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Homotopy Groups of H-Spaces I

by John R. Harper [1])

Introduction

This paper is devoted to a study of the structure of the homotopy groups of H-spaces (Hopf spaces) having the homotopy type of finite CW complexes. The principal motivation is the discovery, beginning with Hilton and Roitberg [14] of "new" H-spaces. The proliferation of further such examples constructed by means of localization techniques naturally leads one to ask if the homotopy groups show any regular features. A secondary motivation is the desire to understand the structure of the homotopy groups of Lie groups by means of algebraic topology. In this respect, we are following a trail initially discovered by Hopf [5].

Before describing the new results, it is worthwhile to mention a few earlier contributions along these lines. In this paragraph X will always denote an H-space having the homotopy of a finite CW complex. First, the Cartan-Serre theorem [18] relates the rank of $\Pi_n(X)$ to the rational homology $H_*(X; Q)$ by means of the Hurewicz homomorphism. The rational homology is known, Hopf [15]. The Cartan-Serre theorem yields that $\Pi_{2n}(X)$ is a finite group. Second, there is a theorem of W. Browder [5] that $\Pi_2(X)=0$. For Lie groups this fact is due to E. Cartan. Third, there is a result of A. Clark [9] that for simply connected associative X, $\Pi_3(X)\neq 0$ and in fact has an infinite cyclic direct summand. This result is related to a theorem of Bott [3] that for compact simply connected Lie groups G, $\Pi_3(G)$ is free abelian. A notable feature in the proofs of the results for H-spaces is the role played by the Hurewicz homomorphism. In much of this paper we continue to focus our attention on the Hurewicz map, but study it in the context of the exact sequence of J. H. C. Whitehead [27].

In a subsequent paper we obtain further results by use of a spectral sequence of Massey and Peterson [21] extending into the unstable range the techniques of Adams [1]. An announcement of some of this work is [11].

This work was initiated while the author was on leave from the University of Rochester, visiting Pontificia Universidade Católica in Rio de Janeiro. It is a pleasure to acknowledge the friendly reception by my colleagues there and especially the efforts of Professors Alberto Azevedo and João Pitombeira de Carvalho who made my visit most pleasant.

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1. Statement of Results

We assume our spaces are connected. We call an H-space *finite* if it has the homotopy type of a finite CW complex. In some of the results we refer to a direct sum of cyclic groups. We do *not* rule out the case that the sum is zero, but avoid further mention of this. We shall also make use of the following well known result [2], [5]. Let X be a finite H-space, and $\tilde{X}$ the fibre of the canonical map

$$X \to K(\Pi_1(X), 1);$$

then $\tilde{X}$ is a finite H-space. In the theorems stated below, X always denotes a finite H-space.

THEOREM 1.1. *$\Pi_4(X)$ is a direct sum of cyclic groups of order 2; furthermore*

$$\dim \Pi_4(X) = \dim \ker Sq^2 : H^3(\tilde{X} : Z_2) \to H^5(\tilde{X} : Z_2).$$

We shall say a space Y is *torsion free* if $H_*(Y; Z)$ is torsion free. For simply connected finite H-spaces X with ΩX torsion free, 1.1 is contained in Bott-Samelson [4].

THEOREM 1.2 *Let X be simply connected and suppose $\Pi_3(X)$ is torsion free. The following sequence is exact:*

$$0 \to \Pi_4(X) \xrightarrow{\eta^*_4} \Pi_5(X) \xrightarrow{h_5} H_5(X; Z) \xrightarrow{\nu_4} \Pi_3(X) \otimes Z_2 \xrightarrow{\eta^*_3} \Pi_4(X) \to 0.$$

Moreover, if ΩX is torsion free, $\ker h_5 = \operatorname{tors} \Pi_5(X)$.

Here h_5 denotes the Hurewicz map and $\operatorname{tors} A$ refers to the torsion subgroup of A. The other maps are defined in section 2.

THEOREM 1.3. *Suppose $\Pi_3(X)$ is torsion free. Let p be a prime. If $p \geqslant 5$, $\Pi_6(X)$ is p-torsion free. The 3-torsion of $\Pi_6(X)$ is of order at most 3 and the 2-torsion of order at most 4.*

THEOREM 1.4. *Suppose either $\Omega\tilde{X}$ is torsion free or $\tilde{X}$ is p-torsion free. Let p be an odd prime. Then*

a) *if $n < 2p$, $\Pi_n(X)$ is p-torsion free*
b) *the p-torsion of $\Pi_{2p}(X)$ has order at most p*
c) $\dim \Pi_{2p}(X) \otimes Z_p = \dim \ker \mathscr{P}^1 : H^3(\tilde{X}; Z_p) \to H^{2p+1}(\tilde{X}; Z_p)$
d) *$\Pi_{2p+1}(X)$ is p-torsion free.*

The final result is aimed at understanding the structure of tors $\Pi_n(X)$. As a crude approximation we make the following definition. Given an integer n, let $P(n)$ denote the set of primes p such that $\Pi_n(X)$ has non-zero p-torsion for some X. Then parts (a) and (d) of 1.4 assert that for X satisfying the hypotheses of 1.4.

$$p \in P(n) \quad \text{implies} \quad n \geqslant 2p$$

$$p \notin P(2p+1).$$

THEOREM 1.5. *Let p be an odd prime. Then* $p \in P(2p+2m)$ *and* $p \in P(4p-4+m)$ *for all* $m \geqslant 0$.

The values of n for which the question "does $p \in P(n)$" is open (for X with $\Omega\tilde{X}$ torsion free) are the $p-3$ odd numbers

$$2p+3, \quad 2p+5, \ldots, 4p-5; \qquad p \quad \text{odd}$$

and apparently large values of n for $p=2$. Using $S^3 \times S^7$ and results of Toda [25] we have $2 \in P(n)$ for $4 \leqslant n \leqslant 26$.

