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Every Odd Dimensional Homotopy Sphere has a Foliation of Codimension One

by ITIRO TAMURA

It is well-known that Reeb constructed a foliation of codimension one on S^3 (Reeb [4]). But, after that, nothing was known of codimension-one foliations of higher dimensional spheres for twenty years. In the circumstances Lawson's recent work is significant. He exhibited foliations of codimension one on each of the (2^k+3) -sphere for $k=1, 2, \dots$ (Lawson [2]).

In this paper we shall prove the following.

THEOREM. *Every odd dimensional homotopy sphere has a foliation of codimension one.*

1. Fiberings over a Circle

Let \tilde{S}^{2m+1} be a $(2m+1)$ -dimensional homotopy sphere ($m \geq 3$) and let F^{2m} be a compact $2m$ -dimensional differentiable manifold imbedded in \tilde{S}^{2m+1} which has the homotopy type of the bouquet of r copies of m -sphere S^m :

$$F^{2m} \simeq \underbrace{S^m \vee S^m \vee \dots \vee S^m}_r.$$

Since the normal bundle of the $(2m-1)$ -dimensional differentiable manifold ∂F , the boundary of F^{2m} , is trivial, the tubular neighborhood of ∂F is $\partial F \times D^2$. Thus $\tilde{S}^{2m+1} - (\partial F \times \text{Int } D^2)$ is a $(2m+1)$ -dimensional differentiable manifold with boundary $\partial F \times S^1$. In the following the intersection $F^{2m} \cap (\tilde{S}^{2m+1} - (\partial F \times \text{Int } D^2))$ is simply denoted as F^{2m} , because they are naturally diffeomorphic.

Let A be the compact $(2m+1)$ -dimensional differentiable manifold (with corner) obtained by splitting $\tilde{S}^{2m+1} - (\partial F \times \text{Int } D^2)$ at F^{2m} . Then $\partial A = F^+ \cup F^- \cup (\partial F \times I)$, where F^+ and F^- are copies of F^{2m} . A has the same homotopy type as $\tilde{S}^{2m+1} - F^{2m}$. It is easy to see that A is simply connected and that, by the Alexander duality, homology groups of A are as follows:

$$H_q(A) = \begin{cases} Z & q = 0, \\ \underbrace{Z + Z + \dots + Z}_r & q = m, \\ 0 & \text{otherwise.} \end{cases}$$

(Homology groups $H_*()$ mean homology groups with integral coefficient group $H_*(; Z)$.)

Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a system of generators of $H_m(F) \cong Z + Z + \dots + Z$ such that each α_i is represented by an imbedded m -sphere $S_i (i=1, 2, \dots, r)$. Let α_i^+ (resp. α_i^-) denote the element of $H_m(F^+)$ (resp. $H_m(F^-)$) corresponding to $\alpha_i \in H_m(F)$. Let α'_i denote the element of $H_{2m}(A)$ corresponding to $\alpha_i \in H_{2m}(F)$ by the Alexander duality ($i=1, 2, \dots, r$). Then $\alpha'_1, \alpha'_2, \dots, \alpha'_r$ form a system of generators of $H_m(A)$.

Let $\iota^+ : F^+ \rightarrow A$ and $\iota^- : F^- \rightarrow A$ be the inclusion maps, and let

$$\iota_*^+ (\alpha_i^+) = \sum_j a_{ij}^+ \alpha'_j, \quad \iota_*^- (\alpha_i^-) = \sum_j a_{ij}^- \alpha'_j \quad (i = 1, 2, \dots, r).$$

Then a_{ij}^+ and a_{ij}^- are expressed by linking numbers as follows.

