Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 47 (1972)

Artikel: Asymptotic Properties of Foliations
Autor: Plante, J.F.
DOI: https://doi.org/10.5169/seals-36378

Asymptotic Properties of Foliations

J. F. Plante

0. Introduction

Let M be a compact differentiable (C^∞) manifold which admits a C^r ($r \geqslant 1$) codimension-one foliation $\mathfrak{F}$ (in the sense of [4]). For convenience we assume that $\mathfrak{F}$ is transversely oriented, i.e., that $\mathfrak{F}$ has a smooth transverse vector field and that M itself is oriented. Sufficient background on the subject of foliations may be found in [4, 6]. One is interested in finding topological restrictions on manifolds admitting foliations and, more specifically, restrictions imposed by certain types of foliations such as those arising from differentiable actions of a Lie group on a manifold. Also, given a specific manifold it is of interest to determine the restrictions that the topology of the manifold imposes on the topology of the leaves of the foliation, e.g., must there exist compact leaves? The principal techniques used in this investigation have been intermediate in nature, that is, neither infinitesimal or asymptotic. These techniques involve considerations of holonomy which may be thought of as a generalization of the Poincaré map of a closed orbit which is used in the study of differentiable flows. It is our purpose here to give some results of a more asymptotic nature. These results are then applied to examples which include invariant foliations of Anosov flows and free and locally free Lie group actions on compact manifolds.

1. Growth of leaves in a foliation

We begin by assuming that M has a fixed continuous Riemannian metric on its tangent bundle which gives rise to a distance function on M. Also the Riemannian metric may be restricted to each leaf, thereby yielding a distance function and a volume element on the leaves in the usual way. For $x \in M$ we let Ω_x, d_x denote the induced volume element and distance function for the leaf through x (which is denoted $L(x)$) and define the disk about x of radius R by $D_R(x) = \{y \in L(x) \mid d_x(x, y) \leqslant R\}$. Now consider the growth function $G: M \times \mathbf{R}^+ \to \mathbf{R}^+$ defined by

$$G(x, R) = \int_{D_R(x)} \Omega_x .$$

Note that G depends on the metric but if H is the growth function for some other metric then by compactness of M there exist constants α, $\beta > 0$ such that $G(x, \alpha R) \leqslant H(x, R) \leqslant G(x, \beta R)$ for all $(x, R) \in M \times \mathbf{R}^+$. The function $G(x, \cdot)$ is called the growth function of $L(x)$ at $x \in M$. In what follows we shall be interested in special restrictions on this growth function.

DEFINITION. $\mathfrak{F}$ is said to have polynomial growth of degree r (denoted $\mathfrak{p}(r)$-growth) at $x \in M$ if $G(x, \cdot)$ dominates some polynomial of degree r. $\mathfrak{F}$ is said to have exponential growth (denoted $\mathfrak{p}(\infty)$-growth) if there exist $A > 0$, $a > 0$ such that $G(x, R) \geqslant A \exp(aR)$.

Remark. It is easily seen that the above definition is independent of metric since M is assumed compact. Furthermore, the type of growth is independent of the point chosen in the leaf.

Now let $\pi_1(M)$ denote the fundamental group of M and consider the function $g: \mathbf{R}^+ \to \mathbf{Z}^+$ defined by

$g(R)$ = number of elements of $\pi_1(M)$ which can be represented by loops of length $\leqslant R$.

As before we say that $\pi_1(M)$ has $\mathfrak{p}(r)$-growth if g dominates a polynomial of degree r and $\mathfrak{p}(\infty)$-growth if g dominates an exponential. These definitions are equivalent to the usual definition of growth for a finitely generated group (see [9] Lemma 4). The following is another characterization.

1.1. PROPOSITION. *Let $\tilde{M}$ be the universal covering space of M (M compact) with induced metric and $V(R)$ denote the volume of the ball of radius R about the basepoint in $\tilde{M}$. Then $\pi_1(M)$ has $\mathfrak{p}(r)$-growth if, and only if, $V(R)$ has $\mathfrak{p}(r)$-growth ($1 \leqslant r \leqslant \infty$).*

Proof. Let K be a compact fundamental domain for the covering projection with finite diameter δ. If $\pi_1(M)$ has $\mathfrak{p}(r)$-growth then the number of fundamental domains which intersect the ball of radius R in $\tilde{M}$ has $\mathfrak{p}(r)$-growth which implies that $V(R+\delta)$ has $\mathfrak{p}(r)$-growth and, hence, so does $V(R)$. On the other hand, if $V(R)$ has $\mathfrak{p}(r)$-growth then $\pi_1(M)$ must have $\mathfrak{p}(r)$-growth since the fundamental domains have a fixed finite volume. This proves (1.1).

