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Analytic First Integrals of Ordinary Differential Equations

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1. Introduction

In a previous paper [3], the author considered complex analytic differential equations

$$\frac{dz_k}{dt} = f_k(t, z_1, \ldots, z_n), \qquad k = 1, \ldots, n. \tag{1}$$

Here the f_k are analytic functions of the $n+1$ complex variables t, $z_1, \ldots, z_n$ in a domain D of the $(n+1)$-dimensional complex space K^{n+1}. It was asked whether such a system has first integrals in the large: that is, whether there exist functions $\varphi(t, z_1, \ldots, z_n)$, analytic in D, which are not identically constant but are constant on each solution of (1). It is known from basic theory that such first integrals exist locally. It was shown by examples that, in general, existence in the large of such first integrals was excluded unless one allowed multiple-valued functions and exceptional sets of measure zero. It was shown that for $n=1$ with $D=K^2$ (f_1 entire), such a multiple-valued first integral in the large does always exist. It was shown by an example that the proof for the case $n=1$ could not be generalized directly to the case of larger n. In the present paper, the result is extended to general n, and it is shown that for $D=K^{n+1}$ there always exist n functionally independent first integrals $\varphi_k(t, z_1, \ldots, z_n)$, in the sense described.

The question can also be phrased in terms of partial differential equations. The integrals φ_k sought are solutions of the first order equation

$$\frac{\partial w}{\partial t} + \sum_{k=1}^{n} f_k(t, z_1, \ldots, z_n) \frac{\partial w}{\partial z_k} = 0. \tag{2}$$

One seeks a complete set of solutions of this equation in the large. The result of the present paper provides such a set, in the sense described, for $D=K^{n+1}$.

When the f_k are independent of t and are analytic in K^n (autonomous case), it is shown that one can find $n-1$ functionally independent first integrals in the large $\varphi_l(z_1, \ldots, z_n)$ $(l=1, \ldots, n-1)$, provided the equilibrium points (the common zeros of the f_k) are countable. The equations $\varphi_l(z_1, \ldots, z_n)=c_l$, $l=1, \ldots, n-1$, then define (almost all) the "trajectories" of (1).

When the f_k are meromorphic in K^{n+1}, the complete set $\varphi_k(t, z_1, \ldots, z_n)$ is again found, under certain restrictions on the zeros and poles of the f_k.

As pointed out in the previous paper, these questions are of interest in mechanics, the first integrals being related to such quantities as energy and momentum.

It will be convenient to write (1) in vector form:

$$\frac{dz}{dt} = f(t, z), \tag{3}$$

where $z = (z_1, \ldots, z_n)$, $f = (f_1, \ldots, f_n)$. We use the norm $|z| = \sum |z_k|$ in K^n.

As in the previous paper, we make use of the following consequence of Fubini's theorem:

Let X and Y be measure spaces, each of which is a countable union of sets of finite measure. Let $g(x, y)$ be a measurable function on $X \times Y$ such that, for almost all x, $g(x, y) = 1$ for almost all y. Then for almost all y, $g(x, y) = 1$ for almost all x.

We shall refer to this principle by simply writing "by Fubini's theorem."

2. A Preliminary Result for the Meromorphic Case

The following theorem will play an essential role in the proof of the main result (Theorem 2 below):

THEOREM 1. *In* (3), *let f be meromorphic in K^{n+1} and suppose that for almost all* (t^0, z^0) (3) *has a solution with initial point* (t^0, z^0) *which can be continued analytically for almost all t. Then there exist n functions $\varphi_k(t, z)$ which are (multiple-valued) analytic in K^{n+1}, can be continued analytically to almost all of K^{n+1}, remain constant on solutions of* (3) *and have nonvanishing functional determinant* $\partial(\varphi_1, \ldots, \varphi_n)/\partial(z_1, \ldots, z_n)$.

This is proved in the same way as Theorem 4 of [3]. One applies the Fubini theorem to show that for almost all t^0 the trajectories through initial points (t^0, z^0) sweep out almost all of K^{n+1}. We then choose a nonexceptional t^0. Then for almost all (t, z) the solution through (t, z) can be continued analytically to t^0, yielding a value z^0 at t^0. One defines the vector $\varphi(t, z)$ to be this value z^0, and thereby obtains n (multiple-valued) analytic functions $\varphi_k(t, z)$, constant on solutions. Since the dependence of solutions on initial values is locally a one-to-one analytic mapping, the functional determinant cannot vanish.

3. The Autonomous Case

We now consider Equation (3) in the *autonomous* case, in which f is independent of t, and assume f is an entire function of z. We seek $n-1$ functionally independent first integrals $\varphi_l(z)$ $(l = 1, \ldots, n-1)$ independent of t. The equilibrium points cause a certain complication, but this is minor. It will be seen that the result for the autonomous case leads easily to that for the nonautonomous case.

