Deformation of Homeomorphisms on Stratified Sets

By L. C. SIEBENMANN

Introduction

Edwards and Kirby have presented in [14] an attractive and powerful method for deforming homeomorphisms of topological manifolds, which elaborates the "torus unfurling" technique of Kirby [20], and offers an alternative to the "meshing" technique of Cernavskii [8] [9]. In this article I shall develop the method further to deal with non-manifolds.[1] In particular I shall prove the new

THEOREM 0. *The topological group $H(X)$ of homeomorphisms of a finite simplicial complex X onto itself is locally contractible.*

This result does not extend to ENR's (euclidean neighborhood retracts). To see this let a space X be obtained from $S^3 = R^3 \cup \infty$ by crushing to a point each of a sequence of mutually disjoint wild non-cellular arcs in $R^3: A_1, A_2, A_3, \ldots$ such that each A_n, $n \geqslant 1$, is a copy of the same wild arc A in the unit ball in R^3 translated by the vector $(4n, 0, 0)$. This X is an ENR; indeed $X \times R$ is homeomorphic to $S^3 \times R = R^4 - 0$ by a result of Andrews and Curtis [4]. Clearly this compactum admits self-homeomorphisms $h: X \to X$ arbitrarily near the identity which nontrivially permute the images of $A_1, A_2, A_3, \ldots$. But no such h is isotopic to the identity because these are isolated points at which X fails to be a manifold. (See also the fish skeleton of §0.)

The treatment of non-manifolds rests roughly speaking on a method for deforming homeomorphisms on $R^m \times cX$, cX being the open cone on X, once one is given such a method on $R^{m+1} \times X$. Then the proof proceeds by induction on the *depth* of X. Here X is regarded as a stratified set, and depth is the greatest difference of dimensions of nonempty strata of X.

Stratified sets are vital to the proof because their open subsets are themselves stratified sets, and often of a lesser depth. Thus it will only clarify matters to deal from the outset with suitable stratified sets. I take this opportunity to introduce classes of pleasant stratified sets that may come to be the topological analogues of polyhedra in the piecewise-linear realm or of Thom's stratified sets in the differentiable realm.

This technique of proof almost automatically provides strong relative and respectful deformation theorems (§4.3, §5.10), which a counterexample (§2.3.1) suggests are

[1] Presumably one could equally well have extended Černavskii's method, but perhaps not so briefly or clearly.

the best possible. For one, the space of those homeomorphisms of a finite simplicial complex X, that respect (alternatively fix pointwise) a given subcomplex, is proved locally contractible. This contains some new information even if X is a manifold cf. [14].

I have compiled a long list of elementary consequences of the deformation theorems proved. Most certainly, they lack glamor, and for several reasons. Many are straightforward enlargements on corollaries drawn by Černavskii, Lees, Edwards and Kirby, Cheeger and Kister, or Gauld. Again many have well known differentiable analogues that can be proved instantly by Thom's device of choosing suitable vector fields and integrating.

This article was inspired by R. D. Edwards' proof (summer 1970) of Černavskii's theorem asserting local contractibility of spaces of embeddings of manifolds in codimension $\geqslant 3$. In it Edwards combined the torus furling and unfurling methods of [14], my inversion device in [31], and a horn device of Černavskii [8]. I noticed that inversion converts this horn device into conjugation by a "horn like" expansion (which has become Θ in §3) in the normal direction, and suddenly the methods were adequate to deform homeomorphisms on $R^m \times$ (cone), and here the inversion device became unnecessary. The text is an expansion of lectures given at Orsay, France, fall 1970. I am indebted again to Edwards for his generous assistance in eradicating errors.

List of Contents

§0. Philosophical Remarks on Deformation Principles (OPTIONAL READING)

For a space X (which is assumed locally compact and Hausdorff throughout this section) the following deformation principle (=axiom or property which may or may not hold) seems to be the center of interest.

$\boxed{\mathscr{D}_1(X)}$ *For each open set $U \subset X$ and each compactum $B \subset U$ the following statement holds:*

$\boxed{\mathscr{D}_1(X; B; U)}$ *If $h: U \to X$ is an open embedding sufficiently near to the identity* *inclusion $i: U \to X$ (for the compact-open topology) then there is a rule assigning to h a homeomorphism $h': X \to X$ equal h on B and equal the identity outside U. For h near i the rule $h \mapsto h'$ can be continuous and send i to* $\mathrm{id} \mid X$.

One might call h' a deformation of the identity $\mathrm{id} \mid X$ induced by the perturbation h of the inclusion $i: U \to X$.

The deformation property $\mathscr{D}(X)$ (see §2) that we will establish for pleasant stratified sets X is similar but more complicated. It is obviously as strong as $\mathscr{D}_1(X)$; in fact it is stronger even for ENR's as the property $\mathscr{D}_1$ holds (while $\mathscr{D}$ fails) for the 3-dimensional ENR of the introduction. This is an easy consequence of $\mathscr{D}_1(R^3 - \cup_i A_i)$. Nevertheless $\mathscr{D}_1$ fails for certain compact ENR's. For a simple example let X be the fish skeleton, the 1-point compactification of $Z_+ \times [-1, 1] \cup [0, \infty) \times 0$. (Here Z_+ is the integers $\geqslant 0$). It is a retract of $[0, \infty) \times [-1, 1] \cup \infty$, a 2-disc. Although $\mathscr{D}_1$ holds for this X, it fails for the cone on X.

If one insists on a powerful deformation principle which may still have very wide applicability, then one might first fix a pointed topological space $(\Lambda, 0)$ of parameters and consider

$\boxed{\mathscr{D}_1^\Lambda(X)}$ *For each open $U \subset X$, each compactum $B \subset U$, and each continuous rule $\varrho: \lambda \mapsto h_\lambda$ mapping a neighborhood of* 0 *in Λ into the space of open embeddings $U \to X$ (with compact open topology) the following holds:*

$\boxed{\mathscr{D}_1^\Lambda(X; B; U)}$ *Suppose $h_0 = i$. Then there exists a neighborhood N of* 0 *in Λ and a continuous rule $\varrho': \lambda \mapsto h'_\lambda$ defined for $\lambda \in N$ such that $h'_\lambda: X \to X$ is a homeomorphism, h'_λ equals h_λ on B, and h'_λ is the identity outside U. (One could adjust ϱ' so that $h'_0 = \mathrm{id} \mid X$.)*

Clearly $\mathscr{D}_1(X)$ implies $\mathscr{D}_1^\Lambda(X)$ for all Λ; and conversely, using embeddings to parameterize themselves. The first question is perhaps: *Does $\mathscr{D}_1^R(X)$ hold for all* ENR's X? (It does hold for the cone on the fish skeleton.)

$\mathscr{D}_1^\Lambda(X)$ should be thought of as an *isotopy extension principal* – for local extension in parameters Λ. Indeed $\mathscr{D}_1^\Lambda(X)$ (in place of $\mathscr{D}(X)$) is quite enough to prove our isotopy extension theorem (§6.5 part I) if we restrict attention to parameter space Λ (called B there); and conversely, for locally connected X, this implies $\mathscr{D}_1^\Lambda(X)$.

Note that if $\mathscr{D}_1^\Lambda(X)$ holds, then the mapping $\bar{\varrho}: \Lambda \times U \to \Lambda \times X$ given by $(\lambda, x) \mapsto$ $\mapsto (\lambda, h_\lambda(x))$ can be shown to be an open embedding by using conditions $\mathscr{D}_1^\Lambda(X; B; U)$ suitably. In the absence of $\mathscr{D}_1^\Lambda(X)$ one needs to assume that X or Λ is locally connected to prove this (Lemma 1.6).

In case $\mathscr{D}_1^\Lambda(X)$ turns out to be often valid we make the

Observations (generalising §6)

1) *With few exceptions*[2]) *each topological application* 6.8 *to* 6.35 *of* $\mathscr{D}$ *is proved on the basis of an isotopy extension principle*[3]) $\mathscr{D}_1^A$, *and does not require* $\mathscr{D}$. (And the appropriate A is easy to spot. Here we are letting the base point in A vary.)

2) In these same applications the hypotheses that certain spaces be locally connected become superfluous when an isotopy extension principle of the form $\mathscr{D}_1^A$ is hypothesized. This occurs because $\mathscr{D}_1^A(X)$ implies that $\bar{\varrho}:(\lambda, x)\mapsto(\lambda, h_\lambda(x))$ as mentioned above, is an open embedding even if X, A are not locally connected.

To conclude, we try to clarify the relation of $\mathscr{D}_1(X)$ to the stronger principle $\mathscr{D}(X)$. The property $\mathscr{D}_1(X; B; U)$ above can be given a relative version $\mathscr{D}_1(X'; A, A', B'; U)$ where A, A' are closed subsets of X and A' is a neighborhood of A. It differs by treating only embeddings h that equal the identity on $A'\cap U$ and in return it promises an h' equal the identity on A.

If to this property $\mathscr{D}_1(X; A, A', B; U)$ we add that for h small, there exists an isotopy h_t, $0\leqslant t\leqslant 1$, of $h_0=\mathrm{id}\mid X$ to $h_1=h'$ so that the rule $h\mapsto\{h_t, 0\leqslant t\leqslant 1\}$ is continuous and each $h_t\mid A=$identity, then we have a property $\tilde{\mathscr{D}}(X; A, A', B; U)$ which, taken for all such A, A', B, U in X, is obviously equivalent to the property $\mathscr{D}(X)$, provided X is locally connected, see §2. Thus the statement $\mathscr{D}(X)$ is two steps more complicated than $\mathscr{D}_1(X)$.

However there are two implications proving that the real difference is slight and vanishes for very pleasant X.

(i) $\mathscr{D}_1(X)$ *implies* $\mathscr{D}_1(X; A, A', B; U)$. Indeed $\mathscr{D}_1(X; B'; U')$ implies it if we set $U'=V-Z$, where $Z=B\cup(X-U)\cup A$ and V is a small open neighborhood of Z, and set B' equal the (compact) frontier of a smaller closed neighborhood of Z. (Draw a diagram!) V should be so small that it lies in a neighborhood of Z of the form $U_1\cup U_2\cup A'$ where U_1 and U_2 are disjoint neighborhoods respectively of B and $(X-U)$. Work on $(h\mid U_1)\cup(\mathrm{id}\mid A)\cup(\mathrm{id}\mid U_2)$ to get the desired h'.

(ii) *For spaces* X *that have a basis of open sets that are (abstractly) open cones on compacta, the property* $\mathscr{D}_1(X)$ *implies* $\mathscr{D}(X)$. The derivation from $\mathscr{D}_1(X)$ of $\tilde{\mathscr{D}}(X; A, A', B; U)$, or $\mathscr{D}(X; A, A', B; U)$ of §2, uses Alexander isotopies on many open cones as follows. One uses properties of type $\mathscr{D}_1(X; A_0, A_0', B_0; U_0)$ to express h as a finite composition $h_s h_{s-1}\dots h_1 h_0$ where $h_0: U\to X$ is the identity near B and h_j, $j=1,2,\dots, s$, is a homeomorphism $X\to X$ fixing points outside a compactum B_j contained in a cone in $U-A$. Each of $h_0,\dots, h_s$ depends continuously on h near $i: U\to X$

---

[2]) All of which come from the second argument in the proof of 6.13, which is not needed if the parameter space B in 6.13 is an ANR. So all exceptions vanish if all parameter spaces involved are ANR's. Alternatively all exceptions vanish if we abandon the isotopies h_t in 6.13, 6.19, 6.20, 6.23. The loss is slight and, incidentally, the normality assumption there vanishes.

[3]) Sometimes in a relative form parallel to $\mathscr{D}(X; \mathscr{S})$ of 6.0. Unfortunately the proof of an implication $\mathscr{D}(X)\Rightarrow\mathscr{D}(X\times B; \mathscr{S})$ in 6.1 fails for $\mathscr{D}_1$. Badly indeed, since $\mathscr{D}_1(X\times R; \mathscr{S})$ easily implies $\mathscr{D}(X)$. In particular $\mathscr{D}_1(X\times R; \mathscr{S})$ fails if X is the fish skeleton.

and equals the identity for $h=i$; also $B_1, \ldots, B_s$ are independent of h. Then we define $h'=h_s \ldots h_1$ and $h_t=h_t^{(s)}\circ h_t^{(s-1)}\circ\cdots\circ h_t^{(1)}$, $0\leqslant t\leqslant 1$, where $h_t^{(j)}$ is the Alexander isotopy, cf. 5.3, of h_j to $\mathrm{id} \mid X$ (arising from the cone containing B_j).

