

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 47 (1972)

Artikel: Parapolicity and existence of bounded biharmonic functions
Autor: LeoSario, / Wang, Cecilia
DOI: <https://doi.org/10.5169/seals-36371>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 21.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Parabolicity and existence of bounded biharmonic functions¹

by LEO SARIO and CECILIA WANG

The existence of bounded biharmonic functions has exhibited interesting dependence on the dimension of the base manifold. Typically, such functions exist on the punctured Euclidean N -space $E^N: 0 < |x| < \infty$ for $N=2$ and for $N=3$, but not for any $N \geq 4$ (Sario-Wang [17]). In the present paper we are interested in the problem: Is there any relation between the parabolicity of a manifold and the existence of bounded biharmonic functions, and does the dimension of the manifold have any bearing on the question.

Denote by H^2B the class of bounded biharmonic functions. In contrast with the case of bounded harmonic functions, which are known not to exist on any parabolic manifold (see e.g. Sario-Nakai [14]), it is possible to endow even the Euclidean plane with a metric which allows H^2B -functions (Nakai-Sario [8]). The process relies on the fact that harmonicity on a Riemann surface, and hence parabolicity, are not affected by a conformal metric, which thus can be freely chosen to bring in H^2B -functions. For manifolds of dimension $N \geq 3$ this process is no longer possible. We shall show that, nevertheless, there exist parabolic manifolds of any dimension which carry H^2B -functions.

That there exist hyperbolic manifolds with H^2B -functions is trivial in view of the Euclidean N -ball. We shall prove that there also exist hyperbolic manifolds of any dimension which do not possess H^2B -functions.

Our study is completed by giving parabolic manifolds of any dimension which do not tolerate H^2B -functions. Thus the totality of Riemannian manifolds for any N is decomposed into four disjoint nonempty classes,

$$O_G^N \cap \tilde{O}_{H^2B}^N, \quad \tilde{O}_G^N \cap \tilde{O}_{H^2B}^N, \quad \tilde{O}_G^N \cap O_{H^2B}^N, \quad O_G^N \cap O_{H^2B}^N,$$

where O_G^N is the class of parabolic N -manifolds, $O_{H^2B}^N$ the class of N -manifolds which do not carry nonharmonic H^2B -functions, and \tilde{O} stands for the complement of a given class O .

1. Consider the punctured N -space

$$M_\alpha^N = \{0 < r < \infty\}, \quad r = |x|, \quad x = (x_1, \dots, x_N),$$

¹⁾ The work was sponsored by the U.S. Army Research Office – Durham, Grant DA-ARO-D-31-124-71-G181, University of California, Los Angeles.

with the metric

$$ds^2 = r^\alpha dr^2 + r^{\alpha+2} d\theta_1^2 + \sum_{i=2}^{N-1} \varphi_i(\theta) d\theta_i^2,$$

α a constant. Here we have utilized the global polar coordinates $(r, \theta_1, \dots, \theta_{N-1})$ of the punctured N -space, and $\varphi_2(\theta), \dots, \varphi_{N-1}(\theta)$ are positive (periodic) functions of $\theta_2, \dots, \theta_{N-1}$ only.

LEMMA 1. $M_\alpha^N \in O_G^N$ for every α .

Proof. Set $\varphi_0 = \prod_{i=2}^{N-1} \varphi_i$. The metric tensor (g_{ij}) is diagonal, with $g^{rr} = r^{-\alpha}$, $g^{\theta_1\theta_1} = r^{-\alpha-2}$, and $\sqrt{g} = r^{\alpha+1} \varphi_0^{\frac{1}{2}}$. For $f(r) \in C^2$, the Laplace-Beltrami operator $\Delta = d\delta + \delta d$ gives

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial r} (\sqrt{g} g^{rr} f') = -r^{-\alpha-1} \varphi_0^{-\frac{1}{2}} \frac{\partial}{\partial r} (r^{\alpha+1} \varphi_0^{\frac{1}{2}} r^{-\alpha} f'),$$

which vanishes if and only if $d(rf')/dr = 0$. Thus every radial harmonic function on M_α^N has the form $f(r) = a \log r + b$. For $b = 0$, and a suitable a , $f(r)$ is the harmonic measure ω_R of the region bounded by $r = 1$ and $r = R$, say. As $R \rightarrow \infty$ or $R \rightarrow 0$, $\omega_R \rightarrow 0$, and $M_\alpha^N \in O_G^N$.