2. The main results

The following result says that if a codimension one foliation does not have a null-homotopic closed transversal then $\pi_1(M)$ grows as fast as any leaf.

2.1. THEOREM. *Let $\mathfrak{F}$ be a codimension one foliation of a compact manifold M. If for some $x \in M$*

(i) *the leaf through x does not intersect a null-homotopic closed transversal, and*

(ii) *the leaf through x has $\mathfrak{p}(r)$-growth ($1 \leqslant r \leqslant \infty$)*

then $\pi_1(M)$ has $\mathfrak{p}(r)$-growth.

Proof. Let M be the universal covering space of $\tilde{M}$ with foliation $\tilde{\mathfrak{F}}$ induced from $\mathfrak{F}$ via the covering projection $\mathfrak{p}: \tilde{M} \to M$. Also let $\tilde{M}$ have the Riemannian metric induced from M and note that $\pi_1(M)$ acts on $\tilde{M}$ as a group of isometries. Let $x \in M$, $\tilde{x} \in \mathfrak{p}^{-1}(x)$ and $D_R(\tilde{x})$ be the disk of radius R in $L(\tilde{x}) \in \tilde{\mathfrak{F}}$. Clearly $L(\tilde{x})$ is a covering space of $L(x)$ (in the leaf topology) and, therefore, $G(\tilde{x}, R) \geq G(x, R)$ for $R \in \mathbf{R}$. Since we assume that $\mathfrak{F}$ is oriented (otherwise take an appropriate two-fold covering) let φ_t be a smooth flow on M transverse to $\mathfrak{F}$ and $\tilde{\varphi}_t$ its lift to $\tilde{M}$. Let $\lambda > 0$ be a fixed positive number so that the set $\bigcup_{|t| \leq \lambda} \tilde{\varphi}_t(D_R(\tilde{x}))$ consists only of points which are at distance $\leq R+1$ from $\tilde{x}$ in the metric on $\tilde{M}$. (This can be done since M compact implies that the vector field $(d/dt)\varphi_t$ is bounded in length.) We claim that if $L(x)$ does not intersect a null-homotopic closed transversal, then the volume of the set $\bigcup_{|t| \leq \lambda} \tilde{\varphi}_t(D_R(\tilde{x}))$ grows at least as fast as $G(\tilde{x}, R)$. First note that the orbit segments $\{\varphi_t(z) \mid z \in D_R(\tilde{x}),\ |t| \leq \lambda\}$ are disjoint for different z. If this were not the case there would be a loop in $\tilde{M}$ of the form $\alpha * \beta$ with α an orbit segment and β in $L(\tilde{x})$, in which case $\mathfrak{p}(a * \beta)$ could be homotoped to a null-homotopic closed transversal for $\mathfrak{F}$. It now follows that the volume of $\bigcup_{|t| \leq \lambda} \tilde{\varphi}_t({}_R(\tilde{x}))$ grows as fast as $G(\tilde{x}, R)$ since the volume is obtained by integrating a positive function over $D_R(\tilde{x})$ which is bounded away from zero since M is compact. We have thus shown that the disk of radius $R+1$ in $\tilde{M}$ has $\mathfrak{p}(r)$-growth which implies the same for the disk of radius R. The result now follows from (1.1).

2.2. COROLLARY. *If a leaf of $\mathfrak{F}$ has $\mathfrak{p}(r)$-growth then $\pi_1(M)$ has $\mathfrak{p}(r)$-growth if either one of the following are satisfied.*

(i) *$\mathfrak{F}$ does not have a vanishing cycle (in the sense of* [6]).

(ii) *$\mathfrak{F}$ does not have a one-sided limit cycle.*

Proof. This follows from (2.1) together with standard Poincaré-Bendixson arguments [4, 6] and the C^1 general-position lemma of [2] (pages 81–84).

