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Generators for Certain Ideals in Regular Rings of Dimension Three

by M. PAVANAN MURTHY¹⁾

Introduction

We prove here the following

THEOREM. *Let A be a regular ring of dimension 3 with $\tilde{K}_0 A = 0$. Let \mathfrak{a} be an unmixed ideal of height 2. Suppose \mathfrak{a} is locally generated by r elements. Then \mathfrak{a} is generated by $r+1$ elements.*

Applying this theorem to $A = K[x_1, x_2, x_3]$, K a field we obtain for instance that if C is a curve in the affine three space A_3 , which is locally a complete intersection (e.g. C non-singular), then the ideal of C is generated by three elements. We also show that this is best possible by giving an example of a non-singular curve in A_3 which is not a complete intersection.

In the case $A = K[x_1, x_2, x_3]$, K algebraically closed and \mathfrak{a} the ideal of a non-singular curve, S. Abhyankar has proved this by quite different methods (see his Montreal Lecture Notes [1]).

A basic tool in the proof is a lemma of Serre [6] which relates projective modules with generators of certain ideals of height 2. In fact for $r \geq 3$, our theorem easily follows from a corollary to Serre's lemma (see corollary to Lemma 1). For $r=2$, we have to make a separate argument using a remark of Bass [3] ($P \oplus A = A^{2n} \Rightarrow P = P' \oplus A$).

We consider here only commutative noetherian rings and finitely generated modules. Most of the time we just use the ring A . For a module M , $\text{hd } M$ denotes its homological dimension. $\dim A$ denotes the Krull dimension of A .

The following lemma is basic for what follows. We include a proof here for the sake of completeness.

LEMMA 1 (Serre [6]). *Let A be a noetherian ring and M a left A -module of homological dimension ≤ 1 . Let $\text{Ext}_A^1(M, A)$ be generated by one element. Then there is an exact sequence*

$$0 \rightarrow A \rightarrow P \rightarrow M \rightarrow 0$$

with P projective.

¹⁾ I am thankful to the Forschungsinstitut, ETH, Zurich, for support when this note was being written.

Proof. Let α generate $\text{Ext}_A^1(M, A)$ and let α correspond to the extension (α) :

$$0 \rightarrow A \rightarrow P \rightarrow M \rightarrow 0$$

Then homing (α) with A , we get the exact sequence

$$\text{Hom}(P, A) \rightarrow \text{Hom}(A, A) \xrightarrow{f} \text{Ext}_A^1(M, A) \rightarrow \text{Ext}_A^1(P, A) \rightarrow 0.$$

By extension theory, $f(1) = \alpha$. Since $\text{Ext}_A^1(M, A)$ is generated by α , we have $\text{Ext}_A^1(P, A) = 0$. Since $\text{hd } M \leq 1$, by (α) , it follows that $\text{hd } P \leq 1$. Since A is noetherian, $\text{hd } P \leq 1$ and $\text{Ext}_A^1(P, A) = 0$ together imply that P is projective.

COROLLARY. *Let A be a noetherian ring and M an A -module of homological dimension ≤ 1 . Let $\text{Ext}_A(M, A)$ be generated by r elements. Then there is an exact sequence*

$$0 \rightarrow A^r \rightarrow P \rightarrow M \rightarrow 0$$

with P projective.

Proof. We prove the corollary by induction on r . For $r = 1$, this is precisely Serre's lemma. Assume that the corollary is true for $r - 1$. Let $\alpha_1, \dots, \alpha_r$ generate $\text{Ext}_A^1(M, A)$. Let α_1 correspond to the extension (α_1) :

$$0 \rightarrow A \rightarrow L \xrightarrow{h} M \rightarrow 0$$

Then we get the exact sequence

$$\text{Hom}(L, A) \rightarrow \text{Hom}(A, A) \xrightarrow{f} \text{Ext}_A^1(M, A) \xrightarrow{g} \text{Ext}_A^1(L, A) \rightarrow 0,$$

where $f(1) = \alpha_1$. Hence $\text{Ext}_A^1(L, A)$ is generated by $r - 1$ elements $g(\alpha_2), \dots, g(\alpha_r)$. The exact sequence (α_1) and the hypothesis that $\text{hd } M \leq 1$ show that $\text{hd } L \leq 1$. Hence by induction hypothesis there is an exact sequence I :

$$0 \rightarrow A^{r-1} \rightarrow P \xrightarrow{k} L \rightarrow 0$$

with P projective. We also have the exact sequence

$$0 \rightarrow \text{Ker}(h \circ k) \rightarrow P \xrightarrow{h \circ k} M \rightarrow 0.$$

By the exact sequences (α_1) and I it easily follows that $\text{Ker}(h \circ k) \approx A^r$. The proof of the corollary is complete.

