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# Every Odd Dimensional Homotopy Sphere has a Foliaton of Codimension One

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It is well-known that Reeb constructed a foliation of codimension one on $S^3$ (Reeb [4]). But, after that, nothing was known of codimension-one foliations of higher dimensional spheres for twenty years. In the circumstances Lawson's recent work is significant. He exhibited foliations of codimension one on each of the $(2^k+3)$-sphere for $k=1, 2, \ldots$ (Lawson [2]).

In this paper we shall prove the following.

THEOREM. *Every odd dimensional homotopy sphere has a foliation of codimension one.*

## 1. Fiberings over a Circle

Let $\tilde{S}^{2m+1}$ be a $(2m+1)$-dimensional homotopy sphere $(m \geqq 3)$ and let $F^{2m}$ be a compact $2m$-dimensional differentiable manifold imbedded in $\tilde{S}^{2m+1}$ which has the homotopy type of the bouquet of $r$ copies of $m$-sphere $S^m$:

$$F^{2m} \simeq \underbrace{S^m \vee S^m \vee \cdots \vee S^m}_{r}.$$

Since the normal bundle of the $(2m-1)$-dimensional differentiable manifold $\partial F$, the boundary of $F^{2m}$, is trivial, the tubular neighborhood of $\partial F$ is $\partial F \times D^2$. Thus $\tilde{S}^{2m+1} - (\partial F \times \operatorname{Int} D^2)$ is a $(2m+1)$-dimensional differentiable manifold with boundary $\partial F \times S^1$. In the following the intersection $F^{2m} \cap (\tilde{S}^{2m+1} - (\partial F \times \operatorname{Int} D^2))$ is simply denoted as $F^{2m}$, because they are naturally diffeomorphic.

Let $A$ be the compact $(2m+1)$-dimensional differentiable manifold (with corner) obtained by splitting $\tilde{S}^{2m+1} - (\partial F \times \operatorname{Int} D^2)$ at $F^{2m}$. Then $\partial A = F^+ \cup F^- \cup (\partial F \times I)$, where $F^+$ and $F^-$ are copies of $F^{2m}$. $A$ has the same homotopy type as $\tilde{S}^{2m+1} - F^{2m}$. It is easy to see that $A$ is simply connected and that, by the Alexander duality, homology groups of $A$ are as follows:

$$H_q(A) = \begin{cases} Z & q = 0, \\ \underbrace{Z + Z + \cdots + Z}_{r} & q = m, \\ 0 & \text{otherwise.} \end{cases}$$

(Homology groups $H_*(\ )$ mean homology groups with integral coefficient group $H_*(\ ;Z)$.)

Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a system of generators of $H_m(F) \cong Z+Z+\cdots+Z$ such that each $\alpha_i$ is represented by an imbedded $m$-sphere $S_i (i=1, 2, \dots, r)$. Let $\alpha_i^+$ (resp. $\alpha_i^-$) denote the element of $H_m(F^+)$ (resp. $H_m(F^-)$) corresponding to $\alpha_i \in H_m(F)$. Let $\alpha_i'$ denote the element of $H_{2m}(A)$ corresponding to $\alpha_i \in H_{2m}(F)$ by the Alexander duality $(i=1, 2, \dots, r)$. Then $\alpha_1', \alpha_2', \dots, \alpha_r'$ form a system of generators of $H_m(A)$.

Let $\iota^+ : F^+ \to A$ and $\iota^- : F^- \to A$ be the inclusion maps, and let

$$\iota_*^+(\alpha_i^+) = \sum_j a_{ij}^+ \alpha_j', \quad \iota_*^-(\alpha_i^-) = \sum_j a_{ij}^- \alpha_j' \quad (i = 1, 2, \dots, r).$$

Then $a_{ij}^+$ and $a_{ij}^-$ are expressed by linking numbers as follows.

Denote by $S_i^+$ (resp. $S_i^-$) a displacement of $S_i$ in $\tilde{S}^{2m+1}$ towards the normal direction of $F^+$ (resp. $F^-$). Then it is easy to see that

$$a_{ij}^+ = Lk(S_i^+, S_j), \quad a_{ij}^- = Lk(S_i^-, S_j).$$

Furthermore it follows from

$$Lk(S_i^+, S_j) = Lk(S_i, S_j^-) = (-1)^{m+1} Lk(S_j^-, S_i),$$

that

$$a_{ij}^+ = (-1)^{m+1} a_{ji}^-$$

Denote by $L(F)$ the following $(r \times r)$ matrix:

$$L(F) = \begin{pmatrix} Lk(S_1^+, S_1) \dots Lk(S_1^+, S_r) \\ \vdots \quad \ddots \quad \vdots \\ Lk(S_r^+, S_1) \dots Lk(S_r^+, S_r) \end{pmatrix}.$$

