\{A_j\}$ and $B=\{B_j\}$ are in $\prod_{0 \leqslant j} GL(R_j)$ and $\alpha: J \to N$ is a function such that both A_j and B_j are in $GL(\alpha_j;\, R_j)$ for all $j \in J$, then clearly $\Delta_\alpha(A\cdot B)=\Delta_\alpha(A)\cdot\Delta_\alpha(B)$. Hence to show that (3.3) gives a well defined homomorphism it suffices to show that if $A_j \in GL(\alpha_j,\, R_j)$ for all j and $\beta: J \to N$ is a function with $\alpha_j < \beta_j$ for all j then $\Delta_\alpha(A) = \Delta_\beta(A)$ in $K_1(R)$. Let $A'=\{A'_j\}$ denote A considered as an element of $\prod_{0\leqslant j} GL \times (\beta_j,\, R_j)$. Let $X_A = \sum_{0 \leqslant j} A_j \in \Gamma(S(\alpha);\, R)^*$ and $X_{A'} = \sum_{0\leqslant j} A'_j \in \Gamma(S(\beta);\, R)^*$. Note that $S(\beta) = S(\alpha) \amalg S(\beta-\alpha)$. Suppose $\Gamma(S;\, R)$ has been made into a sum ring via the decomposition $S = K \amalg L$ where K and L are two copies of S and let $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \Gamma(S;R)$ be as in the proof of (2.4). Choose proper bijections $f: S(\alpha) \to K$ and $g: S(\beta-\alpha) \to L$. This gives a bijection $f \amalg g: S(\alpha) \amalg S(\beta-\alpha) \to K \amalg L$. Let $h: S(\alpha) \to S$ be the proper bijection $\alpha_0 \cdot f$. Then $h \cdot X_A \cdot h^{-1} \oplus 1 = (f \amalg g)\cdot X_{A'} \cdot (f^{-1} \amalg g^{-1})$ in $\Gamma(S;\, R)^*$. Hence $h\cdot X_A \cdot h^{-1} \oplus 1$ is conjugate to

$$(h\cdot X_A \cdot h^{-1} \oplus 1) \oplus 1 = (\mathrm{f} \amalg g)\cdot X_{A'} \cdot (\mathrm{f}^{-1} \amalg g^{-1}) \oplus 1 .$$

This says $\Delta_\alpha(A)=\Delta_\beta(A)$ in $K_1(R)$. q.e.d.

To prove (3.1) we must first show that the sequence is a zero-sequence:

(a) $\Delta \circ (I-S)=0$.

Let $A=\{A_j\} \in \prod_{j>0} GL(\alpha_j,\, R_j)$ represent an element $x \in \prod_{j>0} K_1(R_j)$. Then $\Delta(x) - \Delta(S(x))$ is represented by the blocked germ

$$\sum_{0 \leqslant j} \left(\bigoplus_{|k|=|j|+1} (A_k \otimes R_j \oplus A_k^{-1} \otimes R_j \oplus I) \right)$$

This can be written as a product of commutators using (1.1) and (1.2).

(b) $\partial \circ \Delta = 0$.

Let $A=\{A_j\} \in \prod_{0\leqslant j} GL(\alpha_j,\, R_j)$. Then $\sum_{j \geqslant 0} A_j \in \Gamma(S(\alpha);\, R)^*$ is a germ such that $\sum_{k \leqslant j} A_j \otimes R_k$ is invertible in lR_k. Hence $\varrho([\sum_{k\leqslant j} A_j \otimes R_k])=0$. This says $\partial \circ \Delta = 0$.

(c) $(I-S)\circ\partial=0$.

For each element $[A] \in \Gamma^*$ represented by $A=\{A^\alpha\}$ we have $(I-S)\circ\partial=0$ because of condition (2) in § 2. In view of (1.8) this implies $(I-S)\circ\partial=0$ on $K_1(R)$.

Now we show that (3.1) is exact. For simplicity we assume $J=\{0, 1, 2, 3, \ldots\}$ is the tree with one end. The general case is left to the reader.