EXAMPLES. (1) [9] Suppose $\varphi_t: M \to M$ is a codimension one Anosov flow. This means, in particular, that there is a codimension one C^1 foliation $\mathfrak{F}$ of M which satisfies (i) and (ii) above. In addition, the flow φ_t expands the leaves of $\mathfrak{F}$ exponentially (as t goes either to ∞ or $-\infty$) which implies that the leaves of $\mathfrak{F}$ have exponential growth and, hence, so does $\pi_1(M)$.

(2) The usual Reeb foliation of $S^1 \times D^n$ has linear $(\mathfrak{p}(1))$ growth on the non-compact leaves and the boundary leaf is compact and, therefore, has finite volume. Thus, for the Reeb foliation of $S^1 \times S^n$ the above results are the best possible. However, there are sometimes extra conditions which allow us to increase our estimate for the growth rate of $\pi_1(M)$ (See (2.3) (2.4) below).

(3) Let G be the simply connected Lie group with Lie algebra generated by X, Y,

Z with relations

$$[X, Y] = aZ, \quad a > 0, \quad \left.\begin{array}{l}[X, Z] = bX \\ [Y, Z] = -bY\end{array}\right\} b > 0$$

G is simple and has a uniform discrete subgroup Γ. Let H be the subgroup of G corresponding to the Lie algebra generated by X and Z and let H act on G/Γ by left multiplication. The orbit foliation is analytic and H has exponential growth. By (1.3) (ii) $\Gamma \cong \pi_1(G/\Gamma)$ has exponential growth. By (1.1) G has exponential growth. (G is the universal covering group of $\mathrm{SL}(2, \mathbf{R})$ and this same argument shows that $\mathrm{SL}(2, \mathbf{R})$ has exponential growth [1].)

(4) If $\mathfrak{F}$ is a codimension one foliation of a compact 3-manifold M and φ_t is a flow tangent to $\mathfrak{F}$ such that the volume of $\varphi_t(D)$ has $\mathfrak{p}(r)$-growth $(1 \leqslant r \leqslant \infty)$ for every disk D (contained in a leaf) then $\pi_1(M)$ has $\mathfrak{p}(r)$-growth by (2.2) (i) since, otherwise, by Novikov's theorem [6] would there be a compact leaf (which does not grow).

2.3. PROPOSITION. *If in addition to the previous assumptions we assume that φ_t is volume preserving then if a leaf of $\mathfrak{F}$ has $\mathfrak{p}(r)$-growth but does not intersect any null-homotopic closed transversal then $\pi_1(M)$ has $\mathfrak{p}(r+1)$-growth.*

Proof. The proof is essentially the same as the one above except that we consider the volume of $\bigcup_{|t| \leqslant R} \tilde{\varphi}_t(D_R(\tilde{x}))$. The details are left to the reader.

It would be interesting to know when the extra hypothesis of (2.3) can be satisfied. It cannot be for example, if $\mathfrak{F}$ has a compact leaf which bounds in M.

2.4. PROPOSITION. *If $\mathfrak{F}$ is a foliation of class C^2 with trivial holonomy and if $L \in \mathfrak{F}$ has $\mathfrak{p}(r)$-growth $(1 \leqslant r \leqslant \infty)$ then $\pi_1(M)$ has $\mathfrak{p}(r+1)$-growth.*

Proof. By the theorem of Sacksteder [11] there is a continuous vector field X on M transverse to $\mathfrak{F}$ and tangent to a flow which leaves $\mathfrak{F}$ invariant. It is also known that the universal covering space of M is of the form $\tilde{L} \times \mathbf{R}$ where $\tilde{L}$ is the universal covering space of a leaf $L \in \mathfrak{F}$. Further, $\mathfrak{F}$ is determined by the continuous one-form η defined by $\eta(X) = 1$, $\eta \mid T\mathfrak{F} = 0$, which is closed. By well-known arguments [12, 8] η may be perturbed (in the class of closed one forms) to get a closed form ξ such that kernel ξ is transverse to X and ξ has rational periods. The foliation $\mathfrak{F}'$ determined by ξ is, therefore, transverse to X and has compact leaves. Let K be one of these compact leaves and let ψ_t be the reparametrization of the X-flow which leaves $\mathfrak{F}'$ invariant. Then the map $K \times \mathbf{R} \to M$ given by $(k, t) \to \psi_t(k)$ is a covering projection [8]. Lifting everything now to the universal covering space we see that $\tilde{L}$ is diffeomorphic to the universal covering space $\tilde{K}$ of K via projection along flow lines of $\tilde{\varphi}_t$. Thus, $\tilde{K}$ has $\mathfrak{p}(r)$-growth since L does. Now since a fundamental domain Δ in $\tilde{K}$ is invariant under $\tilde{\psi}_s$ (that is $\tilde{\psi}_s(\Delta \times \{0\}) = \Delta \times \{s\}$) for some $s > 0$ we see that the set $\bigcup_{|t| \leqslant R} D_R(\tilde{x})$ has