LEMMA 2. For every α , $\cos(\alpha+2)\theta_1 \in H^2 B(M_\alpha^N)$.

Proof. Since the φ_i 's are independent of θ_1 ,

$$\begin{aligned} \Delta \cos(\alpha+2)\theta_1 &= -r^{-\alpha-1} \varphi_0^{-\frac{1}{2}} \frac{\partial}{\partial \theta_1} \left[r^{\alpha+1} \varphi_0^{\frac{1}{2}} r^{-\alpha-2} \frac{d}{d\theta_1} \cos(\alpha+2)\theta_1 \right] \\ &= (\alpha+2)^2 r^{-\alpha-2} \cos(\alpha+2)\theta_1, \end{aligned}$$

and

$$\begin{aligned} \Delta^2 \cos(\alpha+2)\theta_1 &= -(\alpha+2)^2 \left\{ r^{-\alpha-1} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} (r^{-\alpha-2} \cos(\alpha+2)\theta_1) \right] \right. \\ &\quad \left. + \frac{\partial}{\partial \theta_1} \left[r^{-\alpha-2} \frac{\partial}{\partial \theta_1} (r^{-\alpha-2} \cos(\alpha+2)\theta_1) \right] \right\} \\ &= -(\alpha+2)^2 [(\alpha+2)^2 r^{-2\alpha-4} \cos(\alpha+2)\theta_1 - (\alpha+2)^2 r^{-2\alpha-4} \times \cos(\alpha+2)\theta_1] = 0. \end{aligned}$$

We have proved:

THEOREM 1. For every N , $O_G^N \cap \tilde{O}_{H^2 B}^N \neq \emptyset$.

2. Consider the space discussed in [13]

$$E_\alpha^N = \{0 < r < \infty\}, \quad r = |x|, \quad x = (x_1, \dots, x_N),$$

with the metric $ds = r^\alpha |dx|$, $\alpha \in \mathbf{R}$.

LEMMA 3. $E_\alpha^N \in O_G^N$ if and only if $\alpha = -1$.

Proof. The metric tensor is given by

$$ds^2 = r^{2\alpha} dr^2 + r^{2\alpha+2} \varphi_1(\theta) d\theta_1^2 + \cdots + r^{2\alpha+2} \varphi_{N-1}(\theta) d\theta_{N-1}^2,$$

where $\theta = (\theta_1, \dots, \theta_{N-1})$ and $\varphi_1, \dots, \varphi_{N-1}$ are trigonometric functions of θ . Set $\varphi_0 = \prod_1^{N-1} \varphi_i$. Then $\sqrt{g} = r^{N-1+N\alpha} \varphi_0^{\frac{1}{2}}(\theta)$, $g^{rr} = r^{-2\alpha}$, and for $f(r) \in C^2$,

$$\begin{aligned} \Delta f(r) &= -r^{-N+1-N\alpha} \varphi_0^{-\frac{1}{2}} \frac{\partial}{\partial r} (r^{N-1+(N-2)\alpha} \varphi_0^{\frac{1}{2}} f') \\ &= -r^{-2\alpha} \{f'' + [N-1+(N-2)\alpha] r^{-1} f'\}. \end{aligned}$$

This vanishes if and only if

$$f(r) = a \int_1^r r^{-N+1-(N-2)\alpha} dr + b.$$

The rest of the proof is as for Lemma 1.

