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# Generators for Certain Ideals in Regular Rings of Dimension Three

by M. PAVANAN MURTHY<sup>1)</sup>

## Introduction

We prove here the following

**THEOREM.** *Let  $A$  be a regular ring of dimension 3 with  $\tilde{K}_0 A = 0$ . Let  $\mathfrak{a}$  be an unmixed ideal of height 2. Suppose  $\mathfrak{a}$  is locally generated by  $r$  elements. Then  $\mathfrak{a}$  is generated by  $r+1$  elements.*

Applying this theorem to  $A = K[x_1, x_2, x_3]$ ,  $K$  a field we obtain for instance that if  $C$  is a curve in the affine three space  $\mathbf{A}_3$ , which is locally a complete intersection (e.g.  $C$  non-singular), then the ideal of  $C$  is generated by three elements. We also show that this is best possible by giving an example of a non-singular curve in  $\mathbf{A}_3$  which is not a complete intersection.

In the case  $A = K[x_1, x_2, x_3]$ ,  $K$  algebraically closed and  $\mathfrak{a}$  the ideal of a non-singular curve, S. Abhyankar has proved this by quite different methods (see his Montreal Lecture Notes [1]).

A basic tool in the proof is a lemma of Serre [6] which relates projective modules with generators of certain ideals of height 2. In fact for  $r \geq 3$ , our theorem easily follows from a corollary to Serre's lemma (see corollary to Lemma 1). For  $r = 2$ , we have to make a separate argument using a remark of Bass [3] ( $P \oplus A = A^{2n} \Rightarrow P = P' \oplus A$ ).

We consider here only commutative noetherian rings and finitely generated modules. Most of the time we just use the ring  $A$ . For a module  $M$ ,  $\text{hd } M$  denotes its homological dimension.  $\dim A$  denotes the Krull dimension of  $A$ .

The following lemma is basic for what follows. We include a proof here for the sake of completeness.

**LEMMA 1** (Serre [6]). *Let  $A$  be a noetherian ring and  $M$  a left  $A$ -module of homological dimension  $\leq 1$ . Let  $\text{Ext}_A^1(M, A)$  be generated by one element. Then there is an exact sequence*

$$0 \rightarrow A \rightarrow P \rightarrow M \rightarrow 0$$

*with  $P$  projective.*

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*Proof.* Let  $\alpha$  generate  $\text{Ext}_A^1(M, A)$  and let  $\alpha$  correspond to the extension  $(\alpha)$ :

$$0 \rightarrow A \rightarrow P \rightarrow M \rightarrow 0$$

Then homing  $(\alpha)$  with  $A$ , we get the exact sequence

$$\text{Hom}(P, A) \rightarrow \text{Hom}(A, A) \xrightarrow{f} \text{Ext}_A^1(M, A) \rightarrow \text{Ext}_A^1(P, A) \rightarrow 0.$$

By extension theory,  $f(1) = \alpha$ . Since  $\text{Ext}_A^1(M, A)$  is generated by  $\alpha$ , we have  $\text{Ext}_A^1(P, A) = 0$ . Since  $\text{hd } M \leq 1$ , by  $(\alpha)$ , it follows that  $\text{hd } P \leq 1$ . Since  $A$  is noetherian,  $\text{hd } P \leq 1$  and  $\text{Ext}_A^1(P, A) = 0$  together imply that  $P$  is projective.

**COROLLARY.** *Let  $A$  be a noetherian ring and  $M$  an  $A$ -module of homological dimension  $\leq 1$ . Let  $\text{Ext}_A(M, A)$  be generated by  $r$  elements. Then there is an exact sequence*

$$0 \rightarrow A^r \rightarrow P \rightarrow M \rightarrow 0$$

with  $P$  projective.