THEOREM 2. *Let the following differential equation be given:*

$$\frac{dz}{dt} = f(z). \tag{4}$$

where f is analytic in K^n. Let the zeros of f form a countable set. Then there exist $n-1$ multiple-valued analytic functions $\varphi_1(z), \ldots, \varphi_{n-1}(z)$, which can be continued analytically to almost all of K^n, such that each $\varphi_l(z)$ is constant on each solution of (4):

$$\sum_{k=1}^{n} f_k(z) \frac{\partial \varphi_l}{\partial z_k} = 0, \qquad l = 1, \ldots, n-1 \tag{5}$$

and such that the matrix

$$\left(\frac{\partial \varphi_l}{\partial z_k} \right) \tag{6}$$

has rank $n-1$.

Proof. Each solution of (4) is of form

$$z = \psi(t), \tag{7}$$

where $\psi(t)$ is analytic. The solution is first defined in a neighborhood in the t-plane, but may then be continued analytically on certain paths in the finite t-plane to yield a complete solution, not capable of further continuation; in general, this complete solution is a multiple-valued vector function in the t-plane. Among the solutions are the equilibrium solutions: $\psi(t) \equiv \text{const}$. By assumption, there are only countably many such solutions.

LEMMA. *Let a path γ be defined in the t-plane by a continuous function $t = t(u)$, $0 \leqslant u < 1$. Let a solution* (7) *be defined at $t(0)$ and continuable along γ. Let $\psi\,[t(u)]$ have a finite limit z° as $u \to 1-$.*

(a) *If also $t(u)$ has a finite limit t_1 as $u \to 1-$, then $\psi(t)$ can be continued along $\gamma \cup \{t_1\}$ up to t_1, so that ψ has no singularity at t_1.*

(b) *If $t(u)$ does not have a finite limit as $u \to 1-$, then z^0 is a zero of f.*

Proof. Assertion (a) follows from a classical theorem of Painlevé (see [1, p. 11]). For (b) we remark that if z^0 is not a zero of f, then at least one component of f, say f_n, has no zero in some neighborhood of z^0. In this neighborhood z_n can then be used as parameter on the solutions and, in a sufficiently small neighborhood, the basic existence theorem shows that one can express all solutions by single-valued analytic functions

$$z_k = z_k(z_n), \qquad k = 1, \ldots, n-1; \qquad t = t(z_n); \qquad |z_n - z_n^0| < \varepsilon. \tag{8}$$

It follows that our solution $\psi(t)$ must be so expressible, for $t = t(u)$, and u sufficiently close to 1. But then, by the uniqueness of solutions, the condition $\psi(t(u)) \to z^0$ can hold only if the corresponding solution goes through z^0: that is, $z_k(z_n^0) = z_k^0$ for $k=1, \dots, n-1$. Since $t(z_n)$ is also analytic at z_n^0, we conclude that $t(u)$ must have a finite limit as $u \to 1$. This contradicts our assumption. Accordingly, z^0 is a zero of f.

We now consider a complete solution (7). As remarked above, ψ is in general multiple-valued and hence there is an associated Riemann surface R over the finite t-plane. We form the universal covering surface R^* of R, and represent R^* as a disc $|\zeta| < \varrho$, $\varrho = 1$ or ∞, in the ζ-plane. Our many-valued function ψ is then replaced by a pair of single-valued analytic functions $t = \chi(\zeta)$, $z = \psi^*(\zeta)$ in $|\zeta| < \varrho$. If $\varrho = \infty$, then $\psi^*(\zeta)$ is entire and hence either ψ is identically constant (equilibrium solution) or else at least one component $\psi_k^*(\zeta)$ takes on all complex values except perhaps one, by the Picard theorem.

We now consider the case $\varrho = 1$ in more detail. We claim that at least one of the functions $\psi_k^*(\zeta)$ must be of unbounded type in the unit circle. For if not, then all $\psi_k^*(\varrho)$ are of bounded type and hence have finite radial limits almost everywhere on $|\zeta| = 1$. Since the zeros of f form a countable set, it follows from the theorem of F. and M. Riesz that $\psi^*(\zeta)$ has a finite radial limit z^0, not a zero of f, for almost all $e^{i\varphi}$ on $|\zeta| = 1$. For such a choice of $e^{i\varphi}$, a radius $\zeta = ue^{i\varphi}$ corresponds to a path $t = t(u)$, $0 \leqslant u < 1$ in the t-plane and the assertion that ψ has a finite limit is equivalent to the statement that $\psi(t(u))$ has a finite limit on the path as in the Lemma above. If $t(u)$ also has a finite limit as $u \to 1-$, then by (a) there is no singularity as $u \to 1-$ and the continuation leads to an interior point of the Riemann surface, hence an interior point of R^*; this contradicts the fact that $\zeta = u\, e^{i\varphi}$ approaches the boundary of R^* as $u \to 1-$. Hence $t(u)$ has no finite limit. Therefore, by (b), z^0 must be a zero of f, contrary to our choice of $e^{i\varphi}$ on $|\zeta| = 1$. Accordingly, we have a contradiction and at least one of $\psi_1^*(\zeta), \dots, \psi_n^*(\zeta)$ is of unbounded type.

