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Artikel: A Definition of Exotic Characteristic Classes of Spherical Fibrations

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A Definition of Exotic Characteristic Classes of Spherical Fibrations[1])

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1. Introduction

The object of this paper is to define certain characteristic cohomology classes for spherical fibrations which are zero on vector bundles and to show that the classes defined are not always zero. For an introductory survey of characteristic classes and spherical fibrations the reader is referred to [12] and the references therein.

Briefly there is a 'structure group' G and a classifying space BG for spherical fibrations and similar spaces (SG and BSG) for oriented spherical fibrations. SG has the homotopy type of $\lim_{n\to\infty}(\Omega^n S^n)_1$, where $(\Omega^n S^n)_1$ is the space of degree 1 base-point preserving maps of S^n to itself. The cohomology of all four spaces has recently been computed by Milgram ([10]), May ([8]) and Tsuchiya ([18]). My object is to define certain classes $e_k \in H^{rp^k-1}(\mathrm{BSG}; \mathrm{Z_p})$ for p an odd prime (where $r=2p-2$) and $e_k \in H^{2^{k-1}-1}(\mathrm{BG}; \mathrm{Z}_2)$ which I will refer to as exotic classes. In order to simplify notation I will only deal with the case of p odd, but all of the theorems herein can be proved for $p=2$ with the obvious changes in notation. All cohomology groups will have $\mathrm{Z_p}$ coefficients unless otherwise indicated. The definition given here is similar to one given by Peterson in [13] and to a definition of e_1 given by Gitler-Stasheff in [4].

In each case the exotic classes are defined in terms of twisted secondary cohomology operations (TSCO's) acting on the Thom class $u \in H^*\mathrm{MSG}$, where MSG is the Thom space of the universal bundle over BSG. TSCO's were introduced by Thomas ([16]) and axiomatized by McClendon ([9]). They are a generalization of ordinary secondary operations to the category of topological pairs (X, V) over a fixed space Y. The analogue of the Steenrod algebra in this category is $A(Y)$ where A is the Steenrod algebra and $A(Y)=H^*Y \otimes A$ as a vector space with the multiplication appropriate to defining an $A(Y)$ module structure on $H^*(X, V)$. TSCO's are derived from relations in $A(Y)$ just as ordinary secondary operations are derived from relations in A. Indeed, ordinary secondary operations can be regarded as TSCO's for the special case $Y=pt$.

Now the Thom space of any oriented spherical fibration can be regarded as a pair over BSG, so relations in $A(\mathrm{BSG})$ could be used to define characteristics classes on suitable spherical fibrations. In [13] Peterson defined an algebra injection $\theta: A \to A(\mathrm{BSG})$ with the property that $\theta(a)$ annihilates the Thom class u of MSG

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(and hence all Thom classes), and if $a \in A$ $\dim a > 0$. Hence any relation in A can be used to define a characteristic class in H^*BSG modulo a certain indeterminacy. He also showed that in the case of the Adem relation for $P^{(p-1)p^{k+1}}P^{p^{k-1}}$ (which will be denoted by R_k) the indeterminacy is zero in H^*MSG, so one could use this construction to define exotic classes in H^*BSG. However, as in the case of ordinary secondary operations, one must make a choice in deriving an operation from a relation in A(BSG) and two such choices can differ by any primary operation, e.g. by multiplication by any class in H^*BSG. Practically speaking this choice is another form of indeterminacy which in the above case is undesirably large. TSCO's and Peterson's use of them are dealt with in more detail in Chapter 2.

The object of Chapter 3 is to give a definition of exotic classes for which there is less indeterminacy in the choice involved. Let $Z = \prod_{i>0} K(Z_p, ir)$ and let $q: \mathrm{BSG} \to Z$ be the map corresponding to the total *Wu* class in H^*BSG ($q_i u = P^i u$ in H^*MSG). The main result of Chapter 3 is:

THEOREM 3.1.1. *There are relations* $\tilde{R}_k$ *in* $A(Z)$ *such that* $q^*(\tilde{R}_k) = \theta(R_k) + \alpha_k$ *where* α_k *is a sum of terms of the form* $0 \cdot P^i$. □

This is proved by a direct computation involving the natural Hopf algebra structures of A(BSG) and A(Z). It can also be shown by homological methods that the R_k along with the relation $\Delta\Delta = 0$ ($\Delta \in A$ being the Bochstein operation) are the only indecomposable relations in $\theta(A) \subset A_S$BSG) which can be lifted to A(Z). This will not be done here since it is irrelevant to the problem at hand.

Then we can define the exoctic class $e_k \in H^{p^{k}r-1}$ BSG by $e_k u = \phi_k^Z u \in H^*$MSG, where ϕ_k^Z is the TSCO associated with the relation $\tilde{R}_k$ in A(Z). Two possible choices of $\phi_k^Z u$ differ only by multiplication by an element in $\mathrm{Im}\, q^* \subset H^*$BSG, and we have

PROPOSITION 3.1.2. $\mathrm{Im}\, q^* = Z_p[q_i : i > 0] \otimes E[\Delta q_i : i > 0]$ *where* $E[\cdot]$ *denotes as usual the exterior algebra on the indicated generators.* □

Hence two possible choices of e_k differ by an ordinary characteristic class. Furthermore ϕ_k^Z as an operation has indeterminacy zero on Thom classes. This definition also has the advantage of enabling one to relate the exotic classes to the action of the ordinary secondary operation ϕ_k^A associated with R_k. We have

COROLLARY 3.1.4. $\phi^A u = \phi^Z u$ *if u is any Thom class on which* ϕ^A *is defined.* □

The proof uses the diagram

$$\begin{array}{ccc} \overline{\mathrm{BSG}} & \xrightarrow{\tilde{q}} & PZ \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \zeta} \\ \mathrm{BSG} & \xrightarrow{q} & Z \end{array} \tag{1.1}$$

where ζ is the path fibration and ρ is the induced fibration. Now TSCO's are natural with respect to test spaces and $\zeta^*(\phi^Z)=\phi^A$. The result follows from the fact that $\widetilde{\mathrm{BSG}}$ is the classifying space for bundles for which $\phi^A u$ is defined.

The object of Chapter 4 is to show that the classes defined herein are nonzero. We have

THEOREM 4.1.1. *There exists a spherical fibration ξ over a space X such that $e_k(\xi)\neq 0\,\forall k$ and $q_i(\xi)=0\,\forall i$.* $\square$

COROLLARY. *The exotic characteristic classes $e_k \in H^{p^k r-1}$* BSG *of Definition* 3.1.3 *are nonzero modulo* $\mathrm{Im}\, q^*$ *for all k.* $\square$

The space X is $\Sigma\Omega\widetilde{\mathrm{BSG}}$ and ξ is induced by the obvious map to BSG. The proof of Theorem 4.1.1 consists of relating the operation ϕ^A to Dyer-Lashof homology operations in H^*G which are known to be nonzero.

