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On the Parametrization of Quasiconformal Mappings with Invariant Boundary Points in an Annulus

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Introduction

In two recent papers [5, 6] Reich and Strebel have considered the class of Q-quasiconformal selfmappings of the closed unit disc with invariant boundary points. Here under a Q-quasiconformal mapping of the closure D of a domain bounded by disjoint Jordan curves (in particular by one Jordan curve) we mean any homeomorphism which is, when restricted to $\operatorname{int} D$, a Q-quasiconformal mapping. The class in question, which will be denoted by F_Q, is very natural from a variational point of view and plays an important role in the Teichmüller theory of extremal quasiconformal mappings. This class was introduced and first investigated by Teichmüller [7].

In [6] Reich and Strebel proved the following two theorems:

THEOREM A. *Suppose that Φ is holomorphic in the open unit disc, Φ' has zeros at $z_j, j=1, 2,\dots$ (we do not exclude the case where $\Phi'(z)\neq 0$, $|z|<1$),*

$$\iint\limits_{|\zeta|\leqslant 1} \frac{\overline{\Phi'(\zeta)}}{\Phi'(\zeta)} \frac{d\xi\, d\eta}{z-\zeta} = 0 \quad \text{for} \quad |z| = 1 \quad \text{and} \quad z = z_j, \quad j = 1, 2, \dots (\zeta = \xi + i\eta), \tag{1}$$

and

$$\left|\frac{\Phi'(w) - \Psi_w(w, t)}{t\Phi'(w)}\right| \leqslant M, 1 < M < +\infty, \quad \text{for} \quad |w| < 1, \tag{2}$$

where

$$\Psi(w, t) = \Phi(w) + t\left[\overline{\Phi(w)} - \frac{1}{\pi}\Phi'(w) \iint\limits_{|\zeta|\leqslant 1} \frac{\overline{\Phi'(\zeta)}}{\Phi'(\zeta)} \frac{d\xi\, d\eta}{w-\zeta}\right]. \tag{3}$$

Then the homeomorphic selfmapping f of the closed unit disc, uniquely determined (cf. e.g. [1], p. 204) *by the conditions*

$$f_{\bar z} = \mu(z, t) f_z, \quad \mu(z, t) = t\overline{\Phi'(z)}/\Phi'(z), \quad 0 \leqslant t < 1, \tag{4}$$

and $f(s_k, t)=s_k$, $|s_k|=1$, $k=1, 2, 3$, $0\leqslant t<1$, belongs to $F_Q(t)$, $Q(t)=(1+t)/(1-t)$, and satisfies the relations $f(z_j, t)=z_j, j=1, 2, \dots$, and

$$\Psi(f(z, t), t) = \Phi(z) + t\overline{\Phi(z)}. \tag{5}$$

Moreover,

$$\iint\limits_{|\zeta|\leqslant 1} \frac{\overline{\Psi_\zeta(\zeta, t)}}{\Psi_\zeta(\zeta, t)} \frac{d\xi\, d\eta}{w-\zeta} = 0 \quad \text{for} \quad |w| = 1 \quad \text{and} \quad w = z_j, \quad j = 1, 2, \ldots \tag{6}$$

THEOREM B. *Under the hypotheses of Theorem* A *with* $t \leqslant 1/2M$ *the function* f *appearing in that theorem is uniquely determined as a solution* $w = f(z, t)$ *of the differential equation*

$$w_t = \frac{1}{\pi} \iint\limits_{|\zeta|\leqslant 1} \frac{\varphi(\zeta, t)}{w-\zeta}\, d\xi\, d\eta \tag{7}$$

with the initial condition $f(z, 0) = z$, $|z| \leqslant 1$, *where*

$$\varphi(w, t) = (1-t^2)^{-1}\, \overline{\Psi_w(w, t)}/\Psi_w(w, t). \tag{8}$$

Denote by F_Q^* the subclass of F_Q consisting of functions satisfying the hypotheses of Theorem A with $t \leqslant (Q-1)/(Q+1)$. It is easily seen that in Theorem B the classes F_Q^* are restricted to $Q < 3$. This restriction, however, can be rejected and we have (cf. [4])

THEOREM C. *The conclusion of Theorem* B *is valid for* $0 \leqslant t < 1$.

In this paper we obtain an analogue of Theorem C in the case of annuli, which is itself a counterpart of the general parametrization theorems established by the present author [2, 3]. The paper is concluded by some suggestions concerning further generalizations of this result.