§1. Locally Cone-Like TOP Stratified Sets

We will use a simple notion of topological stratified set suggested by work of Cerf or of Armstrong and Zeeman [6]. It is just a filtered topological space having a few pleasant properties. Recall that abstracting the differentiable properties of algebraic varieties, Thom has produced a notion of differentiable stratified set [35] [24]. It is a filtered space *with lots of extra equipment* – certain manifold structures and submersions. Although the definition is complex, it is so formal that one can define piecewise-linear and topological stratified sets in the sense of Thom. Our CS sets are simpler, cruder and (as unfiltered spaces) slightly more general [33].

DEFINITION 1.1. A *stratified set* X will in this article be a metrizable space X equipped with a filtration $X\supset\cdots\supset X^{(n)}\supset X^{(n-1)}\supset\cdots\supset X^{(-1)}=\emptyset$ by closed subsets $X^{(n)}$, $n\geqslant -1$, (called skeleta) such that, for each $n\geqslant 0$, the components of $X^{(n)}-X^{(n-1)}$ are open in $X^{(n)}-X^{(n-1)}$. A *vertex* of X is a point in $X^{(0)}$.

It is a *TOP stratified set* if, for each $n\geqslant 0$, $X^{(n)}-X^{(n-1)}$ is an n-manifold without boundary, which is called the (total) n-stratum of X. The symbol TOP signifies that topological manifolds are involved.

Let X be a compact stratified set. Note that the *open cone* cX, obtained from $X\times[0,\infty)$ by smashing $X\times 0$ to a point, has a natural stratification $(cX)^{(n)}=c(X^{(n-1)})$, $n\geqslant 1$, and $(cX)^0=$cone point. The cone on the empty set is a point. Likewise there is an evident join of two compact stratified sets; and a product of *any* two.

Every open-subset of a stratified set is a stratified set.

An *isomorphism* $h:X\to Y$ of stratified sets is a homeomorphism of topological spaces such that $h(X^{(n)})=Y^{(n)}$ for all $n\geqslant 0$. The symbols $\approx$, $\cong$ denote respectively homeomorphism, and isomorphism of stratified sets.

DEFINITION 1.2. A stratified set X is *locally cone-like* if for each point x in X, with x in $X^{(n)}-X^{(n-1)}$ say, there exists an open neighborhood U of x in $X^{(n)}-X^{(n-1)}$, a *compact* stratified set of finite formal dimension[4]) L called a *link* of x in X, and an isomorphism of $U\times cL$ onto an open neighborhood of x in X. (Regard U as stratified with $U=U^n-U^{(n-1)}$.)

Notation. A locally cone-like TOP stratified set will be called a *CS set.*

---

[4]) The formal dimension of L is $\sup\{n \mid L^{(n)}-L^{(n-1)}\neq\emptyset\}$.

Remark. [5]) Similarly one defines piecewise linear CS sets by working from category of polyhedra and piecewise-linear maps rather than the category of metric spaces and continuous maps.

Remark. We have not assumed that the links L in a CS set are TOP stratified sets although one might conjecture that they can always be even CS sets by a suitable choice. Certainly cL may be a TOP stratified set although L is not one. To see this let L be the ENR mentioned in the introduction.

Remark. Examples of Milnor [25] show that another link L' for x in X may not be homeomorphic to L even if both L and L' are closed manifolds, in view of the topological invariance of torsions for manifolds [21] [22]. Contrast this with the fact that in piecewise-linear CS sets links are unique up to piecewise linear isomorphism [19]. We must be content with a fattened uniqueness theorem proved as 4.12 and 4.13 below.

EXAMPLES OF CS SETS 1.3. 0) A topological m-manifold X. Here $X^{(n)}=X$ for $n\geqslant m$, $X^{(m-1)}=\partial X=$(the boundary of X), and $X^{(i)}=\emptyset$ for $i\leqslant m-2$.

1) A locally finite simplicial complex X. Here $X^{(n)}$ is the union of simplices of dimension $\leqslant n$.

2) A polyhedron X with its intrinsic stratification in the sense of Armstrong and Zeeman [6]. Here $X^{(n)}$ is the intersection of all simplicial n-skeleta for piecewise linear triangulations of X. This is a piecewise-linear CS set.

3) A differentiable stratified set X in the sense of Thom [35] [24]. Here $X^{(n)}$ is the union of all of Thom's strata of dimension $\leqslant n$.

4) A manifold pair (X, Y) where $Y\neq\emptyset$ is locally flat in X. Supposing $\partial X=\emptyset=\partial Y$ we form just two non-empty strata, $X-Y$ and Y. The equipment required to make this a TOP Thom stratification certainly would include a normal microbundle to Y in X, which may not exist [26A].

5) In [33] I construct a compact CS set that is locally triangulable but not a simplicial complex. Also I construct a CS set that is not locally triangulable.

Questions 1.4. (Stratification conjectures). Is a metrizable topological space X of finite dimension homeomorphic to a CS set if and only if for each point $x\in X$ there is a compactum K, an open neighborhood V of x in X, and a homeomorphism $V\approx cK$ carrying x to the cone point? Does the space underlying a CS set have a unique intrinsic (minimal) stratification in the sense of Armstrong and Zeeman? Its n-skeleton should be the intersection of all possible n-skeleta of CS stratifications of the space.

DEFINITION 1.5. The *depth* $d(X)$ of a stratified set X is

$$d(X)=\sup\{m-n \mid X^{(m)}-X^{(m-1)}\neq\emptyset\neq X^{(n)}-X^{(n-1)}\}$$

[5]) Other species of stratified sets are experimented with later on. But one can still conjecture that each coinsides with the CS sets or else with the more inclusive species the WCS sets in §5 below.

Note that $d(X \times M) = d(X)$ if M has one stratum. More generally $d(X \times Y) = d(X) + d(Y)$, for any two stratified sets. Also, if L is a link for x in X, $d(U \times (cL - v)) < d(U \times cL) \leqslant d(X)$, v being the vertex of the cone cL.

To convince himself of the utility of depth (as distinct from dimension), the reader can give a trivial proof of Sullivan's discovery [37] that a compact piecewise linear CS set has zero euler characteristic if each nonempty stratum has odd dimension. Hint: Carve out a nice regular neighborhood of the stratum of lowest dimension (a closed manifold), then double to get a compact piecewise linear CS set of lesser depth. Now calculate by induction on depth.

It is appropriate to recall here that if X and Y are locally compact and locally connected Hausdorff spaces and $f_t: X \to Y$, $t \in I$, is a path of open embeddings[6]) for the compact open-topology, then the continuous map $f: I \times X \to I \times Y$ defined by the rule $f(t, x) = (t, f_t(x))$ is itself an open embedding. Thus f_t is automatically what is called an isotopy (through embeddings). This is relatively easy to see if there is an compactum $K \subset X$ such that each f_t fixes all points outside K (and X need merely be Hausdorff). In general one shows roughly that a continuous family of open embeddings cannot suddenly uncover points. The details are collected in a lemma.

LEMMA 1.6 (known but not well known cf. [11]). *Consider a continuous map $f: B \times F \to B \times F'$ of cartesian products of topological spaces, such that f respects the first factor projection p_1 onto B, i.e. $p_1 f = p_1$. Suppose that for each point $b \in B$ the map $f_b: F \to F'$ defined by $f_b(x) = p_2 f(b, x)$ is an open embedding.*

Then f is an open embedding in case either condition (I) *or* (II) *holds:*

(I) *F' is Hausdorff and locally compact, and B is locally connected*

(II) *F' is Hausdorff locally compact, and locally connected.*

Proof of 1.6 (by border watching): It will suffice to show that, given a point x in an open set U of F with compact closure $\bar{U}$, and a point $a \in B$, one can find a neighborhood N of a in B and a neighborhood V of $f_a(x)$ such that $f_b(U) \supset V$ for all b in N.

Case I. Let C, D be compact neighborhoods of x in U with $C \subset \mathring{D}$ (=interior of D in F). Choose a connected neighborhood N of a in B so small that

1) $p_2 f(N \times C) \subset f_a \mathring{D}$

2) $p_2 f(N \times \delta U) \subset F' - f_a D$,

where δU is the (compact!) frontier of U in F; i.e. $f_b(\delta U) \subset F' - f_a D$ for all $b \in N$.

We define $V = f_a C$. For each $y \in C$ and $b \in N$, the set $p_2 f(N \times y) \subset F'$ misses the set $f_b(\delta U) = \delta f_b(U)$ by 1) and 2), i.e. $p_2 f(N \times y) \subset f_b(U) \cup (F' - \overline{f_b U})$. As $p_2 f(N \times y)$ is connected and has the point $f_b(y)$ in common with $f_b(U)$, we must have $p_2 f(N \times y) \subset f_b(U)$, for each y. So $V \equiv f_a(C) \subset f_b U$ as required.

---

[6]) An *open* map is one that carries open sets onto open sets.

Case II. Let $C \subset U$ be a compact neighborhood of x that is connected. There exists a neighborhood N of a in B so small that for each b in N

1) $f_b(x)$ lies in the neighborhood $V = f_a(C)$ of $f_a(x)$, and

2) $f_b(\delta U)$ lies in $F' - f_a(C)$ where δU denotes the (compact!) frontier of U. (Observe that $f_b(\delta U) = \delta f_b(U)$).

To complete the proof we show that $f_b(U) \supset V \equiv f_a(C)$ for all b in N. Since $f_a C$ is connected, and $f_a C \subset F' - \delta f_b U = f_b U \cup (F' - \overline{f_b U})$, we conclude $f_a C \subset f_b U$ from the fact that $f_b(x) \in f_b U \cap f_a C$.

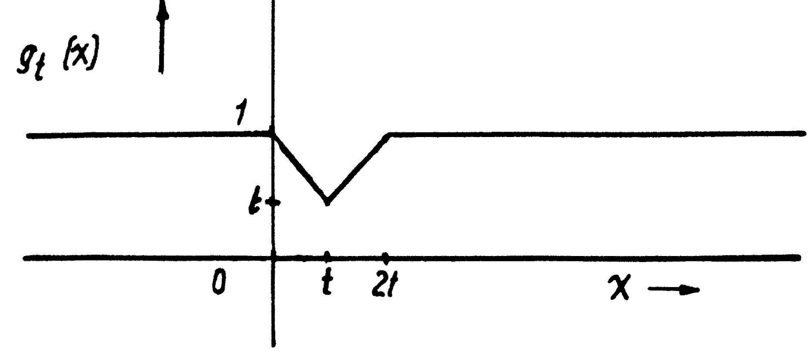
Remark 1.6.1 (data of 1.6). *If $f: B \times F \to B \times F'$ is open, and K is a compactum in $f_a(F)$, then $K \subset f_b(F)$ for all b near a.* The proof is trivial.

Remark 1.6.2 (data of 1.6). *If f is open and there exists a compactum $C \subset F$ such that for $x \notin C$, $f_b(x)$ is independent of $b \in B$, then $f_b F = f_a F$ for all b near a.* (Proof: For b near a, 1.6.1 assures $f_b F \supset f_a C$, hence $f_b F \supset f_a F$; and $f_b F \subset f_a F$ is more trivial.) *Hence, if B is connected, then $f_b F = f_a F$ for all $b \in B$.* (Proof: $b \sim c$ iff $f_b F = f_c F$ is now an open equivalence relation on B).

COROLLARY 1.7. *Let $h: F \to F'$ be an open embedding of locally compact locally connected Hausdorff spaces. Let C be an compactum in F. If $g: F \to F'$ is another open embedding sufficiently near h in the compact-open topology then $h(C) \subset g(F)$. Further, if $g = h$ outside C, and g is sufficiently near h, then $h(F) = g(F)$.*

Proof of 1.7. Let B be the set of open embeddings $F \to F'$, equipped with the compact-open topology. Define $f: B \times F \to B \times F'$ by $f: (g, x) \mapsto (g, g(x))$. Then 1.6 applies; so f is open. Now the above two remarks complete the proof.