$\mathfrak{p}(r+1)$-growth where $\tilde{x}\in\tilde{K}\subset\tilde{M}$ and $D_R(\tilde{x})$ is the disk of radius R in $\tilde{K}$. This completes the proof of (2.4).

Remark. In all of the proofs of this section we used only the $\mathfrak{p}(r)$-growth of $\tilde{L}$ rather than that of L.

3. Locally free Lie group actions

Let G be a connected Lie group and M a compact manifold. A Lie group action $\Phi: G\times M\to M$ is called *locally free* if the isotropy group at each point $x\in M$, $I_x = \{g\in G \mid \Phi(g, x)=x\}$ is a discrete subgroup. In this section we determine some restrictions on such actions when $\dim G=(\dim M)-1$.

The following result was pointed out to the author by R. Roussarie.

3.1. LEMMA. *If $\Phi: G\times M\to M$ is a C^1 locally free action where* $(\dim G)+1=\dim M$ *then the orbit foliation of Φ does not have any vanishing cycles.*

Proof. Without loss of generality we assume that G is simply connected. Suppose that the orbit foliation of Φ has a vanishing cycle through $x_0\in M$. The loop representing the vanishing cycle is of the form $t\to\Phi(\gamma(t), x_0)$ where $\gamma:[0,1]\to G$ is a path in G and $\gamma(0)=e$ (identity of G) and $\gamma(1)\neq e$ (since $G\to L(x_0)$ given by $g\to\Phi(g, x_0)$ is a covering projection and the loop $t\to\Phi(\gamma(t), x_0)$ is not null homotopic in $L(x_0)$). Let τ be a small segment of arc transverse to the foliation which contains x_0 as one endpoint and such that for $x\in\tau$ $(x\neq x_0)$ the deformation of the original loop from $L(x_0)$ to $L(x)$ is null-homotopic in $L(x)$. Clearly, these null-homotopic loops can be taken in the form $t\to\Phi(\gamma*\delta_x(t), x)$ where $\delta_x:[0,1]\to G$ is a continuous family of paths such that $\delta_x(0)=\gamma(1)$ and as $x\to x_0$ (in τ), length $\delta_x\to 0$. Now for $x\neq x_0$ the path $\gamma*\delta_x$ is closed (since G is the universal covering space of $L(x)$) and hence, $\delta_x(1)=e$ for $x\neq x_0$ $(x\in\tau)$. But by the continuous dependence of $\delta_x(1)$ on x we have $\delta_{x_0}(1)=e$ which contradicts the fact that $\delta_{x_0}(t)=\gamma(1)\neq e$ for all $t\in[0,1]$. This proves (3.1).

We say that an action $\Phi: G\times M\to M$ is *free* if the isotropy group at every point is trivial.

3.2 LEMMA. *Let $\Phi: G\times M\to M$ be as in* (3.1) *and let $\tilde{\Phi}: G\times\tilde{M}\to\tilde{M}$ be the induced action on the universal covering space of M. Then $\tilde{\Phi}: G\times\tilde{M}\to\tilde{M}$ is a free action.*

Proof. Using the arguments of [2] (pages 81–84) and [6] (Theorem 6.1 (3)) this follows immediately from (3.1).

We now take a metric on M such that on orbits it corresponds to a fixed left invariant metric on G. In the obvious way we define the notion of $\mathfrak{p}(r)$-growth for the group G.

3.3. THEOREM. *If $\Phi: G\times M\to M$ is a locally free C^1 action where* dim $G=$ $=(\dim M)-1$, *and if G has* $\mathfrak{p}(r)$*-growth then so does* $\pi_1(M)$.