The exotic classes for $p=2$ appear to be related to the Kervaire invariant. Using the techniques of Section 3 it should be possible to relate the exotic classes to the classes $i^*(k_{2j-2})$ in Theorem 4.3 of [1]. For p odd it follows from the work of Tsuchiya [17] that the exotic classes are nonzero in H^*BSPL, so they may be regarded as some sort of smoothing obstructions. A formula for the first exotic class of a $(pr-1)$ dimensional manifold was given by David Frank in [3].

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2. Preliminaries

2.1. *Hopf Algebra Notation and the Algebra $A(Y)$*

First I must establish some notation. Throughout this paper all cohomology and homology groups will have coefficients in the field $\mathrm{Z_p}$ (p a prime) unless otherwise indicated, and all Hopf algebras considered will be graded, connected, associative and coassociative with ground ring $\mathrm{Z_p}$. To simplify notation I will assume p is odd, although *all of the theorems in this paper can be proved analogously for $p=2$ modulo the obvious changes in notation.* If R is a Hopf algebra, $\mu_R: R\otimes R\to R$ will denote the product, $\mu_R^n: R^{\otimes n}\to R$ the iterated product, $\psi_R: R\to R\otimes R$ the coproduct, $\psi_R^n: R\to R^{\otimes n}$ the iterated product, $\varepsilon_R: R\to \mathrm{Z_p}$ the augmentation, $I(R)=\ker \varepsilon_R$ the augmentation ideal, $\eta_R: \mathrm{Z_p}\to R$ the unit, and $\chi_R: R\to R$ the canonical antiautomorphism. Subscripts will be omitted whenever possible. If $r\in R$, $\psi(r)$ will be denoted by $\Sigma r'\otimes r''$, $\psi^n(r)$ by $\Sigma r'\otimes r''\otimes r'''\otimes\cdots r^{(n)}$, and $\chi(r)$ by $\bar{r}$. If M is a left R-module, let $\sigma_{R,M}: R\otimes M\to M$ denote the module structure map and if N is a right R-comodule let $\tau_{N,R}: N\to N\otimes R$

denote the comodule structure map: For right modules and left comodules the subscripts will be reversed. If S is an algebra over R (i.e. if S is an R module with a multiplication such that for $s_1, s_2 \in S$, $r(s_1 s_2) = \Sigma r'(s_1) r''(s_2)$), the semitensor product $R(S)$ is $S \otimes R$ with the following multiplication:

$$(s_1 \otimes r_1)(s_2 \otimes r_2) = \sum (-1)^{|r''_1|\,|s_2|} s_1 r'_1(s_2) \otimes r''_1 r_2$$

where $r_1, r_2 \in R$. If R and S are cocommutative Hopf algebras, so is $R(S)$ and the coproduct is given by

$$\psi(s \otimes r) = \sum (s' \otimes r') \otimes (s'' \otimes r'').$$

The conjugation is given by $\overline{s \otimes r} = \Sigma(-1)^{|r''|\,|s|} \bar{r}'(\bar{s}) \otimes \bar{r}''$. If Y is a space then H^*Y is an algebra over the Steenrod algebra A and the semitensor product $A(H^*Y)$ will be denoted simply by $A(Y)$. If $f: X \to Y$ is a continuous map and $V \subset X$ then $H^*(X, V)$ has an $A(Y)$-module structure defined by $(y \otimes a)\, x = f^*(y) \cup a(x)$ for $a \in A$, $x \in H^*(X, V)$ and $y \in H^*Y$. This module structure is the motivation for considering $A(Y)$.

Now $A(Y)$ can be regarded as the algebra of stable primary operations for the category of pairs of spaces over Y. An object in this category is a map $f: X \to Y$ and a subspace $V \subset X$, all denoted by (X, V, f). A morphism g in the category is a commutative diagram

$$\begin{tikzcd} V \arrow[r, phantom, "\subset"] \arrow[d, "g|V"'] & X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ V' \arrow[r, phantom, "\subset"] & X' \arrow[ur, "f'"'] & \end{tikzcd}$$

The map f induces an $A(Y)$-module structure on $H^*(X, V)$ as described above. This structure will be used in what follows but f will be suppressed in the notation. For precise definitions and properties of this category, see McClendon [9] references therein. The elements in $A(Y)$ will be referred to as twisted primary operations over Y.

2.2. *Twisted Second Cohomology Operations*

McClendon has given axioms for higher order cohomology operations (called twisted operations) which generalize those given by Maunder [6] for the case $Y = pt.$, i.e. for the ordinary category of pairs of spaces. I will give McClendon's axioms (in a slightly modified form) for secondary operations. Let

$$C: C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

be a chain complex of free $A(Y)$-modules. Let

$$\alpha\colon \mathrm{Hom}^k_{A(Y)}(C_1, H^*(X, V)) \to \mathrm{Hom}^k_{A(Y)}(C_2, H^*(X, V))$$

$$\beta\colon \mathrm{Hom}^k_{A(Y)}(C_0, H^*(X, V)) \to \mathrm{Hom}^k_{A(Y)}(C_1, H^*(X, V))$$

be defined by $\alpha = \mathrm{Hom}(d_2, 1)$, $\beta = \mathrm{Hom}(d_1, 1)$ for all k.

DEFINITION. If M and N are modules, an *additive relation* $r\colon M \rightharpoonup N$ is a submodule R of $M \otimes N$ and

$$\mathrm{Def}\, r = \{m \in M \colon \exists n \in N \text{ such that } \langle m, n\rangle \in R\} = \text{the domain of definition of } r$$

$$\mathrm{Ind}\, r = \{n \in N \colon \langle 0, n\rangle \in R\}. = \text{the indeterminacy of } r$$

DEFINITION. A *twisted secondary cohomology operation* (TSCO) ϕ associated with C is an additive relation

$$\phi\colon \mathrm{Hom}^k_{A(Y)}(C_0, H^*(X, V)) \rightharpoonup \mathrm{Hom}^{k-1}_{A(Y)}(C_2, H^*(X, V))$$

defined for all k and for all pairs (X, V, f) over Y and satisfying the following axioms:

1) $\mathrm{Def}\,\phi = \ker \beta$ and $\mathrm{Ind}\,\phi = \mathrm{im}\,\alpha$.

2) Naturality. Let $g(X, V, f) \to (X', V', f')$ be a map of pairs over Y i.e. $g\colon (X, V) \to (X', V')$ and $f'g \simeq f$, and let $\varepsilon \in H^*(X', V')$. Then $\varepsilon \in \mathrm{Def}\,\phi = g^*\varepsilon \in \mathrm{Def}\,\phi$ and $g^*\phi(\varepsilon) \subset \phi(g^*\varepsilon)$.

