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Generators for Certain Ideals in Regular Rings of Dimension Three

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Introduction

We prove here the following

THEOREM. *Let A be a regular ring of dimension* 3 *with* $\tilde{K}_0 A = 0$. *Let* $\mathfrak{a}$ *bean unmixed ideal of height* 2. *Suppose* $\mathfrak{a}$ *is locally generated by* r *elements. Then* $\mathfrak{a}$ *is generated by* $r+1$ *elements.*

Applying this theorem to $A = K[x_1, x_2, x_3]$, K a field we obtain for instance that if C is a curve in the affine three space $\mathbf{A}_3$, which is locally a complete intersection (e.g. C non-singular), then the ideal of C is generated by three elements. We also show that this is best possible by givinh an exvmple of a non-singular curve in $\mathbf{A}_3$ which is not a complete intersection.

In the case $A = K[x_1, x_2, x_3]$, K algebraically closed and $\mathfrak{a}$ the ideal of a non-singular curve, S. Abhyankar has proved this by quite different methods (see his Montreal Lecture Notes [1]).

A basic tool in the proof is a lemma of Serre [6] which relates projective modules with generators of certain ideals of height 2. In fact for $r \geqslant 3$, our theorem easily follows from a corollary to Serre's lemma (see corollary to Lemma 1). For $r=2$, we have to make a separate argument using a remark of Bass [3] ($P \oplus A = A^{2n} = > P = = P' \oplus A$).

We consider here only commutative noetherian rings and finitely generated modules. Most of the time we just use the ring A. For a module M, hd$M$ denotes its homological dimension. dim$A$ denotes the Krull dimension of A.

The following lemma is basic for what follows. We include a proof here for the sake of completeness.

LEMMA 1 (Serre [6]). *Let A be a noetherian ring and M a left A-module of homological dimension* $\leqslant 1$. *Let* $\operatorname{Ext}^1_A(M, A)$ *be generated by one element. Then there is an exact sequence*

$$0 \to A \to P \to M \to 0$$

with P projective.

[1]) I am thankful to the Forschungsinstitut, ETH, Zurich, for support when this note was being written.

Proof. Let α generate $\operatorname{Ext}_A^1(M, A)$ and let α correspond to the extension (α):

$$0 \to A \to P \to M \to 0$$

Then homing (α) with A, we get the exact sequence

$$\operatorname{Hom}(P, A) \to \operatorname{Hom}(A, A) \xrightarrow{f} \operatorname{Ext}_A^1(M, A) \to \operatorname{Ext}_A^1(P, A) \to 0.$$

By extension theory, $f(1)=\alpha$. Since $\operatorname{Ext}_A^1(M, A)$ is generated by α, we have $\operatorname{Ext}_A^1(P, A)=0$. Since $\operatorname{hd} M \leqslant 1$, by (α), it follows that $\operatorname{hd} P \leqslant 1$. Since A is noetherian, $\operatorname{hd} P \leqslant 1$ and $\operatorname{Ext}_A^1(P, A)=0$ together imply that P is projective.

COROLLARY. *Let A be a noetherian ring and M an A-module of homological dimension $\leqslant 1$. Let $\operatorname{Ext}_A(M, A)$ be generated by r elements. Then there is an exact sequence*

$$0 \to A^r \to P \to M \to 0$$

with P projective.

Proof. We prove the corollary by induction on r. For $r=1$, this is precisely Serre's lemma. Assume that the corollary is true for $r-1$. Let $\alpha_1, \ldots, \alpha_r$ generate $\operatorname{Ext}^1(M, A)$. Let α_1 correspond to the extension (α_1):

$$0 \to A \to L \xrightarrow{h} M \to 0$$

Then we get the exact sequence

$$\operatorname{Hom}(L, A) \to \operatorname{Hom}(A, A) \xrightarrow{f} \operatorname{Ext}_A^1(M, A) \xrightarrow{g} \operatorname{Ext}_A^1(L, A) \to 0,$$

where $f(1)=\alpha_1$. Hence $\operatorname{Ext}_A^1(L, A)$ is generated by $r-1$ elements $g(\alpha_2), \ldots, g(\alpha_r)$. The exact sequence (α_1) and the hypothesis that $\operatorname{hd} M \leqslant 1$ show that $\operatorname{hd} L \leqslant 1$. Hence by induction hypothesis there is an exact sequence I:

$$0 \to A^{r-1} \to P \xrightarrow{k} L \to 0$$

with P projective. We also have the exact sequence

$$0 \to \operatorname{Ker}(h \circ k) \to P \xrightarrow{h \circ k} M \to 0.$$

By the exact sequences (α_1) and I it easily follows that $\operatorname{Ker}(h \circ k) \approx A^r$. The proof of the corollary is complete.