The reader will have noted the frequent use of the hypothesis "ΩX is torsion free". This seems essential in many of our proofs. It is a theorem of Bott [3] that for compact simply connected Lie groups G, ΩG is torsion free. Whether or not Bott's result extends to finite H-spaces seems an especially sensitive point at which to study the relation of finite H-spaces with Lie groups.

2. The Whitehead Sequence

In this section we prove the first three results of section 1. The focal point of our arguments is the exact sequence of J. H. C. Whitehead [27]. We also cite Hilton [12] as a source for many useful facts about this sequence and related homotopy theory. The sequence is:

$$\cdots \to H_{n+1}(Y; Z) \xrightarrow{\nu_n} \Gamma_n(Y) \xrightarrow{\lambda_n} \Pi_n(Y) \xrightarrow{h_n} H_n(Y; Z) \to \cdots$$

where we assume Y is simply connected, and a CW complex. The group $\Gamma_n(Y)$ is defined by

$$\Gamma_n(Y) = \operatorname{im} \Pi_n(Y^{n-1}) \to \Pi_n(Y^n)$$

where Y^k is the k-skeleton. The map λ_n is induced by the inclusion $Y^n \subset Y$, h_n is the Hurewicz map and ν_n is a certain connecting homomorphism. The following fact is useful:

LEMMA 2.1 ([12], [27]). *If Y is $n-1$-connected, then $\Gamma_{n+1}(Y)\cong H_n(Y:Z_2)$ and v_{n+1} is the composition*

$$H_{n+2}(Y;Z)\xrightarrow{r} H_{n+2}(Y;Z_2)\xrightarrow{Sq^2_*} H_n(Y:Z_2)$$

where r is reduction mod 2, *and Sq^2_* is adjoint to Sq^2.*

For the structure of the low dimensional groups, we need:

LEMMA 2.2. *Let X be a simply connected finite* H-*space; then $H_4(X:Z)=0$.*

Proof. We assume X is a CW complex. From Browder [5] we have X 2-connected. Let $x\in H_4(X:Z)$ be non-zero. From Hopf [15] we have $x\in \text{tors}\, H_4$. Hence there is a prime p and non-zero elements $y\in H_4(X:Z_p)$, $z\in H_5(X:Z_p)$ with

$$d^r\{z\}=\{y\}$$

in the homology Bockstein spectral sequence of X. But $\{y\}$ is primitive fo rdimensional reasons. Thus by Theorem 6.1 of [5], $\{y\}$ has infinite implications contradicting the finiteness of X.

We remark that 2.2 is an immediate corollary of a result of Weingram [26]. Weingram proves that for finite H-spaces, h_{2n} is the 0 map. This will certainly be one of the useful tools for studying the structure of the higher homotopy groups. In this work we try to avoid using it, because it is interesting to see how this fact is consequence of other properties of H-spaces, at least in low dimensions. We say some more along these lines in the last section.

We now prove Theorem 1.1. Applying 2.2 to $\tilde{X}$ yields:

$$\Gamma_4(\tilde{X})\xrightarrow{v_4}\Pi_4(\tilde{X})\to 0.$$

From 2.1 we get Π_4 as a direct sum of cyclic groups of order 2 and

$$\dim \Pi_4=\dim \operatorname{coker} v_4.$$

From 2.2 it follows that r is epic in the composition for v_4, hence

$$\dim \operatorname{coker} v_4=\dim \operatorname{coker} Sq^2_*=\dim \ker Sq^2.$$

We now turn to the details of Theorem 1.2. We let

$$n_k: S^{k+1}\to S^k, \quad k\geqslant 3$$

denote the essential map. By means of the Hurewicz Theorem and 2.1 we identify Γ_4 with $\Pi_3\otimes Z_2$ for 2-connected spaces. This explains the maps in the sequence of

1.2. We need some lemmas. For the remainder of this section we assume that $\Pi_3(X)$ *is torsion free*. In view of Bott's results this does not exclude any Lie group. Moreover, it is a property of all known finite H-spaces. It is also understood that X is a finite H-space.

LEMMA 2.3. *The 5-skeleton $\tilde{X}^5$ is a bouquet of types S^3, $S^3 \cup_{\eta_3} e^5 : S^5$.*
Proof. Inspection of Hilton [12] p. 129.
For 1.2 the relevant segment of the Whitehead sequence is

$$H_6 \xrightarrow{\nu_6} \Gamma_5 \xrightarrow{\lambda_5} \Pi_5 \xrightarrow{h_5} H_5 \xrightarrow{\nu_4} \Gamma_4 \xrightarrow{\lambda_4} \Pi_4 \to 0.$$

LEMMA 2.4. $\operatorname{Tors} H_6(\tilde{X}:Z)=0$.
Proof. Suppose not. Then there is a prime p and non-zero elements $x \in H_6(\tilde{X}:Z_p)$, $y \in H_7(\tilde{X}:Z_p)$ with

$$d^r\{y\} = \{x\}.$$

Since Π_3 is torsion free, we have $H_4(\tilde{X}:Z_p)=0$, hence $\{y\}$ is primitive for dimensional reasons. The differential yields infinite implications, a contradiction.

LEMMA 2.5. *ν_6 is monic, hence $h_6=0$.*
Proof. Since Π_6 is finite, this follows from 2.4.

LEMMA 2.6. $\lambda_5 \Pi_5 \cong \Pi_4$.
Proof. From 2.3 we obtain $\tilde{X}^4$ is a bouquet of spheres S^3. We write

$$\tilde{X}^4 = \bigvee_{i=1}^{n} S_i^3,$$

A formula of Chang [8] or [13] gives

$$\Pi_5(\tilde{X}^4) = \sum_{i=1}^{n} \Pi_5(S_i^3) + \sum_{1 \leqslant i < j \leqslant n} [S_i^3, S_j^3]$$

where $[S^3, S^3]$ denotes an infinite cyclic group generated by a Whitehead product in $\Pi_5(S^3 \vee S^3)$. From 2.4, 2.5, the fact that $\Pi_5(S^3)$ is finite and that λ_5 annihilates Whitehead products we obtain

$$\nu_6 H_6 = \sum_{i,j} [S_i^3, S_j^3].$$

Now $\Pi_5(S^3)=Z_2$ generated by $\eta_3 \circ \eta_4$, hence $\lambda_5 \Gamma_5$ is a direct sum of groups Z_2, one summand for each sphere S_i^3 on which a 5-cell is not attached. Since $S^3 \cup_{\eta_3} e^5$ carries Sq^2 non-trivially, the lemma follows from the dimension formula in 1.1.