3) Suspension: $\Sigma_Y \phi = -\phi \Sigma_Y$ where Σ_Y is the suspension map in the category of pairs over Y (see McClendon [9] p. 188).

4) Peterson-Stein relation. Let $V' \subset V \subset X \xrightarrow{b} Y$, $\eta \in H^*(X, V')$ and let $H^*(V, V') \xleftarrow{j^*} H^*(X, V') \xleftarrow{j^*} H^*(X, V)$ be the exact sequence of the triple (X, V, V'), with $\delta\colon H^*(V, V') \to H^*(X, V)$.

Then $i^*\eta \in \ker \beta \Rightarrow \alpha j^{*-1} \beta \eta \subset -\delta \phi i^* \eta$.

McClendon has proved the following:

THEOREM 2.2.1. (Existence) *For any chain complex* C: $C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$ *there exists an associated* TSCO ϕ. □

THEOREM 2.2.2. (Quasi-uniqueness) *If* ϕ_0 *and* ϕ_1 *are two* TSCO's *associated with* C, *then* $\exists d\colon C_2 \to C_0$ *such that* $\mathrm{Hom}_{A(Y)}(d, 1)(\varepsilon) \in \phi_0(\varepsilon) - \phi_1(\varepsilon)$ *for each* $\varepsilon \in \mathrm{Def}\,\phi_0 = \mathrm{Def}\,\phi_1$, *i.e.* ϕ_0 *and* ϕ_1 *differ by a twisted primary operation.* □

THEOREM 2.2.1. (Naturality with respect to Y) *Let* $w\colon Y \to Y'$ *be a map,* C' *a chain complex of free* $A(Y')$*-modules, and* $C = A(Y) \otimes_{A(Y')} C'$. *There is a natural* *isomorphism*

$$\gamma: \mathrm{Hom}_{A(Y')}(C', H^*(X, V)) \to \mathrm{Hom}_{A(Y)}(C, H^*(X, V))$$

and the TSCO $\phi = \gamma\phi'\gamma^{-1}$ *is associated with C. Moreover*

$$\mathrm{Ind}\,\phi \supset \mathrm{Ind}\,\phi' \text{ and } \mathrm{Def}\,\phi \subset \mathrm{Def}\,\phi'. \quad \square$$

In particular of $Y = pt.$ then ϕ is an untwisted Adams-Maunder operation. This fact is of crucial importance in this paper. In section 3 I use the fact that an appropriate two stage Postnikov system is a universal example for Adams-Mauder operations. Similar universal examples exist for TSCO's (see McClendon [9] and Thomas [16]) but will not be used here.

2.3. *Peterson's operation*

Now I will recall a TSCO defined by Peterson in [13]. Let the $q_i \in H^*\mathrm{BSG}$ denote the ith *Wu* class of the universal bundle, which is defined by $q_i u = P^i u \in H^*\mathrm{MSG}$, where $P^i \in A$ is the ith Steenrod reduced power, and let $\bar{q}_i \in H^*\mathrm{BSG}$ be the ith *Wu* class of the Whitney inverse of the universal bundle. Let $\theta: A \to A(\mathrm{BSG})$ be defined by $\theta(\Delta) \subset 1 \otimes \Delta$, where $\Delta \in A$ is the Bochstein, and $\theta(P^i) = \Sigma q_t \otimes P^{i-t}$. Peterson showed that this definition makes sense and that θ is an injection of Hopf algebras. Now we can make $A(\mathrm{BSG})$ into a right A-module with structure map $\sigma_{A(\mathrm{BSG}), A} = \mu_{A(\mathrm{BSG})}(1 \otimes \theta)$, so if C^A is any chain complex of free A-modules, $C^{\mathrm{BSG}} = A(\mathrm{BSG}) \otimes_A C^A$ is a chain complex of free $A(\mathrm{BSG})$-modules. The C^A I want to consider has the form

$$0 \to C_2^A \xrightarrow{d_2} C_1^A \xrightarrow{d_1} C_0^A \to 0$$

where $C_0^A = A$; C_1^A has as A-basis the set $\{p_i : i > 0, \dim p_i = ir\}$; an A-basis of C_2^A is the set $\{e_k : k > 0, \dim e_k = p^k r\}$; $d_1 p_i = P^i$; and $d_2 e_k = p_{\Sigma_t=1}^{k-1} a_{k,i} P^{p^k - i} P^i$, where $a_{k,i} \in Z_p$ and $a_{k, p^{k-1}} = 1$ such that $\Sigma a_{k,i} P^{p^k - i} P^i = 0$ is the Adem relation for $P^{(p-1)p^{k-1}} P^{p^{k-1}}$ which will be denoted by R_k.

McClendon's theory then gives a family of TSCO's of the form

$$\phi^{\mathrm{BSG}}: \mathrm{Hom}^n_{A(\mathrm{BSG})}(C_0^{\mathrm{BSG}}, H^*(X, V)) \to \mathrm{Hom}^{n-1}_{A(\mathrm{BSG})}(C_2^{\mathrm{BSG}}, H^*(X, V))$$

where (X, V) is a pair over BSG. We can regard MSG as the pair (BSG, ESG) where ESG is the total space of the universal spherical fibration over BSG. Hence ϕ^{BSG} is defined on a subset of $H^*\mathrm{BSG}$. Peterson proved

PROPOSITION 2.3.1. $\phi^{\mathrm{BSG}} u$ *is defined and has zero indeterminacy.*

Proof.

$$\operatorname{Def}\phi^{\mathrm{BSG}} = \bigcap_{i>0} \ker\theta(P^i) \quad \text{and} \quad \theta(P^i)\,u = \sum \bar{q}_t P^{i-t}u = \sum \bar{q}_t q_{i-t}u = 0$$

so $u \in \operatorname{Def}\phi$.

$$\begin{aligned}\operatorname{Ind}\phi^{\mathrm{BSG}}u &= \bigcup_{\substack{k>0\\ 0<i<p^{k-1}}} \theta(P^{p^k-i})\,(H^{ir-1}\mathrm{BSG})\,u = \bigcup_{k,i}\sum_s \bar{q}_s P^{p^k-i-s}(H^{ir-1}\mathrm{BSG})\,u\\ &= \bigcup_{k',i}\sum_{s,t} \bar{q}_s (P^{p^k-i-s-t}H^{ir-1}\mathrm{BSG})\,P^t u = \bigcup_{k,i} (P^{p^k-i-t}H^{ir-1}\mathrm{BSG})\,\theta(P^t)\,u\\ &= \bigcup_{k,i} (P^{p^k-i}H^{ir-1}\mathrm{BSG})\,u = 0 \quad \text{since} \quad i \leqslant p^{k-1}. \quad \square\end{aligned}$$

Hence we could use ϕ^{BSG} to define the total exotic class, but this would be too imprecise since two choices of ϕ^{BSG} may differ by any twisted primary operation over BSG, e.g. by multiplication by any element in $H^*\mathrm{BSG}$. I will avoid this difficulty in section 3.1 by replacing ϕ^{BSG} by ϕ^{Z}, a TSCO associated with a certain chain complex over $A(\mathrm{Z})$ (see p. 1.3 for the definition of Z). Two choices of $\phi^{\mathrm{Z}}u$ will differ only by multiplication by an ordinary characteristic class.