For a ring A, we denote $\operatorname{Max}(A)$ its maximal ideal spectrum and by $\dim \operatorname{Max}(A)$, the dimension of $\operatorname{Max}(A)$. If A is an integral domain with quotient field K and M an A-module, we recall that $\operatorname{rank} M = \dim_K(K \otimes_A M)$.

LEMMA 2. *Let A be a noetherian domain. Let M be an A-module of* $\operatorname{hd} M \leqslant 1$. *Let M be an A-module of rank n. Let $\mathfrak{a}$ be the annihilator of* $\operatorname{Ext}_A^1(M, A)$. *Suppose that*

$\dim \mathrm{Max}(A/\mathfrak{a}) \leqslant d$. *If M is locally generated by r elements, then* $\mathrm{Ext}_A^1(M, A)$ *is generated by $r+d-n$ elements.*

Proof. By Swan [7], we need only prove that $\mathrm{Ext}_A^1(M, A)$ is locally generated by $r-n$ elements. So, we may assume A is local. Since $\mathrm{hd}\, M \leqslant 1$ M is generated by r elements and rank $M=n$, we get an exact sequence (since projectives are free over local rings)

$$0 \to A^{r-n} \to A^r \to M \to 0.$$

Homing this with A, we see that $\mathrm{Hom}(A^{r-n}, A) \to \mathrm{Ext}_A^1(M, A) \to 0$ is exact. This shows that $\mathrm{Ext}_A^1(M, A)$ is generated by $r-n$ elements.

COROLLARY. *Let A be a regular domain of dimension* 3. *Let $\mathfrak{a}$ be an unmixed ideal of height* 2. *If $\mathfrak{a}$ is locally generetad by r elements, then* $\mathrm{Ext}_A^1(\mathfrak{a}, A)$ *is generated by r elements.*

Proof. Using the well known fact that depth $M + \mathrm{hd}\, M = \dim A$ for a regular local ring A, one easily sees that $\mathfrak{a}$ is unmixed of height 2 implies $\mathrm{hd}\, \mathfrak{a} \leqslant 1$. Since $\mathrm{Ext}_A^1(\mathfrak{a}, A) \approx \approx \mathrm{Ext}_A^2(A/\mathfrak{a}, A)$, it follows that annihilator $\mathfrak{b}$ of $\mathrm{Ext}_A^1(\mathfrak{a}, A)$ contains $\mathfrak{a}$. Hence $\dim A/\mathfrak{b} \leqslant \dim A/\mathfrak{a} \leqslant 1$, since $\mathfrak{a}$ is unmixed of height 2 and $\dim A = 3$. Now the corollary follows from Lemma 2.

Let A be a ring and P an A-module. We recall that $s \in P$ is *unimodular* if s generates a free direct summand of P, isomorphic to A. For $x \in \mathrm{Max}(A)$, we denote by $s(x)$ the image of s under the canonical map $P \to P/xP$.

LEMMA 3. *Let A be a noetherian ring of dimension $\leqslant 1$ and P a projective A-module of rank* 2. *If s_1, s_2, s_3 generate P, then there exist $\lambda_2, \lambda_3 \in A$ such that $s_1 + \lambda_2 s_2 + \lambda_3 s_3$ is unimodular.*

Proof. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be the minimal prime ideals of A. Choose $\mathfrak{M}_1, \ldots, \mathfrak{M}_r \in \mathrm{Max}(A)$ such that $\mathfrak{M}_i \supset \mathfrak{P}_i$. Since P is of rank 2, $P/(\Pi \mathfrak{M}_i)P$ is $A/\Pi \mathfrak{M}_i$ free of rank 2. Using Chineese Remainder Theorem we can find easily $a, b \in A$ such that if we set $s_1' = s_1 + a s_3$, $s_2' = s_2 + b s_3$, then $s_1'(\mathfrak{M}_i)$, $s_2'(\mathfrak{M}_i)$ are linearly independent over $A/\mathfrak{M}_i$, $1 \leqslant i \leqslant r$. Thus we may assume s_1, s_2 are linearly independent at $\mathfrak{M}_i$, $1 \leqslant i \leqslant r$. Then the set $T = \{\mathfrak{M} \in \mathrm{Max}(A) \mid s_1(\mathfrak{M}), s_2(\mathfrak{M}) \text{ are linearly dependent}\}$ is a closed set [5, p. 6] which does not contain any irreducible component of $\mathrm{Max}(A)$. Since $\dim A \leqslant 1$, it follows that T is finite. Let $U = \{\mathfrak{M} \in \mathrm{Max}(A) \mid s_1(\mathfrak{M}) = 0\}$. Then $U \subset T$. Since P is of constant rank 2 and s_1, s_2, s_3 generate P it follows that $s_2(\mathfrak{M}) \neq 0$, $\mathfrak{M} \in U$. Choose $f \in A$ such that $f(\mathfrak{M}) \neq 0$, $\mathfrak{M} \in U$ and $f(\mathfrak{M}) = 0$, $\mathfrak{M} \in T - U$. Clearly $s_1 + f s_2$ is unimodular.