Suppose now that $L(F)$ is unimodular. Then the homomorphisms

$$\iota_*^+ : H_m(F^+) \to H_m(A), \iota_*^- : H_m(F^-) \to H_m(A)$$

are isomorphisms. This shows, since $F^+$, $F^-$ and $A$ are simply connected, that $\iota^+$ and $\iota^-$ are homotopy equivalence. Thus, according to the relative $h$-cobordism theorem (Smale [5], Corollary 3.2), the following holds:

$$(A, \partial F \times I) = (F^{2m}, \partial F) \times I.$$

This implies the following proposition which is a differential topological version of so-called Milnor fibering. (See also Tamura [6].)

PROPOSITION 1. *If the matrix $L(F)$ is unimodular, then there exists a fibering $\tilde{S}^{2m+1} - (\partial F \times \mathrm{Int} D^2) \to S^1$ having $F$ as a fibre.*

## 2. Construction of Fiberings

Let $X$, $Y$ denote the following matrices

$$X=\begin{pmatrix} 2 & & & 1 & & & & \\ & 2 & 1 & & & & 0 & \\ & 1 & 2 & 1 & & & & \\ 1 & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ 0 & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}, \quad Y=\begin{pmatrix} 2 & & & 1 & & & & & \\ & 2 & 1 & & & & & & 0 \\ & 1 & 2 & 1 & & & & & \\ 1 & & 1 & 2 & 1 & & & & \\ & & & 1 & 2 & 1 & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & 1 & 2 & 1 & \\ 0 & & & & & & 1 & 2 & 1 \\ & & & & & & & 1 & 2 \end{pmatrix}.$$

As is well known, $X$ is positive definite and unimodular. The rank of $(9\times 9)$ matrix $Y$ is 8 and its elementary divisor is (1, 1, 1, 1, 1, 1, 1, 1).

Let $\Delta$ denote the diagonal of $S^{2n}\times S^{2n}(n\geqq 2)$ and let $N$ be a tubular neighborhood of $\Delta$ in $S^{2n}\times S^{2n}$. Then $N$ has the homotopy type of $S^{2n}$ and the self-intersection number of a generator of $H_{2n}(N)\cong Z$ is 2. Let $W(X)$ be the parallelizable compact oriented $4n$-dimensional differentiable manifold formed from $N_1, N_2,\ldots, N_8$ (8 copies of $N$), by plumbing $N_i$ and $N_{i+1}$ $(i=2, 3,\ldots, 7)$, and $N_1$ and $N_4$. Then $W(X)$ has the homotype types of $\underbrace{S^{2n}\vee S^{2n}\vee\cdots\vee S^{2n}}_{8}$. The orientation of $W(X)$ is chosen so that

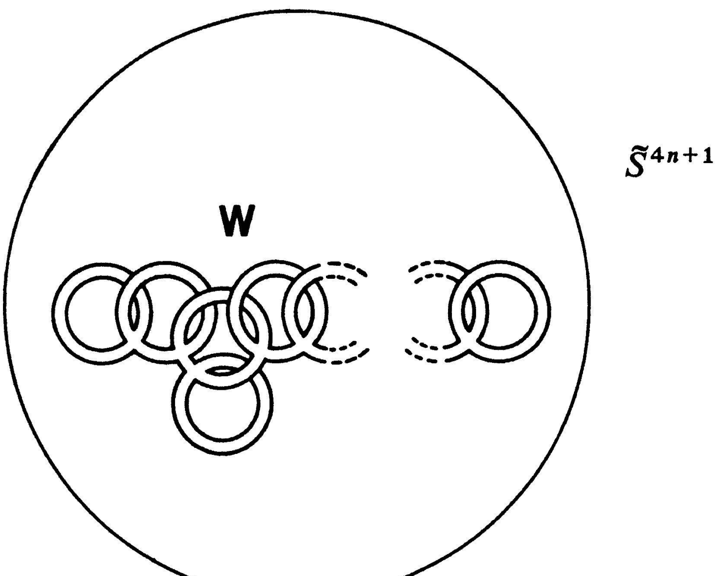


Fig. 1.

the matrix of intersection numbers of $H_{2n}(W(X))$ is $X$. Similarly parallelizable compact oriented $4n$-dimensional differentiable manifolds $W(-X)$ and $W(Y)$ both of which have the homotopy type of bouquets of $2n$-spheres, are defined. The matrix of intersection numbers of $H_{2n}(W(-X))$ (resp. $H_{2n}(W(Y))$) is $-X$ (resp. $Y$).