*Proof.* We prove the corollary by induction on  $r$ . For  $r = 1$ , this is precisely Serre's lemma. Assume that the corollary is true for  $r - 1$ . Let  $\alpha_1, \dots, \alpha_r$  generate  $\text{Ext}_A^1(M, A)$ . Let  $\alpha_1$  correspond to the extension  $(\alpha_1)$ :

$$0 \rightarrow A \rightarrow L \xrightarrow{h} M \rightarrow 0$$

Then we get the exact sequence

$$\text{Hom}(L, A) \rightarrow \text{Hom}(A, A) \xrightarrow{f} \text{Ext}_A^1(M, A) \xrightarrow{g} \text{Ext}_A^1(L, A) \rightarrow 0,$$

where  $f(1) = \alpha_1$ . Hence  $\text{Ext}_A^1(L, A)$  is generated by  $r - 1$  elements  $g(\alpha_2), \dots, g(\alpha_r)$ . The exact sequence  $(\alpha_1)$  and the hypothesis that  $\text{hd } M \leq 1$  show that  $\text{hd } L \leq 1$ . Hence by induction hypothesis there is an exact sequence  $I$ :

$$0 \rightarrow A^{r-1} \rightarrow P \xrightarrow{k} L \rightarrow 0$$

with  $P$  projective. We also have the exact sequence

$$0 \rightarrow \text{Ker}(h \circ k) \rightarrow P \xrightarrow{h \circ k} M \rightarrow 0.$$

By the exact sequences  $(\alpha_1)$  and  $I$  it easily follows that  $\text{Ker}(h \circ k) \approx A^r$ . The proof of the corollary is complete.

For a ring  $A$ , we denote  $\text{Max}(A)$  its maximal ideal spectrum and by  $\dim \text{Max}(A)$ , the dimension of  $\text{Max}(A)$ . If  $A$  is an integral domain with quotient field  $K$  and  $M$  an  $A$ -module, we recall that  $\text{rank } M = \dim_K(K \otimes_A M)$ .

**LEMMA 2.** *Let  $A$  be a noetherian domain. Let  $M$  be an  $A$ -module of  $\text{hd } M \leq 1$ . Let  $M$  be an  $A$ -module of rank  $n$ . Let  $\alpha$  be the annihilator of  $\text{Ext}_A^1(M, A)$ . Suppose that*

$\dim \text{Max}(A/\mathfrak{a}) \leq d$ . If  $M$  is locally generated by  $r$  elements, then  $\text{Ext}_A^1(M, A)$  is generated by  $r+d-n$  elements.

*Proof.* By Swan [7], we need only prove that  $\text{Ext}_A^1(M, A)$  is locally generated by  $r-n$  elements. So, we may assume  $A$  is local. Since  $\text{hd } M \leq 1$   $M$  is generated by  $r$  elements and rank  $M = n$ , we get an exact sequence (since projectives are free over local rings)

$$0 \rightarrow A^{r-n} \rightarrow A^r \rightarrow M \rightarrow 0.$$

Homing this with  $A$ , we see that  $\text{Hom}(A^{r-n}, A) \rightarrow \text{Ext}_A^1(M, A) \rightarrow 0$  is exact. This shows that  $\text{Ext}_A^1(M, A)$  is generated by  $r-n$  elements.

**COROLLARY.** *Let  $A$  be a regular domain of dimension 3. Let  $\mathfrak{a}$  be an unmixed ideal of height 2. If  $\mathfrak{a}$  is locally generated by  $r$  elements, then  $\text{Ext}_A^1(\mathfrak{a}, A)$  is generated by  $r$  elements.*

*Proof.* Using the well known fact that  $\text{depth } M + \text{hd } M = \dim A$  for a regular local ring  $A$ , one easily sees that  $\mathfrak{a}$  is unmixed of height 2 implies  $\text{hd } \mathfrak{a} \leq 1$ . Since  $\text{Ext}_A^1(\mathfrak{a}, A) \approx \text{Ext}_A^2(A/\mathfrak{a}, A)$ , it follows that annihilator  $\mathfrak{b}$  of  $\text{Ext}_A^1(\mathfrak{a}, A)$  contains  $\mathfrak{a}$ . Hence  $\dim A/\mathfrak{b} \leq \dim A/\mathfrak{a} \leq 1$ , since  $\mathfrak{a}$  is unmixed of height 2 and  $\dim A = 3$ . Now the corollary follows from Lemma 2.

Let  $A$  be a ring and  $P$  an  $A$ -module. We recall that  $s \in P$  is *unimodular* if  $s$  generates a free direct summand of  $P$ , isomorphic to  $A$ . For  $x \in \text{Max}(A)$ , we denote by  $s(x)$  the image of  $s$  under the canonical map  $P \rightarrow P/xP$ .