We now consider complex scalars $c_1, \dots, c_n$ and form the vector space V of all linear combinations $\sum_1^n \bar{c}_k \psi_k^*(\zeta)$, $|\zeta| < 1$. The functions of bounded type form a subspace of V (see [4, p. 162, p. 179]). We have just seen that at least one of the functions $\psi_k^*(\zeta)$ is of unbounded type. It follows that those vectors $c = (c_1, \dots, c_n)$ for which $\sum \bar{c}_k \psi_k^*(\zeta)$ is of bounded type form a subspace of V of dimension *less* than n; therefore for almost all c in K^n (with reference to $2n$-dimensional Lebesgue measure), $\sum \bar{c}_k \psi_k^*(\zeta)$ is of unbounded type; that is, for almost all vectors c, the projection of $\psi^*(\zeta)$ on the one-dimensional subspace of K^n generated by c is a function $w = \psi^{*c}(\zeta)$ of unbounded type. Therefore w takes on almost all complex values (in fact, all except a set of capacity 0 … see [4, p. 202]).

A similar conclusion obtains in the case $\varrho = \infty$, with bounded type and unbounded type replaced by constant and nonconstant entire function. If $\psi(t)$ is not identically constant, then for almost all c, $w = \psi^{*c}(\zeta)$ takes on almost all complex values.

Now let $\sigma(z^0, c, w) = 1$ if the solution ψ of (4) through z^0 for $t = 0$ is such that

$\psi^{*c}(\zeta)$ can be continued to take on the value w; let $\sigma(z^0, c, w)=0$ otherwise. Then σ is measurable and we have just shown that, except for the countable set of zeros of f, for each z^0, for almost all c, $\sigma(z^0, c, w)=1$ for almost all w. It now follows from the Fubini theorem that, for almost all c, for almost all z^0, $\sigma(z^0, c, w)=1$ for almost all w. We choose one such value c. Without loss of generality, we can assume that $c=(0, \ldots, 0, 1)$; that is, that $z_n=\psi_n^*(\zeta)$ attains almost all complex values, for almost every initial point z^0.

We now use z_n as new independent variable, and consider our trajectories in K^n as solutions of the meromorphic differential equations

$$\frac{dz_l}{dz_n}=\frac{f_l(z_1, \ldots, z_n)}{f_n(z_1, \ldots, z_n)}, \quad l=1, \ldots, n-1. \tag{9}$$

Each such solution is given by (multiple-valued) functions

$$z_l=q_l(z_n), \quad l=1, \ldots, n-1, \tag{10}$$

obtained by eliminating t from $z_1=\psi_1(t), \ldots, z_n=\psi_n(t)$ by solving the last equation for t; from the properties given above, it follows that, for almost every initial point z^0, the corresponding solution (10) can be continued analytically to reach almost all values z_n. Therefore Theorem 1 can be applied, with n replaced by $n-1$ and z_n by t, and there must exist $n-1$ functions $\varphi_k(z_1, \ldots, z_n)$, $k=1, \ldots, n-1$ with the properties (5) and (6). Thus Theorem 2 is proved.

4. Extensions

We first consider the non-autonomous case.

THEOREM 3. *Let the differential equation*

$$\frac{dz}{dt}=f(z, t) \tag{11}$$

be given, where $z=(z_1, \ldots, z_n), f=(f_1, \ldots, f_n)$ *and* f *is analytic in all of* K^{n+1}. *Then there exist* n *(multiple-valued) analytic functions* $\Phi_k(z, t)$, $k=1, \ldots, n$, *which can be continued to almost all of* K^{n+1}, *which are constant along each solution of* (11) *and whose functional matrix*

$$(\partial\Phi_k/\partial z_l) \quad (k=1, \ldots, n, \quad l=1, \ldots, n+1, z_{n+1}=t)$$

has rank n.

Proof. If we set $z_{n+1}=t$, then (11) becomes an autonomous system

$$\frac{dz_1}{dt}=f_1(z_1, \ldots, z_{n+1}), \ldots, \frac{dz_n}{dt}=f_n(z_1, \ldots, z_{n+1}), \frac{dz_{n+1}}{dt}=1,$$

with no equilibrium points. If we apply Theorem 2 to this system, we obtain the n integrals $\Phi_k(z_1, \ldots, z_{n+1})$ satisfying the conditions stated.