Proof. This follows from (2.2), (3.1), (3.2) and the remark following (2.4).

Remark. Since the non abelian 2-dimensional Lie group has exponential growth and a finitely generated abelian group has at most polynomial growth, (3.3) generalizes Theorem 3.3(ii) of [7].

The following is an interesting special case of the above result.

3.4. PROPOSITION. *Let N be a simply-connected nilpotent Lie group which possesses a uniform discrete subgroup Γ and let $\Phi: N\times M\to M$ be a C^1 locally free action where M is compact and* $(\dim M)-1=\dim N=n$. *Then $\pi_1(M)$ has $\mathfrak{p}(n)$-growth and if N is not abelian $\pi_1(M)$ has $\mathfrak{p}(n+1)$-growth.*

Proof. If N is abelian then it has $\mathfrak{p}(n)$-growth but not $\mathfrak{p}(k)$-growth for any $k>n$, and if N is not abelian then Γ is also not abelian [5, 1] and N has $\mathfrak{p}(n+1)$-growth by (1.1) and [14] (page 427). Thus, (3.4) follows from (3.3).

The following is proved in [3] under the assumption that N is abelian and that the action is of class C^2. We let T^q denote the q-dimensional torus.

3.5. COROLLARY. *Let V be a simply-connected manifold of dimension $k>1$ and N a nilpotent group as in* (3.4). *Then there is no locally free C^1 action $N\times(V\times T^{n-k+1})\to V\times T^{n-k+1}$.*

Proof. This follows since N has $\mathfrak{p}(n)$-growth but $\pi_1(V\times T^{n-k+1})=\pi_1(T^{n-k+1})$ has at most $\mathfrak{p}(n-k+1)$-growth.

Remark. If we assume that the action is C^2 in (3.4) then we can show that $\pi_1(M)$ has $\mathfrak{p}(n+1)$-growth even if N is abelian. By a result of Sacksteder [11], if the action is C^2 and has no compact orbit then M is a torus bundle over a torus where the sum of the dimensions is $\dim M$. On the other hand, if the action has a compact leaf, Chatelet and Rosenberg have shown that M is a bundle over S^1 with fiber T^n. In either case $\pi_1(M)$ has $\mathfrak{p}(n+1)$-growth.

4. Free Lie group actions

In this section we consider free actions $\Phi: G\times M\to M$. The following is clear.

4.1. LEMMA. *If $\Phi: G\times M\to M$ is a free action of class C^1 then the orbit foliation of Φ has trivial holonomy.*

4.2. PROPOSITION. *Let $\Phi: G\times M\to M$ be a C^2 locally free action where* $\dim M=(\dim G)+1$. *If the orbit foliation of Φ has trivial holonomy and G has $\mathfrak{p}(r)$-growth then $\pi_1(M)$ has $\mathfrak{p}(r+1)$-growth.*

Proof. (4.2) follows from (2.4), and the remark following (2.4).

4.3. THEOREM. *Let $\Phi: G \times M \to M$ be a C^2 free action where* $(\dim G)+1 = \dim M$. *If G has* $\mathfrak{p}(r)$*-growth then $\pi_1(M)$ has* $\mathfrak{p}(r+1)$*-growth.*

Proof. (4.3) follows from (4.1) and (4.2).

4.4. COROLLARY. *If G is an n-dimensional Lie group satisfying either of the following two conditions then G does not act freely of class C^2 on a compact manifold of dimension $n+1$.*

(i) *G is homeomorphic to $\mathbf{R}^n$ and has* $\mathfrak{p}(n+1)$*-growth.*

(ii) *G is simply-connected and has exponential growth.*

Proof. Suppose there is a free C^2 action $\Phi: G \times M \to M$ and assume (i). It is known [6, 11, 8] that an $(n+1)$-dimensional compact manifold which is C^2-foliated by n-planes has fundamental group which is free abelian on $(n+1)$ generators and such a group has $\mathfrak{p}(n+1)$-growth but not $\mathfrak{p}(r)$-growth for any $r > n+1$ ([14]). But (2.4) implies that $\pi_1(M)$ has $\mathfrak{p}(n+2)$-growth which implies that Φ cannot be free. Now suppose Φ satisfies (ii). In this case the assumption that Φ is free implies that $\pi_1(M)$ is free abelian and, hence, does not have exponential growth which contradicts (2.4). This completes the proof of (4.4).