For a ring A , we denote $\text{Max}(A)$ its maximal ideal spectrum and by $\dim \text{Max}(A)$, the dimension of $\text{Max}(A)$. If A is an integral domain with quotient field K and M an A -module, we recall that $\text{rank } M = \dim_K(K \otimes_A M)$.

LEMMA 2. *Let A be a noetherian domain. Let M be an A -module of $\text{hd } M \leq 1$. Let M be an A -module of rank n . Let α be the annihilator of $\text{Ext}_A^1(M, A)$. Suppose that*

$\dim \text{Max}(A/\mathfrak{a}) \leq d$. If M is locally generated by r elements, then $\text{Ext}_A^1(M, A)$ is generated by $r + d - n$ elements.

Proof. By Swan [7], we need only prove that $\text{Ext}_A^1(M, A)$ is locally generated by $r - n$ elements. So, we may assume A is local. Since $\text{hd } M \leq 1$ M is generated by r elements and $\text{rank } M = n$, we get an exact sequence (since projectives are free over local rings)

$$0 \rightarrow A^{r-n} \rightarrow A^r \rightarrow M \rightarrow 0.$$

Homming this with A , we see that $\text{Hom}(A^{r-n}, A) \rightarrow \text{Ext}_A^1(M, A) \rightarrow 0$ is exact. This shows that $\text{Ext}_A^1(M, A)$ is generated by $r - n$ elements.

COROLLARY. *Let A be a regular domain of dimension 3. Let \mathfrak{a} be an unmixed ideal of height 2. If \mathfrak{a} is locally generated by r elements, then $\text{Ext}_A^1(\mathfrak{a}, A)$ is generated by r elements.*

Proof. Using the well known fact that $\text{depth } M + \text{hd } M = \dim A$ for a regular local ring A , one easily sees that \mathfrak{a} is unmixed of height 2 implies $\text{hd } \mathfrak{a} \leq 1$. Since $\text{Ext}_A^1(\mathfrak{a}, A) \approx \text{Ext}_A^2(A/\mathfrak{a}, A)$, it follows that annihilator \mathfrak{b} of $\text{Ext}_A^1(\mathfrak{a}, A)$ contains \mathfrak{a} . Hence $\dim A/\mathfrak{b} \leq \dim A/\mathfrak{a} \leq 1$, since \mathfrak{a} is unmixed of height 2 and $\dim A = 3$. Now the corollary follows from Lemma 2.

Let A be a ring and P an A -module. We recall that $s \in P$ is *unimodular* if s generates a free direct summand of P , isomorphic to A . For $x \in \text{Max}(A)$, we denote by $s(x)$ the image of s under the canonical map $P \rightarrow P/xP$.

LEMMA 3. *Let A be a noetherian ring of dimension ≤ 1 and P a projective A -module of rank 2. If s_1, s_2, s_3 generate P , then there exist $\lambda_2, \lambda_3 \in A$ such that $s_1 + \lambda_2 s_2 + \lambda_3 s_3$ is unimodular.*

Proof. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ be the minimal prime ideals of A . Choose $\mathfrak{M}_1, \dots, \mathfrak{M}_r \in \text{Max}(A)$ such that $\mathfrak{M}_i \supset \mathfrak{P}_i$. Since P is of rank 2, $P/(\prod \mathfrak{M}_i)P$ is $A/\prod \mathfrak{M}_i$ free of rank 2. Using Chinese Remainder Theorem we can find easily $a, b \in A$ such that if we set $s'_1 = s_1 + as_3$, $s'_2 = s_2 + bs_3$, then $s'_1(\mathfrak{M}_i), s'_2(\mathfrak{M}_i)$ are linearly independent over A/\mathfrak{M}_i , $1 \leq i \leq r$. Thus we may assume s_1, s_2 are linearly independent at \mathfrak{M}_i , $1 \leq i \leq r$. Then the set $T = \{\mathfrak{M} \in \text{Max}(A) \mid s_1(\mathfrak{M}), s_2(\mathfrak{M}) \text{ are linearly dependent}\}$ is a closed set [5, p. 6] which does not contain any irreducible component of $\text{Max}(A)$. Since $\dim A \leq 1$, it follows that T is finite. Let $U = \{\mathfrak{M} \in \text{Max}(A) \mid s_1(\mathfrak{M}) = 0\}$. Then $U \subset T$. Since P is of constant rank 2 and s_1, s_2, s_3 generate P it follows that $s_2(\mathfrak{M}) \neq 0, \mathfrak{M} \in U$. Choose $f \in A$ such that $f(\mathfrak{M}) \neq 0, \mathfrak{M} \in U$ and $f(\mathfrak{M}) = 0, \mathfrak{M} \in T - U$. Clearly $s_1 + fs_2$ is unimodular.