Exactness at $\prod_{j>0} K_0(R_j)$

Let $x=\{\langle P_j \rangle - \langle R_n^{n_j} \rangle\}_{j>0}$ be an element killed by $I-S$. For convenience set $P_0=0$ and $n_0=0$. Then for $j \geqslant 0$ we have

$$\langle P_j \rangle - \langle R_j^{n_j} \rangle = \langle P_{j+1} \otimes R_j \rangle - \langle R_j^{n_{j+1}} \rangle$$

In particular there are positive integers m_j such that

$$P_j \oplus R_{j+1}^{n_j} \oplus R_j^{m_j} \cong (P_{j+1} \otimes R_j) \oplus R_j^{n_j} \oplus R_j^{m_j}.$$

Choose finitely generated projective modulus Q_j' over R_j such that $P_j \oplus Q_j$ is free over R_j. Let $Q_j = Q_j' \oplus R_j^{m_j}$. Then

$$P_j \oplus Q_j \oplus R_j^{n_{j+1}} \cong R_j^{n_j} \oplus Q_j \oplus (P_{j+1} \otimes R_j)$$

In particular the module on the right hand side of the equation is free over R_j. Define the germ $[A]$ where $A = \{A^j\}$ by letting

$$A^j = \begin{tikzcd}[R_j^{n_j} \oplus \arrow[d, "0"] & Q_j \oplus \arrow[d, "id"] & (P_{j+1} \otimes R_j)] \oplus \arrow[dr, "id"] & {[R_j^{n_j+1}} \oplus \arrow[dl, "id"'] & (Q_{j+1} \otimes R_j) \oplus \arrow[d, "id"'] & (P_{j+2} \otimes R_j)] \oplus \cdots \\ {[P_j} \oplus & Q_j \oplus & R^{n_j+1}] \oplus & {[(P_{j+1}} \otimes R_j) \oplus & (Q_{j+1} \otimes R_j) \oplus & R^{n_j+2}] \oplus \cdots\end{tikzcd}$$

Note that for $j=0$ the matrix A^0 is in lR_0 because $P_0 = 0 = R_0^{n_0}$. For $j>0$ the matrix A^j is only invertible modulo a finite matrix. It is clear that $\varrho(A^j) = \langle P_j \rangle - \langle R_j^{n_j} \rangle$. Hence $\partial([A]) = x$.

Exactness at $K_1(R)$

Let $E = \{e_j\}$ denote the standard tree associated to J. For $0 \leqslant p, q \leqslant \infty$ let $E(p, q) = \{e_j \mid p \leqslant j < q\}$ and let $F(p, q; R')$ denote the free module generated over a ring by R' by the set $E(p, q)$.

To show exactness it suffices to show that for any germ $[A] \in \Gamma^*$ with $\partial[A] = 0$ there is a blocked matrix $B = \Sigma B_i \in [\Gamma^*, \Gamma^*]$ such that $[B \cdot A]$ has a blocked representative $\sum_j M^j$ where M_j is an invertible square matrix over R_j.

So let $\partial[A] = 0$. Then using (1.5) we find a sequence of integers $0 = n_{-1} = m_0 < n_0 < m_1 < n_1 < m_2 < n_2 < \cdots$ and a representative $A = \{\alpha^j\}$ of $[A]$ such that the following conditions hold:

(1) For $0 \leqslant j$, α^j is defined on $F(n_{j-1}, \infty; R_j)$, $\alpha^i = \alpha^j \otimes R_i$ on $F(n_{j-1}, \infty; R_i)$ when $0 \leqslant i < j$, $\alpha^j(F(n_{j-1}, \infty; R_j)) \supset F(m_j, \infty; R_j)$, and $\alpha^j(F(n_j, \infty; R_j)) \subset F(m_j, \infty; R_j)$.

(2) Let $Q_0 = F(m_0, m_1; R_0)$ and for $j>0$ let $Q_j = \alpha^j(F(n_j, \infty; R_j)) \cap F(m_j, m_{j+1}; R_j)$. Then for $j>0$, Q_j is free and is isomorphic to $F(n_j, m_{j+1}; R_j)$.

(3) For $j>0$ let $P_j = \alpha^j(F(n_{j-1}, n_j; R_j)) \cap F(m_j, m_{j+1}; R_j)$. Then P_j is free and is isomorphic to $F(m_j, n_j; R_j)$

(4) $F(m_j, m_{j+1}; R_j) = P_j \oplus Q_j$ and for $j>0$, $\alpha^j(F(n_j, n_{j+1}; R_j)) = Q_j \oplus (P_{j+1} \oplus \oplus R_j)$. See the diagram below.