We also note that the above argument yields

LEMMA 2.7. *The inclusion* $0 \to \lambda\Gamma_5 \to \Pi_5$ *is the monomorphism* $0 \to \Pi_4 \xrightarrow{\eta_{4*}} \Pi_5$.

LEMMA 2.8. *The identification* $\Gamma_4 \cong \Pi_3 \otimes Z_2$ *yields the commutative diagram*

$$\begin{array}{ccc} \Gamma_4 & \xrightarrow{\lambda_4} & \Pi_4 \\ & \searrow\!\!\!\searrow \quad \nearrow_{\eta^*_3} & \\ & \Pi_3 \otimes Z_2 & \end{array}$$

Proof. Since $\tilde{X}^4 = \tilde{X}^3$ and η_3 generates $\Pi_4(S^3)$, the result is immediate.

At this point we have obtained the exact sequence of 1.2. We now look at $\ker h_5$. Of course by Cartan-Serre [18] we have

Remark 2.9. $\ker h_n \subset \operatorname{tors} \Pi_n$. Thus we are interested in the opposite inclusion for $n = 5$. Essentially the argument is that if h_5 maps a torsion element in a non-zero manner, then ΩX is not torsion free. The following lemma is useful.

LEMMA 2.10. *Let X be simply connected and ΩX torsion free. Then* $QH^6(X; Z_2) = 0$ *and*

$$Sq^1 H^5(X: Z_2) = Sq^3 H^3(X: Z_2).$$

Proof. Since $H^*(\Omega X; Q)$ is a polynomial algebra on even dimensional generators, the hypothesis on ΩX implies $H^*(\Omega X; Z_2)$ is 0 in odd dimensions. Let $x \in H^6(X; Z_2)$. In the Serre spectral sequence for $\Omega X \to PX \to X$, x can only be hit by a differential

$$d_3: E_3^{3,2} \to E_3^{6,0}.$$

Since $E_2 = E_3$ we obtain $x = y \cdot d_3(z)$ for some $y \in H^3(X; Z_2)$ and $z \in H^2(\Omega X; Z_2)$. Now let $u \in H^5(X; Z_2)$. Dimensional considerations imply u is primitive. Hence $Sq^1 u$ is primitive and decomposable. It follows that $Sq^1 u \in Sq^3 H^3$. The opposite inclusion $Sq^3 H^3 \subset Sq^1 H^5$ follows from the Adem relation $Sq^3 = Sq^1 Sq^2$.

LEMMA 2.11. *Under the hypotheses of* 2.10, *the* 2-*torsion of* $H^6(X; Z)$ *is of order at most* 2.

Proof. Suppose not. Then there are elements $x \in H^5(X; Z_2)$ and $y \in H^6(X; Z_2)$ related by a higher order Bochstein β by

$$y = \beta x \bmod Sq^1 H^5.$$

Since x is primitive, y is primitive modulo

$$Sq^1 H^5 (X \times X, X \vee X; Z_2)$$

which is easily seen to be 0. From 2.10 we obtain

$$y = Sq^3 z = Sq^1 Sq^2 z .$$

Thus $0 \in \beta x$, a contradiction.

Before giving the next lemma, we quote a result which is used not only in its proof but frequently in other parts of this paper.

THEOREM 2.12 (Browder [6]). *Suppose X is simply connected and ΩX is torsion free, then $H_i(X; Z_p)$ is p-torsion free for $i \leqslant 2p$, p a prime.*

In fact Browder proves much more in this direction, but 2.12 suffices for our needs.

In view of 2.9, the rest of 1.2 will follow from

LEMMA 2.13. *If X is simply connected and ΩX torsion free, then* $h_5 \operatorname{tors} \Pi_5(X) = 0$.

Proof. Combining 2.11 and 2.12 we see that $\operatorname{tors} H_5$ is a direct sum of cyclic groups of order 2. Suppose $\alpha \in \operatorname{tors} \Pi_5$ with $h_5(\alpha) \neq 0$. Then $h_5(\alpha)$ generates some cyclic subgroup of H_5 of order 2. Hence we can find elements $x \in H_5(X:Z_2)$, $y \in H_6(X:Z_2)$ such that

$$Sq^1_* y = x$$

and

$$x = \operatorname{mod} 2 \text{ image of } h_5(\alpha) .$$

This latter statement implies

$$Sq^2_* x = 0$$

But $Sq^2_* x = Sq^3_* y$ by the Adem relations. Now let $x' \in H^5(X; Z_2)$, $y' \in H^6(X:Z_2)$ be elements such that

$$\langle x', x \rangle \neq 0, \qquad y' = Sq^1 x' .$$

By 2.10, there is $z \in H^3(X:Z_2)$ such that

$$Sq^1 x' = Sq^3 z .$$

Thus

$$0 \neq \langle x', x\rangle = \langle x', Sq^1_* y\rangle = \langle Sq^1 x', y\rangle$$
$$= \langle Sq^3 z, y\rangle = \langle z, Sq^2_* x\rangle,$$

which implies $Sq^2_* x \neq 0$, a contradiction.

We conclude this section with a proof of Theorem 1.3. Looking at the Whitehead sequence and using 2.5 (since Π_3 is assumed to be torsion free), we obtain

$$\Gamma_6 \xrightarrow{\lambda_6} \Pi_6 \to 0.$$

The following information is contained in Hilton [12].

LEMMA 2.14. *$\Pi_6(S^3)=Z_{12}$, $\Pi_6(S^3 \cup_{\eta_3} e^5)=Z_6$, and $\Pi_6(S^5)=Z_2$.*

By 2.3 these are the only groups contributing to Γ_6 via $\lambda\Gamma_6$, since "cross terms" in $\Pi_6(\tilde{X}^5)$ are annihilated by λ_6 in $\Pi_6(\tilde{X}^6)$.