THEOREM. *Let A be a regular integral domain of dimension* 3 *with $\tilde{K}_0 A = 0$. Let $\mathfrak{a}$ be an ideal unmixed of height* 2. *Suppose $\mathfrak{a}$ is locally generated by r elements. Then $\mathfrak{a}$ is generated by $r+1$ elements.*

We recall that $\tilde{K}_0 A=0$ is equivalent to saying that for any finitely generated projective A-module P, $P \oplus A^m \approx A^n$ for some m, n.

Proof of the theorem. By the corollary to Lemma 2, we see that $\mathrm{Ext}_A^1(\mathfrak{a}, A)$ is generated by r elements. Consequently by the corollary to Lemma 1 (since $\mathrm{hd}\,\mathfrak{a} \leqslant 1$), we get an exact sequence

$$0 \to A^r \to P \to \mathfrak{a} \to 0.$$

with P a projective A-module of rank $r+1$. If $r \geqslant 3$, then rank $P \geqslant 4$. Since $\tilde{K}_0 A=0$ and $\dim A=3$, it follows from Bass' cancellation theorem [2, p. 184, (3.5)] that P is free. Hence $\mathfrak{a}$ is generated by $r+1$ elements. Thus the theorem is proved in case $r \geqslant 3$.

Now we consider the case $r=2$. In this case P is a projective module of rank 3. Again by [2, p. 184, (3.5)], $P \oplus A \approx A^4$. Hence by [3], P admits a free direct summand of rank 1: $P=P' \oplus A\alpha$. We have the exact sequence

$$0 \to A^2 \to P' \oplus A\alpha \xrightarrow{f} \mathfrak{a} \to 0. \qquad (*)$$

Since A is a regular ring and $\mathfrak{a}$ is an ideal of height 2, locally generated by two elements, it follows that $\mathfrak{a}_{\mathfrak{M}}$ is generated by an $A_{\mathfrak{M}}$-sequence of length 2 for any maximal ideal $\mathfrak{M} \supset \mathfrak{a}$. Hence $\mathfrak{a}/\mathfrak{a}^2$ is a locally free $A/\mathfrak{a}$-module of rank 2. Tensoring the exact sequence $(*)$ by $A/\mathfrak{a}$, we get the exact sequence

$$A^2 \to \bar{P}' \oplus \bar{A}\bar{\alpha} \xrightarrow{\bar{f}} \mathfrak{a}/\mathfrak{a}^2 \to 0$$

where $\bar{M}=M/\mathfrak{a}M$ for an A-module M and for $x \in M$, $\bar{x}$ denotes residue class of x modulo $\mathfrak{a}M$. Since P' is stably free, so is the $\bar{A}$-module $\bar{P}'$. Since dimension of $\bar{A} \leqslant 1$, it follows by [2, p. 170 §2], that $\bar{P}' \approx \bar{A}^2$. Let $\alpha_1, \alpha_2 \in P'$ be such that $\bar{\alpha}_1, \bar{\alpha}_2$ generated $\bar{P}'$. Then $\bar{f}(\alpha), \bar{f}(\bar{\alpha}_1), \bar{f}(\bar{\alpha}_2)$ generate $\mathfrak{a}/\mathfrak{a}^2$. Since $\mathfrak{a}/\mathfrak{a}^2$ is a projective $A/\mathfrak{a}$-module of rank 2, by Lemma 3, there exist $\bar{a}_1, \bar{a}_2 \in \bar{A}$ such that $\bar{f}(\bar{a}+\bar{a}_1\bar{\alpha}_1+\bar{a}_2\bar{\alpha}_2)$ is unimodular in $\mathfrak{a}/\mathfrak{a}^2$. Since α generates a free direct summand of rank 1 with supplement P', it follows that $\alpha+a_1\alpha_1+a_2\alpha_2$ also generates a free direct summand of rank 1.

The upshot of the above discussion is that we may assume by changing α to $\alpha+a_1\alpha_1+a_2\alpha_2$, that the class of $f(\alpha)$ in $\mathfrak{a}/\mathfrak{a}^2$ is unimodular. Set $f(\alpha)=a$. We claim that $\mathfrak{a}/Aa$ is projective ideal of rank 1 in A/aA. To show this we observe that since $\mathfrak{a}/\mathfrak{a} \approx \bar{A}\bar{a} \oplus D$ for some D, it follows that for any maximal ideal $\mathfrak{M} \supset \mathfrak{a}$, a can be chosen as one of the two generators for $\mathfrak{a}_{\mathfrak{M}}$. Since any two generators of $\mathfrak{a}_{\mathfrak{M}}$ form an $A_{\mathfrak{M}}$-sequence, it follows that $\mathfrak{a}/Aa$ is locally generated by one element which is even a non-zero-divisor. Thus $\mathfrak{a}/Aa$ is a projective A/aA-module of rank 1.