**LEMMA 3.** *Let  $A$  be a noetherian ring of dimension  $\leq 1$  and  $P$  a projective  $A$ -module of rank 2. If  $s_1, s_2, s_3$  generate  $P$ , then there exist  $\lambda_2, \lambda_3 \in A$  such that  $s_1 + \lambda_2 s_2 + \lambda_3 s_3$  is unimodular.*

*Proof.* Let  $\mathfrak{P}_1, \dots, \mathfrak{P}_r$  be the minimal prime ideals of  $A$ . Choose  $\mathfrak{M}_1, \dots, \mathfrak{M}_r \in \text{Max}(A)$  such that  $\mathfrak{M}_i \supset \mathfrak{P}_i$ . Since  $P$  is of rank 2,  $P/(\prod \mathfrak{M}_i)P$  is  $A/\prod \mathfrak{M}_i$  free of rank 2. Using Chinese Remainder Theorem we can find easily  $a, b \in A$  such that if we set  $s'_1 = s_1 + as_3$ ,  $s'_2 = s_2 + bs_3$ , then  $s'_1(\mathfrak{M}_i), s'_2(\mathfrak{M}_i)$  are linearly independent over  $A/\mathfrak{M}_i$ ,  $1 \leq i \leq r$ . Thus we may assume  $s_1, s_2$  are linearly independent at  $\mathfrak{M}_i$ ,  $1 \leq i \leq r$ . Then the set  $T = \{\mathfrak{M} \in \text{Max}(A) \mid s_1(\mathfrak{M}), s_2(\mathfrak{M}) \text{ are linearly dependent}\}$  is a closed set [5, p. 6] which does not contain any irreducible component of  $\text{Max}(A)$ . Since  $\dim A \leq 1$ , it follows that  $T$  is finite. Let  $U = \{\mathfrak{M} \in \text{Max}(A) \mid s_1(\mathfrak{M}) = 0\}$ . Then  $U \subset T$ . Since  $P$  is of constant rank 2 and  $s_1, s_2, s_3$  generate  $P$  it follows that  $s_2(\mathfrak{M}) \neq 0$ ,  $\mathfrak{M} \in U$ . Choose  $f \in A$  such that  $f(\mathfrak{M}) \neq 0$ ,  $\mathfrak{M} \in U$  and  $f(\mathfrak{M}) = 0$ ,  $\mathfrak{M} \in T - U$ . Clearly  $s_1 + fs_2$  is unimodular.

**THEOREM.** *Let  $A$  be a regular integral domain of dimension 3 with  $\tilde{K}_0 A = 0$ . Let  $\mathfrak{a}$  be an ideal unmixed of height 2. Suppose  $\mathfrak{a}$  is locally generated by  $r$  elements. Then  $\mathfrak{a}$  is generated by  $r+1$  elements.*

We recall that  $\tilde{K}_0 A = 0$  is equivalent to saying that for any finitely generated projective  $A$ -module  $P$ ,  $P \oplus A^m \approx A^n$  for some  $m, n$ .

*Proof of the theorem.* By the corollary to Lemma 2, we see that  $\text{Ext}_A^1(\mathfrak{a}, A)$  is generated by  $r$  elements. Consequently by the corollary to Lemma 1 (since  $\text{hd } \mathfrak{a} \leq 1$ ), we get an exact sequence

$$0 \rightarrow A^r \rightarrow P \rightarrow \mathfrak{a} \rightarrow 0.$$

with  $P$  a projective  $A$ -module of rank  $r+1$ . If  $r \geq 3$ , then  $\text{rank } P \geq 4$ . Since  $\tilde{K}_0 A = 0$  and  $\dim A = 3$ , it follows from Bass' cancellation theorem [2, p. 184, (3.5)] that  $P$  is free. Hence  $\mathfrak{a}$  is generated by  $r+1$  elements. Thus the theorem is proved in case  $r \geq 3$ .