COUNTEREXAMPLES 1.8. Lemma 1.6 breaks down for lack of local compactness for the family of homeomorphisms of separable Hilbert space, $f: [0, 1] \times l_2 \to$ $\to [0, 1] \times l_2$ given in [5, §5]. This example suggests the following finite-dimensional example $f: [0, 1] \times F \to [0, 1] \times F$ where F is the origin union the positive half plane, $F = \{(0, 0)\} \cup \{(x, y) \in R^2; x > 0\}$. The formula $f(t, x, y) = (t, x, g_t(x)\, y)$ defines f where $g_t: R \to R$ has the graph



for $0 < t \leqslant 1$ and $g_0 =$identity.

Lemma 1.6 breaks down, for lack of local connectedness, on the map $f: B\times F \to B\times F$ defined as follows. Let F be the subset of the 2-point compactified line $R\cup\{-\infty, +\infty\}$ given by $Z\cup\{+\infty\}$. Let $B=Z_+\cup\{+\infty\}$. For $b\in Z_+$ (=integers $\geqslant 0$), we define $f_b: F\to F$ to be the cyclic permutation $n\to n-1$ modulo 2b on the segment $[-b, b)$ of F and set f_b equal the identity elsewhere. For $b=+\infty$, we define $f_b(n)=n-1$ for $n\in Z$ and $f_b(+\infty)=+\infty$. To show f^{-1} is not continuous, note that $f_b(-b)=b-1$ for all $b\in Z_+$.

§2. The Deformation Theorem

We will often work within the category LOC of continuous maps of locally compact, locally connected, Hausdorff topological spaces. Let X be a space in LOC. Our study of homeomorphisms centers on the following deformability statement $\mathscr{D}(X)$, which generalizes the statement of Theorem 0. Our first goal is to prove it if X is any CS set.

$\boxed{\mathscr{D}(X)}$ *Let $A\subset A'$ be closed subsets of X such that A' is a neighborhood of A. Let $B\subset X$ be compact, and let $U\subset X$ be an open neighborhood of B. Then the following statement always holds.*

$\boxed{\mathscr{D}(X; A, A', B; U)}$ *If $h: U\to X$ is an open embedding equal to the identity inclusion $i: U\to X$ on $A'\cap U$ and h is sufficiently near to $i: U\to X$ (say for the compact-open topology), then there exists an isotopy h_t, $0\leqslant t\leqslant 1$, of h through open embeddings $h_t: U\to X$ such that $h_1=i$ on $A\cup B$, and $h_t=h$ on A and outside some compactum K in U (independent of t and even of h). Further, the isotopy is standard in the sense that it is constructed to be a continuous function of h (for the same topology) as h varies sufficiently near i. Also $h_t=i$ in case $h=i$.*

By 1.6.2, it is inevitable that $hU=h_tU$, $0\leqslant t\leqslant 1$. Thus, for example, if $g_t: X\to X$ is defined for h near i by $g_t(x)=x$ for $x\notin hU$ and $g_t(x)=hh_t^{-1}(x)$ for $x\in hU$, then g_t is an isotopy through homeomorphisms fixing A and points outside hK, from $g_0=\mathrm{id}\mid X$ to a homeomorphism g_1 equal h on B. One can think of the rule $h\rightsquigarrow g_t$ as assigning to a perturbation h of $i: U\to X$ a deformation g_t of the identity homeomorphism $\mathrm{id}\mid X$ to coincide with h near $B\subset U$. This is the description of $\mathscr{D}(X)$ which suggested the title of this article.

Convention. When we speak of a statement $\mathscr{D}(X; A, A', B; U)$ for certain sets X, A, A', B, U the assumptions made above about these sets are automatically presumed to hold unless some contrary statement is explicitly added.

Remark 2.1. Note that as U and A' become smaller (while X, A and B remain unchanged) the statement $\mathscr{D}(X; A, A', B; U)$ becomes stronger. Thus in testing $\mathscr{D}(X)$,

U may as well have compact closure $\bar{U}$, and h have a continuous extension to $\bar{U}$. From this we easily conclude that in case X is metric one might as well use the uniform (epsilon) topology on the open embeddings $U \to X$.

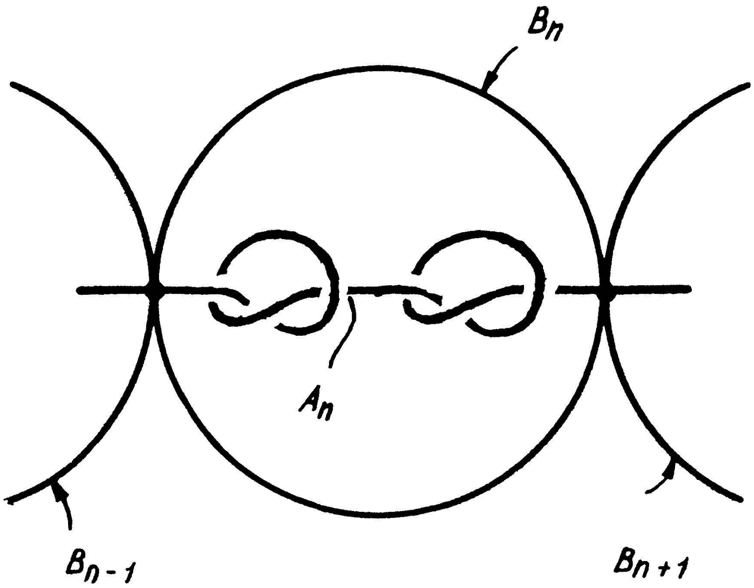
Remark 2.2. *$\mathscr{D}(X)$ is true if (and only if) X is covered by open sets U such that $\mathscr{D}(U)$ holds.* To see this first prove that, for U_1 and U_2 open in X, $\mathscr{D}(U_1)$ and $\mathscr{D}(U_2)$ together imply $\mathscr{D}(U_1 \cup U_2)$, — by taking advantage of A, A' in $\mathscr{D}(U_i;\ A,\ A',\ B,\ U)$.

DEFORMATION THEOREM 2.3. *Suppose X is a CS set. Then the statement $\mathscr{D}(X)$ is true.*

Our main task will be to prove this (in §2 and §3) and then generalize it. A complement concerning subspaces is proved in §4, and results are extended to locally *weakly* cone-like stratified sets in §5.

Remark 2.3.1. Supposing X is a manifold, Černavskii asks whether $\mathscr{D}(X; A, A', B; U)$ is true when $A = A'$ in defiance of our convention that A' be a neighborhood of A. Theorem 4.3 below implies that it is true if $A = A'$ is locally tame in X. However, it is not true in general even if A is a manifold. For example $\mathscr{D}(S^3;\ A,\ A,\ S^3;\ S^3)$ is untrue if $A \subset S^3 = R^3 \cup \infty$ is the wild circle described as follows.

Let B_n be the ball of radius $\frac{1}{2}$ about the point $(n, 0, 0)$, $n = 0, \pm 1, \pm 2, \ldots$, and form an arc A_n in B_n from $(n - \frac{1}{2}, 0, 0)$ to $(n + \frac{1}{2}, 0, 0)$ in which two trefoil knots are tied



One arranges that $A_n \cap \partial B_n = \partial A_n$, where ∂ means boundary, so that $A = \{\cup_n A_n\} \cup \infty$ is a circle in $S^3 = R^3 \cup \infty$ with one wild point, at ∞.

There is a homeomorphism h_n of $S^3 = R^3 \cup \infty$ fixing A and the complement of B_n such that h_n induces a map of $\pi_1(S^3 - A)$ distinct from the identity. Namely h_n can come from interchanging the two trefoils in B_n (slide one through the other). Now h_n

converges to id $| S^3$ as n tends to infinity. This shows that the group $H(S^3 \operatorname{rel} A)$ of homeomorphisms of S^3 fixing A is not locally connected.

Proof of Deformation Theorem. The proof exploits the Alexander isotopy on cones and induction, in a simple way illustrated by the following trivial

*Test Case of $\mathscr{D}(X; A, A', B; U)$: where $X = U = cM$ is the cone on a compact topological manifold M without boundary; $A = A' = \emptyset$; $B =$*vertex of *cM.*

Proof of test case: If $h: cM \to cM$ is near the identity, then

$$h(M \times (\tfrac{1}{2}, \infty)) \subset cM - B = M \times (0, \infty),$$

and $\mathscr{D}(M \times (0, \infty))$, (which is known to be true [14]), provides an isotopy of h to an embedding $h': cM \to cM$ which is the identity near $M \times 1$. Define $h'': cM \to cM$ by

$$h''(x) = \begin{cases} h'(x) & \text{for} \quad x \in c_1 M, \quad \text{the quotient of } M \times [0,1] \text{ in cM} \\ x & \text{for} \quad x \in M \times [1, \infty). \end{cases}$$

Then the Alexander isotopy H_t, $0 \leqslant t \leqslant 1$, of h'' to identity (see [3] or §5) permits us to define

$$h_t = H_t h''^{-1} h, \quad 0 \leqslant t \leqslant 1,$$

which is the required isotopy of h. The rule $h \rightsquigarrow h'$ is continuous by hypothesis $\mathscr{D}(M \times (0, \infty))$. So the rule $h \rightsquigarrow h_t$, $0 \leqslant t \leqslant 1$, clearly is too. Also $h_t = \mathrm{id}$ if $h = \mathrm{id}$.

Thus test case is proved.

Setting the above case aside we now reduce the Deformation Theorem to cases where $X = R^m \times (\text{cone})$, which will be proved in §3.

First note that the case where X is of finite dimension really includes the general case, because X is locally of finite dimension. (See Remark 2.1 or 2.2.)

We deal with the case of finite dimension by induction on depth. *Thus suppose inductively that $\mathscr{D}(Y)$ is true for each CS set Y of depth $< d < \infty$.* We proceed to prove $\mathscr{D}(Z)$ for any CS set Z of depth d. *Beware that the inductive assumption persists until the end of this section* (2.5 *and* 2.6 *excepted*).

As usual, it will suffice to prove handle lemmas.

Notations 2.4. $B^m = [-1, 1]^m \subset R^m$ and $\mathring{B}^m = (-1, 1)^m \subset R^m$. If $S \subset R^m$, $\lambda S = \{\lambda x \in R^m \mid x \in S\}$. We identify $R^m = R^k \times R^n$ when $k + n = m$. If cL is an open cone, $c_\lambda L = [0, \lambda) \times L / 0 \times L \subset cL$ and $\bar{c}_\lambda L = [0, \lambda] \times L / 0 \times L \subset cL$, is its closure. The vertex of cL is denoted by v.

2.5. HANDLE LEMMA (index 0). *The statement $\mathscr{D}(X; A, A', B; U)$ is true in case: $X = R^m \times cL$, where L is a compact stratified set and $\mathscr{D}(R^m \times (cL - v))$ holds; $A = A' = \emptyset$; $B = B^m \times \bar{c}_1 L$; $U = 10\mathring{B}^m \times c_{10} L$.*

2.6. HANDLE LEMMA (index k). *The same as index* 0 *except that*

$A = A' = (R^k - \mathring{B}^k) \times R^n \times cL, \qquad k + n = m.$

Observe that for $k>0$, A' is *not* a neighborhood of A.

First note that the handle lemmas applied stepwise finitely often to some handles of a suitable array of small handles in R^m will prove

ASSERTION (1). *Let $X = R^m \times cL$ be CS depth d; let $B \subset R^m \times v$ be a compactum; and let U be an open neighborhood of B.*

Let closed sets $A \subset A'$ X be given with A' a neighborhood of A. Then there exists a compact neighborhood B' of B in U such that $\mathscr{D}(X;\ A,\ A',\ B';\ U)$ holds true.

The handles can correspond to the simplices of a linear triangulation of a neighborhood of B in $R^n \times v$ so fine that no closed star meets both A and $R^n - A'$. Cf. arguments in [14], [31, §3].

ASSERTION (2). *$\mathscr{D}(X)$ is true if $X = R^m \times cL$ is a CS set of depth d.*

Proof of (2): According to Assertion (1), there exists a compact neighborhood B' of $B \cap (R^m \times v)$ in X such that $\mathscr{D}(X;\ A'',\ A',\ B';\ U)$ holds for $A \subset\subset A'' \subset\subset A'$. Here $S \subset\subset T$ means closure $(S) \subset$ interior (T).