1. Formulation of the Main Result

Our main result may be formulated as follows:

THEOREM 1. *Suppose that the hypotheses of Theorem* A *are fulfilled and that*

$$\frac{\overline{\Phi'(z)}}{\Phi'(z)} = \frac{z^2}{\bar{z}^2} \left[\frac{\overline{\Phi'(s)}}{\Phi'(s)} \right]_{s = r^{2\nu}/\bar{z}} \quad \text{for} \quad r^{2\nu} \leqslant |z| < r^{2\nu-1}, \quad \nu = 1, 2, \ldots, \tag{9}$$

$$\frac{\overline{\Phi'(z)}}{\Phi'(z)} = \left[\frac{\overline{\Phi'(s)}}{\Phi'(s)} \right]_{s = z/r^{2\nu}} \quad \text{for} \quad r^{2\nu+1} \leqslant |z| < r^{2\nu}, \quad \nu = 1, 2, \ldots, \tag{10}$$

where $0 < r < 1$. *Then the following statements hold:*

(i) *The differential equation*

$$\varrho' = \frac{1}{2\pi} \iint\limits_{\varrho \leqslant |\zeta| \leqslant 1} \varrho \left[\frac{\varphi(\zeta, t)}{\zeta^2} + \frac{\overline{\varphi(\zeta, t)}}{\bar{\zeta}^2} \right] d\xi \, d\eta, \tag{11}$$

where φ *is given by* (8), *has a unique solution* $\varrho = R(t)$, $0 \leqslant t < 1$, *subject to the initial condition* $R(0) = r$.

(iii) *The homeomorphic mapping* f *of the annulus* $\{z : r \leqslant |z| \leqslant 1\}$ *onto* $\{w : R^*(t) \leqslant |w| \leqslant 1\}$, *uniquely determined* (cf. [2], p. 26) *by the conditions* (4) *and* $f(s_0, t) = s_0$, $|s| = 1$ (*for an* s_0), $0 \leqslant t < 1$, *is the restriction of a mapping of the class* $F_{Q(t)}$, $Q(t) = (1+t)/(1-t)$, *to the annulus* $\{z : r \leqslant |z| \leqslant 1\}$, *and satisfies the relations* $f(z_j, t) = z_j$ *provided* $|z_j| \geqslant r$, $j = 1, 2, \ldots$, *and* (5). *Moreover,* $R^*(t) = R(t)$, $0 \leqslant t < 1$.

(iii) *Relation* (6) *holds for* $|w| = 1$ *and* $w = z_j$, $|z_j| \geqslant r$, $j = 1, 2, \ldots$

(iv) *The function* f *is uniquely determined as a solution* $w = f(z, t)$ *of the differential equation*

$$w_t = \frac{1}{\pi} \iint\limits_{R(t) \leqslant |\zeta| \leqslant 1} \left\{ \frac{\varphi(\zeta, t)}{w - \zeta} + \sum_{\nu=1}^{+\infty} R^{4\nu}(t) \left[\frac{\varphi(\zeta, t)}{w - R^{2\nu}(t)\,\zeta} - \frac{1}{\bar{\zeta}^3} \frac{\overline{\varphi(\zeta, t)}}{R^{2\nu}(t) - w\bar{\zeta}} \right] \right\} d\xi \, d\eta \tag{12}$$

with the initial condition $f(z, 0) = z$, $r \leqslant |z| \leqslant 1$, *where* $R^{2\nu}(t) = [R(t)]^{2\nu}$ *and* φ *is given by* (8).

In order to prove Theorem 1 we cannot apply our parametrization theorems for annuli established in [2, 3] since in the case of annuli we have no counterpart of Theorem A and we leave this problem open (cf. the conjecture in Section 5). Thus we will utilize Theorems A and C, and some of our results for annuli which do not require any generalization of Theorem A.

For more clearness we divide our proof into parts corresponding to the statements posed in Theorem 1. We always assume that the hypotheses of Theorem A are fulfilled and that Φ satisfies (9) and (10).