THEOREM. *Let A be a regular integral domain of dimension 3 with $\tilde{K}_0 A = 0$. Let \mathfrak{a} be an ideal unmixed of height 2. Suppose \mathfrak{a} is locally generated by r elements. Then \mathfrak{a} is generated by $r + 1$ elements.*

We recall that $\tilde{K}_0 A = 0$ is equivalent to saying that for any finitely generated projective A -module P , $P \oplus A^m \approx A^n$ for some m, n .

Proof of the theorem. By the corollary to Lemma 2, we see that $\text{Ext}_A^1(\mathfrak{a}, A)$ is generated by r elements. Consequently by the corollary to Lemma 1 (since $\text{hd } \mathfrak{a} \leq 1$), we get an exact sequence

$$0 \rightarrow A^r \rightarrow P \rightarrow \mathfrak{a} \rightarrow 0.$$

with P a projective A -module of rank $r+1$. If $r \geq 3$, then $\text{rank } P \geq 4$. Since $\tilde{K}_0 A = 0$ and $\dim A = 3$, it follows from Bass' cancellation theorem [2, p. 184, (3.5)] that P is free. Hence \mathfrak{a} is generated by $r+1$ elements. Thus the theorem is proved in case $r \geq 3$.

Now we consider the case $r=2$. In this case P is a projective module of rank 3. Again by [2, p. 184, (3.5)], $P \oplus A \approx A^4$. Hence by [3], P admits a free direct summand of rank 1: $P = P' \oplus A\alpha$. We have the exact sequence

$$0 \rightarrow A^2 \rightarrow P' \oplus A\alpha \xrightarrow{f} \mathfrak{a} \rightarrow 0. \quad (*)$$

Since A is a regular ring and \mathfrak{a} is an ideal of height 2, locally generated by two elements, it follows that $\mathfrak{a}_{\mathfrak{M}}$ is generated by an $A_{\mathfrak{M}}$ -sequence of length 2 for any maximal ideal $\mathfrak{M} \supset \mathfrak{a}$. Hence $\mathfrak{a}/\mathfrak{a}^2$ is a locally free A/\mathfrak{a} -module of rank 2. Tensoring the exact sequence (*) by A/\mathfrak{a} , we get the exact sequence

$$A^2 \rightarrow \bar{P}' \oplus \bar{A}\bar{\alpha} \xrightarrow{\bar{f}} \mathfrak{a}/\mathfrak{a}^2 \rightarrow 0$$

where $\bar{M} = M/\mathfrak{a}M$ for an A -module M and for $x \in M$, \bar{x} denotes residue class of x modulo $\mathfrak{a}M$. Since P' is stably free, so is the \bar{A} -module \bar{P}' . Since dimension of $\bar{A} \leq 1$, it follows by [2, p. 170 §2], that $\bar{P}' \approx \bar{A}^2$. Let $\alpha_1, \alpha_2 \in P'$ be such that $\bar{\alpha}_1, \bar{\alpha}_2$ generate \bar{P}' . Then $\bar{f}(\alpha), \bar{f}(\bar{\alpha}_1), \bar{f}(\bar{\alpha}_2)$ generate $\mathfrak{a}/\mathfrak{a}^2$. Since $\mathfrak{a}/\mathfrak{a}^2$ is a projective A/\mathfrak{a} -module of rank 2, by Lemma 3, there exist $\bar{a}_1, \bar{a}_2 \in \bar{A}$ such that $\bar{f}(\bar{a} + \bar{a}_1\bar{\alpha}_1 + \bar{a}_2\bar{\alpha}_2)$ is unimodular in $\mathfrak{a}/\mathfrak{a}^2$. Since α generates a free direct summand of rank 1 with supplement P' , it follows that $\alpha + a_1\alpha_1 + a_2\alpha_2$ also generates a free direct summand of rank 1.

The upshot of the above discussion is that we may assume by changing α to $\alpha + a_1\alpha_1 + a_2\alpha_2$, that the class of $f(\alpha)$ in $\mathfrak{a}/\mathfrak{a}^2$ is unimodular. Set $f(\alpha) = a$. We claim that \mathfrak{a}/Aa is projective ideal of rank 1 in A/aA . To show this we observe that since $\mathfrak{a}/\mathfrak{a} \approx \bar{A}\bar{\alpha} \oplus D$ for some D , it follows that for any maximal ideal $\mathfrak{M} \supset \mathfrak{a}$, a can be chosen as one of the two generators for $\mathfrak{a}_{\mathfrak{M}}$. Since any two generators of $\mathfrak{a}_{\mathfrak{M}}$ form an $A_{\mathfrak{M}}$ -sequence, it follows that \mathfrak{a}/Aa is locally generated by one element which is even a non-zero-divisor. Thus \mathfrak{a}/Aa is a projective A/aA -module of rank 1.