Now we consider the case  $r=2$ . In this case  $P$  is a projective module of rank 3. Again by [2, p. 184, (3.5)],  $P \oplus A \approx A^4$ . Hence by [3],  $P$  admits a free direct summand of rank 1:  $P = P' \oplus A\alpha$ . We have the exact sequence

$$0 \rightarrow A^2 \rightarrow P' \oplus A\alpha \xrightarrow{f} \mathfrak{a} \rightarrow 0. \quad (*)$$

Since  $A$  is a regular ring and  $\mathfrak{a}$  is an ideal of height 2, locally generated by two elements, it follows that  $\mathfrak{a}_{\mathfrak{M}}$  is generated by an  $A_{\mathfrak{M}}$ -sequence of length 2 for any maximal ideal  $\mathfrak{M} \supset \mathfrak{a}$ . Hence  $\mathfrak{a}/\mathfrak{a}^2$  is a locally free  $A/\mathfrak{a}$ -module of rank 2. Tensoring the exact sequence  $(*)$  by  $A/\mathfrak{a}$ , we get the exact sequence

$$A^2 \rightarrow \bar{P}' \oplus \bar{A}\bar{\alpha} \xrightarrow{f} \mathfrak{a}/\mathfrak{a}^2 \rightarrow 0$$

where  $\bar{M} = M/\mathfrak{a}M$  for an  $A$ -module  $M$  and for  $x \in M$ ,  $\bar{x}$  denotes residue class of  $x$  modulo  $\mathfrak{a}M$ . Since  $P'$  is stably free, so is the  $\bar{A}$ -module  $\bar{P}'$ . Since dimension of  $\bar{A} \leq 1$ , it follows by [2, p. 170 §2], that  $\bar{P}' \approx \bar{A}^2$ . Let  $\alpha_1, \alpha_2 \in P'$  be such that  $\bar{\alpha}_1, \bar{\alpha}_2$  generated  $\bar{P}'$ . Then  $f(\alpha), f(\bar{\alpha}_1), f(\bar{\alpha}_2)$  generate  $\mathfrak{a}/\mathfrak{a}^2$ . Since  $\mathfrak{a}/\mathfrak{a}^2$  is a projective  $A/\mathfrak{a}$ -module of rank 2, by Lemma 3, there exist  $\bar{a}_1, \bar{a}_2 \in \bar{A}$  such that  $f(\bar{a} + \bar{a}_1\bar{\alpha}_1 + \bar{a}_2\bar{\alpha}_2)$  is unimodular in  $\mathfrak{a}/\mathfrak{a}^2$ . Since  $\alpha$  generates a free direct summand of rank 1 with supplement  $P'$ , it follows that  $\alpha + a_1\alpha_1 + a_2\alpha_2$  also generates a free direct summand of rank 1.

The upshot of the above discussion is that we may assume by changing  $\alpha$  to  $\alpha + a_1\alpha_1 + a_2\alpha_2$ , that the class of  $f(\alpha)$  in  $\mathfrak{a}/\mathfrak{a}^2$  is unimodular. Set  $f(\alpha) = a$ . We claim that  $\mathfrak{a}/Aa$  is projective ideal of rank 1 in  $A/aA$ . To show this we observe that since  $\mathfrak{a}/\mathfrak{a} \approx \bar{A}\bar{a} \oplus D$  for some  $D$ , it follows that for any maximal ideal  $\mathfrak{M} \supset \mathfrak{a}$ ,  $a$  can be chosen as one of the two generators for  $\mathfrak{a}_{\mathfrak{M}}$ . Since any two generators of  $\mathfrak{a}_{\mathfrak{M}}$  form an  $A_{\mathfrak{M}}$ -sequence, it follows that  $\mathfrak{a}/Aa$  is locally generated by one element which is even a non-zero-divisor. Thus  $\mathfrak{a}/Aa$  is a projective  $A/aA$ -module of rank 1.

By the exact sequence  $(*)$ , we get (since  $f(\alpha) = a$ ) the exact sequence

$$0 \rightarrow A^2 \rightarrow P' \rightarrow \mathfrak{a}/Aa \rightarrow 0.$$

Tensoring this sequence with  $A/aA$ , we have the exact sequence

$$\frac{A^2}{aA^2} \xrightarrow{g} \frac{P'}{aP'} \rightarrow \mathfrak{a}/Aa \rightarrow 0.$$

Since  $\mathfrak{a}/Aa$  is projective, we have

$$\frac{P'}{aP'} \approx \text{Img} \oplus \mathfrak{a}/Aa$$

Taking  $\wedge^2$  both sides and observing that  $\wedge^2 P' \approx A$  (since  $P'$  is stably free) and so  $\wedge^2(P'/aP') \approx A/\mathfrak{a}A$ , it follows that

$$\mathfrak{a}/Aa \otimes \text{Img} \approx A/\mathfrak{a}A \quad \text{i.e.} \quad \mathfrak{a}/Aa \approx (\text{Img})^{-1}.$$

Since  $\text{Img}$  is generated by two elements, it easily follows that  $\mathfrak{a}/Aa$  is generated by two elements. Hence  $\mathfrak{a}$  is generated by three elements. The proof of the theorem is completely established.