2. Proof of (i)

It is known ([2], Theorem 6, where in the first line of this theorem "a function" should be read as "a measurable function" and in the proof, on p. 48, line 15, "$d \to \infty$ $(c > 0)$" should be read as "$c \to 0+$ $(c < d)$") that if ψ is a measurable function of w and τ, defined for $|w| \leqslant 1$ and $0 \leqslant \tau \leqslant T$, and bounded by $\frac{1}{2}$, then the differential equation (11) with φ replaced by ψ has a unique solution $\varrho = R(T_0\tau)$, $0 \leqslant \tau \leqslant T$, subject to the initial condition $R(0) = r$. In our case, if we put $t = T_0\tau$, (11) and (8) become

$$\varrho_\tau = \frac{1}{2\pi} \iint\limits_{\varrho \leqslant |\zeta| \leqslant 1} \varrho \left[(1/\zeta^2)\, T_0 \varphi(\zeta, T_0\tau) + (1/\bar{\zeta}^2)\, T_0 \overline{\varphi(\zeta, T_0\tau)} \right] d\xi \, d\eta \tag{13}$$

and

$$T_0\varphi(w, T_0\tau) = T_0(1 - T_0^2\tau^2)^{-1}\overline{\Psi_w(w, T_0\tau)}/\Psi_w(w, T_0\tau),$$

respectively. Since Ψ is holomorphic, in order to assure the existence of a unique solution $\varrho = R(T_0\tau)$ of (13) subject to the initial condition $R(0)=r$, we have to assume that $T_0/(1-T_0^2/\tau^2) \leqslant \frac{1}{2}$. It is easilys een that if we take T_0, $T_0>0$, sufficiently small, we can get t, $t<1$, arbitrarily close to 1, and this completes the proof of (i).

3. Proof of (ii)

By the theorem on existence and uniqueness of quasiconformal mappings in doubly connected domains (cf. [2], p. 26), for each t, $0 \leqslant t < 1$, there is exactly one number $R^*(t)$, $0 < R^*(t) < 1$, and a quasiconformal mapping f of the annulus $\{z: r \leqslant |z| \leqslant 1\}$ onto $\{w: R^*(t) \leqslant |w| \leqslant 1\}$, determined uniquely by the conditions (4) and $f(s_0, t) = s_0$, $|s_0| = 1$ (for an s_0), $0 \leqslant t < 1$. In order to apply Theorem A let us continue f into the inner disc $\{z: |z| < r\}$ by the formulae

$$f^*(z, t) = R^{*2\nu}(t)/\overline{f(r^{2\nu}/\bar{z}, t)} \quad \text{for} \quad r^{2\nu} \leqslant |z| < r^{2\nu-1}, \quad \nu = 1, 2, \ldots, \tag{14}$$

$$f^*(z, t) = R^{*2\nu}(t) f(z/r^{2\nu}, t) \quad \text{for} \quad r^{2\nu+1} \leqslant |z| < r^{2\nu}, \quad \nu = 1, 2, \ldots. \tag{15}$$

Obviously, we admit $f^*(z, t) = f(z, t)$ for $r \leqslant |z| \leqslant 1$ and $f^*(0, t) = 0$. It is easily seen that f^* is the function uniquely determined in Theorem A, so, by this theorem, it belongs to $F_{Q(t)}$, $Q(t) = (1+t)/(1-t)$, and satisfies the relations $f^*(z_j, t) = z_j$, $j = 1, 2 \ldots$, and (5), where f is replaced with f^*.

In order to complete the proof of (ii) we have to show that $R^*(t) = R(t)$, $0 \leqslant t < 1$. To this end we apply Theorem 3A of [3]. This theorem asserts among others that if g maps $Q(t)$-quasiconformally the annulus $\{z: r \leqslant |z| \leqslant 1\}$ onto $\{w: R^*(t) \leqslant |w| \leqslant 1\}$ so that $g(1, t) = 1$, and is generated by the complex dilatation $\nu(z, t) = t\nu(z, T)$, $0 \leqslant t \leqslant T$, then R^* is a solution $\varrho = R^*(t)$ of the differential equation (11) with the initial condition $R^*(0) = r$, where φ is replaced with ψ, given by the formula

$$\psi(w, t) = \frac{\nu(g^{-1}(w, t), T)}{1 - t^2|\nu(g^{-1}(w, t), T)|^2} \exp(-2i \arg g_w^{-1}(w, t)).$$