Now by using [1, Cor. 1.10, p. 353] and rechoosing the m_j and n_j (if necessary) it is possible to find isomorphisms

$$h_j : P_j \to F(m_j, n_j; R_j)$$

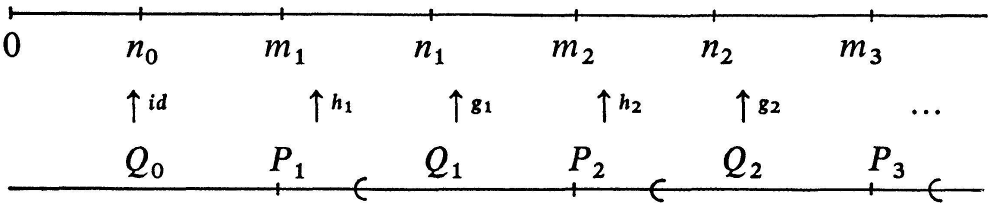
and

$$g_j : Q_j \to F(n_j, m_{j+1}; R_j)$$

whenever $j > 0$ such that

$$h_j \oplus g_j : F(m_j, m_{j+1}; R_j) \to F(m_j, m_{j+1}; R_j)$$

is of the form $[u_j, v_j]$ where u_j and v_j are in $GL(m_{j+1} - m_j; R_j)$. Let $g_0 = id \in GL \times (m_1 - m_0; R_0)$, and $h_0 = 0$. Let $B = \{B^j\}$ where $B^j = \sum_{j \leqslant k} (h_k \oplus g_k) \otimes R_j$. Note that B is blocked with respect to the tree $\{E(m_j, \infty)\}$ and in fact $B = [u, v]$ where $u = \sum_{0 \leqslant j} v_j$ and $v = \sum_{0 \leqslant j} v_j$. The germ $[L \cdot A]$ is blocked with the respect to tree of infinite sets $\{E(n'_j, \infty)\}$ where $n'_0 = 0$ and $n'_j = n_j$ for $j > 0$. q.e.d.



Exactness at $\prod_{j \geqslant 0} K_1(R_j)$

Step I. Let $G_0 \xleftarrow{f_0} G_1 \xleftarrow{f_1} G_2 \leftarrow \cdots$ be an inverse system of abelian groups and let $f_{ij} : G_j \to G_i$ be the composition $G_j \to G_{j-1} \to \cdots \to G_i$. An *amalgamation* of $\alpha = \{\alpha_j\} \in \prod_{0 \leqslant j} G_j$ is a sequence $\beta = \{b_j\} \in \prod_{0 \leqslant j} G_j$ obtained from α by choosing a sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$ and letting $\beta_j = \sum_i f_{ji}(x_i)$ where $n_j \leqslant i < n_{j+1}$.

LEMMA 3.4. *The element* $\alpha = \{\alpha_j\} \in \prod_{0 \leqslant j} G_j$ *lies in the image of* $I - S$ *if some amalgamation* $\beta = \{\beta_j\}$ *of* α *lies in the image of* $I - S$.

Proof. Suppose there is an element $X = \{X_j\} \in \prod_{0 < j} G_j$ such that $X - S(X) = \beta$ where β is obtained from α as above. Then

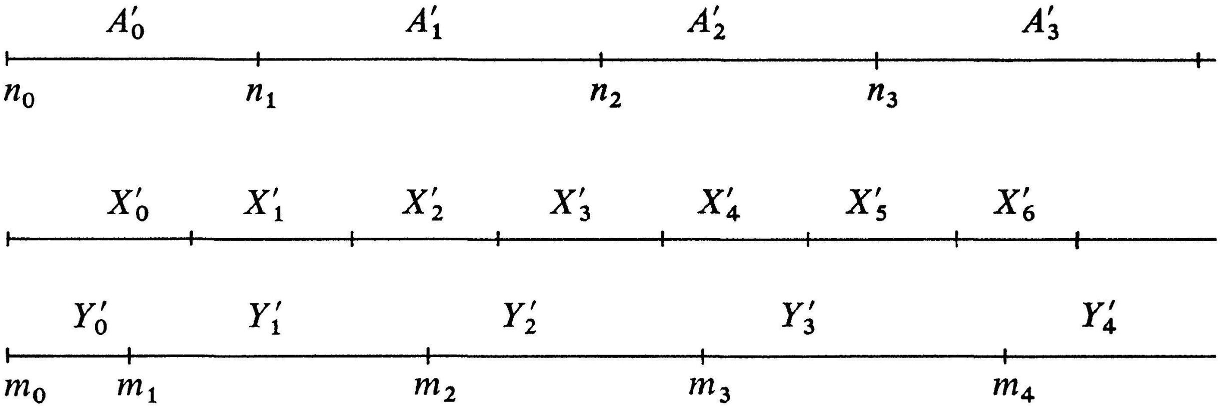
$$\beta_0 = -S(X_1) \quad \text{and} \quad \beta_j = X_j - S(X_{j+1}) \quad \text{for} \quad j > 0.$$