Since by [2, p. 639, (3.5)]  $\tilde{K}_0(K[x_1, \dots, x_n]) = 0$ , where  $K$  is a field, we have

**COROLLARY.** *Let  $\mathfrak{a} \subset K[x_1, x_2, x_3]$  be an unmixed ideal of height 2. Suppose  $\mathfrak{a}$  is locally generated by  $r$  elements. Then  $\mathfrak{a}$  is generated by  $r+1$  elements.*

A particular case of the Corollary which may be of interest to us

**COROLLARY 2.** *Let  $C$  be a closed affine curve in  $\mathbf{A}_3$  which is locally a complete intersection (e.g.  $C$  non-singular), then the ideal  $I(C)$  of  $C$  is generated by three elements.*

**Remark.** (i) One can multiply examples of three dimensional regular rings with trivial  $\tilde{K}_0$  for instance, by using the following well known facts

a)  $\tilde{K}_0 A = 0$ , if  $A$  is a principal ideal domain or a local ring.

b) (Grothendieck), If  $A$  is regular,  $K_0 A \rightarrow K_0 A[x]$  is an isomorphism.

c) If  $A$  is regular, then  $K_0 A \rightarrow K_0 S^{-1} A$  is surjective,  $S$  being multiplicatively closed set.

(ii) (Fossum and Claborn) Let  $K$  be a field of characteristic  $\neq 2$  with  $\sqrt{-1} \in K$  or  $K = \mathbf{R}$  and  $A = K[x_0, x_1, x_2, x_3]$ ,  $\sum x_i^2 = 1$ . Then  $\tilde{K}_0 A = 0$ .

**AN EXAMPLE.** The theorem above is best possible in the sense that there do exist non-singular affine curves  $C$  in  $\mathbf{A}_3$  whose prime ideals are *not* generated by two elements. For example, let  $C$  be a complete non-singular curve of genus 2. Let  $\Omega$  denote a divisor in its canonical class. Let  $P \in C$  be such that  $\Omega$  is not linearly equivalent to  $2P$  (such points exist since otherwise the Jacobian variety of  $C$  will be a 2-torsion group!). Consider  $C' = C - \{P\}$ . Then  $C'$  is an affine non-singular curve which can be embedded as a closed set in  $\mathbf{A}_3$ . This we can do for instance by considering the complete linear system<sup>2)</sup>  $|5P|$  or by a well known result which says that any non-singular affine curve can be embedded as a closed set in  $\mathbf{A}_3$ . Let  $\mathfrak{P}$  be the ideal of  $C'$  in  $A = K[x_1, x_2, x_3]$  ( $K$  algebraically closed). We claim that  $\mathfrak{P}$  is not generated

<sup>2)</sup> I am thankful M. S. Narasimhan for pointing this to me.

by two elements. For if  $\mathfrak{P}$  were generated by two elements, then  $\text{Ext}_A^2(A/\mathfrak{P}, A) \approx A/\mathfrak{P}$  (this one can be seen for example by Koszul-resolution for  $A/\mathfrak{P}$ ). Since by [4],  $\text{Ext}_A^2(A/\mathfrak{P}, A)$  is the module of sections of the canonical line bundle  $\Omega_{C'}$  it follows that  $\Omega_{C'}$  is trivial. This implies that a canonical divisor  $\Omega \sim nP$ . Since  $\deg \Omega = 2$ , we have  $\Omega \sim 2P$ . Contradiction.

We do not know if the hypothesis  $\tilde{K}_0(A) = 0$  is essential in our theorem.

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*Added in Proof:* In the case when  $r \geq 3$ , our result is an easy consequence of Swan [7]. Also, it is not difficult to see that in the statement of the theorem one can drop the hypothesis that  $\mathfrak{a}$  be of height 2.