In our case, denote by μ^* the complex dilatation of f^{-1}. Since $\mu^* = -\mu \exp(-2i \arg f_w^{-1})$ (cf. e.g. [1], p. 193), then, by (4), we get

$$\psi(w, t) = -(T/t)\mu^*(w, t)/\{1 - T^2|\mu^*(w, t)|^2\}$$

On the other hand, by (5) and holomorphicity of Ψ, we have

$$\mu^*(w, t) = \frac{\{(1-t^2)^{-1}[\Psi(w, t) - t\overline{\Psi(w, t)}]\}_{\bar{w}}}{\{(1-t^2)^{-1}[\Psi(w, t) - t\overline{\Psi(w, t)}]\}_w} = -t\frac{\overline{\Psi_w(w, t)}}{\Psi_w(w, t)}, \tag{16}$$

whence

$$\psi(w,t)=T(1-T^2t^2)^{-1}\overline{\Psi_w(w,t)}/\Psi_w(w,t). \tag{17}$$

For $T\to 1-$ relation (17) reduces to $\psi=\varphi$, where $0\leqslant t<1$ and φ is given by (8). Finally, by assertion (i), we conclude that $R^*=R$, as desired.

4. Proof of (iii) and (iv)

Assertion (iii) is an immediate consequence of Theorem A, especially formula (6).

We proceed to prove (iv). Let f^* be the mapping determined in Theorem A. Then, by Theorem C, it is a solution of the differential equation (7) with the initial condition $f^*(z,0)=z$, $|z|\leqslant 1$, where φ is given by (8). On the other hand, since Φ satisfies (9) and (10), then, by the uniqueness of f^* (cf. Theorem A), we get

$$f^*(z,t)=R^{2\nu}(t)/\overline{f^*(r^{2\nu}/\bar z,t)} \quad \text{for} \quad r^{2\nu}\leqslant |z|\leqslant r^{2\nu-1}, \quad \nu=1,2,\dots, \tag{18}$$

$$f^*(z,t)=R^{2\nu}(t)f^*(z/r^{2\nu},t) \quad \text{for} \quad r^{2\nu+1}\leqslant |z|\leqslant r^{2\nu}, \quad \nu=1,2,\dots, \tag{19}$$

where $R(t)$ is uniquely determined by (4) with f replaced by f^* (cf. [2], p. 26). Therefore, since, by Theorem A, f^* is one-to-one, we have

$$f^{*-1}(w,t)=r^{2\nu}/\overline{f^{*-1}(R^{2\nu}(t)/\bar w,t)} \quad \text{for} \quad R^{2\nu}(t)\leqslant |w|\leqslant R^{2\nu-1}(t), \quad \nu=1,2,\dots,$$

$$f^{*-1}(w,t)=r^{2\nu}f^{*-1}(w/R^{2\nu}(t),t) \quad \text{for} \quad R^{2\nu+1}(t)\leqslant |w|\leqslant R^{2\nu}(t), \quad \nu=1,2,\dots.$$

Consequently, by (16), where μ^* is the complex dilatation of f^{*-1}, we obtain

$$\frac{\overline{\Psi_w(w,t)}}{\Psi_w(w,t)}=\frac{w^2}{\bar w^2}\left[\frac{\overline{\Psi_s(s,t)}}{\Psi_s(s,t)}\right]_{s=R^{2\nu}(t)/\bar w} \quad \text{for} \quad R^{2\nu}(t)\leqslant |w|\leqslant R^{2\nu-1}(t), \quad \nu=1,2,\dots, \tag{20}$$

$$\frac{\overline{\Psi_w(w,t)}}{\Psi_w(w,t)}=\left[\frac{\overline{\Psi_s(s,t)}}{\Psi_s(s,t)}\right]_{s=w/R^{2\nu}(t)} \quad \text{for} \quad R^{2\nu+1}(t)\leqslant |w|\leqslant R^{2\nu}(t), \quad \nu=1,2,\dots. \tag{21}$$