In greater detail we can proceed as follows. Choose the basepoint of $\tilde{X}$ to be the homotopy identity. Suppose that the k-skeleton $\tilde{X}^k$ can be expressed

$$\tilde{X}^k = Y \vee Z$$

where the wedge is at the identity. Then the inclusion $\tilde{X}^k \subset \tilde{X}$ determining λ can be factored through the folding map F

$$Y \vee Z \to \tilde{X} \vee \tilde{X} \xrightarrow{F} \tilde{X}.$$

Let $m: \tilde{X} \times \tilde{X} \to \tilde{X}$ be the multiplication. Then we have a commutative diagram

$$\begin{array}{ccccccc}
0 \to \Pi_{k+2}(Y \times Z, Y \vee Z) & \to & \Pi_{k+1}(Y \vee Z) & \to & \Pi_{k+1}(Y \times Z) & \to 0 \\
 & & \downarrow & & \downarrow & \\
 & & \Pi_{k+1}(\tilde{X} \vee \tilde{X}) & \to & \Pi_{k+1}(\tilde{X} \times \tilde{X}) & \\
 & & \downarrow F_* & \swarrow m_* & & \\
 & & \Pi_{k+1}(\tilde{X}) & & &
\end{array}$$

in which the top row is split exact. Thus λ annihilates the "cross terms", those coming from Π_{k+2}.

3. The 5-skeleton and $\dim \Pi_4$

In this section we consider some further details about the structure of Π_4. The first result is a Peterson-Stein formula which might be of use in applying 1.1. In fact, we use it to show that no principal $SO(3)$ bundle over S^5 is an H-space. This question was asked by G. Mislin in connection with his work on low dimensional H-spaces. The second result of this section concerns $\mathrm{im}\, h_5$.

Recall that $\tilde{X}$ was constructed so as to yield a fibration

$$\tilde{X} \xrightarrow{p} X \xrightarrow{f} K(\Pi_1, 1).$$

Browder [7] uses this fibration to calculate the cohomology of $\tilde{X}$. For the group Z_2, he obtains (additively)

$$H^*(\tilde{X}; Z_2) \cong H^*(X; Z_2)/I \otimes E$$

where I is the ideal generated by $\mathrm{im}\, f^*$ and E is a certain exterior algebra on generators of dimension $2^i - 1$, $i \geqslant 2$. The explicit calculations of [7] along with the definition of functional cohomology operations by means of universal examples, e.g. [22], yield

LEMMA 3.1. *Let* $x \in H^{2^i-1}(\tilde{X}; Z_2)$, *x not in* $\mathrm{im}\, p^*$. *Then there exists* $y \in H^1(\tilde{X}; Z_2)$ *such that*

$$x \in Sq^k \ldots Sq^2 Sq^1_p(y), \quad k = 2^{i-1} - 1.$$

This fact is contained in Peterson-Stein [22].

Let φ be the stable secondary cohomology operation associated with the relation

$$Sq^2(Sq^2 Sq^1) = 0.$$

Then a Peterson-Stein formula [22] immediately yields

PROPOSITION 3.2. *Let* $x \in H^3(\tilde{X}; Z_2)$, *x not in* $\mathrm{im}\, p^*$. *Let* $y \in H^1(X: Z_2)$ *be the element given by* 3.1. *Then*

$$Sq^2 x = p^* \varphi(y) \text{ modulo } p^* Sq^2 H^3(X; Z_2).$$

As an application of 3.2, we have

PROPOSITION 3.3. *No principal* $SO(3)$*-bundle over* S^5 *is an* H*-space.*

Proof. Such bundles are classified by $\alpha \in \Pi_4(SO(3)) = Z_2$. Denote the total space by E. If $\alpha = 0$, then $E \simeq SO(3) \times S^5$ and is not an H-space. If $\alpha \neq 0$, then the homotopy exact sequence yields $\Pi_4(E) = 0$. Now, as algebras over Z_2,

$$H^*(E) = Z_2[x_1, x_5]/(x_1^4, x_5^2)$$

and

$$H^*(\tilde{E}) = E[x_3, p^*(x_5)]$$

where E denotes an exterior algebra and x_3 is determined by the relation $x_1^4=0$. Since $\Pi_4=0$, we have $Sq^2x_3=p^*(x_5)$. From 3.2 it follows that

$$x_5 = \varphi(x_1) \bmod Sq^2H^3(E).$$

But $Sq^2H^3(E)=0$. Since φ is a stable operation and x_1 is primitive, this absence of indeterminacy implies x_5 is primitive. Thus $H^*(E)$ is primitively generated, so a result of E. Thomas [23] yields

$$x_5 \in Sq^2H^3(E)$$

a contradiction. Thus E is not an H-space.

We now turn to the exact sequence of Theorem 1.2 and particularly $\operatorname{im} h_5$.

PROPOSITION 3.4. *Let X be a simply connected finite* H-*space with ΩX torsion free. The following statements are equivalent:*

a) $\operatorname{im} h_5=2H_5(X;Z)$;

b) $\dim \Pi_4(X)=\dim H_3(X;Z_2)-\dim H_5(X;Z_2)$;

c) *the* 5-*skeleton of X is a bouquet of types S^3 and $S^3\cup_{\eta_3} e^5$*;

d) $H^5(X;Z_2)=Sq^2H^3(X;Z_2)$.

Proof. a)$\Rightarrow$b): As in the proof of 2.13 we write

$$H_5(X\ \ Z) = F \oplus T$$

where F is free and T is a direct sum of groups Z_2. Then $\operatorname{im} h_5=2F$ and $H_5(X;Z_2)=F\otimes Z_2\oplus T$ is mapped monomorphically by v_4.

b)$\Rightarrow$c): We are ruling out types S^5 in the 5-skeleton in view of 2.3. The only way an S^5 could appear in a non-trivial way and be compatible with b) is by having a 6-cell attached to X^5 by a map having odd degree on the S^5 in question. But this yields odd torsion in H_5, contradicting 2.12.

c)$\Rightarrow$d): Immediate.

d)$\Rightarrow$a): As in 2.1 we have $\operatorname{im} h_5=\ker r=2H_5$.