EXAMPLES. (a) If N is an n-dimensional non abelian nilpotent group which possesses a uniform discrete subgroup then (as in the proof of (3.4)) N has $p\,(n+1)$-growth and, hence, cannot act C^2 freely on a compact $(n+1)$-manifold.

(b) The universal covering group of $\mathrm{SL}(n, \mathbf{R})$ $(n \geq 2)$ satisfies (ii) and the universal covering group of $\mathrm{SL}(2, \mathbf{R})$ also satisfies (i) ((1.1) and [13]).

(c) The 2-dimensional non abelian group satisfies (i) and (ii). Thus, (4.4) generalizes Theorem 3.1 of [7].

It is easy to give free actions of $\mathbf{R}^n$ on T^{n+1} by taking as generators appropriate one parameter subgroups of T^{n+1}. In view of the above examples one might conjecture that a non abelian Lie group cannot act freely in codimension one. This, however, turns out to be false as shown by the following example.

EXAMPLE. Let S be the simply-connected solvable 3-dimensional Lie group which corresponds to the Lie algebra generated by X, Y, Z with relations

$$[X, Y] = 0, \quad [X, Z] = aY, \quad [Y, Z] = -aX, \quad a > 0.$$

In [1] it is shown that S has a uniform discrete subgroup Γ such that S/Γ is diffeomorphic to T^3. Extend the action of S on S/Γ (by left multiplication) in the obvious way to an action $S \times T^4 \to T^4$ where we think of T^4 as $(S/\Gamma) \times S^1$. The orbit foliation $\mathfrak{F}$ of this action may be perturbed to give a C^∞ foliation $\mathfrak{F}'$ whose leaves are all diffeomorphic to $\mathbf{R}^3$ and which is still transverse to the circles $\{x\} \times S^1 \subset (S/\Gamma) \times S^1$.

Let W be the vector field (with flow of period 1) in the S^1 direction. W commutes with X, Y, Z and by projecting X, Y, Z onto $\mathfrak{F}'$ along W-orbits we obtain an action $S \times T^4 \to T^4$ with $\mathfrak{F}'$ as orbit foliation. This action is free since the leaves of $\mathfrak{F}'$ are simply-connected.

REFERENCES

[1] AUSLANDER, L., GREEN, L., and HAHN, F., *Flows on homogeneous spaces*, Annals of Math. Studies No. 53, Princeton (1963).
[2] FRANKS, J., *Anosov diffeomorphisms*, Proceedings of Symposia in Pure Math. *14* (1970), 61–93.
[3] GARANÇON, M., *Le rang de certaines variétés closes*, Annales de l'Institut Fourier *20* (1) (1970), 1–19.
[4] HAEFLIGER, A., *Variétées feuilletées*, Ann. Scuola Norm. Sup. Pisa. (3) *16* (1962), 367–397.
[5] MALCEV, A., *On a class of homogeneous spaces*, (Russian) Isvest. Akad. Nauk *13* (1949), 9–32, (A.M.S. translation 1951).
[6] NOVIKOV, S., *Topology of foliations*, (Russian) Trudy Mosk. Mat. Obsh. *14* (1965) (A.M.S.) translation 1467).
[7] PLANTE, J., *Diffeomorphisms with invariant line bundles*, Inventiones Math. *13* (1971), 325–334.
[8] ——, *Anosov flows*, Amer. J. of Math. *94* (3) (1972).
[9] PLANTE, J. and THURSTON, W., *Anosov flows and the fundamental group*, Topology *11* (1972), 147–150.
[10] ROSENBERG, H., ROUSSARIE, R. and WEIL, D., *A classification of closed orientable* 3-*manifolds of rank* 2, Annals of Math. *91* (1970), 449–464.
[11] SACKSTEDER, R., *Foliations and pseudogroups*, Amer. J. of Math. *8* (1965), 79–102.
[12] TISCHLER, D., *On fibering certain foliated manifolds over* S^1, Topology *9* (1970), 153–154.
[13] TOMTER, P., *Anosov flows on infrahomogeneous spaces*, Proceedings of Symposia in Pure Math. *14* (1970), 299–328.
[14] WOLF, J., *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, Journal of Differential Geometry *2* (1968), 421–446.

University of Wisconsin, Madison