3. The Definition of Exotic Characteristic Classes

3.1. *Statement of Results*

The object of this section is to give a definition of exotic characteristic classes modulo ordinary characteristic classes which will enable one to construct (in chapter 4) a spherical fibration for which the exotic characteristic classes can be shown to be nonzero. The main tool is

THEOREM 3.1.1. *There exists a chain complex C^{Z} of free $A(\mathrm{Z})$-modules such that $C^{\mathrm{BSG}} = A(\mathrm{BSG}) \otimes_{A(\mathrm{Z})} C^{\mathrm{Z}}$.*

Proof. See section 3.3.

PROPOSITION 3.1.2. $\operatorname{Im} q^* = \mathrm{Z}_p[q_i : i>0] \otimes E[\Delta q_i : i>0]$ *where* $E[\cdot]$ *as usual denotes the exterior algebra on the indicated generators.*

Proof. See section 3.2.

DEFINITION 3.1.3. *Let ϕ^{Z} be a* TSCO *associated with C^{Z}. Define the total exotic class $e \in H^*\mathrm{BSG}/\operatorname{Im} q^*$ by $eu = \phi^{\mathrm{Z}}\,u$ where $u \in H^*\mathrm{MSG}$ is the Thom class. Let e_k denote the $(p^k r - 1)$-dimensional component of e.*

Remarks. This definition makes sense since by Theorem 3.1.1. $\phi^{\mathrm{Z}}u = \phi^{\mathrm{BSG}}u$ for a suitable choice of ϕ^{BSG} and therefore $\phi^{\mathrm{Z}}u$ is defined with indeterminacy zero by

Theorem 2.2.3. Two possible choices of ϕ^Z differ by a twisted primary operation over Z. Such an operation applied to u gives an element in $(\mathrm{Im}\, q^*)\, u$ and hence e is independent of the choice of ϕ^Z, i.e. the exotic characteristic classes are defined modulo ordinary characteristic classes.

COROLLARY 3.1.4. *Let ϕ^A be an ordinary secondary operation associated with C^A, then $\phi^A \tilde{u} = \phi^Z \tilde{u} = e\tilde{u}$, where $\tilde{u}$ is the Thom class of $\overset{\frown}{\mathrm{MSG}}$, the Thom space of $\overset{\frown}{\mathrm{BSG}}$* (defined on p.1.4).

Proof. $C^A = A \otimes_{A(\mathrm{Z})} C^Z$, where the $A(\mathrm{Z})$-module structure on A is derived from diagram 0.1 and $\overset{\frown}{\mathrm{MSG}}$ can be regarded as a pair over $PZ \simeq pt.$, so the result follows from Theorem 2.2.3. $\square$

Remarks. $\phi^Z \tilde{u}$ does not depend on the choice of ϕ^Z since all twisted primary operations over Z vanish on $\tilde{u}$. Corollary 3.1.4. will be used in the next section to show $\phi^Z u \neq 0$.

3.2. *$A(\mathrm{Z})$ as a Hopf Algebra*

The object of this section is to prove that $A(\mathrm{Z})$ is a Hopf algebra which is an extension of a certain Hopf algebra $A(B) \subset A(\mathrm{BSG})$ by a bicommutative Hopf algebra D. (See Gugenheim [5] and Singer [14] for the definition and basic properties of Hopf algebra extensions.)

Let $B = \mathrm{Im}\, q^* \subset H^*\mathrm{BSG}$. The structure of B is given by Proposition 3.1.2. which I will now prove

Proof of 3.1.2. Clearly $\mathrm{Z}_p[q_i] \otimes E[\Delta q_i] \subset B$ so it suffices to show that

$$a(q_n) \in \mathrm{Z}_p[q_i] \otimes E[\Delta q_i] \qquad \forall a \in A$$

so it suffices in turn to show this for $a \in A$ indecomposable. It is obvious for $a = \Delta$. Now I will show $P^k q_n \in \mathrm{Z}_p[q_i]$ by induction on $k+n$. To start the induction we have $P^0 q_0 = 1 \in \mathrm{Z}_p[q_i]$. Now $qu = Pu$ so the Cartan formula gives

$$P^k(q_n u) = \sum_{i=0}^{k} (P^i q_n)(P^{k-i} u) = \sum_{i=0}^{k} (P^i q_n)(q_{k-i} u). \qquad 3.2.11$$

If $k \geqslant pn$, $P^k q_n = 0$ so I will assume $k < pn$. Then there is an Adem relation

$$\begin{aligned} P^k(q_n u) = P^k P^n u &= \sum_{i=0}^{\left[\frac{k+u}{\{p+1\}}\right]} c_i P^{k+u-i} P^i u \; (C_i \in \mathrm{Z}_p) \\ &= \sum_i c_i P^{k+n-i}(q_i u) = \sum_{i,j} c_i (P^{k+n-i-j} q_i)(P^j u) = \sum_{i,j} c_i (P^{k+n-i-j} q_i)(q_j u). \end{aligned} \qquad 3.2.12$$

Equating 3.2.11 with 3.2.12 gives a recursive formula for $P^k q_n \in \mathrm{Z}_p[q_i]$. $\square$

Moreover B is an A-algebra and $\mathrm{Im}\,\theta \subset A(B) \subset A(\mathrm{BSG})$.

In order to proceed further, I must recall the definition of the cotensor product $M \overset{R}{\otimes} N$, where M and N are right and left comodules over the coalgebra R respectively,

$$M \overset{R}{\otimes} N = \ker(\tau_{M,R} \otimes 1 - 1 \otimes \tau_{R,N}): M \otimes N \to M \otimes R \otimes N.$$

Next observe that B and $H^*\mathrm{Z}$ are bicommutative A-Hopf algebras with coproducts $\psi_B(q) = q \otimes q$ and $\psi_{H^*\mathrm{Z}}(\iota) = \iota \otimes \iota$ where $\iota = 1 + \sum_{i>0} \iota_{ir}$ and $\iota_{ir} \in H^{ir}\, K(\mathrm{Z_p}, ir)$ is the fundamental class. Moreover $q^*: H^*\mathrm{Z} \to B$ is an A-Hopf algebra map.