CONJECTURE 3.5. *If X is* 3-*connected (and possibly require ΩX torsion free) then X is* 6-*connected.*

The statements of 3.4 are true if $H^*(X;Z_2)$ is primitively generated by a result of Thomas [23]. Then 3.5 is true as noted by Thomas in [24]. In a sequel to this paper we intend to establish the truth of the statements in 3.4.

4. Odd Torsion in the Homotopy Groups

In this section we supply proofs of 1.4 and 1.5. We make use of the technique of localization as developed in [20] by Mimura, Nishida and Toda.

Since $\tilde{X}$ is a finite H-space, it is enough to work only with simply connected finite H-spaces to prove our assertions. Hence in this section X will always denote a *simply connected* finite H-space. We use Q_p to denote the integers localized at p and continue to use Z_p for Z/pZ. The space X_p is X localized at p. Without explicit mention to the contrary, p is assumed to be an *odd prime*.

Our program in this section is as follows. Given a suitable space X, there is a space X_p and a map

$$l: X \to X_p$$

with the property that the maps in homotopy and homology (with integer coefficients) induced by l localize these objects; that is

$$l_*: \Pi_*(X) \to \Pi_*(X_p)$$

and

$$l_*: H_*(X; Z) \to H_*(X_p; Z),$$

carry isomorphisms

$$\Pi_*(X) \otimes Q_p \cong \Pi_*(X_p)$$

and

$$H_*(X; Z) \otimes Q_p \cong H_*(X_p; Z).$$

We obtain some specific information about the cohomology of X and X_p in order to build a few stages of the Postnikov system of X_p. From this we can read off information about the p-torsion in $\Pi_*(X)$. By this means 1.4 is obtained. Theorem 1.5 is proved by exhibiting examples. Here a simple application of the mixing method is used.

We go to the details. First some analogues of 2.10 and 2.11 are needed. In Lemmas 4.1, 4.2 and 4.3 we assume ΩX *is torsion free.*

LEMMA 4.1. *$\beta H^{2p+1}(X; Z_p) = \beta\mathscr{P}^1 H^3(X; Z_p)$ modulo decomposables.*

Proof. From 2.12 and universal coefficients we obtain that $H^i(X; Z)$ is p-torsion free for $i \leqslant 2p+1$. Hence in dimensions $\leqslant 2p+1$, $H^*(X; Z_p)$ is isomorphic as an algebra to an exterior algebra on odd dimensional generators

$$E[x_1, \ldots, x_k].$$

In the Serre spectral sequence for $\Omega X \to PX \to X$ we have elements $y_i \in H^*(\Omega X; Z_p)$ transgressing to the x_i. If $x_i \in H^{2p+1}(X; Z_p)$ is decomposable, then $\beta x = 0$ for dimensional reasons. Thus we need consider the action of β only on algebra generators in H^{2p+1}. Let y_k transgress to $x_k \in H^{2p+1}$ and assume $\beta x_k \neq 0$. Since $\beta y_k = 0$ we have βx_k hit by some differential starting off the edge of the spectral sequence. However, inspection of the spectral sequence shows that in the range we are considering, the only non-edge transgressive elements are those given by the Kudo transgression theorem [16], that is

$$d_{2p-1}(-y^{p-1} \otimes w) = \beta \mathscr{P}^1 w$$

where

$$d_2 y = w, \quad y \in H^2(\Omega X), \quad w \in H^3(X).$$

This gives the lemma.

LEMMA 4.2. *The p-torsion of $H^{2p+2}(X; Z)$ has order at most p.*

Proof. Suppose there is higher torsion. Then we have elements $x \in H^{2p+1}(X; Z_p)$, $y \in H^{2p+2}(X; Z_p)$ such that in the cohomology Bockstein spectral sequence $E_*(X)$

$$\{y\} = d_r\{x\}, \quad \text{some} \quad r \geqslant 2.$$

Since $E_1 = E_\infty$ in dimensions $\leqslant 2p$ and d_r is a map of Hopf algebras, it follows that $\{y\}$ is primitive. Since $\{y\}$ has degree $2p+2$, the Milnor-Moore sequence [18] yields that $\{y\}$ is indecomposable. In terms of cohomology operations we have

$$y = \beta_r x \bmod \beta H^{2p+1}$$

where β_r is the r-th order Bockstein, and y is indecomposable. Since ΩX is torsion free, $H^*(\Omega X; Z_p)$ is 0 in odd dimensions. Then the same argument as in 4.1 gives

$$y = \beta \mathscr{P}^1 w \text{ mod decomposables}.$$

Since r$\geqslant$2, we obtain $\{y\}$ is decomposable, a contradiction.

LEMMA 4.3. *Let $x \in H^{2p+2}(X; Z_p)$ be indecomposable. Then $x = \beta \mathscr{P}^1 w$* mod *decomposables for some $w \in H^3(X; Z_p)$.*

Proof. From [5] we have $\beta x = 0$. Since $E_\infty(X)$ in the cohomology Bockstein spectral sequence is an exterior algebra on odd dimensional generators, we have