For $j \geqslant 1$ define $Y_j - X_j + \sum_i f_{ji}(\alpha_i)$ where $j \leqslant i < n_j$: Then $\alpha_0 = -S(Y_1)$ and $\alpha_j = Y_j - S(Y_{j+1})$ for $j > 0$.

Step II. Now let $G = \prod_{J \leqslant j} K_1(R_j)$ and $G^+ = \prod_{0 < j} K_1(R_j)$. We show exactness by taking an element in G which is killed by Δ and finding some amalgamation of it which pulls back to G^+.

Let $z \in G$ be in $\ker \Delta$ and let $z = \{A_j\}$ where $A_j \in GL(n_j, R_j)$ for $j \geqslant 0$. Then $\Delta(z)$ is represented by the blocked germ $[A]$ where $A = \sum_{j \geqslant 0} A_j$ and the support of A_j is $E(P_j, P_{j+1})$ where $P_0 = 0$ and $P_j = n_0 + \cdots + n_{j-1}$ for $j > 0$. To say that $\Delta(z) = 0$ implies that some stabilization $[(A \oplus 1) \oplus \cdots \oplus 1]$ lies in $[\Gamma^*, \Gamma^*]$. Now $(A \oplus 1) \oplus \cdots \oplus 1$ is a blocked matrix of the form $\sum A'_j$ where A'_j is some large square matrix conjugate to A_j in $GL(R_j)$. Hence $\{A'_j\}$ also represents z so we may as well assume that the matrix $A = \sum_{0 \leqslant j} A_j$ is itself in $[\Gamma^*, \Gamma^*]$ and, in fact, that it is in $E_n(\Gamma)$ for some n. By (2.7) we know that $[A] \cdot [X] \cdot [Y] = 1$ where $x = \sum_{0 \leqslant j} M^j(X)$ and $Y = \sum_{0 \leqslant j} M^j(Y)$ are blocked and the square matrices $M^j(X)$ and $M^j(Y)$ are products of elementary matrices over R_j.

Use amalgamation as in (2.7) to write $A = \Sigma A'_j$, $X = \Sigma X'_j$ and $Y = \Sigma Y'_j$ as shown schematically by the following diagram:



The integers m_i and n_i are intertwined sequences $0 = m_0 = n_0 < m_1 < n_1 < m_2 < \cdots$.

Let $B = \sum_{j \geqslant 0} B_j$ be the blocked matrix defined by

$$B_0 = A'_0 \cdot X'_0 \cdot Y'_0$$

and for $j > 0$

$$B_j = A'_j \cdot (X'_{2j} \otimes R_j).$$

Note that $B_j = A'_j$ when considered as elements of $K_1(R_j)$. Also, the sequence $\{B_j\}$ considered as an element of G is an amalgamation of the original element $z \in G$ in the sense of Step I.