3. To show that there exist hyperbolic N -manifolds without $H^2 B$ -functions, it will be convenient to choose $\alpha = -\frac{3}{4}$ in No. 2. Then the equation $\Delta f(r) = 0$ has a solution

$$\sigma(r) = \begin{cases} \log r & \text{for } N = 2, \\ r^{-(N-2)/4} & \text{for } N \neq 2, \end{cases}$$

and the general solution is $a\sigma + b$. Let $S_{nm}(\theta)$ be the surface spherical harmonics, $n = 1, 2, \dots$, and $m = 1, \dots, m_n$, where m_n is determined by

$$(1+x)(1-x)^{-N+1} = \sum_{n=0}^{\infty} m_n x^n.$$

We have $\Delta S_{nm} = n(n+N-2)r^{-\frac{1}{2}}S_{nm}$, and the equation $\Delta(f(r)S_{nm}(\theta)) = 0$ has the general solution $f(r) = ar^{p_n} + br^{q_n}$, where

$$p_n, q_n = \frac{1}{2} [-(N-2)/4 \pm \sqrt{(N-2)^2/16 + 4n(n+N-2)}].$$

For a fixed r , any harmonic function $h(r, \theta)$ on $E_{-3/4}^N$ is C^∞ on the (Euclidean)

unit sphere, with an eigenfunction expansion

$$h(r, \theta) = f_0(r) + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} f_{nm}(r) S_{nm}(\theta).$$

Given $0 < r_1 < r_2 < \infty$, choose constants a_{nm} , b_{nm} , a , b , such that for $i=1, 2$,

$$a_{nm} r_i^{p_n} + b_{nm} r_i^{q_n} = f_{nm}(r_i), \quad a\sigma(r_i) + b = f_0(r_i).$$

Then h has the expansion

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n}) S_{nm} + a\sigma(r) + b$$

on $r=r_1$ and $r=r_2$, hence by the harmonicity on $r_1 \leq r \leq r_2$. The uniqueness is verified by choosing $0 < r'_1 < r_1 < r_2 < r'_2 < \infty$, which gives on $r'_1 \leq r \leq r'_2$ an expansion that on $r_1 \leq r \leq r_2$ must coincide with the above.

4. By a straightforward computation of Δ we find that the equation $\Delta f(r)=1$ has a solution $s(r)=-(8/N)r^{1/2}$, the general solution being $s+a\sigma+b$. Similarly, the equation $\Delta f(r)=\sigma(r)$ has a solution

$$\tau(r) = \begin{cases} s(r)(\log r - 4) & \text{for } N=2, \\ -2\log r & \text{for } N=4, \\ \frac{8}{N-4} r^{-(N-4)/4} & \text{for } N \neq 2, 4, \end{cases}$$

and the general solution is $\tau+a\sigma+b$. Set $P_n=N/8+p_n$, $Q_n=N/8+q_n$. The equation $\Delta u=r^{p_n}S_{nm}$ is satisfied by

$$u_{nm} = -\frac{1}{P_n} r^{p_n+\frac{1}{2}} S_{nm},$$

and the equation $\Delta v=r^{q_n}S_{nm}$ by

$$v_{nm} = -\frac{1}{Q_n} r^{q_n+\frac{1}{2}} S_{nm}.$$

Given a biharmonic function u on $E_{-3/4}^N$, let

$$\Delta u = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n}) S_{nm} + a\sigma(r) + b,$$

and set

$$u_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} a_{nm} u_{nm} + b_{nm} v_{nm} + a\tau(r) + bs(r).$$

Then $u = u_0 + k$, where k is a harmonic function

$$k = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm} r^{p_n} + d_{nm} r^{q_n}) S_{nm} + c\sigma(r) + d.$$

In fact, the compact convergence of u_0 is entailed by that of Δu , and the statement follows by Nos. 3–4.

5. We are ready to state:

LEMMA 4. $E_{-3/4}^N \in O_{H^2 B}^N$ for every N .

Proof. Let $u \in H^2 B(E_{-3/4}^N)$. Clearly $|(u, \varphi)| \leq \sup |u|(1, |\varphi|)$ for all $\varphi \in L^1$, in particular for the family of functions $\varphi_t = \varphi_t(r) S_{nm}$, where φ_1 is a fixed function $\in C$, $\varphi_1 \geq 0$, $\text{supp } \varphi_1 \subset (1, 2)$, and $\varphi_t(r) = \varphi_1(r + 1 - t)$ for $t \geq 1$. By the orthogonality of $\{S_{nm}\}$,

$$(u, \varphi_t) = \int_t^{t+1} (C_1 a_{nm} r^{p_n + \frac{1}{2}} + C_2 b_{nm} r^{q_n + \frac{1}{2}} + C_3 c_{nm} r^{p_n} + C_4 d_{nm} r^{q_n}) \varphi_t(r) r^{N/4 - 1} dr.$$