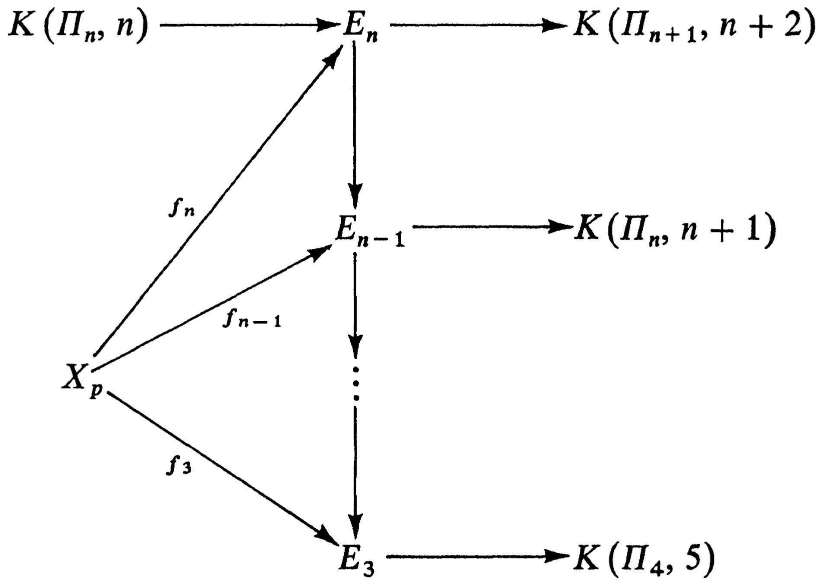
$$\{x\} = \sum \{u_i\} \{v_i\}$$

in E_∞ and hence in E_2 by 4.2. Hence there is a $z \in H^{2p+1}$ such that

$$x - \sum u_i v_i = \beta z .$$

We now turn to the Postnikov system of X_p. In the range of dimensions being considered, the calculations are with simple (essentially) 2-stage systems. The arguments proving 1.4 are pitched to the hypothesis that ΩX is torsion free. If X is p-torsion free, then $H^*(X; Z_p)$ is an exterior algebra on odd dimensional generators and the problems (for which 4.1–4.3 are used) don't arise.

We index the system as pictured, where f_n^* is an isomorphism in dimension $\leqslant n$ and a monomorphism in dimension $n+1$.



We need the following facts in our study of the k-invariants. Let G be an abelian groups. From 2.12 and Hopf's Theorem we obtain

$$H^i(X; G) \cong H^i(S^{n_1} \times \cdots \times S^{n_k}; G) \quad \text{for} \quad i \leqslant 2p, \quad \text{where} \quad n_i \quad \text{is odd}. \tag{1}$$

Recall that G is *p-local* if $G \cong G \otimes Q_p$. If G is p-local and m odd we have

$$H^i(Q_p, m; G) = 0 \quad \text{for} \quad m < i < m + 2(p-1). \tag{2}$$

Proof (2). We know $H_m(Z, m; Z) = Z$ and $H_i(Z, m; Z) \otimes Q_p = 0$ for i as in (2). Thus for $i > m+1$, (2) follows from the fact that

$$K(Q_p, m) \cong K(Z, m)_p ,$$

the fundamental localization property and universal coefficients. When $i=m+1$, we obtain

$$H^{m+1}(Q_p, m; G) \cong \mathrm{Ext}(Q_p, G).$$

Let

$$0 \to G \to A \to Q_p \to 0$$

be a representative extension. If we tensor the sequence with Q_p, then the facts that tensoring with Q_p preserves exactness, and that G is p-local, along with the 5-lemma yield that A is p-local, and hence a Q_p-module. Thus the sequence splits, and we obtain $\mathrm{Ext}(Q_p, G)=0$, giving (2).

We shall also need

$$H^{2p+1}(Q_p, 3; Z_p) \cong Z_p \quad \text{generated by } \mathscr{P}^1 \iota_3 \tag{3}$$

$$H^{2p+1}(Q_p, 3; Z_{p^k}) \simeq Z_p \tag{4}$$

$$H^{2p+2}(Q_p, 3; Z_p) \cong Z_p \quad \text{generated by } \beta\mathscr{P}^1 \iota_3 . \tag{5}$$

These could be obtained as above. However, the following lemma yields them instantly and is also useful for other purposes.

LEMMA 4.4. *Let G be a finite abelian group. Then $l: X \to X_p$ induces a natural isomorphism $\tilde{H}^*(X_p; G) \cong \tilde{H}^*(X_p; G \otimes Q_p) \cong \tilde{H}^*(X; G \otimes Q_p)$. If $G=Z_p$, the isomorphism is as algebras over the Steenrod algebra.*

As this fact is probably well known, we defer a proof to the appendix.

We now construct part of the Postnikov system of X_p.

LEMMA 4.5. *For $n \leqslant 2p-1$, E_n is a product of $K(Q_p, n_i)$ where the $n_i (\leqslant 2p-1)$ are as in* (1).

Proof. The statement is true for $n=3$. We use induction on the height of the system. Suppose $n<2p-1$ and E_n decomposes as stated. From (2) and the fact that the k-invariants are primitive, it follows that the k-invariant

$$E_n \to K(\Pi_{n+1}, n+2)$$

is trivial. Thus $E_{n+1} \simeq E_n \times K(\Pi_{n+1}, n+1)$, and f_{n+1}^* is an isomorphism in dimension $n+1<2p$, and monic in dimension $n+2 \leqslant 2p$. Thus (1) and 4.4 imply that Π_{n+1} is torsion free; that is, Π_{n+1} is a direct sum of groups Q_p. The induction can proceed.

Note that 4.5 gives part (a) of 1.4, i.e. Π_n is p-torsion free for $n<2p$.

Let F be the fibre of the map

$$K(Q_p, 3) \xrightarrow{\mathscr{P}^1 \iota_3} K(Z_p, 2p+1).$$

LEMMA 4.6. *E_{2p} is a product of spaces of types $K(Q_p, n_i)$ and F.*

Proof. We decompose the k-invariant by expressing Π_{2p} as a direct sum of cyclic groups

$$E_{2p-1} \xrightarrow{k_j} X_j K(Z_{p^k}, 2p+1).$$

We first observe that no $k_j = 0$, for if so, then the factor $K(Z_{p^k}, 2p)$ in E_{2p} would yield via f_{2p}^* an indecomposable

$$x = f_{2p}^*(\iota_{2p}) \in H^{2p}(X_p; Z_p)$$

in violation of (1) and 4.4. Suppose there is some $k_j \neq 0$ associated with $k > 1$. Then from (2) we see that $k_j \in H^{2p+1}(E_3; Z_{p^k})$. Applying (4) we see that $H^{2p+1}(E_3; Z_{p^k})$ is a direct sum of groups Z_p. Hence there is an element

$$y \in H^{2p}(E_{2p}; Z_{p^k})$$

restricting to $p\iota_{2p}$ in the fibre. Thus y is indecomposable, yielding a contradiction as above. Thus $k=1$ and (3) yields $k_j = \mathscr{P}^1 \iota_3$ by making a change of basis in $H^3(E_{2p-1}; Z_p)$ if necessary.