Choose a compactum B'' so that $B \cap (R^m \times v) \subset\subset B'' \subset\subset B'$. Since $B - \mathring{B}''$ is a compactum in $X - R^m \times v$, which has depth $<d$, the inductive hypothesis that $\mathscr{D}(X - R^m \times v)$ holds shows that

$\mathscr{D}(X;\ A \cup B'',\ A'' \cup B',\ B - \mathring{B}'';\ U)$

holds true (cf. Remark 2.1). This is the same statement as

$\mathscr{D}(X;\ A \cup B'',\ A'' \cup B',\ B;\ U)$

But together with $\mathscr{D}(X;\ A'',\ A',\ B';\ U)$ this implies $\mathscr{D}(X;\ A,\ A',\ B;\ U)$ simply by composing isotopies. So assertion (2) is established.

If X is a CS set of depth d, it is covered by open sets, each isomorphic to $R^m \times cL$ for suitable R^m and L, and each of depth $\leqslant d$. Since each $\mathscr{D}(R^m \times cL)$ holds by Assertion (2), $\mathscr{D}(X)$ also holds in view of Remark 2.2. Thus the Deformation Theorem is established by induction on depth assuming, of course, the proof of the handle lemmas in the next section.

§3. Proof of Handle Lemmas 2.5, 2.6.

We will use the uniform epsilon topology of standard metrics throughout.

First consider index 0. We proceed to construct a sequence g_1, g_2, g_3, g_4, g_5 of open embeddings all of which we stipulate must

1) *be equal* h *on* $B^m \times \bar{c}_1 L$,[7])
2) *be a continuous function of* h *as* h *varies near* i, *and*
3) *be the identity when* $h = \text{identity}$.

These properties will usually *not* be explicitly mentioned again as the construction proceeds.

$\boxed{g_1 : T^m \times c_9 L \to T^m \times cL}$, where T^m is the m-torus $R^m/8Z^m$ which contains $3B^m$ quotient of $3B^m \subset R^m$. This g_1 is constructed by wrapping up h, m-times successively, once along each co-ordinate axis of R^m, using each time the "wrapping-up" idea of R. Edwards in a form that we will state as 4.9 in the next section. The reader unacquainted with this method should examine closely 4.9 or [14, §8.1], as the construction of g_1 is basic.

$\boxed{g_2 : T^m \times cL \to T^m \times cL}$, a homeomorphism equal the identity outside $T^m \times c_2 L$. It is obtained by applying $\mathscr{D}(T^m \times (cL - v))$, which is equivalent to $\mathscr{D}(R^m \times (cL - v))$ by Remark 2.2, to the restriction $T^m \times (c_9 L - \bar{c}_1 L) \xrightarrow{g_1} T^m \times (cL - v)$. This is defined when h, and hence g_1, is near the identity.

This lets us alter g_1 outside $T^m \times c_1 L$ to be the identity near the frontier of $T^m \times c_2 L$. Then by fiat we change it to equal the identity outside $T^m \times c_2 L$ thus obtaining g_2. By 1.7, g_2 is a homeomorphism.

$\boxed{g_3 : R^m \times cL \to R^m \times cL}$ is the homeomorphism covering g_2 that is near the identity and equal the identity outside $R^m \times c_3 L$.

$\boxed{g_4 : R^m \times cL \to R^m + cL}$, will be defined to be $\Theta^{-1} g_3 \Theta$ where Θ is a "horn" homeomorphism of $R^m \times cL$ defined as follows.

Let $\gamma_t : cL \to cL$, $0 \leqslant t \leqslant 1$, be defined by $\gamma_t q(u, y) = q((1-t)u, y)$ where $q : [0, \infty) \times \times L \to cL$ is the quotient map. It is an isotopy for $t < 1$, and $\gamma_1(cL) = v$.

Let $\beta : [0, \infty) \to [0, 1)$ map $[0, 2]$ to 0 and give a homeomorphism $[2, \infty) \to [0, 1)$. Now define $\Theta(x, z) = (x, \gamma_{\beta(|x|)}^{-1}(z))$ for $(x, z) \in R^m \times cL$.

Observe that g_4 has properties 1), 2), 3) and also three more:

4) It is bounded on the R^m factor, i.e.

$$\sup\{|x - p_1 g_4(x, z)| ; (x, z) \in R^m \times cL\} < \infty$$

where p_1 is projection to R^m.

5) For each neighborhood N of v in cL there exists a radius $R(N)$ such that $g_4(x, z) = (x, z)$ if $|x| > R(N)$ and $z \notin N$.

6) It equals the identity outside $R^m \times c_4 L$.

---

[7]) To make this always meaningful we will identify $3B^m \times cL$ with its quotient in $T^m \times cL$.

$\boxed{g_5 : R^m \times cL \to R^m \times cL}$, is defined using a ray preserving embedding $j : R^m \to R^m$ onto $5\mathring{B}^m$ that is the identity on $2B^m$.

Let $J = j \times (\mathrm{id} \mid cL) : R^m \times cL \to R^m \times cL$ and define g_5 by

$$\begin{cases} g_5(x) = Jg_4 J^{-1}(x) & \text{for} \quad x \in \mathrm{Image}(J) \\ g_5(x) = x & \text{otherwise}. \end{cases}$$

In view of 4) and 5) it is a homeomorphism. And it clearly equals the identity outside $5B^m \times c_5 L$.

We now use g_5 to construct h_t, $0 \leqslant t \leqslant 1$. Being the product $(c\partial B^m) \times (cL)$ of two open cones, $R^m \times cL$ is naturally an open cone. So there is an Alexander isotopy G_t, $0 \leqslant t \leqslant 1$, of $g_5 = G_0$ to the identity (see remarks preceding 5.4 below), and G_t will certainly fix the complement of $5B^m \times c_6 L$.
Define

$$h_t = G_t g_5^{-1} h, \qquad 0 \leqslant t \leqslant 1.$$

Now $h_1 = g_5^{-1} h$ certainly equals the identity on $B^m \times \bar{c}_1 L$. Also $h_t = h$ outside $h^{-1}(5B^m \times c_6 L)$, which lies in the compactum $9B^m \times \bar{c}_9 L \subset 10\mathring{B}^m \times c_{10} L = U$ if h is near the identity. Thus the rule $h \rightsquigarrow h_t$ establishes the handle lemma for index 0.

About the handle lemma for index $k>0$. We will indicate the changes required in the above proof for index 0.

To the stipulations 1), 2), 3) about $g_1, \ldots, g_5$, add that each g_i

0) *be equal the identity on* $A = (R^k - \mathring{B}^k) \times R^n \times cL$ *or on* $\hat{A} = (T^k - \mathring{B}^k) \times T^n \times cL$ *(whichever set meets the domain of* g_i*).*

To assure this, one need only change the construction in two minor ways.

First, in changing g_1 [8]) to be the identity near the frontier of $T^m \times c_2 L$, one must at the outset alter g_1 to g_1' equal the identity on a small neighborhood of $\hat{A} \cap T^m \times \times Fr(c_2 L)$. Here Fr indicates frontier in cL. This is to be done by conjugating g_1 by a fixed small self-homeomorphism of $T^m \times cL$ that maps $\hat{A}$ into $\hat{A}$, maps $\hat{A} \cap T^m \times \times Fr(c_2 L)$ into the interior of $\hat{A}$ and fixes $T^m \times c_1 L$ and the complement of $T^m \times c_3 L$. Then by building g_2 from g_1' (rather than g_1) we can assure 0).

Second, one must alter the construction of g_3 from g_2. Observe that the universal covering $g_3' : R^m \times cL \to R^m \times cL$ of g_2 near the identity equals the identity on the frontier $(\partial B^k) \times R^m \times cL$ of A but not on the translates of $B^k \times R^m \times cL$ by $8Z^k \subset 8Z^m$. So we just set $g_3 =$ identity on A and set $g_3 = g_3'$ outside A.

After these two modifications the construction of h_t becomes valid for index $k>0$. At all other points of the construction, condition 0) is automatically verified.

The handle lemmas have now been proved.

---

[8]) For which 0) is assured as we wrap up trivially along the first k factors of R^n.

Deformation of Homeomorphisms on Stratified Sets (Continued)

by L. C. SIEBENMANN

§4. Deformations Respecting Subsets

DEFINITION 4.1. A *sub stratified set* Y of a stratified set X consists of a *closed* subspace Y of the space X equipped with the filtration $Y^{(n)} = X^{(n)} \cap Y$, such that, for each n, $Y^{(n)} - Y^{(n-1)}$ is open (as well as closed) in $X^{(n)} - X^{(n-1)}$. We write $Y < X$ to indicate this.

DEFINITION 4.2. Let U be an open subset of any space X. We will say that an open embedding $h: U \to X$ *thoroughly respects* a subset $C \subset X$ if h gives by restriction an open embedding $U \cap C \to C$.

We shall prove the

4.3. COMPLEMENT TO THE DEFORMATION THEOREM. *Let X be a CS set. The statement $\mathscr{D}(X)$ remains valid when the following statement is appended to it.*

$\boxed{\mathscr{R}(X)}$ *For each $Y < X$ it is true that whenever $h: U \to X$ thoroughly respects Y or pointwise fixes Y, then h_t does likewise for all t.*

As preparation for the proof of this complement we insert some lemmas. In them X will denote a stratified set.

LEMMA 4.4. *A closed subset Y of a TOP stratified set X is a sub stratified set of X if (and only if) $Y^{(n)} - Y^{(n-1)} \equiv Y \cap (X^{(n)} - X^{(n-1)})$ is a topological n-manifold without boundary for each $n \geqslant 0$.*

Proof 4.4. Clearly $Y^{(n)} - Y^{(n-1)}$ is closed in $X^{(n)} - X^{(n-1)}$. Also $Y^{(n)} - Y^{(n-1)}$ is open in $X^{(n)} - X^{(n-1)}$ by invariance of domain.

We can now find all sub stratified sets of X. Consider the space $\hat{X}$ (not Hausdorff in general) obtained on dividing X by the equivalence relation which equates pairs of points that belong to the same connected (open and closed!) component of some stratum of X. It has one point for each such component.

LEMMA 4.5. *Let $q: X \to \hat{X}$ be the (continuous) quotient map. A subset Y of X constitutes a sub TOP stratified set of X if and only if $Y = q^{-1}C$ where C is a closed subset of $\hat{X}$.*

Definitions have been expressly chosen to make this trivially true.

LEMMA 4.6. *Let M be a connected manifold without boundary. Then, for each $Z < X \times M$, one has $Z = Y \times M$ for some $Y < X$.*

Proof of 4.6. In view of 4.5 this amounts to observing that $\hat{X} \approx (X \times M)^\wedge$ by sending the quotient of a stratum component S to the quotient of $S \times M$.

LEMMA 4.7. *Let $X = R^m \times cL$, where L is a compact stratified set. The sub TOP stratified sets of X are precisely the sets $R^m \times cK$ for $K < L$.*

Proof of 4.7. Suppose $Y < R^m \times cL$. Then $Y - R^m \times v < R^m \times (cL - v) \cong R^{m+1} \times L$, v being the vertex of cL. Hence $Y - R^m \times v = R^m \times (cK - v) \cong R^{m+1} \times K$, for some $K < L$ by Lemma 4.6. It follows easily that $Y = R^m \times cK$ as required.

An immediate consequence of this lemma is the

LEMMA 4.8. *If X is a CS set, then every sub-stratified set Y of X is itself a CS set. (Call it a sub CS set.)*

We now give

Proof of complement 4.3. We assert that the construction of h_t, $0 \leqslant t \leqslant 1$, from $h: U \to X$ described in proving the original deformation theorem *automatically* satisfies $\mathscr{R}(X)$ if a little extra care is taken in specifying it.

More specifically, the proof of $\mathscr{D}(X)$ can be repeated with the addition of a statement of the form $\mathscr{R}(X)$ to *each* statement of the form $\mathscr{D}(X)$, or $\mathscr{D}(X; A, A', B; U)$ encountered in it. Only the complemented handle lemmas require further proof.