Let $W=W(-X)\natural W(Y)$ be the boundary connected sum of $W(-X)$ and $W(Y)$. $W$ is a parallelizable compact oriented $4n$-dimensional differentiable manifold. Let us imbed $W$ into $\tilde{S}^{4n+1}$ as indicated in the Fig. 1, by 17 copies of naturally imbedded $S^{2n}\times S^{2n}$ which osculate consecutively, so that unnecessary linking numbers do not occur in the matrix $L(W)$ (cf. Tamura [6], section 2).
Then it is easy to see that the matrix $L(W)$ of linking numbers is given by

$$L(W)=\begin{pmatrix} P & \\ & Q \end{pmatrix},$$

where

$$P=\begin{pmatrix} -1 & & & -1 & & & & \\ & -1 & -1 & & & & & \\ & & -1 & -1 & & & & \\ & & & -1 & -1 & & & \\ & & & & -1 & -1 & & \\ & & & & & -1 & -1 & \\ & & & & & & -1 & -1 \\ & & & & & & & -1 \end{pmatrix}, \quad Q=\begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & 1 & 1 & & & & & & \\ 1 & & 1 & 1 & & & & & \\ & & & 1 & 1 & & & & \\ & & & & 1 & 1 & & & \\ & & & & & 1 & 1 & & \\ & & & & & & 1 & 1 & \\ & & & & & & & 1 & 1 \end{pmatrix}.$$

Thus, by Proposition 1, the following holds.

PROPOSITION 2. *There exists a fibering* $\tilde{S}^{4n+1}-(\partial W\times \operatorname{Int} D^2)\to S^1$ *having* $W$ *as a fibre* ($n\geqq 2$).

Let $\hat{\Delta}$ denote the diagonal of $S^{2n-1}\times S^{2n-1}$ ($n\geqq 2$) and let $\hat{N}$ be a tubular neighborhood of $\hat{\Delta}$ in $S^{2n-1}\times S^{2n-1}$. Let us imbed $\hat{N}$ into $\tilde{S}^{4n-1}$ by imbedding $S^{2n-1}\times S^{2n-1}$ into $\tilde{S}^{4n-1}$ naturally. Then the matrix $L(\hat{N})$ of linking numbers is given by $L(\hat{N})=(1)$. Thus, by Proposition 1, the following holds.

PROPOSITION 3. *There exists a fibering* $\tilde{S}^{4n-1}-(\partial\hat{N}\times \operatorname{Int} D^2)\to S^1$ *having* $\hat{N}$ *as a fibre* ($n\geqq 2$).

This fibering corresponds to the Milnor fibering of $z_0^2+z_1^2+\cdots+z_{2n-1}^2=0$.

## 3. Boundary of the Fibre $W$

Let $M$ denote the boundary of the fibre $W$ in Proposition 2. Then $M$ is an orientable closed $(4n-1)$-dimensional differentiable manifold. It follows by the Poincaré-

Lefschetz duality that

$$H_q(W, M) \cong H^{4n-q}(W),$$

and that the natural homomorphism

$$H_{2n}(W) \to H_{2n}(W, M) \cong \operatorname{Hom}(H_{2n}(W), Z)$$

is determined by $\begin{pmatrix} -X & \\ & Y \end{pmatrix}$, the matrix of intersection numbers of $H_{2n}(W)$. Thus the following is a direct consequence of the homology exact sequence of $(W, M)$:

$$H_q(M) = \begin{cases} Z & q = 0, 2n-1, 2n, 4n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $W$ is obtained from $W(-X) \natural W(X)$ by attaching a handle $D^{2n} \times D^{2n}$:

$$W = (W(-X) \natural W(X)) \bigcup_g (D^{2n} \times D^{2n}),$$

where $g: \partial D^{2n} \times D^{2n} \to \partial(W(-X) \natural W(X))$ is an attaching map. The boundary $\partial(W(-X) \natural W(X))$ is the natural $(4n-1)$-sphere (Kervaire-Milnor [1]), and the following decomposition holds:

$$W = W(-X) \natural W(X) \natural (D^{4n} \bigcup_g (D^{2n} \times D^{2n})).$$

According to the $h$-cobordism theorem (Smale [5]), $B = D^{4n} \bigcup_g (D^{2n} \times D^{2n})$ is the total space of a $2n$-disk bundle $\xi$ over $S^{2n}$, and its differentiable structure is compatible with the bundle structure. Thus $M = \partial W = \partial B$ is the total space of an $S^{2n-1}$-bundle over $S^{2n}$ associated with $\xi$. Let $\alpha \in \pi_{2n-1}(SO(2n))$ be the characteristic map of $\xi$. Since $B$ is parallelizable, $\xi$ is stably trivial and, thus, $\alpha$ belongs to the kernel of $\pi_{2n-1}(SO(2n)) \to \pi_{2n-1}(SO(2n+1))$. Let us consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{2n}(S^{2n}) \arrow[r, "\partial"] \arrow[dr, "\times 2"'] & \pi_{2n-1}(SO(2n)) \arrow[r] \arrow[d, "p_*"] & \pi_{2n-1}(SO(2n+1)) \arrow[r] & \cdots \\ & & \pi_{2n-1}(S^{2n-1}), & & \end{tikzcd}$$

consisting of the homotopy exact sequence of the fibering $SO(2n+1) \to SO(2n+1)/SO(2n) = S^{2n}$ and the homomorphism induced by the projection $p: SO(2n) \to SO(2n)/SO(2n-1) = S^{2n-1}$. Let $\iota_{2n}, \iota_{2n-1}$ be generators of $\pi_{2n}(S^{2n}), \pi_{2n-1}(S^{2n-1})$ respectively. Since $\alpha \in \partial(\pi_{2n}(S^{2n}))$, $\alpha = \partial(c\iota_{2n})$ for an integer $c$. If $c \neq 0$, $p_* \partial(c\iota_{2n}) = \pm 2c\iota_{2n-1} \neq 0$ and, thus, the Euler class of $\xi$ is not zero. This implies, by using the Thom-Gysin exact sequence, that $H_{2n-1}(M) = H_{2n-1}(\partial B) \not\cong Z$, which is a contradiction. Thus $c = 0$ and $\xi$ is a trivial bundle. This yields the following.

LEMMA 1. *The boundary of $W$ is $S^{2n-1}\times S^{2n}$.*

## 4. Proof of Theorem

Let $E$ be a compact connected $(2m+1)$-dimensional differentiable manifold such that $E$ is a total space of a fibering over $S^1$ and $\partial E$ is connected. Then it is well known that there exists a foliation of codimension one on $E$ having $\partial E$ as the only compact leaf (cf. Lawson [2]).

LEMMA 2. *Suppose that $S^{2n+1}$ has a foliation of codimension one $(n\geqq 2)$, then the following holds:*

(i) *Any $(4n+1)$-dimensional homotopy sphere $\tilde{S}^{4n+1}$ has a foliation of codimension one.*

(ii) *Any $(4n-1)$-dimensional homotopy sphere $\tilde{S}^{4n-1}$ has a foliation of codimension one.*

*Proof.* Let $\gamma$ be a closed smooth curve in $S^{2n+1}$ which is transversal to the leaves. The existence of such $\gamma$ is a classical fact. The tubular neighborhood of $\gamma$ is $S^1\times D^{2n}$. The foliation on $S^{2n+1}$ can be modified so that its restriction on $S^{2n+1}-(S^1\times \mathrm{Int} D^{2n})$ $=S^{2n-1}\times D^2$ is a foliation having the boundary as a compact leaf.
Now, by Proposition 2, $\tilde{S}^{4n+1}-(\partial W\times \mathrm{Int} D^2)$ has a foliation of codimension one such that $\partial W\times S^1$ is the only compact leaf. On the other hand, since $\partial W=S^{2n-1}\times S^{2n}$ by Lemma 1, $\partial W\times D^2$ has a foliation of codimension one which is induced by the projection $\partial W\times D^2\to S^{2n-1}\times D^2$ from the foliation of $S^{2n-1}\times D^2$. This completes the proof of (i).
Making use of Proposition 3 and the projection $\partial\hat{N}\times D^2\to\hat{\Delta}\times D^2=S^{2n-1}\times D^2$, the proof of (ii) is completely analogous to that of (i).

*Remark.* Lemma 2, (ii) is (a slightly generalized form of) a result of Lawson [2].

Let $\tilde{S}^{2m+1}$ be a (2m+1)-dimensional homotopy sphere. In case $m=1, 2$, the existence of a foliation of codimension one is proved by Novikov [3] and Lawson [2] respectively. Suppose that, for $2\leqq m<q$, $\tilde{S}^{2m+1}$ has a foliation of codimension one. Then, if $q$ is even (resp. odd), the existence of a foliation of codimension one of $\tilde{S}^{2q+1}$ is assured by Lemma 2 (i) (resp. (ii)). This completes the proof of the theorem by induction.

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