Relations (7), (8), (20), and (21) yield

$$\begin{aligned} w_t=&\frac{1}{\pi}\iint\limits_{R(t)\leqslant|\zeta|\leqslant 1}\frac{\varphi(\zeta,t)}{w-\zeta}\,d\xi\,d\eta+\frac{1}{\pi}\sum_{\nu=1}^{+\infty}\Bigg\{\iint\limits_{R^{2\nu+1}(t)\leqslant|\zeta|\leqslant R^{2\nu}(t)}\frac{\varphi(\zeta/R^{2\nu}(t),t)}{w-\zeta}\,d\xi\,d\eta\\ &+\iint\limits_{R^{2\nu}(t)\leqslant|\zeta|\leqslant R^{2\nu-1}(t)}\frac{\overline{\varphi(R^{2\nu}(t)/\bar\zeta,t)}}{w-\zeta}\,d\xi\,d\eta\Bigg\}=\frac{1}{\pi}\iint\limits_{R(t)\leqslant|\zeta|\leqslant 1}\frac{\varphi(\zeta,t)}{w-\zeta}\,d\xi\,d\eta\\ &+\frac{1}{\pi}\sum_{\nu=1}^{+\infty}R^{4\nu}(t)\iint\limits_{R(t)\leqslant|\zeta|\leqslant 1}\left[\frac{\varphi(\zeta,t)}{w-R^{2\nu}(t)\zeta}-\frac{1}{\bar\zeta^3}\frac{\overline{\varphi(\zeta,t)}}{R^{2\nu}(t)-w\bar\zeta}\right]d\xi\,d\eta. \end{aligned}$$

It can easily be checked with help of the well known Weierstrass' test that the series of integrands in the above formula is uniformly convergent. Since all integrals appearing in that formula are continuous with respect to w (cf. e.g. [8], p. 27), then for any w of $\{w: R(t) \leqslant |w| \leqslant 1\}$ we can interchange the order of integration and summation, and this implies that the function f, defined as the restriction of f^* to $\{z: r \leqslant |z| \leqslant 1\}$, is a solution $w = f(z, t)$ of (12) with the initial condition $f(z, 0) = z$, $r \leqslant |z| \leqslant 1$, as desired.

Consider now the differential equation (12) with the initial condition $f(z, 0) = z$, $r \leqslant |z| \leqslant 1$, where φ is given by (8). From the first part of the proof of (iv) it follows that if f^* is a solution $w = f^*(z, t)$ of (7) with the initial condition $f^*(z, 0) = z$, $|z| \leqslant 1$, and it satisfies (18) and (19), then the function f, defined as the restriction of f^* to $\{z: r \leqslant |z| \leqslant 1\}$, is a solution $w = f(z, t)$ of (12) with the initial condition $f(z, 0) = z$, $r \leqslant |z| \leqslant 1$. On the other hand, if f is a solution $w = f(z, t)$ of (12) with the initial condition $f(z, 0) = z$, $r \leqslant |z| \leqslant 1$, then f^*, defined by (14) and (15) with $R^* = R$ and, obviously, $f^*(z, t) = f(z, t)$, $r \leqslant |z| \leqslant 1$, $f^*(0, t) = 0$, is a solution of (7) with the initial condition $f^*(z, 0) = z$, $|z| \leqslant 1$. Hence, by Theorem C, f^* is the restriction of a mapping of the class $F_{Q(t)}$, $Q(t) = (1+t)/(1-t)$, namely f^*, to the annulus $\{z: r \leqslant |z| \leqslant 1\}$, as desired. This completes the proof.

5. Remarks and Conclusions

The author conjectures that the statements of Theorem 1 remain valid if we assume that Φ is defined as a holomorphic function satisfying (2) for $r < |z| < 1$ only. He also conjectures that if we assume (1) for $|z| = r$ as well, we get $f(z, t) = [R(t)/r]z$ for $|z| = r$ and (6) for $|w| = R(t)$. These questions are hoped to be answered in a subsequent paper.

Finally, in the same way as above, we can prove in the case $r = 0$, i.e. in the case of mappings of F_Q^* with the additional invariant point 0, the following analogue of Theorem 1.

THEOREM 2. *Suppose that the hypotheses of Theorem* A *are fulfilled and that* (1) *holds also for* $z = 0$. *Then the following statements hold.*

(i) *The homeomorphic selfmapping of the closed unit disc, uniquely determined* (cf. e.g. [1], p. 204) *by the conditions* (4) *and* $f(s_0, t) = s_0$, $|s_0| = 1$ (for an s_0), *and* $f(0, t) = 0$, $0 \leqslant t \leqslant 1$, *belongs to* $F_{Q(t)}$, $Q(t) = (1+t)/(1-t)$, *and satisfies the relations* $f(z_j, t) = z_j$, j = 1,2,..., and (5).

(ii) *Relation* (6) *holds for* $|w| = 1$, $w = 0$, *and* $w = z_j$, $j = 1, 2, \ldots$

(iii) *The function f is uniquely determined as a solution* $w = f(z, t)$ *of the differential equation* (7) *with the initial condition* $f(z, 0) = z$, $|z| \leqslant 1$, *where* φ *is given by* (8).

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