Denote by S_i^+ (resp. S_i^-) a displacement of S_i in \tilde{S}^{2m+1} towards the normal direction of F^+ (resp. F^-). Then it is easy to see that

$$a_{ij}^+ = Lk(S_i^+, S_j), \quad a_{ij}^- = Lk(S_i^-, S_j).$$

Furthermore it follows from

$$Lk(S_i^+, S_j) = Lk(S_i, S_j^-) = (-1)^{m+1} Lk(S_j^-, S_i),$$

that

$$a_{ij}^+ = (-1)^{m+1} a_{ji}^-$$

Denote by $L(F)$ the following $(r \times r)$ matrix:

$$L(F) = \begin{pmatrix} Lk(S_1^+, S_1) & \dots & Lk(S_1^+, S_r) \\ \vdots & \ddots & \vdots \\ Lk(S_r^+, S_1) & \dots & Lk(S_r^+, S_r) \end{pmatrix}.$$

Suppose now that $L(F)$ is unimodular. Then the homomorphisms

$$\iota_*^+ : H_m(F^+) \rightarrow H_m(A), \quad \iota_*^- : H_m(F^-) \rightarrow H_m(A)$$

are isomorphisms. This shows, since F^+, F^- and A are simply connected, that ι^+ and ι^- are homotopy equivalence. Thus, according to the relative h -cobordism theorem (Smale [5], Corollary 3.2), the following holds:

$$(A, \partial F \times I) = (F^{2m}, \partial F) \times I.$$

This implies the following proposition which is a differential topological version of so-called Milnor fibering. (See also Tamura [6].)

PROPOSITION 1. *If the matrix $L(F)$ is unimodular, then there exists a fibering $\tilde{S}^{2m+1} - (\partial F \times \text{Int } D^2) \rightarrow S^1$ having F as a fibre.*

2. Construction of Fiberings

Let X, Y denote the following matrices

$$X = \begin{pmatrix} 2 & & & & & & & & \\ & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & 1 & 2 & 1 & & & \\ 1 & & & & 1 & 2 & 1 & & \\ & & & & & 1 & 2 & 1 & \\ & & & & & & 1 & 2 & 1 \\ & & & & & & & 1 & 2 \\ 0 & & & & & & & & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & & & & & & & & \\ & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & 1 & 2 & 1 & & & \\ 1 & & & & 1 & 2 & 1 & & \\ & & & & & 1 & 2 & 1 & \\ & & & & & & 1 & 2 & 1 \\ & & & & & & & 1 & 2 & 1 \\ & & & & & & & & 1 & 2 \\ 0 & & & & & & & & & 0 \end{pmatrix}.$$

As is well known, X is positive definite and unimodular. The rank of (9×9) matrix Y is 8 and its elementary divisor is $(1, 1, 1, 1, 1, 1, 1, 1)$.

Let Δ denote the diagonal of $S^{2n} \times S^{2n}$ ($n \geq 2$) and let N be a tubular neighborhood of Δ in $S^{2n} \times S^{2n}$. Then N has the homotopy type of S^{2n} and the self-intersection number of a generator of $H_{2n}(N) \cong \mathbb{Z}$ is 2. Let $W(X)$ be the parallelizable compact oriented $4n$ -dimensional differentiable manifold formed from N_1, N_2, \dots, N_8 (8 copies of N), by plumbing N_i and N_{i+1} ($i=2, 3, \dots, 7$), and N_1 and N_4 . Then $W(X)$ has the homotopy types of $\underbrace{S^{2n} \vee S^{2n} \vee \dots \vee S^{2n}}_8$. The orientation of $W(X)$ is chosen so that