Here and later the C 's are constants, not always the same. Clearly $\int_t^{t+1} \varphi_t(r) dr$ is constant as $t \rightarrow \infty$. If some $a_{nm} \neq 0$, then

$$(u, \varphi_t) \sim C t^{p_n + N/4 - \frac{1}{2}}, \quad \text{whereas} \quad (1, |\varphi_t|) = O(t^{N/4 - 1}).$$

A fortiori, we have a contradiction for n such that $p_n + N/4 - \frac{1}{2} > N/4 - 1$, that is, $p_n > -\frac{1}{2}$. Since $p_n > 0$ for all n , we obtain $a_{nm} = 0$ for all n, m .

If some $c_{nm} \neq 0$, we infer by $q_n + \frac{1}{2} < p_n$ for all n, N that

$$(u, \varphi_t) \sim C t^{p_n + N/4 - 1}$$

as $t \rightarrow \infty$. Every n such that $p_n > 0$ is ruled out, and we conclude again that $c_{nm} = 0$ for all n, m .

Now choose $\varphi_t(r) = \varphi_1(r/t)$, with φ_1 as before, and $0 < t \leq 1$. Then $\text{supp } \varphi_t \subset (t, 2t)$ and $\int_t^{2t} \varphi_t(r) dr = Ct$. If some $d_{nm} \neq 0$, then

$$(u, \varphi_t) \sim C t^{q_n + N/4} \quad \text{and} \quad (1, |\varphi_t|) \sim O(t^{N/4})$$

as $t \rightarrow 0$. Inequality $q_n + N/4 < N/4$ gives a contradiction, and by $q_n < 0$ we deduce that

for all n, m . In the same manner we see that $b_{nm}=0$ if $q_n+\frac{1}{2}<0$, that is, for all n, m .

Thus the function u reduces to $a\tau(r)+bs(r)+c\sigma(r)+d$. Since τ, s, σ are linearly independent and unbounded, we have $a=b=c=0$, and u is a constant.

We combine Lemmas 3 and 4 to conclude:

THEOREM 2. *For every N , $\tilde{O}_G^N \cap O_{H^2B}^N \neq \emptyset$.*

6. The existence of hyperbolic N -manifolds with H^2B -functions is given by the Euclidean N -ball. It remains to find a parabolic N -manifold without H^2B -functions.

LEMMA 5. $E_{-1}^N \in O_{H^2B}^N$ for every N .

Proof. The proof arrangement is the same as in Nos. 3–5, and we only point out the changes. We now have $\sigma(r)=\log r$ for every N , $p_n=-q_n=\sqrt{n(n+N-2)}$, and the expansion of a harmonic function h is as before. As to biharmonic functions, $s(r)=-\frac{1}{2}(\log r)^2$, $\tau(r)=-\frac{1}{6}(\log r)^3$, both for every N , and

$$u_{nm} = -\frac{1}{2p_n} r^{p_n} \log r \cdot S_{nm}, \quad v_{nm} = \frac{1}{2p_n} r^{-p_n} \log r \cdot S_{nm}.$$

With this notation, there is again no change in the expansion of a biharmonic function u .

If some $a_{nm} \neq 0$, we have for $\varphi_t = \varrho_t(r)S_{nm}$, $\varrho_t(r) = \varrho_1(r+1-t)$,

$$(u, \varphi_t) \sim C \int_t^{t+1} r^{p_n} \log r \cdot r^{-1} \varrho_t(r) dr \sim Ct^{p_n-1} \log t, \quad (1, |\varphi_t|) = O(t^{-1})$$

as $t \rightarrow \infty$. Therefore $a_{nm}=0$ for $p_n-1 > -1$, that is, for all n, m . That $c_{nm}=0$ for all n, m is concluded in the same manner.