Let

$$K = \sum_{0 \leqslant j} K_j$$

where

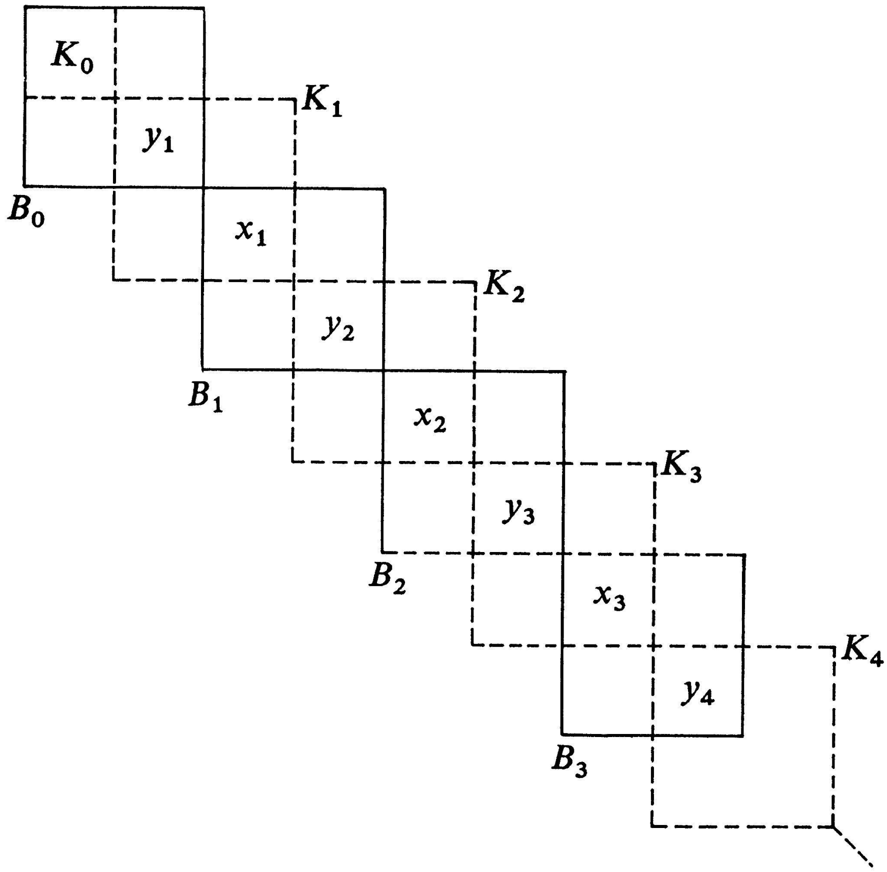
$$K_0 = id \in GL(m_1 - m_0, R_0)$$

and for $j>0$

$$K_j = Y_j^{-1} \cdot (X_{2j-1}^{-1} \otimes R_j)$$

we have $[B]=[K]$ in Γ^*.

The support of B_j is $E(n_j, n_{j+1})$ and the support of K_j is $E(m_j, m_{j+1})$. For short set $\alpha_j = E(n_j, n_{j+1})$ and $\kappa_j = E(m_j, m_{j+1})$. The matrices A and K overlap as in the following diagram:



The solid squares are the $\alpha_j \times \alpha_j$ and the dashed squares are the $\kappa_j \times \kappa_j$. Each square $\alpha_j \times \alpha_j$ is the union of two diagonal squares $(\alpha_j \times \alpha_j) \cap (\kappa_j \times \kappa_j)$ and $(\alpha_j \times \alpha_j) \cap \cap (\kappa_{j-1} \times \kappa_{j-1})$ and two off diagonal rectangles $(\alpha_j \cap \kappa_{j+1}) \times (\alpha_j \cap \kappa_j)$ and $(\alpha_j \cap \kappa_j) \times (\alpha_j \cap \kappa_{j+1})$.

Recall that $B^j = \sum_{j \leqslant k} B_j \otimes R_k$ and similarly for K^j. Since $B^0 = K^0$, each of the square matrices $B_j \otimes R_0$ and $K_j \otimes R_0$ has zero entries in the off diagonal rectangles. Since $B^j = K^j +$ finite matrix when $j>0$ we see that for each such j the square matrices $B_l \otimes R_j$ and $K_l \otimes R_j$ are zero in the off diagonal rectangles for l large enough. By amalgamating the B_j's and the K_j's again and absorbing (when necessary) the K_j^{-1}'s into the new blocks of the amalgamated B_j's we can assume that $B = \sum_{0 \leqslant j} B_j$ and $K = \sum_{0 \leqslant j} K_j$ satisfy the following properties:

(a) each B_j and K_j has zero entries in the off diagonal rectangles.

(b) $K_0 = id$, and for $j > 0$ and $l \geqslant j$, $K_l = B_l$ in the squares $(\alpha_l \times \alpha_l) \cap (\kappa_l \times \kappa_l)$ and $K_l \otimes R_{l-1} = B_{l-1}$ in the square $(\alpha_{l-1} \times \alpha_{l-1}) \cap (\kappa_l \times \kappa_l)$.