By the exact sequence (*), we get (since $f(\alpha) = a$) the exact sequence

$$0 \rightarrow A^2 \rightarrow P' \rightarrow \mathfrak{a}/Aa \rightarrow 0.$$

Tensoring this sequence with A/aA , we have the exact sequence

$$\frac{A^2}{aA^2} \xrightarrow{g} \frac{P'}{aP'} \rightarrow \mathfrak{a}/Aa \rightarrow 0.$$

Since $\mathfrak{a}/A\mathfrak{a}$ is projective, we have

$$\frac{P'}{aP'} \approx \text{Img} \oplus \mathfrak{a}/A\mathfrak{a}$$

Taking $\overset{2}{\wedge}$ both sides and observing that $\overset{2}{\wedge} P' \approx A$ (since P' is stably free) and so $\overset{2}{\wedge} (P'/aP') \approx A/A\mathfrak{a}$, it follows that

$$\mathfrak{a}/A\mathfrak{a} \otimes \text{Img} \approx A/aA \quad \text{i.e.} \quad \mathfrak{a}/aA \approx (\text{Img})^{-1}.$$

Since Img is generated by two elements, it easily follows that \mathfrak{a}/aA is generated by two elements. Hence \mathfrak{a} is generated by three elements. The proof of the theorem is completely established.

Since by [2, p. 639, (3.5)] $\tilde{K}_0(K[x_1, \dots, x_n])=0$, where K is a field, we have

COROLLARY. *Let $\mathfrak{a} \subset K[x_1, x_2, x_3]$ be an unmixed ideal of height 2. Suppose \mathfrak{a} is locally generated by r elements. Then \mathfrak{a} is generated by $r+1$ elements.*

A particular case of the Corollary which may be of interest to us

COROLLARY 2. *Let C be a closed affine curve in \mathbf{A}_3 which is locally a complete intersection (e.g. C non-singular), then the ideal $I(C)$ of C is generated by three elements.*

Remark. (i) One can multiply examples of three dimensional regular rings with trivial \tilde{K}_0 for instance, by using the following well known facts

a) $\tilde{K}_0 A = 0$, if A is a principal ideal domain or a local ring.

b) (Grothendieck), If A is regular, $K_0 A \rightarrow K_0 A[x]$ is an isomorphism .

c) If A is regular, then $K_0 A \rightarrow K_0 S^{-1}A$ is surjective, S being multiplicatively closed set.

(ii) (Fossum and Claborn) Let K be a field of characteristic $\neq 2$ with $\sqrt{-1} \in K$ or $K = \mathbf{R}$ and $A = K[x_0, x_1, x_2, x_3]$, $\sum x_i^2 = 1$. Then $\tilde{K}_0 A = 0$.

AN EXAMPLE. The theorem above is best possible in the sense that there do exist non-singular affine curves C in \mathbf{A}_3 whose prime ideals are *not* generated by two elements. For example, let C be a complete non-singular curve of genus 2. Let Ω denote a divisor in its canonical class. Let $P \in C$ be such that Ω is not linearly equivalent to $2P$ (such points exist since otherwise the Jacobian variety of C will be a 2-torsion group!). Consider $C' = C - \{P\}$. Then C' is an affine non-singular curve which can be embedded as a closed set in \mathbf{A}_3 . This we can do for instance by considering the complete linear system²⁾ $|5P|$ or by a well known result which says that any non-singular affine curve can be embedded as a closed set in \mathbf{A}_3 . Let \mathfrak{B} be the ideal of C' in $A = K[x_1, x_2, x_3]$ (K algebraically closed). We claim that \mathfrak{B} is not generated

²⁾ I am thankful M. S. Narasimhan for pointing this to me.

by two elements. For if \mathfrak{B} were generated by two elements, then $\text{Ext}_A^2(A/\mathfrak{B}, A) \approx A/\mathfrak{B}$ (this one can be seen for example by Koszul-resolution for A/\mathfrak{B}). Since by [4], $\text{Ext}_A^2(A/\mathfrak{B}, A)$ is the module of sections of the canonical line bundle $\Omega_{C'}$, it follows that $\Omega_{C'}$ is trivial. This implies that a canonical divisor $\Omega \sim nP$. Since $\deg \Omega = 2$, we have $\Omega \sim 2P$. Contradiction.

We do not know if the hypothesis $\tilde{K}_0(A) = 0$ is essential in our theorem.

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Added in Proof: In the case when $r \geq 3$, our result is an easy consequence of Swan [7]. Also, it is not difficult to see that in the statement of the theorem one can drop the hypothesis that \mathfrak{a} be of height 2.