This lemma gives parts (b) and (c) of 1.4. The proof of 1.4 is concluded by

LEMMA 4.7. $\operatorname{Tors} \Pi_{2p+1}(X_p) = 0$.

Proof. We decompose the k-invariant by:

$$E_{2p} \to K(\Pi_{2p+1}, 2p+2) \to K(\operatorname{tors} \Pi_{2p+1}, 2p+2)$$

and express the composite

$$E_{2p} \xrightarrow{k_j} X_j K(Z_{p_k}, 2p+2).$$

If some $k_j = 0$, we obtain via f_{2p+1}^* an element

$$x \in H^{2p+1}(X_p, Z_{p^k})$$

with non-zero k-th order Bockstein, $\beta_k x \neq 0$. Applying 4.1, 4.2 and 4.3 we obtain $k=1$ and

$$\beta x = \beta \mathscr{P}^1 y + \text{decomposables}, \qquad y \in H^3(X_p; Z_p).$$

Let $i: K(Z_p; 2p) \to F$ be the inclusion of the fibre. Since p is odd, a simple calculation yields

$$i^* H^{2p+2}(F; Z_{p^k}) = 0.$$

(This would break down if $p=2$; we could have $Sq^2 \iota_4 \in \operatorname{im} i^*$). Thus using (2), (5) and 4.6 we see that if some $k_j \neq 0$ and we compute with Z_p-coefficients, then we obtain

$$x \in H^{2p+2}(E_{2p+1}; Z_p)$$

restricting to $\beta \iota_{2p+1}$ in the fibre. Applying f_{2p+1}^*, we obtain an indecomposable element

$$f_{2p+1}^*(x) \in H^{2p+2}(X_p; Z_p)$$

which is not in the image of any Bockstein. This contradicts 4.3 (using 4.4).

We remark that the reader familiar with the Massey-Peterson spectral sequence [21] can see that essentially what we are proving is that $E_2 = E_\infty$ in the range being considered. Lemmas 4.1–4.3 provide enough information so that 1.4 is a consequence of calculating the Ext for this spectral sequence. The absence of differentials is essentially for dimensional reasons.

We now prove 1.5. Recall that $P(n)$ is the set of primes p such that $\Pi_n(X)$ has p-torsion for some finite H-space X. We show that for odd primes p

(a) if $n \geqslant p$, then $p \in P(2n)$,

(b) if $n \geqslant 4p-3$, then $p \in P(n)$.

These yield 1.5. Recall [19] that

$${}_p\Pi_{2m+2p}(S^{2m+3}) \cong Z_p$$

and

$${}_p\Pi_{2m+4p-3}(S^{2m+3}) \cong Z_p$$

for all $m \geqslant 0$. Hence mixing $SU(m+2)$ localized at all primes except p with $S^3 \times S^5 \times \cdots \times S^{2m+3}$ localized at p yields (a) and (b).

5. The Hurewicz Map and Finite H-Spaces

In this section we return briefly to the theme of the second section, that the Hurewicz map has a fundamental influence on the structure of the homotopy groups of a finite H-space. We denote the map by

$$h_n\colon \Pi_n(X) \to H_n(X; Z).$$

First we quote a result of Weingram referred to earlier.

THEOREM [26]. *Let X be a finite* H-*space. Then* $\ker h_{2n} = \operatorname{tors} \Pi_{2n}(X)$.
Note that we are combining the earlier statement of Weingram's result with the Cartan-Serre result that Π_{2n} is finite.

It is our feeling that this result is part of a more general phenomenon and make

CONJECTURE 5.1. *Let X be a simply connected finite* H-*space with ΩX torsion free. Then* $\ker h_n = \operatorname{tors} \Pi_n(X)$.

Of course for $n=3$ there is no content. For $n=5$ it is proved in 2.13. To prove 5.1 for $n=7$ we have only to check the prime 2 because 1.4 rules out odd torsion in Π_7. In the remainder of this section X is as in 5.1.

LEMMA 5.2. $QH^8(X; Z_2)=0$.

Proof. Since ΩX is torsion free, $H^*(\Omega X; Z_2)$ is 0 for odd dimensions, and in the Serre spectral sequence for $\Omega X \to PX \to X$ we have $E_2 = E_3$. Thus the only terms from which $E_*^{8,0}$ can be hit are

$$E_3^{5,2} \quad \text{and} \quad E_5^{3,4}.$$

If $u \otimes v \in E_3^{5,2}$, then

$$d_3(u \otimes v) = u \cdot d_3(v)$$

a decomposible in $H^8(X)$. We shall prove

$$E_5^{3,4} = 0.$$

We define elements α_y and β_y in $H^*(\Omega X)$ corresponding to $y \in H^3(X)$. Given $y \in H^3(X)$, α_y is the element in $H^2(\Omega X)$ such that

$$d_3(\alpha_y) = y.$$

If $y^2=0$, then there is an element β_y in $H^4(\Omega X)$ such that

$$d_3(\beta_y) = y \otimes \alpha_y.$$

A Z_2-basis for $H^2(\Omega X)$ is $\{\alpha_y\}$ as y runs over $H^3(X)$. A Z_2-basis for $E_3^{3,4}$ consists of

$$\{y \otimes \alpha_y^2, y' \otimes \alpha_y^2, y \otimes \beta_y, y' \otimes \beta_y, y'' \otimes \alpha_{y'}\alpha_y\}$$

where $y' \neq y$, y'' arbitrary.