Now choose $\varrho_t(r) = \varrho_1(r/t)$, $t \rightarrow 0$. If some $b_{nm} \neq 0$, then

$$(u, \varphi_t) \sim Ct^{-p_n} \log t, \quad \text{and} \quad (1, |\varphi_t|) = O(1).$$

Thus all n with $-p_n < 0$ are ruled out, and we have $b_{nm}=0$ for all n, m . Similarly all $d_{nm}=0$.

The function u again reduces to the radial terms of its expansion, and as before we infer that u is a constant.

We have established:

THEOREM 3. *For every N , $O_G^N \cap O_{H^2B}^N \neq \emptyset$.*

7. We may combine our results in the following form:

THEOREM 4. *The totality $\{R^N\}$ of Riemannian N -manifolds decomposes, for every N , into the four disjoint nonempty classes*

$$\{R^N\} = O_G^N \cap O_{H^2B}^N + O_G^N \cap \tilde{O}_{H^2B}^N + \tilde{O}_G^N \cap O_{H^2B}^N + \tilde{O}_G^N \cap \tilde{O}_{H^2B}^N.$$

We append a bibliography of recent work in the field.

BIBLIOGRAPHY

- [1] KWON, Y. K., SARIO, L., and WALSH, B., *Behavior of biharmonic functions on Wiener's and Royden's compactifications*, Ann. Inst. Fourier (Grenoble) 21 (1971), 217–226.
- [2] NAKAI, M., *Dirichlet finite biharmonic functions on the plane with distorted metrics* (to appear).
- [3] NAKAI, M. and SARIO, L., *Biharmonic classification of Riemannian manifolds*, Bull. Amer. Math. Soc. 77 (1971), 432–436.
- [4] ——, *Quasiharmonic classification of Riemannian manifolds*, Proc. Amer. Math. Soc. 31 (1972), 165–169.
- [5] ——, *Dirichlet finite biharmonic functions with Dirichlet finite Laplacians*, Math. Z. 122 (1971), 203–216.
- [6] ——, *A property of biharmonic functions with Dirichlet finite Laplacians*, Math. Scand. (to appear).
- [7] ——, *Existence of Dirichlet finite biharmonic functions*, Ann. Acad. Sci. Fenn. (to appear).
- [8] ——, *Existence of bounded biharmonic functions*, J. Reine Angew. Math. (to appear).
- [9] ——, *Existence of bounded Dirichlet finite biharmonic functions*, Ann. Acad. Sci. Fenn. Ser. A.I. No. 505 (1972), 1–12.
- [10] ——, *Biharmonic functions on Riemannian manifolds*, Continuum Mechanics and Related Problems of Analysis, Nauka (to appear).
- [11] O'MALLA, H., *Dirichlet finite biharmonic functions on the unit disk with distorted metrics*, Bull. Amer. Math. Soc. (to appear).
- [12] SARIO, L., *Lectures on biharmonic and quasiharmonic functions on Riemannian manifolds*, Duplicated lecture notes 1968–70, University of California, Los Angeles.
- [13] ——, *Riemannian manifolds of dimension $N \geq 4$ without bounded biharmonic functions* (to appear).
- [14] SARIO, L. and NAKAI, M., *Classification Theory of Riemann Surfaces*, Springer, 1970, 446 pp.
- [15] SARIO, L. and WANG, C., *The class of (p, q) -biharmonic functions*, Pacific J. Math. (to appear).
- [16] ——, *Counterexamples in the biharmonic classification of Riemannian manifolds* (to appear).
- [17] ——, *Generators of the space of bounded biharmonic functions*, Math. Z. (to appear).
- [18] ——, *Quasiharmonic functions on the Poincaré N -ball* (to appear).
- [19] ——, *Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball* (to appear).
- [20] ——, *Negative quasiharmonic functions* (to appear).
- [21] SARIO, L., WANG, C., and RANGE, M., *Biharmonic projection and decomposition*, Ann. Acad. Sci. Fenn., Ser. A.I. No. 494 (1971), 1–14.
- [22] WANG, C. and SARIO, L., *Polyharmonic classification of Riemannian manifolds*, Kyoto Math. J. 12 (1972), 129–140.

Received 6th June 1972