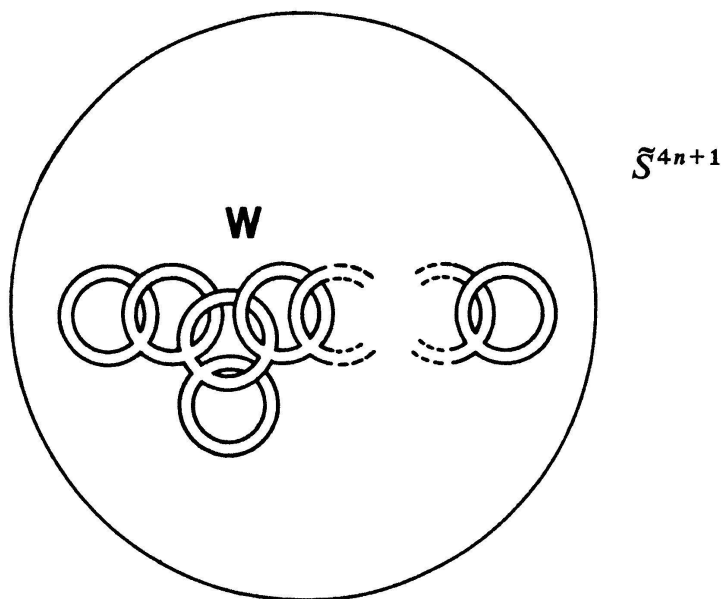


Fig. 1.

Then it is easy to see that the matrix $L(W)$ of linking numbers is given by

$$L(W) = \begin{pmatrix} P & \\ & Q \end{pmatrix},$$

$$P = \begin{bmatrix} -1 & & & & & & & & & \\ & -1 & -1 & & & & & & & \\ & & -1 & -1 & & & & & & \\ & & & -1 & -1 & & & & & \\ & & & & -1 & -1 & & & & \\ & & & & & -1 & -1 & & & \\ & & & & & & -1 & -1 & & \\ & & & & & & & -1 & -1 & \\ & & & & & & & & -1 & -1 \\ & & & & & & & & & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & 1 & & & & & & \\ & & & 1 & 1 & & & & & \\ & 1 & & & 1 & 1 & & & & \\ & & & & 1 & 1 & & & & \\ & & & & & 1 & 1 & & & \\ & & & & & & 1 & 1 & & \\ & & & & & & & 1 & 1 & \\ & & & & & & & & 1 & 1 \\ & & & & & & & & & 1 & 1 \end{bmatrix}.$$

Let M denote the boundary of the fibre W in Proposition 2. Then M is an orientable closed $(4n-1)$ -dimensional differentiable manifold. It follows by the Poincaré-

Lefschetz duality that

$$H_q(W, M) \cong H^{4n-q}(W),$$

and that the natural homomorphism

$$H_{2n}(W) \rightarrow H_{2n}(W, M) \cong \text{Hom}(H_{2n}(W), Z)$$

is determined by $\begin{pmatrix} -X & \\ & Y \end{pmatrix}$, the matrix of intersection numbers of $H_{2n}(W)$. Thus the following is a direct consequence of the homology exact sequence of (W, M) :

$$H_q(M) = \begin{cases} Z & q = 0, 2n-1, 2n, 4n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously W is obtained from $W(-X) \natural W(X)$ by attaching a handle $D^{2n} \times D^{2n}$:

$$W = (W(-X) \natural W(X)) \bigcup_g (D^{2n} \times D^{2n}),$$

where $g: \partial D^{2n} \times D^{2n} \rightarrow \partial(W(-X) \natural W(X))$ is an attaching map. The boundary $\partial(W(-X) \natural W(X))$ is the natural $(4n-1)$ -sphere (Kervaire-Milnor [1]), and the following decomposition holds:

$$W = W(-X) \natural W(X) \natural (D^{4n} \bigcup_g (D^{2n} \times D^{2n})).$$

According to the h -cobordism theorem (Smale [5]), $B = D^{4n} \bigcup_g (D^{2n} \times D^{2n})$ is the total space of a $2n$ -disk bundle ξ over S^{2n} , and its differentiable structure is compatible with the bundle structure. Thus $M = \partial W = \partial B$ is the total space of an S^{2n-1} -bundle over S^{2n} associated with ξ . Let $\alpha \in \pi_{2n-1}(SO(2n))$ be the characteristic map of ξ . Since B is parallelizable, ξ is stably trivial and, thus, α belongs to the kernel of $\pi_{2n-1}(SO(2n)) \rightarrow \pi_{2n-1}(SO(2n+1))$. Let us consider the diagram