THEOREM 4. *Let the differential equations*

$$\frac{dz_k}{dt} = \frac{f_k(z)}{g(z)}, \quad k = 1, \ldots, n \tag{12}$$

be given, where $f_k(z) = f_k(z_1, \ldots, z_n)$ and $g(z) = g(z_1, \ldots, z_n)$ are analytic in K^n and $g(z) \not\equiv 0$. Let $f(z) = (f_1(z), \ldots, f_n(z))$ have only a countable set of zeros. Then first integrals $\varphi_1(z), \ldots, \varphi_{n-1}(z)$ exist as in Theorem 2.

For we can choose $\varphi_1, \ldots$ for the corresponding equation (4), as in Theorem 2. Then (5) holds, so that also

$$\sum_{k=1}^{n} \frac{f_k(z)}{g(z)} \frac{\partial \varphi_l}{\partial z_k} = 0, \quad l = 1, \ldots, n-1$$

and hence φ_l is constant on each solution of (12). In fact, the solutions of (12) have the same trajectories as (4), with a new "time" parameter $\tau = \int g(z)\, dt$.

For a general autonomous meromorphic system

$$\frac{dz_k}{dt} = \frac{f_k(z)}{g_k(z)} \tag{13}$$

one can always write the equations in form (12) by reducing the fractions to a common denominator. If, after this reduction, the numerators have only countably many common zeros, then Theorem 4 is applicable. For a nonautonomous system analogous to (12):

$$\frac{dz_k}{dt} = \frac{f_k(z, t)}{g(z, t)}, \quad k = 1, \ldots, n, \tag{14}$$

one can introduce t as $(n+1)-$st dependent variable, as in the proof of Theorem 3, and obtain the equivalent autonomous system

$$\frac{dz_1}{d\tau} = f_1(z, t), \ldots, \quad \frac{dz_n}{d\tau} = f_n(z, t), \quad \frac{dt}{d\tau} = g(z, t) \tag{15}$$

and hence, if the functions $f_1, \ldots, f_n$, g have only countably many common zeros, we obtain n independent first integrals $\Phi_1(z, t), \ldots, \Phi_n(z, t)$.

5. Examples

In the paper [3] various examples are given. In particular, it is pointed out that for the first order equation $z'=P(z)$, where P is a polynomial of degree 4, in general the first integral has logarithmic branch points and the typical solution is a curve dense in tz-space. By raising the degree of P or replacing $P(z)$ by $\sin z^2$, for example, the complexity can be increased.

The following example is also instructive. (Its relevance to the problem at hand was pointed out to the author by T. Kimura.) It is well known that each "linear-polymorphic" function satisfies a differential equation of the form

$$[z]_\zeta = R(\zeta), \tag{16}$$

where $[z]_\zeta$ is the Schwarzian derivative and $R(\zeta)$ is rational (see [2], p. 444). Since (16) is of third order, it can be written in the form (1) with $k=3$ and the f_k rational. The general solution of (16) is given by

$$z = \frac{ah(\zeta)+b}{ch(\zeta)+d}, \tag{17}$$

where $h(\zeta)$ is one particular solution. For proper choice of $R(\zeta)$, $h(\zeta)$ is the inverse of the elliptic modular function and is hence continuable everywhere except for logarithmic branch points over three points of the ζ-plane. Each solution (17) has then a similar character, and hence Theorem 1 is applicable. One can in fact obtain a complete set of three first integrals φ_1, φ_2 φ_3 from (17) by differentiating twice and solving for a, b, c, with d set equal to 1. The resulting functions φ_k are many-valued and can be continued to almost all of K^4. In particular, the solutions with $d=0$ have been lost; their graphs fill a set of measure 0.

By taking ζ as dependent variable and z as independent variable, (16) can be rewritten as an algebraic differential equation of third order for $\zeta(z)$. For appropriate choice of $R(\zeta)$, as above, the solutions are given by

$$\zeta = \psi\left(\frac{az+b}{cz+d}\right), \tag{18}$$

where ψ is the elliptic modular function. The third order differential equation can again be replaced by a system of first order equations (here, autonomous). Because of the nature of the elliptic modular function, the solutions of this system are single-valued but each is defined only in a half-plane or a circle. Thus the hypotheses of Theorem 1 are not satisfied. However, the function ψ is of unbounded type ([4], p. 201) and hence one has the situation encountered in the proof of Theorem 2. Thus the existence of first integrals is assured. These can be obtained explicitly as above by inverting (18), differentiating twice and solving for a, b, c, with d set equal to 1. The fact that ζ is preferable to z as independent variable, so that one may be able to apply Theorem 1, is indicated by the proof of Theorem 2.

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