By the exact sequence $(*)$, we get (since $f(\alpha)=a$) the exact sequence

$$0 \to A^2 \to P' \to \mathfrak{a}/Aa \to 0.$$

Tensoring this sequence with A/aA, we have the exact sequence

$$\frac{A^2}{aA^2} \xrightarrow{g} \frac{P'}{aP'} \to \mathfrak{a}/Aa \to 0.$$

Since $\mathfrak{a}/Aa$ is projective, we have

$$\frac{P'}{aP'} \approx \mathrm{Img} \oplus \mathfrak{a}/Aa$$

Taking $\overset{2}{\wedge}$ both sides and observing that $\overset{2}{\wedge}P' \approx A$ (since P' is stably free) and so $\overset{2}{\wedge}(P'/aP') \approx A/Aa$, it follows that

$$\mathfrak{a}/Aa \otimes \mathrm{Img} \approx A/aA \quad \text{i.e.} \quad \mathfrak{a}/aA \approx (\mathrm{Img})^{-1}.$$

Since Img is generated by two elements, it easily follows that $\mathfrak{a}/aA$ is generated by two elements. Hence $\mathfrak{a}$ is generated by three elements. The proof of the theorem is completely established.

Since by [2, p. 639, (3.5)] $\tilde{K}_0(K[x_1,\dots, x_n])=0$, where K is a field, we have

COROLLARY. *Let $\mathfrak{a} \subset K[x_1, x_2, x_3]$ be an unmixed ideal of height* 2. *Suppose $\mathfrak{a}$ is locally generated by r elements. Then $\mathfrak{a}$ is generated by $r+1$ elements.*

A particular case of the Corollary which may be of interest to us

COROLLARY 2. *Let C be a closed affine curve in $\mathbf{A}_3$ which is locally a complete intersection (e.g. C non-singular), then the ideal $I(C)$ of C is generated by three elements.*

Remark. (i) One can multiply examples of three dimensional regular rings with trivial $\tilde{K}_0$ for instance, by using the following well known facts

a) $\tilde{K}_0 A=0$, if A is a principal ideal domain or a local ring.

b) (Grothendieck), If A is regular, $K_0 A \to K_0 A[x]$ is an isomorphism .

c) If A is regular, then $K_0 A \to K_0 S^{-1} A$ is surjective, S being multiplicatively closed set.

(ii) (Fossum and Claborn) Let K be a field of characteristic $\neq 2$ with $\sqrt{-1} \in K$ or $K=\mathbf{R}$ and $A=K\,[x_0, x_1, x_2, x_3]$, $\sum x_i^2=1$. Then $\tilde{K}_0 A=0$.

AN EXAMPLE. The theorem above is best possible in the sense that there do exist non-singular affine curves C in $\mathbf{A}_3$ whose prime ideals are *not* generated by two elements. For example, let C be a complete non-singular curve of genus 2. Let Ω denote a divisor in its canonical class. Let $P \in C$ be such that Ω is not linearly equivalent to $2P$ (such points exist since otherwise the Jacobian variety of C will be a 2-torsion group!). Consider $C'=C-\{P\}$. Then C' is an affine non-singular curve which can be embedded as a closed set in $\mathbf{A}_3$. This we can do for instance by considering the complete linear system[2] $|5P|$ or by a well known result which says that any non-singular affine curve can be embedded as a closed set in $\mathbf{A}_3$. Let $\mathfrak{P}$ be the ideal of C' in $A=K[x_1, x_2, x_3]$ (K algebraically closed). We claim that $\mathfrak{P}$ is not generated

[2]) I am thankful M. S. Narasimhan for pointing this to me.

by two elements. For if $\mathfrak{P}$ were generated by two elements, then $\mathrm{Ext}_A^2(A/\mathfrak{P}, A) \approx A/\mathfrak{P}$ (this one can be seen for example by Koszul-resolution for $A/\mathfrak{P}$). Since by [4], $\mathrm{Ext}_A^2(A/\mathfrak{P}, A)$ is the module of sections of the canonical line bundle $\Omega_{C'}$ it follows that $\Omega_{C'}$ is trivial. This implies that a canonical divisor $\Omega \sim nP$. Since $\deg \Omega = 2$, we have $\Omega \sim 2P$. Contradiction.

We do not know if the hypothesis $\tilde{K}_0(A) = 0$ is essential in our theorem.

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Added in Proof: In the case when $r \geqslant 3$, our result is an easy consequence of Swan [7]. Also, it is not difficult to see that in the statement of the theorem one can drop the hypothesis that $\mathfrak{a}$ be of height 2.