As for these handle lemmas add to their proof in §3 as follows. For the construction of $g_1, \ldots, g_5$ add the extra stipulation that

(*) *For each $R^m \times cK < R^m \times cL$, each g_i thoroughly respects* [fixes] $R^m \times cK$ or $T^m \times cK$ *(whichever makes sense) whenever h thoroughly respects* [fixes] $R^m \times cK$.

Fortunately the construction as given guarantees (*). This is evident for g_2, g_3, g_4, g_5. To check (*) for $g_1: T^m \times c_q L \to T^m \times cL$ we recall Edwards wrapping up process.

PROPOSITION 4.9. *Let A be a metric space in LOC. Let $A \supset B \supset C$ where B, C are open and the closure $\bar{C}$ of C is a compactum in B.*

Consider open embeddings

$$h: (10\mathring{B}^m) \times B \to R^m \times A.$$

If h is sufficiently near to the inclusion $i: (10\mathring{B}^m) \times B \hookrightarrow R^m \times A$ (say for the uniform topology[9]*) of the metric on $R^m \times A$), then there exists a construction producing from h an open embedding*

$$h': T^m \times C \to T^m \times A$$

where $T^m = R^m/8Z^m$. The rule $h \rightsquigarrow h'$ has the following properties:

(a) *h equals h' on $2B^m \times C$, when we identify $3B^m$ with its quotient in T^m.*

(b) *h' is a continuous function of h (for the same uniform topology).*

(c) *h' is a product with* id $| \, T^m$ *when*
h is a product with id $| \, 10\mathring{B}^m$.

[9]) Equivalently one could use the compact-open topology. Then 4.9 holds without a metric on A.

(d) *For each closed subset* $D \subset A$ *it is true that whenever* h *thoroughly respects or fixes* $R^m \times D$, *then* h' *does likewise for* $T^m \times D$.

The proof of 4.9 can easily be adapted from that of Lemma 8.1 in [14]. Here I simply define h' and leave a diagram to help readers to complete the proof by inspection.

An m-fold application of the case $R^m - R$ yields the general case. Hence we can assume $m=1$ for the proof.

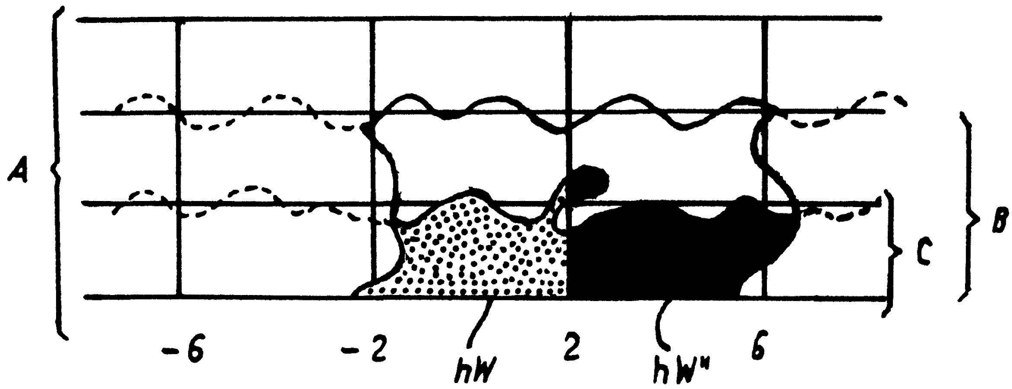


Figure 4a

Writing $q: R \times A \to (R/8Z) \times A \equiv T^1 \times A \equiv S^1 \times A$ for the quotient map define $h'(qx) = qh(x)$ for points qx in $qW \approx W$, where

$$W = \{x \in [-2, 6] \times C \mid p_1 h(x) \leqslant 2\} \subset R^1 \times C.$$

It remains to define h' on $qW'' \approx W''$, where

$$W'' = \{x \in [-2, 6] \times C \mid p_1 h(x) \geqslant 2\} \subset R^1 \times C,$$

as the composition of homeomorphisms

$$qW'' \xrightarrow{q^{-1}} W'' \xrightarrow{h} hW'' \xrightarrow{\alpha \times \mathrm{id}} \hat{W} \xrightarrow{h(\beta \times \mathrm{id})h^{-1}} W' \xrightarrow{q} qW' \subset T^1 \times A.$$

Here $\alpha: R \to R$ is any homeomorphism sending 2 to -6 and fixing points $\geqslant 5$ while $\beta: R \to R$ is any homeomorphism sending 6 to -2 and fixing points $\leqslant -5$, so that $\beta\alpha$ maps [2, 6] linearly onto $[-6, -2]$. To assure the continuity of h^{-1} here as a function of h we need 1.6 and $A \in \mathrm{LOC}$.

This completes the construction of h', valid for h near i. For $A = B = C$ it is the construction I made in [24, §5].

The complement 4.3 is now established.

This section concludes with a digression.

There is a variant of Edward's proposition that is more truly a generalization of the proposition $X \times R \approx Y \times R \Rightarrow X \times S^1 \approx Y \times S^1$ for compacta X, Y, which suggested it. We mention it because it proves a topological analogue in CS sets of the PL invariance of links in polyhedra.

4.10. VARIANT OF 4.9. *Suppose we alter the data of* 4.9 *by considering open imbeddings*

$$h:(-10, 10)\times B\to R\times A^*$$

where A^* is a locally compact metric space distinct from A. Then for certain such h there is a construction producing an open embedding

$$h':T^1\times C\to T^1\times A^*.$$

And the rule $h\rightsquigarrow h'$ satisfies conditions (a), (b), (c), (d) *of* 4.9 (for $m=1$). *The following conditions on h are quite sufficient to guarantee that h' be defined.*

(i) $p_1h:(-10, 10)\times B\to R$ *is near the* 1*st factor projection onto* $(-10, 10)$, *say everywhere within distance* 1 *of it.*

(ii) $h((-10, 10)\times B)\supset[-8, 8]\times p_2h([-2, 6]\times\bar{C})$ *where p_2 is projection* $R\times A^*\to$ $\to A^*$.

The proof of 4.10 is essentially identical to that of 4.9.

A straightforward m-fold application of this variant yields:

PROPOSITION 4.11. *Let C, C^* be locally compact stratified sets with one vertex each v, v^*. Suppose $h:10\mathring{B}^m\times C\to R^m\times C^*$ is an open embedding inducing an isomorphism onto its image, and equal the natural identification $R^m\times v\to R^m\times v^*$ on $10\mathring{B}^m\times v$. Then there is a neighborhood C_0 of v in C and an open embedding $h':T^m\times C_0\to T^m\times C^*$, realizing an isomorphism onto its image, such that $h=h'$ near $B^m\times v$, and h' extends the identification $T^m\times v=T^m\times v^*$.*

COROLLARY 4.12 of 4.11. *In this situation, if C, C^* are open cones cL, cL^*, then*

(i) $T^m\times cL\cong T^m\times cL^*$ by an h' as in 4.11,

(ii) $T^{m+1}\times L\cong T^{m+1}\times L^*$ and

(iii) $R^m\times cL\cong R^m\times cL^*$.

Proof of 4.12 *from* 4.11. In 4.11 we can arrange that $C_0=c_\lambda L$ with $\lambda>0$ (notation of 2.4). Applying the theorem of uniqueness of open cone neighborhoods [23] (which is clearly valid in a stratified version) we find that $h'(T^m\times c_\lambda L)\cong T^m\times cL$ fixing a neighborhood of $T^m\times v^*$.

This proves (i); and (ii), (iii) follow from it.

The salient conclusion to draw from 4.12 is that if L, L^* are two links in X for $x\in X^{(m)}-X^{(m-1)}$, then $T^m\times cL\cong T^m\times cL^*$. More precisely we get

COROLLARY 4.13 of 4.12. STAR UNIQUENESS (cf. 5.12 for WCS sets). *Let X be a CS set and let C, C^* be two open cones on stratified sets. Suppose $R^m\times C$, $R^m\times C^*$ isomorphic to open neighborhoods of one point of $X^{(m)}-X^{(m-1)}$ by embeddings j, j^*.*

Then there exists an isomorphism $\theta: R^m \times C \cong R^m \times C^*$. *Further if* $j(R^m \times v) = j^*(R^m \times v)$, *then* θ *can equal* $(j^*)^{-1} \circ j$ *near* $B^m \times v$ *in* $R^m \times C$, *and on* $R^m \times v$. *Further* θ *can cover an isomorphism* $T^m \times C \cong T^m \times C^*$.

§5. The Generalisation to WCS Sets

DEFINITION 5.1. A stratified set X is *locally weakly cone-like* if it is locally of finite depth, and for each $n \geqslant 0$ and each point x in $X^{(n)} - X^{(n-1)}$ there is a mock open cone C with vertex v (to be defined presently) and a homeomorphism $\theta: R^n \times C \to N$ onto an open neighborhood of x in X so that $\theta^{-1} X^{(n)} = R^n \times v$.

Notation. A locally weakly cone-like TOP stratified set is called a WCS set.

DEFINITION 5.2. *A mock open cone C with vertex v* is a locally compact metric space C equipped with a homotopy $\gamma_t: C \to C$, $0 \leqslant t \leqslant 1$, such that

(1) γ_t, $0 \leqslant t < 1$, is an isotopy of $\mathrm{id} \mid C$ (through homeomorphisms).

(2) $\gamma_0 = \mathrm{id} \mid C$, $\quad \gamma_1(C) = v \in C$, and $\gamma_t(v) = v$ for all t.

Call such a homotopy γ_t a *crush* (of C to v).

Open cones on compacta are the trivial examples.

5.3. NON TRIVIAL EXAMPLES OF MOCK OPEN CONES. Consider $C = W \cup \varepsilon_+$ where

(a) W is an open TOP manifold of dimension $\geqslant 5$ that is proper homotopy equivalent to (or even properly dominated by) $K \times R$ where K is a finite connected complex.

(b) ε_+ is one of two end points ε_-, ε_+ of W.

For this example a crush γ_t of $W \cup \varepsilon_+$ to ε_+ can be constructed by an engulfing argument so that (1) and (2) hold and, for each t, γ_t fixes points outside a compactum in W (depending this time on t). See [28, §2] [30, §7] [32]. If an obstruction $\sigma(\varepsilon_+)$ in $\tilde{K}_0 \pi_1 W$ is non-zero, then $C = W \cup \varepsilon_+$ is not an open cone [27] [28] [21]. All the mock open cones I know of that fail to be open cones, are built from these examples and variants of them where W has boundary.

Observe that if $h: C \to C$ is a self-homeomorphism of a mock-open cone fixing all points outside a compact set $K \subset C$, then there is an *Alexander isotopy* of h to the identity defined by

$$\begin{cases} h_t = \gamma_t h \gamma_t^{-1} & \text{for} \quad t < 1 \\ h_t = \mathrm{id} \mid C & \text{for} \quad t = 1 . \end{cases} \tag{5.3}$$

To see that h_t varies continuously with t observe that h_t fixes $\gamma_t(C - K) = C - \gamma_t K$, and that $\gamma_t(K)$ lies in any prescribed neighborhood of v for t sufficiently near 1.

Note that each WCS set X is locally compact, locally contractible, and locally finite dimensional, and so is locally an ENR (globally if $\dim X < \infty$).

The evident fact that *the product of two mock open cones is a mock open cone* shows that the product of two WCS sets is a WCS sets.

Now we prove

5.4. DEFORMATION THEOREM (generalized). *$\mathscr{D}(X)$ is true for each* WCS *set X.*

The proof for CS sets has been so constructed that after a few changes (mostly notational) our proof for CS sets can be reread verbation. Here are the changes. Substitute "WCS set" for "CS set". Substitute "mock open cone" for "open cone". Substitute a mock open cone C with vertex v for the open cone cL with vertex v, wherever cL is mentioned.
Let $\gamma_t: C \to C$, $0 \leqslant t \leqslant 1$, crush C to v and substitute this γ_t for the γ_t used to define g_4. Let $C_1 \subset C_2 \subset C_3 \subset \ldots$ be any sequence[10]) of open neighborhoods of the vertex v in C with compact closures $\bar{C}_1, \bar{C}_2, \bar{C}_3, \ldots$ such that for each $\lambda \geqslant 1$ and each $t \in [0, 1]$, one has $\gamma_t(\bar{C}_\lambda) \subset C_{\lambda+1}$. Now substitute C_λ for $c_\lambda L$ and $\bar{C}_\lambda$ for $\bar{c}_\lambda L$ for $1 \leqslant \lambda \leqslant 10$.