Note that for some y, $\alpha_y^2 = 0$ is possible. Then we have

$$d_3(\alpha_{y'}\, \alpha_y^2) = y' \otimes \alpha_y^2$$

$$d_3(\alpha_y^3) = y \otimes \alpha_y^2$$

$$d_3(y'' \otimes \alpha_{y'}\, \alpha_y) = y''y' \otimes \alpha_y + y''y \otimes \alpha_{y'} \neq 0$$

$$d_3(\alpha' \otimes \beta_y) = y'y \otimes \alpha_y \neq 0$$

$$d_3(\alpha_y\beta_y + \alpha_y^3) = y \otimes \beta_y.$$

Hence $E_4^{3,4} = 0$.

Conjecture 5.1 for $n = 7$ now follows from

LEMMA 5.3. *$H_7(X; Z)$ is 2-torsion free.*

Proof. If not, then H^8 has 2-torsion and there are elements

$$x \in H^7(X; Z_2), \qquad y \in H^8(X; Z_2)$$

such that

$$d_r\{x\} = \{y\}$$

in the cohomology Bockstein spectral sequence. However, from 5.2 we can write

$$y = \sum u_i v_i$$

and since $H^4(X; Z_2) = 0$, y is not a square and thus $\{y\}$ is not primitive. But $\{x\}$ is primitive for dimensional reasons, hence $d_r\{x\}$ is primitive, a contradiction.

Appendix: Localization and Cohomology

Here we supply a proof of lemma 4.4.

A routine use of the localization formula and universal coefficients yields (for G finitely generated)

$$H_*(X_p; G) \cong H_*(X; G \otimes Q_p) \qquad \text{A.1.}$$

The situation for cohomology is not as simple. Consider S_0^n, the sphere localized at 0. Then

$$H^n(S_0^n; Z) \cong \mathrm{Hom}(Q, Z) = 0.$$

Wilder things can happen as shown by Eilenberg and MacLane [10] or [17], p. 76. However cohomology does enjoy the analogue of A.1, if G is small. We prove

A.2. *Let G be finite and p a prime. Then the localization maps $l: X \to X_p$ and $\lambda: G \to G \otimes Q_p$ yield isomorphisms*

$$\tilde{H}^*(X_p; G) \underset{\cong}{\overset{\lambda_*}{\to}} \tilde{H}^*(X_p; G \otimes Q_p) \underset{\cong}{\overset{l^*}{\to}} \tilde{H}^*(X; G \otimes Q_p).$$

If $G = Z_p$, then the isomorphism is as algebras over the Steenrod algebra.

Proof. Since G is finite we can write

$$G \cong G \otimes Q_P \oplus H$$

where $H \otimes Q_p = 0$. Since $H_*(X_p; Z)$ is p-local, we have

$$\tilde{H}^*(X_P; H) = 0,$$

hence λ_* is an isomorphism. Let $B = G \otimes Q_p$. From universal coefficients we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_*(X_p; Z), B) & \to & H^*(X_p; B) & \to & \mathrm{Hom}(H_*(X_p; Z), B) & \to 0 \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle \beta} & \\ 0 \to & \mathrm{Ext}(H_*(X; Z), B) & \to & H^*(X; B) & \to & \mathrm{Hom}(H_*(X, Z), B) & \to 0 \end{array}$$

where α and β are induced by l. We also have the following commutative diagram:

$$\begin{array}{ccc} H_*(X; Z) & \overset{j}{\to} & H_*(X; Z) \otimes Q_P \\ \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle l_* \otimes 1} \\ H_*(X_p; Z) & \underset{\cong}{\leftarrow} & H_*(X_p; Z) \otimes Q_p \end{array}$$

where $l_* \otimes 1$ is an isomorphism, and $j(x) = x \otimes 1$. Thus if we show that j induces an isomorphism of the Hom and Ext factors, it follows that α and β are isomorphisms. Then l^* is an isomorphism by the 5-lemma, giving A.2.

Since $H_*(X; Z)$ is a graded abelian group whose components are finitely generated, it suffices to prove the statements below. Let A be finitely generated and B be finite and P-local. Then $j^{\#}$ induces isomorphisms

$$\mathrm{Hom}(A \otimes Q_p, B) \to \mathrm{Hom}(A, B), \qquad \text{A.3.}$$

$\operatorname{Ext}(A \otimes Q_p, B) \to \operatorname{Ext}(A, B)$. A.4.

Proof. Define $k: \operatorname{Hom}(A, B) \to \operatorname{Hom}(A \otimes Q_p, B)$ by

$$k(f)(a \otimes q) = f(a) \cdot q$$
$$a \in A, \quad q \in Q_p.$$

Then using the fact that elements of $\operatorname{Hom}(A \otimes Q_p, B)$ are right Q_p-maps, a direct computation shows k is a two sided inverse to $j^{\#}$ in A.3.

Using the fact that A is finitely generated, we write

$$A \cong F \oplus T_P \oplus T_R$$

where F is free, T_P is p-local and $T_R \otimes Q_p = 0$. Then j splits as a sum of maps

$$j_1: F \to F \otimes Q_p$$
$$j_2: T_P \to T_P \otimes Q_p \cong T_P$$
$$j_3: T_R \to T_R \otimes Q_p \cong 0.$$

Hence $j^{\#}$ is a sum of maps

$$j_1^{\#}: \operatorname{Ext}(F \otimes Q_p, B) \to \operatorname{Ext}(F, B) \cong 0$$
$$j_2^{\#}: \operatorname{Ext}(T_P \otimes Q_p, B) \to \operatorname{Ext}(T_P, B)$$

which is an isomorphism, and

$$j_3^{\#}: \operatorname{Ext}(T_R \otimes Q_p, B) \to \operatorname{Ext}(T_R, B) = 0,$$

which is trivially an isomorphism (of 0-groups). Since B is p-local

$$\operatorname{Ext}(F \otimes Q_p, B) = 0$$

(as in the argument for (2) of section 4). Hence $j^{\#}$ is an isomorphism and the proof of A.2 is complete.

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