Now for $j > 0$ let $y_j = K_j$ restricted to $(\alpha_{j-1} \times \alpha_{j-1}) \cap (\kappa_j \times \kappa_j)$ and $x_j = B_j$ restricted to $(\alpha_j \times \alpha_j) \cap (\kappa_j \times \kappa_j)$. Then

$$K_j = \begin{pmatrix} y_j & 0 \\ 0 & x_{j+1} \end{pmatrix}$$

so both y_j and x_j are square invertible matrices over R_j. Since K_j is a product of elementary matrices, $y_j = -x_j$ when considered as elements of $K_1(R_j)$. Since

$$B_j = \begin{pmatrix} x_j & 0 \\ 0 & y_{j+1} \otimes R_j \end{pmatrix}$$

we have $B_j = x_j + y_{j+1} \otimes R_j$ in $K_1(R_j)$. Let $\beta_j \in K_1(R_j)$ be the element of determined by B_j. Then

$$\beta_0 = -s(x_1) \quad \text{in} \quad K_1(R_0)$$

and

$$\beta_j = x_j - s(x_{j+1}) \quad \text{in} \quad K_1(R_j)$$

for $j > 0$. This says that the element $\beta = \{\beta_j\}$ lies in the image of $I - S$. But $\beta \in G$ is an amalgamation of the original element $z \in G$, so z is also in the image of $I - S$ by Step I. This completes the proof of (3.1).

Let $G = \{G_j, \gamma_{ij}\}$ be a tree of groups over J. The *group ring of* G, written $Z[G]$, is the tree of rings over J given by the collection $\{Z[G_j], \gamma_{ij}\}$ where the ring homomorphism $\gamma_{ij}: Z[G_j] \to Z[G_i]$ is the one induced by $\gamma_{ij}: G_j \to G_i$. Let Z_J denote the tree of rings $\{Z_j, \gamma_{ij}\}$ over J where $Z_j = Z$ and $\gamma_{ij} = id$ for all $i, j \in J$. There is a natural "morphism" $i: Z_J \to Z[G]$ of rings over J given by $Z_j \to Z[G_j]$, similarly we have a morphism $e: Z[G] \to Z_J$ given by the evaluation maps $Z[G_j] \to Z_j$. These morphisms of trees of rings induce homomorphisms $i_*: K_1(Z_J) \to K_1(Z[G])$ and $e_*: K_1(Z[G]) \to K_1(Z_J)$ such that $e_* \circ i_* = id: K_1(Z_J) \to K_1(Z_J)$. Let $\tilde{K}_1(Z[G]) = \text{coker}\ (K_1(Z_J) \to K_1(Z[G]))$. Then $K_1(Z[G]) = K_1(Z_J) \oplus \tilde{K}_1(Z[G])$.

Let $\pm G \subset \Gamma(Z[G])^*$ be the subgroup of diagonal germs with group entries; that is, a germ $D = \{D^j\} = \{(d^j_{pq})\}$ lies in $\pm G$ iff for each $j \in J$, $d^j_{p,q} = 0$ for $p \neq q$ and $d^j_{p,p} = \pm g_p$ where $g_p \in G_j$.

Now define the Whitehead group of G as

$$\mathrm{Wh}(G) = \tilde{K}_1(Z[G]) \bmod \langle \pm G \rangle \tag{3.5}$$

where $\langle \pm G \rangle \subset \tilde{K}_1(Z[G])$ is the subgroup generated by the elements of $\pm G$.

Arguing in a similar way to the proof of (3.1) one can derive the following exact sequence which is of interest in the theory of algebraic torsion for infinite simple homotopy types discussed in [4] and [14].

THEOREM 3.6. *There is a five term exact sequence*

$$\prod_{0<j} \mathrm{Wh}(G_j) \xrightarrow{I-S} \prod_{0\leqslant j} \mathrm{Wh}(G_j) \xrightarrow{\Delta} \mathrm{Wh}(G) \xrightarrow{\partial} \prod_{0<j} \tilde{K}_0(G_j) \xrightarrow{I-S} \prod_{0\leqslant j} \tilde{K}_0(G_j).$$

Here $\mathrm{Wh}(G_j)$ is the ordinary Whitehead group of G_j and $\tilde{K}_0(G_j)$ is the reduced group $\tilde{K}_0(G_j)=\mathrm{coker}(K_0(Z)\to K_0(Z[G_j]))$. Here are some examples of the sequences (3.1) and (3.6).