Consider next the problem of making isotopies respect subsets of the WCS set X. The discussion of §4 generalizes trivially if we restrict attention to "orderly" WCS sets.

DEFINITION 5.6. Let C be a mock open cone with vertex v that is simultaneously a stratified set. A crush $\gamma_t: C \to C$, $0 \leqslant t \leqslant 1$, of C to v is *orderly* if, for each $n \geqslant 0$ and for each $t < 1$, γ_t maps $C^{(n)}$ homeomorphically onto $C^{(n)}$. Then C is an *orderly mock open cone.*

An orderly crush γ_t clearly has the property that for each $t < 1$, γ_t maps each component of $C^{(n)} - C^{(n-1)}$ homeomorphically onto itself. From this we immediately deduce:

LEMMA 5.7. *Let $\gamma_t: C \to C$, $0 \leqslant t \leqslant 1$, be an orderly crush of a locally compact stratified set C to v. Then for each $D < C$, γ_t gives an orderly crush of D. In particular D is a mock open cone.*

DEFINITION 5.8. An *orderly* WCS set is defined by replacing mock open cones by orderly mock open cones in the definition of WCS sets (5.1 above), while insisting that θ be an isomorphism of stratified sets.

Remark. Conceivably every WCS set is orderly.

An immediate corollary of 5.7 is

---

[10]) It is important that C_{10} can be as small as we please when one (re)proves assertion (1) below 2.6.

LEMMA 5.9. *If $Y<X$ and X is an orderly WCS set, then Y is also an orderly WCS set.*

5.10. GENERALIZED COMPLEMENT TO DEFORMATION THEOREM. *Let X be an orderly WCS set. The statement $\mathscr{D}(X)$ remains valid when the statement $\mathscr{R}(X)$ of* 4.3 *is appended.*

Proof of 5.10. In the proof of the complement for CS sets given in §4 we merely make the same substitutions as we have made above to prove the Deformation Theorem for WCS sets. The two lemmas 5.7, 5.9 above replace analogous results from §4.

Remark 5.11. If we append only the restricted version of $\mathscr{R}(X)$ where $Y<X$ is fixed, then the assumption that X be orderly can clearly be weakened somewhat.

Remark 5.12. *The star uniqueness result* 4.13 *holds in WCS sets if orderly mock open cones replace open cones.* The proof is as in §4 except that the uniqueness theorem for open cone neighborhoods must be replaced by a generalization in [32].

Question 5.13. *When is a mock open cone $C=W\cup\varepsilon$ of example* 5.3 *stably cone-like?* i.e. *when is $R^m\times C$ locally cone-like for some m?*

Here is one step toward the answer. Suppose that, for some compactum L, $R^n\times cL$ embeds† in $R^n\times C$ sending $R^n\times$(vertex) into $R^n\times$(vertex). By Remark 5.12 and 4.13, we can find an isomorphism $T^n\times C\cong T^n\times cL$ and arrange that it extends the identity-isomorphism $T^n\times(\text{vertex})\cong T^n\times(\text{vertex})$. Hence $T^n\times V\approx T^{n+1}\times L$ where V is W with its two ends glued together as in [30, Chap. II] and the homeomorphism respects the natural projection of each fundamental group to $Z^n\times Z=Z^{n+1}$.

Conversely if $T^n\times V\approx T^{n+1}\times L$ for some compactum L by a homeomorphism respecting fundamental group in this way, then we find $R^n\times C\cong R^n\times cL$ as follows. Passing to a covering of $T^n\times V\approx T^{n+1}\times L$ we get a homeomorphism $R^n\times W\approx R^{n+1}\times L$ commuting with action of Z^{n+1} as covering translations. This extends therefore over suspension spheres to a homeomorphism $\Sigma^n\hat{W}\approx\Sigma^{n+1}L$ where $\hat{W}=W\cup\{\varepsilon_-,\varepsilon_+\}$ is the end compactification of W, cf. [33, Theorem A, p. 77] [33 A]. Since $\Sigma^n\{\varepsilon_-,\varepsilon_+\}$ corresponds to $\Sigma^{n+1}(\emptyset)\approx S^n$ we deduce an open embedding respecting strata $R^n\times cL\to$ $\to R^n\times\{W\cup\varepsilon_+\}=R^n\times C$, which by 4.13 implies $R^n\times C\cong R^n\times cL$. This discussion generalizes immediately to any mock open cone C with vertex v such that $\mathscr{D}(C-v)$ is valid (so that isotopy extensions are available permitting the gluing construction of [30] and the considerations of [32]).

Applying this for $n=1$ we get

PROPOSITION 5.14. *Let $C=W\cup\varepsilon_+$ be a mock open cone from example* 5.3, *with $\dim W\geqslant 5$. Then $C\times R$ is locally cone-like if and only if $\sigma(\varepsilon_+)=x-(-1)^{\dim W}\bar{x}$*

---

† As an open subset.

with $x \in \tilde{K}_0 \pi_1 W$ *where bar indicates the duality involution* [27] *depending only on* $w_1(W): \pi_1 W \to Z_2$.

Proof of 5.14. $C \times R$ is locally cone-like implies $Y \times T^1 \approx L \times T^2$ respecting fundamental groups, as shown above. This occurs iff Y is invertibly cobordant to $L \times S^1$, and W to $L \times R$. Finally, if W is invertibly cobordant to $L \times R$, then [33] one has $\sigma(\varepsilon_+) = x + (-1)^{\dim W} \bar{x}$, for $x \in \tilde{K}_0 \pi_1 W$, and conversely if $\dim W \geqslant 5$. (The element x is the infinite torsion of a cobordism from W to $L \times R$).

§6. Familiar Consequences of $\mathscr{D}(X)$

These consequences are developed in generality to provide a convenient reference for eventual applications.

6.0. The hypothesis of deformability $\mathscr{D}(X)$ is stated in §2. In applications we frequently want a relative form of it concerning a family $\mathscr{S}$ of one or more subsets of X.

$\boxed{\mathscr{D}(X; \mathscr{S})}$ denotes the statement $\mathscr{D}(X)$ modified as follows:

(i) *Restrict attention to open embeddings* $h: U \to X$ *which thoroughly respect each* Y *in* $\mathscr{S}$ (in the sense of §4.2).

(ii) *Demand that the isotopy* h_t *of* h *thoroughly respect each* Y *in* $\mathscr{S}$.

(iii) *For any* Y *in* $\mathscr{S}$, *demand that* $h_t \mid U \cap Y = $ *(inclusion) for all* t, *whenever* $h \mid U \cap Y = $ *(inclusion)*.

The same modifications define $\mathscr{D}(X; A, A', B; U; \mathscr{S})$ when applied to $\mathscr{D}(X; A, A', B; U)$.

Note that if X is a CS set and $\mathscr{S}$ some family of sub CS sets $Y < X$, then $\mathscr{D}(X, \mathscr{S})$ is implied by 4.3 (complement to the deformation theorem).

The next theorem verifies $\mathscr{D}(X, \mathscr{S})$ in a rather different situation.

THEOREM 6.1. *Consider a product* $X \times B$ *in* LOC *and let* $\mathscr{S}$ *be the family* $\{X \times b;\ b \text{ in } B\}$. *Then* $\mathscr{D}(X)$ *implies* $\mathscr{D}(X \times B; \mathscr{S})$.

Proof of 6.1. We have to prove an arbitrary statement $\mathscr{D}(X \times B; A, A', C; U; \mathscr{S})$ given the usual assumptions about A, A', C, U (cf. §2). The reader will easily check that for this it suffices to establish the *Special Case*[11]: *Where* A, A', C, U *are of the form*

$$A = A_0 \times B \cup X \times A_1;$$
$$A' = A'_0 \times B \cup X \times A'_1;$$
$$C = C_0 \times C_1;$$
$$U = U_0 \times U_1.$$

[11]) The proof will apply even when $A_1 = A'_1$. So something stronger than 6.1 holds.

Proof of Special Case. Consider an open embedding $h: U_0 \times U_1 \to X \times B$ which thoroughly respects each set $X \times b$, b in B. Define $h^{(b)}: U_0 \to X$ for b in U_1 by the equation $h(x, b) = (h^{(b)}(x), b)$. If h is near the inclusion $\mathscr{D}(X;\ A_0, A_0', C_0;\ U_0)$ provides an isotopy $h_t^{(b)}$, $0 \leqslant t \leqslant 1$, which depends (uniformly) continuously on $h^{(b)}$, and satisfies certain conditions (listed in §2). In particular $h_t^{(b)}$ is the inclusion for all t if $h^{(b)}$ is – hence if $b \in A_1'$. And $h_t^{(b)}(x) = h^{(b)}(x)$ for x outside a compactum $K_0 \subset U_0$.

Let K_1 be a compact neighborhood of C_1 in U_1 and let $\alpha: U_1 \to [0, 1]$ be a continuous map such that $\alpha(C_1) = 1$ and $\alpha(U_1 - K_1) = 0$. Then for $(x, b) \in U_0 \times U_1$ define

$$h_t(x, b) = (h^{(b)}_{\alpha(b)t}(x), b).$$

It is an isotopy of h and the rule $h \rightsquigarrow h_t$ establishes $\mathscr{D}(X;\ A, A', C;\ U;\ \mathscr{S})$ in this special case.

EXAMPLE 6.2. Beware that $\mathscr{D}(T^2;\ \mathscr{S})$ is false if $\mathscr{S}$ is the foliation of T^2 derived from the lines in R^2 with a given irrational slope.

Now we give Černavskii's generalization [8] of $\mathscr{D}(X;\ A, A', B;\ U)$. It attempts to allow B to be any closed set.

If W, X are metric spaces let $\mathrm{Map}(W, X)$ denote the set of continuous maps (=functions) $f: W \to X$. On it consider two topologies

(a) The compact-open topology

(b) The majorant topology in which base of neighborhoods of f in $\mathrm{Map}(W, X)$ is the collection of sets

$$N_\varepsilon(f) = \{g \in \mathrm{Map}(W, X) \mid d(f(x), g(x)) < \varepsilon(x)\}$$

where ε varies over positive continuous functions $\varepsilon: W \to (0, \infty)$ called majorants.

$\mathrm{Emb}(W, X) \subset \mathrm{Map}(W, X)$ will denote the set of open embeddings of W into X.

A homotopy $h_t: W \to X$, $0 \leqslant t \leqslant 1$, is identified with the continuous function $(t, x) \to h_t(x)$ in $\mathrm{Map}(I \times W, X)$.

Consider the following statement about a metric space X in LOC.

$\boxed{\mathscr{D}'(X)}$ *Let $A \subset A'$ be any closed subsets of X such that A' is a neighborhood of A. Let $B \subset X$ be closed and let $i: U \to X$ be the inclusion of an open neighborhood of B. Finally let $\varepsilon: U \to (0, \infty)$ be a continuous map. Then the following always holds.*

$\boxed{\mathscr{D}'(X;\ A, A', B;\ U;\ \varepsilon)}$ *There exists a continuous map $\delta: U \to (0, \infty)$ and a rule Γ that assigns to each embedding $h: U \to X$ in $N_\delta(i)$ and equal i on A' a homotopy $\Gamma(h) = (h_t: U \to X, 0 \leqslant t \leqslant 1)$ in $\mathrm{Map}(I \times U, X)$ such that*

(1) *$\Gamma(h) = (h_t, 0 \leqslant t \leqslant 1)$ is an isotopy*[12] *through open embeddings $U \to X$ from h to an embedding h_1 equal i on B.*

---

[12] See remarks preceding 1.6.

(2) *$h_t = h$ on A and $h_t \in N_\varepsilon(i)$.*

(3) *$\Gamma: h \mapsto (h_t)$ is a continuous function of h when* Map(U, X) *and* Map$(I \times U, X)$ *are assigned either both the compact open topology, or both the majorant topology. Also $\Gamma(i)$ is the constant isotopy of i.*

Question. Is $\mathscr{D}'(X)$ true if X is separable Hilbert space?

THEOREM 6.3. *$\mathscr{D}(X)$ implies $\mathscr{D}'(X)$ for any metrizable X in* LOC.