$$\begin{array}{ccc} \cdots \rightarrow \pi_{2n}(S^{2n}) \xrightarrow{\partial} \pi_{2n-1}(SO(2n)) & \rightarrow & \pi_{2n-1}(SO(2n+1)) \rightarrow \cdots \\ & \searrow \times 2 & \downarrow p_* \\ & & \pi_{2n-1}(S^{2n-1}), \end{array}$$

consisting of the homotopy exact sequence of the fibering $SO(2n+1) \rightarrow SO(2n+1)/SO(2n) = S^{2n}$ and the homomorphism induced by the projection $p: SO(2n) \rightarrow SO(2n)/SO(2n-1) = S^{2n-1}$. Let ι_{2n}, ι_{2n-1} be generators of $\pi_{2n}(S^{2n}), \pi_{2n-1}(S^{2n-1})$ respectively. Since $\alpha \in \partial(\pi_{2n}(S^{2n}))$, $\alpha = \partial(c\iota_{2n})$ for an integer c . If $c \neq 0$, $p_*\partial(c\iota_{2n}) = \pm 2c\iota_{2n-1} \neq 0$ and, thus, the Euler class of ξ is not zero. This implies, by using the Thom-Gysin exact sequence, that $H_{2n-1}(M) = H_{2n-1}(\partial B) \not\cong Z$, which is a contradiction. Thus $c = 0$ and ξ is a trivial bundle. This yields the following.

LEMMA 1. *The boundary of W is $S^{2n-1} \times S^{2n}$.*

4. Proof of Theorem

Let E be a compact connected $(2m+1)$ -dimensional differentiable manifold such that E is a total space of a fibering over S^1 and ∂E is connected. Then it is well known that there exists a foliation of codimension one on E having ∂E as the only compact leaf (cf. Lawson [2]).

LEMMA 2. *Suppose that S^{2n+1} has a foliation of codimension one ($n \geq 2$), then the following holds:*

(i) *Any $(4n+1)$ -dimensional homotopy sphere \tilde{S}^{4n+1} has a foliation of codimension one.*

(ii) *Any $(4n-1)$ -dimensional homotopy sphere \tilde{S}^{4n-1} has a foliation of codimension one.*

Proof. Let γ be a closed smooth curve in S^{2n+1} which is transversal to the leaves. The existence of such γ is a classical fact. The tubular neighborhood of γ is $S^1 \times D^{2n}$. The foliation on S^{2n+1} can be modified so that its restriction on $S^{2n+1} - (S^1 \times \text{Int } D^{2n}) = S^{2n-1} \times D^2$ is a foliation having the boundary as a compact leaf.

Now, by Proposition 2, $\tilde{S}^{4n+1} - (\partial W \times \text{Int } D^2)$ has a foliation of codimension one such that $\partial W \times S^1$ is the only compact leaf. On the other hand, since $\partial W = S^{2n-1} \times S^{2n}$ by Lemma 1, $\partial W \times D^2$ has a foliation of codimension one which is induced by the projection $\partial W \times D^2 \rightarrow S^{2n-1} \times D^2$ from the foliation of $S^{2n-1} \times D^2$. This completes the proof of (i).

Making use of Proposition 3 and the projection $\partial \hat{N} \times D^2 \rightarrow \hat{A} \times D^2 = S^{2n-1} \times D^2$, the proof of (ii) is completely analogous to that of (i).

Remark. Lemma 2, (ii) is (a slightly generalized form of) a result of Lawson [2].

Let \tilde{S}^{2m+1} be a $(2m+1)$ -dimensional homotopy sphere. In case $m=1, 2$, the existence of a foliation of codimension one is proved by Novikov [3] and Lawson [2] respectively. Suppose that, for $2 \leq m < q$, \tilde{S}^{2m+1} has a foliation of codimension one. Then, if q is even (resp. odd), the existence of a foliation of codimension one of \tilde{S}^{2q+1} is assured by Lemma 2 (i) (resp. (ii)). This completes the proof of the theorem by induction.

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