EXAMPLE 1. Suppose the tree of rings R is the inverse system $B\xleftarrow[f]{}A\xleftarrow[id]{}A\xleftarrow[id]{}A\leftarrow\cdots$. Then $\Gamma(R)=\gamma f$ as in Example 2 of § 2 and the basic sequence (3.1) reduces to the sequence

$$K_1(A)\to K_1(B)\to K_1(\gamma f)\to K_0(A)\to K_0(B). \tag{3.7}$$

Now "classically" there is the exact sequence of Bass

$$K_1(A)\to K_1(B)\to K_0(f)\to K_0(A)\to K_0(B) \tag{3.8}$$

as described in [1].

There is a natural isomorphism of sequences

$$\begin{tikzcd} & & K_1(\gamma f) \arrow[dd, "\theta"'] \arrow[dr] & & \\ K_1(A) \arrow[r] & K_1(B) \arrow[ur] \arrow[dr] & & K_0(A) \arrow[r] & K_0(B). \\ & & K_0(f) \arrow[ur] & & \end{tikzcd} \tag{3.9}$$

We indicate how to construct the isomorphism $\theta: K_1(\gamma f)\to K_0(f)$ and leave it to the reader to check as an exercise that everything is well defined, etc. Recall that $K_0(f)$ is "K_1" of the category of triples (P, α, Q) where P and Q are finitely generated, projective A-modules and $\alpha: P\otimes B\to Q\otimes B$ is a B-linear isomorphism.

Let A^∞ denote the free A-module based on $\{e_1, e_2, e_3, \ldots\}$, let $A^n\subset A^\infty$ denote the free submodule based on $\{e_{n+1}, e_{n+2}, \ldots\}$, and let $A_n\subset A^\infty$ be the free module based on $\{e_1, \ldots, e_n\}$. Similarly for B^∞, B^n, and B_n. Now let $(\beta, \hat{\alpha})\in\gamma f$ where $\beta\in lB$, $\alpha\in lA$ represents $\hat{\alpha}\in\mu A$, and $\alpha\otimes B=\beta+$finite matric. As in § 1 choose an integer n so large that $A^\infty/\alpha(A^n)$ is finitely generated, projective over A. Also choose n so large that $\alpha\otimes B \mid B^n=\beta\mid B^n$. Then let

$$\theta((\beta, \hat{\alpha}))=(A^\infty/A^n, \bar{\beta}, A^\infty/\alpha(A_n)) \quad\text{in}\quad K_0(f)$$

where

$$\bar{\beta}: (A^{\infty}/A^{n}) \otimes B \to (A^{\infty}/\alpha(A_n)) \otimes B$$

is just the map induced by β:

$$(A^{\infty}/A^{n}) \otimes B = B^{\infty}/B^{n} \xrightarrow[\beta]{} B^{\infty}/\alpha(A^{n}) \otimes B = (A^{\infty}/\alpha(A_n)) \otimes B .$$

If G is the tree of groups $G_0 \xleftarrow[f]{} G_1 \xleftarrow[id]{} G_2 \xleftarrow[id]{} \cdots$ and $Z[G_0] \xleftarrow[f]{} Z[G_1] \xleftarrow[id]{} Z[G_2] \leftarrow \cdots$ is the associated tree of rings, the exact sequence (3.6) reduces to

$$\mathrm{Wh}(G_1) \to \mathrm{Wh}(G_0) \to \mathrm{Wh}(G) \to \tilde{K}_0(G_1) \to \tilde{K}_0(G_0) \tag{3.10}$$

Define $\mathrm{Wh}(f) = K_0(f)/\langle \pm g \rangle$ where $\langle \pm g \rangle$ is the subgroup generated by triples of the form $(Z[G_1], \pm g, Z[G_1])$ for $g \in G_0$. There is an exact sequence

$$\mathrm{Wh}(G_1) \to \mathrm{Wh}(G_0) \to \mathrm{Wh}(f) \to \tilde{K}_0(G_1) \to \tilde{K}_0(G_0) . \tag{3.11}$$

As in (3.8) there is an isomorphism $\mathrm{Wh}(G) \to \mathrm{Wh}(f)$ which produces an isomorphism between the sequences (3.10) and (3.11).