Now let $D = H^*\mathrm{Z} \overset{B}{\otimes} \mathrm{Z_p}$ and observe that $D \subsetneq H^*Z$ is a sub-Hopf algebra (Gugenheim [5], Theorem 4.21*). $H^*\mathrm{Z}$ is a free (in the graded sense) commutative algebra on generators $a\iota_{ir}$ where $\iota_{ir} \in H^{ir}\, \mathrm{Z}$ is a fundamental class and $a \in A$ with excess $a < (p-1)\, i$. Hence the sub-Hopf algebra $\mathrm{Z}_p[\iota_{ir}] \otimes E[\Delta \iota_{ir}]$, is a factor of $H^*\mathrm{Z}$ (over $\mathrm{Z_p}$) which q^* maps isomorphically onto B. The inverse γ of this isomorphism is a $\mathrm{Z_p}$ Hopf algebra map and a splitting of the Hopf algebra extension

$$E_{H^*\mathrm{Z}} D \to H^*\mathrm{Z} \underset{\gamma}{\overset{q^*}{\rightleftarrows}} B$$

so the extension is trivial over $\mathrm{Z_p}$ and $H^*\mathrm{Z} \approx D \otimes B$ as $\mathrm{Z_p}$-Hopf algebras. However the splitting is not an A-map and $H^*\mathrm{Z}$ does not split over A.

PROPOSITION 3.2.3. $A(\mathrm{Z}) \overset{(AB)}{\otimes} \mathrm{Z_p} \approx D$.

Proof: An element $z \otimes a \in A(\mathrm{Z})$ ($a \in A$, $z \in \mathrm{Z}$) is in $A(\mathrm{Z}) \overset{(AB)}{\otimes} \mathrm{Z_p}$ iff

$$\begin{aligned} 0 &= \tau_{A(\mathrm{Z}),A(B)}(z \otimes a) \otimes 1 - z \otimes a \otimes 1 \otimes 1 \\ &= \textstyle\sum z' \otimes a' \otimes q^*(z'') \otimes a'' \otimes 1 - z \otimes a \otimes 1 \otimes 1 \otimes 1 \\ &= 0 \quad \text{iff} \quad a = 1 \quad \text{and} \quad q^*(z'') = 0 \quad \forall q'' \neq 1. \end{aligned}$$

But this is precisely the definition of an element in D. □

COROLLARY 3.4. *$D \to A(\mathrm{Z}) \to A(B)$ is an extension of Hopf algebras and $A(\mathrm{Z}) = D \otimes A(B)$ as coalgebras.* □

3.3. *The Proof of Theorem* 3.1.1

The object of this section is to prove theorem 3.1.1 by constructing a relation in $A(\mathrm{Z})$ which maps to the relation in $A(\mathrm{BSG})$ used to define C^{BSG}. The construction will involve the Hopf algebra structure of $A(\mathrm{Z})$ given by Corollary 3.4.

Identifying $A(\mathrm{Z})$ with $D \otimes A(B)$, its product is given by (Singer [14], Prop. 3.4)

$$(d_1 \otimes b_1)(d_2 \otimes b_2) = \textstyle\sum d_1 b_1'(d_2)\, \tau(b_1'' \otimes b_2') \otimes b_1''' b_2''$$

where $d_1, d_2 \in D$; $b_1, b_2 \in A(B)$; $b_1'(d_2)$ is defined by a certain $A(B)$ module structure on D (derived from the A-module structure which D inherits from H^*Z) which will not be needed here; and $\tau: A(B) \otimes A(B) \to D$ (the twist) is a certain coalgebra map which plays an essential role in what follows. To define τ let $\tilde{\gamma} = \eta_D \otimes 1 : A(B) \to \to D \otimes A(B) = A(Z)$. Then $\tau(b_1 \otimes b_2) = \delta \Sigma \tilde{\gamma}(b_1') \tilde{\gamma}(b_2') \overline{\tilde{\gamma}(b_1'' b_2'')}$ where $\delta = 1 \otimes \varepsilon_{A(B)} : D \otimes \otimes A(B) = A(Z) \to D$.

The following calculation is the crucial part of this entire section.

LEMMA 3.3.1. *Let $\theta_m = \theta(P^m) \in A(B)$. Then $\tau(\theta_m \otimes \theta_n) = 0$ if $m \geqslant (p-1)n$.*

Proof. Let $\xi: A \to B$ be the composition $A \xrightarrow{\theta} A(B) = B \otimes A \xrightarrow{1 \otimes \varepsilon_A} B$ ξ is a coalgebra map and we have $\theta(a) = \Sigma \xi(a') \otimes a''$ and (identifying $A(Z)$ with $H^*Z \otimes A$))

$$\begin{aligned} \tau(\theta(a) \otimes \theta(b)) &= \delta \Sigma (\gamma\xi(a') \otimes a'') (\gamma\xi(b') \otimes b'') \overline{(\gamma\xi(a'''b''') \otimes a^{(4)}b^{(4)})} \\ &= \delta \Sigma (\gamma\xi(a')\, a''(\gamma\xi(b')) \otimes a'''b'') (\bar{b}'''\bar{a}^{(4)}\overline{\gamma\xi(a^{(5)}b^{(4)})} \otimes \bar{b}^{(5)}\bar{a}^{(6)}) \\ &= \delta \Sigma \gamma\xi(a')\, a''(\gamma\xi(b'))\, \overline{\gamma\xi(a'''b''')} \otimes 1 = \delta(a(\gamma\xi(b)) \otimes 1) \end{aligned}$$

The last step above follows from the fact that $\overline{\gamma\xi(a'''b''')} \in \operatorname{Im} \gamma$ and $\delta\tilde{\gamma} = 0$ in positive dimensions.

Hence

$$\begin{aligned} \tau(\theta_m \otimes \theta_n) &= \delta(P^m(\gamma\xi(P^n))) = 0 \quad \text{if} \quad m > (p-1)n \\ &= \delta(\gamma\xi(P^n))^p = 0 \quad \text{if} \quad m = (p-1)n. \quad \square \end{aligned}$$