The proof will generalize trivially to prove:

COMPLEMENT 6.4. (Respectful version, same data). *In X single out a class $\mathscr{S}$ of closed subsets, and indicate membership $Y \in \mathscr{S}$ by $Y < X$. Then $\mathscr{D}(X)$ with statement $\mathscr{R}(X)$* (under 4.3) *appended implies $\mathscr{D}'(X)$ with $\mathscr{R}(X)$ appended.*

Again, the (weaker) *hypothesis $\mathscr{D}(X; \mathscr{S})$ implies the* (weaker) *conclusion $\mathscr{D}'(X; \mathscr{S})$. (The latter indicates $\mathscr{D}'(X)$ modified by* (i), (ii), (iii) in 6.0).

Proof of Theorem 6.3. 1) *The case where B is expressible as a disjoint union of compacta $B_1, B_2, \ldots$ with each B_i open in B.*

Proof of 1). Choose disjoint open neighborhoods U_j of B_j in X, each with compact closure in U. By assumption $\mathscr{D}(X;\ A,\ A',\ B_j;\ U_j)$ holds for each j. Then if $h: U \to X$ fixes A' and is (majorant) near i we can define $h_t \mid U_j$ to be the isotopy (fixing points outside a compactum), offered by this statement. This defines $\Gamma: h \to (h_t)$ and it is routine to verify (1), (2), (3).

2) *The general case.*

Proof of 2). Every connected locally compact metric space is separable [36, Appendix 2]. Without loss of generality we assume X connected. Then find a nest $X_0 \subset X_1 \subset X_2 \subset \cdots$ of compacta in X such that $X = \cup_k \mathring{X}_k$ and $X_k \subset \mathring{X}_{k+1}$. Set $X_k = \emptyset$ for $k < 0$ and define $Z_{1,k} = X_{4k+3} - \mathring{X}_{4k}$; $Z_{2,k} = X_{4k+1} - \mathring{X}_{4k-2}$. Define $Z_j = \cup_k Z_{j,k}$. Then Z_j is closed, $X = \mathring{Z}_1 \cup \mathring{Z}_2$, and $Z_{j,k}$ is a compactum open in Z_j.

Choose a closed set A'' in X with $A \subset \mathring{A}''$, $A'' \subset \mathring{A}'$.

Let B' be a closed neighborhood of B with $B' \subset U$ and set $B_j' = B' \cap Z_j$, $B_j = B \cap Z_j$. By 1)

$$\mathscr{D}'(X;\ A'',\ A',\ B_1';\ U;\ \gamma) \tag{*}$$

is valid for any $\gamma: U \to (0, \infty)$. Similarly Step 1) proves

$$\mathscr{D}'(X;\ A \cup (B - \mathring{B}_2),\ A'' \cup B_1',\ B_2;\ U;\ \varepsilon). \tag{**}$$

Composing (on adjacent intervals) isotopies offered by (*) for suitable γ with those offered by (**), we obtain a rule $\Gamma: h \mapsto (h_t)$ establishing $\mathscr{D}'(X;\ A,\ A',\ B;\ U;\ \varepsilon)$. The majorant γ is suitable if it is $\leqslant$ the majorant $\delta = \delta(\varepsilon)$ offered by (**).

This completes the proof of 6.1.

Extension of Isotopies

One of the most useful corollaries of $\mathscr{D}$ is an isotopy extension theorem.

Consider a continuous family $f_t: V \to X$, $t \in B$ of open embeddings. Continuous means that $(t, x) \mapsto (t, f_t(x))$ is a continuous map $f: B \times V \to B \times X$. Given a closed subset $C \subset V$ and $b \in B$ we enquire when there exists a (continuous) family $F_t: X \to X$, $t \in B$, of self-homeomorphisms of X such that $F_t f_b \mid C = f_t \mid C$. To simplify notation we identify V to $f_b V \subset X$. The inclusion $V \to X$ is then f_b.

ISOTOPY EXTENSION THEOREM 6.5. cf. [8] [23A] [14]. *About the above data, make the following suppositions: X is Hausdorff locally compact and locally connected* (i.e. $X \in$ LOC)*; C has compact frontier in V; for each t in B, $f_t(C)$ is closed; $\mathscr{D}(V-C)$ holds true. Finally suppose that either B is locally connected or X has finitely many components.*

(I) *There exists a neighborhood N of b in B and a continuous family $F_t: X \to X$, $t \in N$ of homeomorphisms such that $F_t \mid C = f_t$ for all $t \in N$.*

(II) *If $B = I^n = [0, 1]^n$ or any retract of I^n then N in* (I) *can be all of B.*

(III) *Further if $K \subset I^n$ is a retract of $B = I^n$ containing b and $F_t': X \to X$, $t \in K$, is a continuous family of homeomorphisms with $F_t' \mid C = f_t \mid C$, then the family F_t, $t \in I^n$, provided by* (II) *can be chosen so that $F_t = F_t'$ for $t \in K$.*

EXAMPLE. (justifying the last supposition under 6.5). Statement (I) fails if $X = V = C = \mathbf{Z}$; $B = \{0\} \cup \{1/n;\ n \geqslant 1 \text{ in } \mathbf{Z}\}$; $b = \{0\}$; $f_0 = \mathrm{id} \mid \mathbf{Z}$, $f_{1/n}(x) = x$ for $x \leqslant n$, $f_{1/n}(x) = x+1$ for $x > n$.

The proof will generalize trivially to prove

COMPLEMENT 6.6. (Respectful version). Theorem 6.5 remains valid when the data are modified as follows. Single out a family $\mathscr{S}$ of closed subsets of X, and suppose that each f_t, $t \in B$, thoroughly respects [or fixes] each Y in $\mathscr{S}$. In place of $\mathscr{D}(V-C)$ suppose $\mathscr{D}(V-C;\ \mathscr{S}_0)$, where $\mathscr{S}_0 = \{Y \cap (V-C) \mid Y \text{ in } \mathscr{S}\}$. Finally insist that each F_t or F_t' mentioned thoroughly respect [or fix] each Y in $\mathscr{S}$.

Proof of (I). Find a closed set $D \subset X$ with $C \subset \mathring{D}$ and $D \subset V$ such that $D - \mathring{C}$ is compact and hence D has compact frontier δD in $V - C$. Find an open neighborhood U of δD in $V - C$, with compact closure $\bar{U} \subset V - C$; and let E be a compact neighborhood of δD in U.

Applying $\mathscr{D}(V-C;\ \emptyset,\ \emptyset,\ E;\ U)$ we get a rule that associates, to each embedding $U \xrightarrow{f_t} V - C$ sufficiently near the inclusion, an embedding $g_t: U \to V - C$ such that $g_t = \mathrm{id}$ on $E \supset \delta D$, $g_t = f_t$ outside a compactum in U, and $g_t(U) = f_t(U)$.

Now if t is sufficiently near b in N the map $\bar{U} \xrightarrow{f_t} V - C$ lies within any prescribed neighborhood of the inclusion for the compact-open topology. Thus the rule $f_t \rightsquigarrow g_t$ is defined for t in a small neighborhood N_0 of b.

For t near b define

$$H_t(x) = f_t(x) \quad \text{if} \quad x \in V - U$$
$$H_t(x) = g_t(x) \quad \text{if} \quad x \in U.$$

This is clearly a continuous family of open embeddings $H_t: V \to X$ and $H_t \mid \delta D =$ inclusion. Define $H: N_0 \times V \to X$ by $H(t, x) = H_t(x)$.

ASSERTION 1). *For t near b, $H_t D = D$ (say for all $t \in N \subset B$).*

Proof. $H_t D$ is the union of the closed set $H_t C = f_t C$ and the compactum $H_t(D - \mathring{C})$. So $H_t D$ is closed in X and hence $H_t \mathring{D}$ is closed in $X - \delta D$. Also $H_t \mathring{D}$ is open in $X - \delta D$. So $H_t \mathring{D}$ consists of some of the components of $X - \delta D$.

If B is locally connected, let $N_1 \subset N_0$ be a connected open neighborhood of b. For each x in $\mathring{D}$, the connected set $H(N_1 x)$ lies in the same component of $\mathring{D}$ as does $H(b, x)$. Thus for all t in N_1, $H_t \mathring{D} = H_b \mathring{D} = \mathring{D}$, which proves 1).

In case X has finitely many components (and B is not locally connected) we complete the proof of 1) differently. Recall the

LEMMA 6.5.1. [18, p. 111]. *Let M be a connected, locally connected, and locally compact Hausdorff space. For any compactum K in M and any neighborhood U of K, all but finitely many of the (open!) components of $M - K$ lie in U.* (Hint: δU may as well be compact).

This shows that, for t in N_0, and x in $\mathring{D}$, $H_t(x) = x$ unless x lies in one of finitely many open components $\mathring{D}_1, \ldots, \mathring{D}_k$ of $\mathring{D}$. Choose $x_i \in \mathring{D}_i$, $i \leqslant k$, then choose a neighborhood $N_1 \subset N_0$ of b in B so small that, for t in N_1, $H_t(x_i) \in \mathring{D}_i$, $i \leqslant k$. Then $H_t(\mathring{D}_i) = \mathring{D}_i$, and we conclude that $H_t \mathring{D} = \mathring{D}$ and $H_t D = D$. This completes the proof of 1).

Finally define the continuous family $F_t: X \to X$ for $t \in N$ by

$$F_t(x) = H_t(x) \quad \text{if} \quad x \in D.$$
$$F_t(x) = x \qquad \text{if} \quad x \in X - \mathring{D}.$$

Assertion 1) shows that F_t is a homeomorphism. This completes the proof of (I).

Proof of (II). First use a retraction $\varrho: I^n \to B$ to define f_t for all $t \in I^n$ by setting $f_t = f\varrho(t)$ for $t \notin B$. Find a "Lebesque" number $\varepsilon > 0$ so small that each set in I^n of diameter $< \varepsilon$ lies in some N_a provided by (I) applied with $a \in I^n$ in place of b. Dice I^n into n-cubes of diameter $< \varepsilon$, (lexicographically) ordered $C_1, C_2, C_3, \ldots$ so that if $D_k = \cup \{C_j \mid j < k\}$, then there is a retraction $r = r_k: D_{k+1} \to D_k$.

Now, for each C_k, there is $a \in I^n$ and a family of homeomorphisms $F_t^{(k)}: X \to X$, $t \in C_k$, with $F_t^{(k)} f_a \mid C = f_t \mid C$.

Suppose for an inductive construction that F_t is defined for $t \in D_k$, $(k \geqslant 1)$, so that $F_t f_b = f_t$. Then define

$$F_t' = G_t F_{r(t)}, \quad t \in C_k, \quad \text{where} \quad G_t = F_t^{(k)} (F_{r(t)}^{(k)})^{-1}, \quad t \in C_k.$$

Since $G_t f_{r(t)} \mid C = f_t \mid C$ we have $F_t' f_b \mid C = f_t \mid C, \quad t \in C_k.$

Also $F_{r(t)}' = F_{r(t)}$. So we can define $F_t = F_t'$ for $t \in C_k$ to complete the induction – provided we know that G_t is a *continuous* family of homeomorphisms. But this follows from Lemma 1.6.

Proof of (III). If H_t, $t \in I^n$, provided by (II) satisfies $H_t f_b \mid C = f_t \mid C$ define

$$F_t = H_t H_{\varrho(t)}^{-1} F_{\varrho(t)}', \quad t \in I^n$$

where $\varrho : I^n \to K$ is a retraction.

This completes the proof of the isotopy extension theorem.

6.7. OTHER CATEGORIES. An isotopy extension theorem holds in two more, familiar categories.

PL – The objects are metric spaces equipped each with a maximal piecewise linearly compatible atlas of charts to finite simplicial complexes; the morphisms are piecewise-linear maps.

DIFF. – The objects are smooth C^∞ finite dimensional manifolds possibly with corners (say as in [22]); the morphisms are C^∞ maps.

Indeed the isotopy extension theorem holds true in CAT (=DIFF or PL) when the statement has been modified as follows:

(a) Assume all objects and maps mentioned are CAT.

(b) By CAT open embedding understand a CAT isomorphism onto an open subset.

(c) Omit the hypothesis $\mathscr{D}(V-C)$.

Call the resulting statement $\mathscr{I}(X)$.