Example 2. Let R be the inverse system with two stable ends

$$\to A \to A \to C \leftarrow B \leftarrow B \leftarrow \cdots$$

Then (3.1) reduces to

$$K_1(A) \oplus K_1(B) \to K_1(C) \to K_1(R) \to K_0(A) \oplus K_0(B) \to K_0(C) .$$

Remark. Let $R = \{R_j, \gamma_{ij}\}$ be a tree of rings over J and let $J' \subset J$ be a cofinal subset containing the smallest element $0 \in J$ as in (2.1). We get a tree of rings $R' = \{R'_j, \gamma'_{ij}\}$ by just restricting the indices i and j to be in J'. Then the isomorphism (2.1) induces an isomorphism

$$K_1(R) \xrightarrow{\cong} K_1(R') . \tag{3.12}$$

Furthermore, if $R = Z[G]$ and $R' = Z[G']$ are trees of group rings, then there is an isomorphism

$$\mathrm{Wh}(G) \xrightarrow{\cong} \mathrm{Wh}(G') . \tag{3.13}$$

Thus, while the exact sequences (3.1) and (3.6) are different for R and R', the middle term stays the same.

§ 4. A Definition of $K_1(f)$

As an interesting sideline to the main emphasis of this paper we note that the eaxct sequence

$$K_1(A) \to K_1(B) \to K_0(f) \to K_0(A) \to K_0(B)$$

of any ring homomorphism $f: A \to B$ can be extended to an exact sequence

$$K_2(A) \to K_2(B) \to K_1(f) \to K_1(A) \to K_1(B) \tag{4.1}$$

Here the K_2 is the one defined by Milnor in [8] and, by definition, we take

$$K_1(f) = K_2(\gamma f) \tag{4.2}$$

When $f: A \to B$ is a surjection $K_2(\gamma f)$ is naturally isomorphic to the relative $K_1(f)$ defined by Bass [1] and the sequence (4.1) is naturally isomorphic to the usual one as constructed in [8]. In fact, in [16] and [5] the exactness of (4.1) is established and it is shown that the sequence can be extended indefinitely to the left using the higher K_i's of Quillen [10]. Since [16] and [5] supersede our original argument we simply indicate here the proof of

PROPOSITION 4.3. *For any surjection $f: A \to B$ there is a natural isomorphism $\theta: K_2(\gamma f) \overset{\approx}{\to} K_1(f)$.*

To define the homomorphism θ, let $z \in K_2(\gamma f)$ be represented by the word $\prod x_{i_\alpha j_\alpha}(b_\alpha, a_\alpha) \in St(\gamma f)$ where $b_\alpha \in lB$ and $a_\alpha \in \mu A$. Choose a lifting $a'_\alpha \in lA$ of $a_\alpha \in \mu A$ such that $f(a'_\alpha) = b_\alpha$. This can be done because f is a surjection. Now the matrix $M_z = \prod e_{i_\alpha j_\alpha}(a'_\alpha) \in GL(lA)$ actually lies in the subgroup $GL(mA)$ of $GL(lA) = E(lA)$ because $\prod e_{i_\alpha j_\alpha}(a_\alpha) = id$ in $E(\mu A)$, which is isomorphic to $E(lA) \bmod GL(mA)$. See [16]. Furthermore, since $\prod e_{i_\alpha j_\alpha}(b_\alpha) = id$ in $GL(lB)$ and $f(a'_\alpha) = b_\alpha$, the matrix M_z lies in the kernel of $GL(mA) \to GL(mB)$. Hence we can define

$$\theta(z) = \langle M_z \rangle \in K_1(f).$$

Here we are using the natural isomorphism $GL(A) \cong GL(mA)$. See [16].

The argument showing θ is well defined is essentially is a combination of the arguments of Lemma 6.1 of [8] and Lemma 1.2 of [16]. One now checks that there is a transformation of exact sequences.

$$\begin{tikzcd}
 & & K_2(\gamma f) \arrow[dd, "\theta"] \arrow[dr] & & \\
K_2(A) \arrow[r] & K_2(B) \arrow[ur] \arrow[dr] & & K_1(A) \arrow[r] & K_1(B). \\
 & & K_1(f) \arrow[ur] & &
\end{tikzcd}$$

Hence the "five-lemma" says θ is an isomorphism.

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