Proof of 3.1.1. Let $r: I(A(B)) \otimes I(A(B)) \to I(A(Z)) \otimes I(A(Z))$ be defined by

$$\begin{aligned} r(a \otimes b) = &\sum_{\dim b'' > 0} (\overline{\tau(a' \otimes b')} \otimes a) \otimes (1 \otimes b'') \\ + &\sum_{\dim a'' > 0} (\overline{\tau(a' \otimes b)} \otimes 1) \otimes (1 \otimes a'') \end{aligned}$$

for $a, b \in A(B)$. It is straightforward (bearing in mind that τ is a coalgebra map) that $\mu_{I(A(Z))} r(a \otimes b) = 1 \otimes ab - \overline{\tau(a \otimes b)} \otimes 1$. Hence by the above lemma $r(\theta_m \otimes \theta_n) = 1 \otimes \theta_m \theta_n$ for $m \geqslant (p-1)n$. To define $C_z: C_2^z \xrightarrow{d_2} C_1^z \xrightarrow{d_1} C_0^z = A(Z)$ let $\{t_i : i > 0, \dim t_i = ir\}$ be an $A(Z)$-basis for C_1^z, $\{e_K : k > 0 \dim e_K = p^k r\}$ a basis for C_2^z, $d_1 t_i = \tilde{\gamma}(\theta_i)$ and

$$d_2 e_k = \sum_{\substack{u \geqslant 0 \\ i, v > 0}} a_{k,i} \overline{(\tau(P^{p^k - i - u} \otimes P^{i-v}) \otimes P^u)}\, t_v + \sum_{i, u > 0} a_{k,i} \overline{(\tau(P^{p^k - i - u} \otimes P^i) \otimes 1)}\, t_u$$

Then it is straightforward that $d_1 d_2 = 0$ and $C^A = A \otimes_{A(Z)} C^Z$. $\square$

Remark. Note that $d_2 e_k \in C_1^z$ has terms involving t_i for $0 < i < p^k$ where as $d_2 e_K \in C_1^A$ only involves p_i for $0 < i < p^{k-1}$, so there are some extra terms of the form $0 \cdot p^i, p^{k-1} < i < p^k$, in the relation one would use to define the ordinary operation for e_k, but these terms will not change the value of the operation on any cohomology class on which it is defined.

The Nontriviality of Exotic Characteristic Classes

4.1. *The Main Theorem and Dyer-Lashoff Operations*

In this section I will prove that the exotic classes are nontrivial, i.e.

THEOREM 4.1.1. *There exists a spherical fibration ξ over a space X such that the operation ϕ^A is defined on the Thom class of ξ and is nontrivial in every possible dimension, i.e. $e_K(\xi)\neq 0 \forall k$.* $\square$

The proof will involve certain information about H_*QS^0 where $QS^0=\lim_{n\to\infty}\Omega^n S^n$, so I will begin by recalling some basic properties of the Dyer-Lashoff homology operations as axiomatized by May [7]. In the following theorem of May the modifications necessary for $p=2$ will be indicated in square brackets:

THEOREM 4.1.2. *Let B be an infinite loop space. There exist natural homomorphisms $Q^i: H_*B\to H_*B$ of degree ir [of degree i]. They are axiomatized by the properties:*

1) *$Q^0(\phi)=\phi$ and $Q^i(\phi)=0$ for $i>0$, where $\phi\in H_0B$ is the identity element for the loop product in B.*
2) *$Q^i(x)=0$ if $2i<\dim x$ [if $i<\dim x$]*
3) *$Q^i(x)=x^p$ if $2i=\dim x$ [if $i=\dim x$]*
4) *$\sigma_* Q^i=Q^i\sigma_*$, where $\sigma_* I(H_*\Omega B)\to H_*B$ is the homology suspension.*

The operations also satisfy the properties:

5) *Cartan formula: $Q^s(xy)=\sum_{i=0}^{s} Q^i(x)\,Q^{s-i}(y)$ and*

$$\psi Q^s(x)=\sum_{i=0}^{s}\Sigma Q^i(x')\otimes Q^{s-i}(x'')$$

6) *Nishida relations. Let $P^s_* H_*B\to H_*B$ of degree $(-sr)$ be the dual of $P^s\in A$ and if $p>2$ let β be the Bochstein in homology. Then*

$$P^s_* Q^r=\sum_i(-1)^{i+s}\binom{(p-1)(r-s)}{s-pi}Q^{r-s+i}P^i_*$$

$$P^s_*\beta Q^r=\sum_i(-1)^{i+s}\binom{(p-1)(r-s)-1}{s-pi}\beta Q^{r-s+i}P^i_*$$

$$+\sum_i(-1)^{i+s}\binom{(p-1)(r-s)-1}{s-pi-1}Q^{r-s+i}P^i_*\beta. \quad \square$$

These operations generate a Hopf algebra R with the property

THEOREM 4.1.3. (May [8]) *Let* $[d]\in H_0QS^0$ *be the generator for the component corresponding to maps of degree d. Then R acts freely on* $[1]$. $\square$

Remark. If $x\in H_*QS^0$ is in the component of degree d, then Q^ix is in the component of degree pd, and $SG\subset QS^0$ is the component of degree 1. The product in H_*QS^0 will be the loop product; the composition product will not be used here.

4.2. *Proof of the Main Theorem*

Now I am ready to define ξ. Consider the following diagram

$$\begin{tikzcd}
\Omega K_2 \arrow[rr, "i_2"] & & E_2 \arrow[d, "p_2"] & \\
& & E_1 \arrow[r, "\phi^{A} p_1{}^{*}\iota_n"'] \arrow[d, "p_1"] & K_2 = \prod_{k>0} K(\mathrm{Z}_p, n+rp^k-1) \\
M \arrow[uu, "h"] \arrow[r, "t"] & S^n \arrow[uur, "x''_n"] \arrow[ur, "x'_n"'] \arrow[r, "x_n"'] & K(\mathrm{Z}, n) \arrow[r, "\Pi P^i\iota_n"'] & K_1 = \prod_{i>0} K(\mathrm{Z}_p, n+ri)
\end{tikzcd} \tag{4.2.01}$$

where $x_n\in H^n(S^n, \mathrm{Z})$ is a generator, p_1 and p_2 are the principal fibrations induced by the indicated maps, x'_n and x''_n are the unique liftings of x_n, i_2 is the inclusion of the fibre and M is the fibre production of x''_n and i_2, and n is sufficiently large. Hence ΩE_1 is the fibre of t and we have fibre squares.

$$\text{a)}\ \begin{array}{ccc} M & \longrightarrow & PE_1 \\ {\scriptstyle t}\downarrow & & \downarrow \\ S^n & \xrightarrow{x'_n} & E_1 \end{array} \qquad \text{b)}\ \begin{array}{ccc} \Omega^n M & \longrightarrow & \Omega^n PE_1 \\ {\scriptstyle \Omega^n t}\downarrow & & \downarrow \\ \Omega^n S^n & \xrightarrow{\Omega^n x'_n} & \Omega^n E_1 \simeq \mathrm{Z}\times\Omega^{n+1}K_1 \end{array} \tag{4.2.02}$$

where PE_1 is the path space of E_1 and the equivalence on the right is a homotopy equivalence but not an H-homotopy equivalence. M is n-connected so Ω^nM is connected and its image under Ω^nt will lie in the degree zero component of Ω^nS^n, which is canonically homotopy equivalent to SG, so I have a map $g:\Omega^nM\to \mathrm{SG}$ which induces an orientable spherical fibration ξ on $X=\Sigma\Omega^nM$. Now I need two lemmas which will be proved in 4.3:

LEMMA 4.2.1. *The class* $[-p^2]\,\beta Q^{(p-1)p^i}\beta Q^{p^i}([1])\in H_*\Omega^nS^n$ *is in the image of* $(\Omega^nt)_*$. $\square$

LEMMA 4.2.2. *Let* $\tilde{b}_i\in H_{rp^{i+1}-2}\,\Omega^nE_2$ *be the fundamental class. Then*

$\tilde{b}_i=[-p^2]\,\beta Q^{p^i(p-1)}\beta Q^{p^i}([1])$. $\square$

Now we have a map $\Omega^nM\xrightarrow{\Omega^nt}\Omega^nS^n$, which defines the bundle ξ over $\Sigma\Omega^nM$. This map has an adjoint $\Sigma^n\Omega^nM\xrightarrow{\tilde{t}}S^n$ and it is straightforward that the Thom space $T\xi$ is homotopy equivalent to $\Sigma C^{\tilde{t}}$, where $C^{\tilde{t}}$ is the mapping one of $\tilde{t}$. Now consider the following diagram:

$$\begin{array}{ccccccc}
\Omega K_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{p_2} & E_1 & \xrightarrow{\phi^A} & K_2 \\
{\scriptstyle h}\uparrow & & {\scriptstyle x''_n}\downarrow & & \| & & \uparrow{\scriptstyle \Sigma h} \\
M & \xrightarrow{t} & S^n & \xrightarrow{x'_n} & E_1 & \xrightarrow{v} & \Sigma M \\
{\scriptstyle \alpha}\uparrow & & \| & & \uparrow{\scriptstyle \alpha'} & & \uparrow{\scriptstyle \Sigma\alpha} \\
\Sigma^n\Omega^n M & \xrightarrow{\tilde{t}} & S^n & \xrightarrow{\tilde{x}'_n} & C_{\tilde{t}} & \xrightarrow{\tilde{v}} & \Sigma^{n+1}\Omega^n M
\end{array}$$

where α is the adjoint of the identity, the rows are fiber sequences in the stable range, and α', $\tilde{x}'_n$, v, and $\tilde{v}$ are the obvious maps. I must show that every fundamental class in H_*K_2 lies in

$$\mathrm{Im}\,(\phi^A\alpha')_* = \mathrm{Im}(\Sigma\,(h\alpha)\,\tilde{v})_* = \Sigma\,\mathrm{Im}\,(h\alpha)_* = \Sigma^{n+1}\,(\Omega^n h)_*.$$

This leads us to the square

$$\begin{array}{ccc}
\Omega^{n+1}K_2 & \xrightarrow{\Omega^n i_2} & \Omega^n E_2 \simeq \Omega^{n+1}K_2 \times \Omega^{n+1}K_1 \times Z \\
{\scriptstyle \Omega^n h}\uparrow & & \uparrow{\scriptstyle \Omega^n x''_n} \\
\Omega^n M & \xrightarrow{\Omega^n t} & \Omega^n S^n
\end{array}$$

where I must show that the fundamental classes in $H_*\Omega^{n+1}K_2$ are in the image of $(\Omega^n h)_*$, i.e. in the notation of Lemma 4.2.2, I must show $\tilde{b}_i \in \mathrm{Im}\,(\Omega^n i_2 h)_* \subset H_*\Omega^n E_2$, but

$$\begin{aligned}
\tilde{b}_i &= [-p^2]\,\beta Q^{p^i(p-1)}\beta Q^{p^i}[1] && \text{by 4.2.2} \\
&\in \mathrm{Im}\,(\Omega^n x''_n t)_* && \text{by 4.2.1}
\end{aligned}$$

and theorem 4.1.1 is proved. □

4.3. *The Proof of Lemmas* 4.2.1. *and* 4.2.2

Proof of Lemma 4.2.1. I will use the Eilenberg-Moore spectral sequence for H-spaces as described by Moore-Smith in [11]. In their terminology the degree zero component of 4.2.02 b) is a Hopf fibre square and there is a spectral sequence converging to $H_*\Omega^n M$ with

$$\begin{aligned}
E^2 &= \mathrm{Cotor}^{H^*\Omega^{n+1}K_1}(H_*\,(\Omega^n S^n)_0, Z_p) \\
&\cong (H_*\,(\Omega^n S^n)_0 \overset{\mathrm{Im}(\Omega^n x'_n)_*}{\otimes} Z_p) \otimes \mathrm{Cotor}^{H^*\Omega^{n+1}K_1}(Z_p, Z_p)
\end{aligned}$$

where $(\Omega^n S^n)_0$ is the component of degree zero. The indicated isomorphism is proved in [11]. Hence $H_*(\Omega^n S^n)_0 \otimes^{\mathrm{Im}\,(\Omega^n x'_n)_*} Z_p$ lives to E^∞ so it suffices to show that it contains

$$[-p^2]\,\beta Q^{(p-1)p^i}\beta Q^{p^i}[1], \quad \text{i.e. that} \quad (1\otimes(\Omega^n x')_*)\,\psi([-p^2]\,\beta Q^{(p-1)p^i}\beta Q^{p^i}[1]) = [-p^2]\,\beta Q^{(p-1)p^i}\beta Q^{p^i}[1]\otimes[0].$$

By the Cartan formula, the Adem relations, and the fact that $\psi[d]=[d]\otimes[d]$ we have

$$\begin{aligned}\psi(Q^{(p-1)p^i}\beta Q^{p^i}[1]) &= \sum_{j,k} Q^j\beta Q^k[1]\otimes Q^{(p-1)p^i-j}Q^{p^i-k}[1]\\ &\quad + Q^jQ^k[1]\otimes Q^{(p-1)p^i-j}\beta Q^{p^i-k}[1]\\ &= \sum_j Q^{(p-1)j}\beta Q^j[1]\otimes Q^{(p-1)(p^i-j)}Q^{p^i-j}[1]\\ &\quad + Q^{(p-1)j}Q^j[1]\otimes Q^{(p-1)(p^i-j)}\beta Q^{p^i-j}[1]\end{aligned}$$

so

$$\begin{aligned}&\psi([-p^2]\,\beta Q^{(p-1)p^i}\beta Q^{p^i}[1])\\ &\quad=\sum_j [-p^2]\,\beta Q^{(p-1)j}\beta Q^j[1]\otimes[-p^2]\,Q^{(p-1)(p^i-j)}Q^{p^i-j}[1]\\ &\quad\quad+[-p^2]\,Q^{(p-1)j}Q^j[1]\otimes[-p^2]\,\beta Q^{(p-1)(p^i-j)}\beta Q^{p^i-j}[1].\end{aligned}$$

To complete the proof I will show that

$$(\Omega^n x_n')_* Q^iQ^j[1] = (\Omega^n x_n')_* Q^i\beta Q^j[1] = 0 \qquad \forall i, j.$$

Consider the inclusion of the fibre $\Omega^n i_1:\Omega^{n+1}K_1\to\Omega^n E_1$. Clearly $Q^j[1]\in H_{r_j}\Omega^n E_1$ is in $\operatorname{Im}(\Omega^n i_1)_*$ so for the iterated operation to be nontrivial on $[1]\in H_0E_1$, we must have a Q^i acting nontrivially in $H_*\Omega^{n+1}K_1$, but Dyer-Lashoff operations are always trivial in a product of Eilenberg-MacLane spaces with the product infinite loop space structure, as follows from the following

PROPOSITION 4.3.01. *Dyer-Lashoff operations are all trivial in $H_*K(\pi, m)$, where $\pi = Z$ or Z_p.*

Proof. The Dyer-Lashoff operations on $K(\pi, n)$ are defined in terms of a certain Σ_p-equivariant map (see [2])

$$\theta_p: W_{\Sigma_p}\times(K(\pi, n))^p\to K(\pi, n)$$

where Σ_p is the symmetric group on p letters, W_{Σ_p} is a free acyclic Σ_p-complex on which Σ_p is acting, Σ_p acts on $(K(\pi, n))^p$ by permutation of factors and the composition

$$(K(\pi, n))^p\to W_{\Sigma_p}\times(K(\pi, n))^p\xrightarrow{\theta_p}K(\pi, n)$$

is p-fold multiplication in $K(\pi, n)$. Since $K(\pi, n)$ is an abelian group the map θ_p can be chosen to factor through the projection onto $(K(\pi, n))^p$ and it follows that the Dyer-Lashoff operations are trivial on $H_*K(\pi, n)$. □

Now in order to prove lemma 4.2.2 I will need

LEMMA 4.3.1. *Let* $\tilde{h}_i \in H_{(p^i r-1)}(\Omega^n E_1)$ *be the fundamental class. Then* $h_i = \beta Q^{p^i}[1]$.

Proof of 4.3.1. For any simply-connected space X the Eilenberg-Moore spectral sequence gives

$$\mathrm{Tor}_{H^*X}(Z_p, Z_p) \Rightarrow H^*\Omega X$$

and this is functorial on X. Let $E_{1,i}$ be the fibre of $K(Z, n) \xrightarrow{Pp^i\iota_n} K(Z_p, n+rp^i)$. For the fiber inclusion $\Omega^{n-2p^i-1}(E_{1,i}) \xrightarrow{p_{1,i}} K(Z, n))$ we get

$$\begin{array}{ccc} \mathrm{Tor}_{H^*K(Z, 2p^i+1)}(Z_p, Z_p) & \Rightarrow & H^*K(Z, 2p^i) \\ \downarrow & & \downarrow \\ \mathrm{Tor}\, H^*\Omega^{n-2p^i-1}E_1(Z_p, Z_p) & \Rightarrow & H^*\Omega^{n-2p^i}E_1. \end{array}$$

The upper spectral sequence has been studied by Larry Smith in [15] where it was shown that $\mathrm{Tor}_{H^*K(Z, 2p^i+1)}(Z_p, Z_p) \subset \Gamma(x_{p^i})$, where $x_{p^i} \in \mathrm{Tor}^{-1, 2p^i+1}_{H^*K(Z, 2p^i+1)}(Z_p, Z_p)$ and $\Gamma(\cdot)$ denotes the divided polynominal algebra; and that $d_{p-1}\gamma_p(x_{p^i}) = C_i \Delta P^{p^i} x_{p^i}$, $C_i \neq 0 \in Z_p$, where $\gamma_p(\cdot)$ is the divided pth power. Hence

$$\begin{aligned} d_{p-1}((\Omega^{n-2p^i-1}p_{1,i})^*(x_{p^i})) &= C_i \Delta P^{p^i}(\Omega^{n-2p^i-1}p_{1,i})^*(x_{p^i}) \\ &= 0 \end{aligned}$$

so $\gamma_p((\Omega^{n-2p^i-1}p_{1,i})^*(x_{p^i}))$ lives to E_∞ and the fundamental class $\tilde{x}_{p^i} \in H_{2p^i}$ $\Omega^{n-2p^i}E_{1,i}$ has $\tilde{x}^p_{p^i} = Q^{p^i}\tilde{x}_{p^i} \neq 0$. Now $\Omega^n E_{1,i} = Z \times K(Z_p, rp^i - 1)$ and $Q^{p^i}[1]$ is the only nonzero class in $\dim rp^i$ so $\beta Q^{p^i}[1] = q_{1,i}(\Omega^n 2p_i)_* \tilde{h}_i$ where $q_{1,i}: E_1 \to E_{1,i}$ is some lifting of $p_1: E_1 \to K(Z_p, n)$. Hence $\beta Q^{p^i}[1] \neq 0 \in H^*\Omega^n E_1$. Now $H_*\Omega^n E_1 \approx H_*(\Omega^{n+1}K_1 \times Z)$ is a divided polynomial algebra on certain primitive generators related to each other by the action of the Steenrod algebra. $\beta Q^{p^i}[1]$ is such a generator so it is equal to $\tilde{h}_i$ if it is annihilated by every nontrivial Steenrod operation, and this is a direct consequence of the Nishida relations. □

Proof of Lemma 4.2.2. Let E_3 be the fibre of $p_1 p_2: E_2 \to K(Z, n)$. I will prove the lemma by analyzing the Dyer-Lashoff operations on $\Omega^n E_3$. E_3 is also the fiber of

$$\prod_{j \geq 1} K(Z_p, n + rj - 1) = \Omega K_1 \xrightarrow{\Sigma a_{i,j}\iota_{n-rpj-1}} K_2 = \prod_{i \geq 0} K(Z_p, n + rp^{i+1} - 1)$$

where $a_{i,j} \in A$, $\dim a_{i,j} = r(p^{i+1} - j)$ and $a_{i,p^i} = P^{(p-1)p^i}$. Let $E_{3,i}$ be the fibre of $K(Z_p, n+rp^i-1) \xrightarrow{P^{(p-1)p^i}\iota} (Z_p, n+rp^{i+1}-1)$. Using maps $E_3 \to E_{3,i}$ and an argument similar to that of the proof of 4.2.1 one can show that $\tilde{\imath}_{rp^{i+1}-2}=\beta Q^{(p-1)p^i}\tilde{\imath}_{rp^i-1}\in H_*\Omega^n E_3$ where $\tilde{\imath}_k$ is the fundamental homology class in dimension k. Then the fibre inclusion $\Omega^n E_3\to\Omega^n E_2$ sends $\tilde{\imath}_{rp^i-1}$ to $\tilde{h}_i$ and the result follows. $\square$

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