However the CAT proof is radically different. See [24] for DIFF, [18] for PL.

Respectful CAT versions $\mathscr{I}(X; \mathscr{S})$ (parallel to 6.6 with the assumption $\mathscr{I}(V-C; \mathscr{S}_0)$ suppressed) can be produced by reinforcing the existing CAT proofs. One has to determine what families $\mathscr{S}$ will work (See [24] [2]). From $\mathscr{I}(X)$ one can at least prove $\mathscr{I}(X \times B; \mathscr{S})$, where $\mathscr{S} = \{X \times b \mid b \in B\}$, by making use of (III), cf. proof of 6.1.

As topological companions for DIFF and PL we use two categories

T_0 – The category of continuous maps of topological spaces.

LOC – The category of continuous maps of locally compact locally connected Hausdorff spaces.

Submersions

Next we present a proof that a proper submersion is often a bundle map.

DEFINITION 6.8. A CAT map (CAT = DIFF, PL, T_0, LOC) $p:E\to X$ is a CAT *submersion* if for each y in E, there is a CAT object U, an open neighbourhood N of $p(y)$ in X, and a CAT open imbedding $f: U\times N\to E$ onto a neighborhood of y such that pf is the projection $U\times N\to N\subset X$. Then $F=p^{-1}p(y)$ is a CAT object. When f is normalised so that U is an open subset of F and $f(u, p(F))=u$ for u in $U\subset F$, then we call f a *product chart* about U for p.

Recall that $p:E\to X$ is a CAT *bundle* if for each x in X, there is a CAT product chart $f: p^{-1}(x)\times N\to E$ about the fiber $p^{-1}(x)$ of p, such that Image $(f)=p^{-1}(N)$.

UNION LEMMA 6.9. (CAT = DIFF, PL or T_0). *Data: $p:E\to X$ a* CAT *submersion; $F=p^{-1}(x_0)$ for a point x_0 in X; A, B compacta in F; U, V open neighborhoods of A, B in F; f: $U\times N'\to E$ and $g: V\times N''\to E$ product charts about U and V for p. If* CAT $=T_0$, *suppose F is in* LOC *and $\mathscr{D}(F)$ holds.*

Given this data one can find a CAT *product chart $h: W\times N\to E$ about an open neighborhood W of $A\cup B$ in F. Further one can choose h so that $h=f$ near $A\times x_0$ and $h=g$ near $(B-U)\times x_0$.*

As usual the proof will establish a respectful version. With the data of the union lemma single out a class $\mathscr{S}$ of closed subsets of E and indicate $Y\in\mathscr{S}$ by $Y<E$. A product chart $f: U\times N\to E$ about U for p is said to *respect* $\mathscr{S}$ if for each $Y<E$, $f^{-1}Y=(Y\cap U)\times N$ and f gives by restriction to $f^{-1}Y$ a product chart about $Y\cap U$ for $Y\xrightarrow{p}X$.

FIRST COMPLEMENT 6.10. (Respectful version, CAT = DIFF, PL or T_0). *The union lemma continues to hold when modified as follows: Assume f and g respect $\mathscr{S}$; suppose that $\mathscr{D}(F;\ \mathscr{S}_F)$ holds where $\mathscr{S}_F=\{F\cap Y\mid Y$ in $\mathscr{S}\}$; finally, insist that h respect $\mathscr{S}$.*

SECOND COMPLEMENT 6.11. (CAT $=T_0$). *The union lemma continues to hold when further modified as follows: For fixed $Y<E$ suppose that, in $(Y\cap F)\times X$, f and g agree whenever both are defined, namely on $(Y\cap U\cap V)\times(N'\cap N'')$. Then insist that h agree with f in $(Y\cap F)\times X$ wherever both are defined, and that h agree similarly with g.*

The reader will verify these complements by using the complemented isotopy extension theorem (6.5, 6.6) to generalize the following.

The proof of the union lemma uses another lemma which is a direct consequence of the CAT isotopy extension theorem 6.5, 6.7 (form I)).

LEMMA 6.12. (CAT = DIFF, PL or T_0). *Consider projection $p_2:F\times B\to B$, and* *identify F to $F\times b$ for some point b in B. Let $U\subset F$ be open bounded and $C\subset U$ be compact. Let $f: U\times N\to F\times B$ be a product chart for p_2 about U. In case* CAT$=T_0$, *suppose that $F\in$*LOC *and that $\mathscr{D}(F)$ holds.*

Then there exists a product chart $g: F\times N'\to F\times B$ about F for p_2 such that $f=g$ near $C\times b$, and $g=$(identity) outside $K\times N'$ where K is some compact neighborhood of C in U. (Do not confuse B here with B in the Union Lemma.)

Proof of Union Lemma. Let $A'\subset U$, $B'\subset V$ be compact neighborhoods of A, B respectively in F. Using lemma 6.12 we can find a product chart for p about V $g': V\times\hat{N}\to E$ such that $g'=g$ on $(V-U)\times\hat{N}$ and $g'=f$ on $(A'\cap B')\times\hat{N}$. Let $W=\mathring{A}'\cup\mathring{B}'$, $N=N'\cap\hat{N}$ and define $h: W\times N\to E$ by $h\mid\mathring{A}'\times N=f\mid\mathring{A}'\times N$ and $h\mid\mathring{B}'\times N=g'\mid\mathring{B}'\times N$. This is the required product chart provided it is injective. At least h is locally injective. And $h\mid(W\times b)$ is injective – being the inclusion. It follows that, after W and N are cut down if necessary, h will be an embedding. To see this, check that for each compactum $W_1\subset W$ the set of double points of $h\mid W_1\times N$ is closed in $W\times N$, and disjoint from the compactum $(A\cup B)\times x_0$. A *double* point $x\in K$ of any map $f: K\to L$ is one such that $f(x)=f(y)$ for some $y\neq x$ in K. This completes the proof.

The following addition to Lemma 6.12 will be recast as a uniqueness theorem (6.19) for "transverse" normal microbundles to closed leaves in a foliation.

ADDENDUM 6.13 to 6.12. *Given the data of* 6.12, *let A be a closed subset of F such that $f: U\times N\to F\times B$ equals the identity near $A\times B$.*
(α) *Then g can equal the identity near $A\times b$.*
(β) *If* CAT$=T_0$, *suppose now that B is Hausdorff and normal. Then one can find a* CAT *isotopy h_t, $0\leqslant t\leqslant 1$, of* id $\mid F\times B$ *through* CAT *automorphisms such that*

(i) *$h_t(x)=x$ for x in $F\times b$, near $A\times b$, and outside $U\times N'$,*

(ii) $p_2h_t=p_2$,

(iii) *$h_1=g=f$ near $C\times b$.*

Proof of Addendum 6.13. Part (α), being easy, is left to the reader.

For part (β) we use two arguments. The first applies if there is a CAT homotopy θ_t, $0\leqslant t\leqslant 1$, of id $\mid N'$ fixing the complement of a compactum in N', to a CAT map $\theta_1: N'\to N'$ such that $\theta_1^{-1}(b)$ is a neighborhood of b. This certainly exists if CAT$=$DIFF or PL. Define the CAT isomorphism $g[a]: F\to F$ for a in N' by the equation $g(x,a)=(g[a](x),a)$. Then define the CAT isotopy $h_t: F\times B\to F\times B$, $0\leqslant t\leqslant 1$, by

$$h_t(x,a)=(g[\theta_t(a)]^{-1}\,g[a](x),a),\quad \text{for } a \text{ in } N'$$

$$h_t(x,a)=(x,a)\qquad\qquad ,\quad \text{for } a \text{ in } B-N'.$$

The wanted properties are evident.

The second argument applies when CAT$=T_0$. We use $\mathscr{D}(F; A'', A', \bar{U}; X)$ where $A''\subset A'$ are closed neighborhoods of $(F-U)\cup A$ so that $g=$(identity) near $A'\times b$. This produces a certain isotopy $g[a]_t$, $0\leqslant t\leqslant 1$, of $g[a]$ for a in a small open neighborhood N'' of b in N'. Using the normality of B, find $\theta: B\to[0, 1]$ a continuous function equal 1 near b and equal 0 near $B-N''$.

Now define $h_t: F\times B\to F\times B$, $0\leqslant t\leqslant 1$, by

$$
\begin{aligned}
h_t(x, a) &= (\{g[a]_{\theta(a)t}\}^{-1}\circ g[a]\,(x), a) && \text{for } a \text{ in } N''\\
h_t(x, a) &= (x, a) && \text{for } a \text{ in } B-N''
\end{aligned}
$$

This completes 6.13.

COROLLARY 6.14. (to union lemma) CAT$=$DIFF, PL or T_0. *Let $p: E\to X$ be a* CAT *submersion that is closed*[13]) (i.e. *p maps closed sets onto closed sets*). *Suppose $p^{-1}(x)$ is compact for each x in X. For* CAT$=T_0$ *provide that for each x in X, $p^{-1}(x)$ is in* LOC *and $\mathscr{D}(p^{-1}(x))$ holds. Then p is a* CAT *bundle map.*

This is explained more precisely by

COROLLARY 6.15. (to union lemma). *Adopt the data of* 6.9, *and suppose F is sigma-compact. There always exists an open neighborhood Y of $F\times x_0$ in $F\times X$ and a* CAT *open embedding $h: Y\to E$ with $ph=$(projection)$: Y\subset F\times X\xrightarrow{p_2}X$. Hence if F is compact there exists a product chart about F, namely $h\,|\,F\times N$, where N is a small neighborhood of x_0 in X.*

Proof of 6.15. The union lemma 6.9 implies that there is a product chart about a neighborhood of each compactum in F. When F is sigma-compact we have compacta $C_1\subset C_2\subset C_3\subset\cdots$ with $C_i\subset C_{i+1}$ and $\cup\, C_i=F$, and about each C_i we have a product chart. Applying the union lemma (especially its last part) in an infinite induction, we alter these to agree and give h.

Proof of 6.14 *from* 6.15. For each x in X, 6.15 says there is a product chart $g: p^{-1}(x)\times N\to E$ about $p^{-1}(x)$ for p. The set $S=p(E\text{-image}(g))$ is closed in X since p is closed, and it does not contain x. Let $N'=N\cap\{X-S\}$. Then $p^{-1}(x)\times N'\xrightarrow{g}E$ has image equal $p^{-1}(N')$ not less. Indeed if $p(y)\in N'$, then $p(y)\notin S$ so $y\in\text{Image}\,(g)$.

Remark 6.16. For CAT$=T_0$ the first complement 6.10 of the union lemma gives an evident respectful version of this corollary. It closely resembles Thom's 1st isotopy theorem [35], [24] or again Rourke's theorem [26] about covering the track of a PL isotopy.

[13]) A continuous map of metric (or compactly generated Hausdorff) spaces is closed if it is proper in the sense that the preimage of each compactum is compact. See [34]. Conversely for any topological spaces, a closed map with all point preimages compact is necessarily proper, cf. [7; Chap I, §10, p. 115, p. 118 ex. 4, p. 164]. The proofs are trivial.

What the second complement 6.11 makes of this corollary is best stated as an isotopy extension theorem.

THEOREM 6.17. EXTENSION OF LOCALLY FLAT ISOTOPIES. *Adopt the category* LOC. *Let* $f_t: M \to Q$, $t \in B \equiv [0, 1]^n$, *be a continuous family of closed embeddings such that the map* $f: M \times B \to Q \times B$ *given by* $(x, t) \to (f_t(x), t)$ *is closed, and locally flat in the following sense. For each point* (y, a) *in* $M \times B$ *there is an open neighborhood* $f(y, a)\, U \times N$ *of* $f(y, a)$ *and a local product chart* (see 6.8) $g: U \times N \to Q \times B$ *for* p_2: $Q \times B \to B$ *respecting* $f(M \times B)$ (see 6.10) *so that* $g_t f_a(y) = f_t(y)$ *for all* (y, t) *in* $f^{-1}(U) \times N$.

For each t in B assume $\mathscr{D}(Q; \{f_t M\})$ *holds. (Recall that this is a local question in each pair* $(Q, f_t M)$ – cf. 2.2; *so it is independant of t.)*

Provide that, for each point x outside a certain compactum in M, $f_t(x)$ *is constant as t runs through B.*

Then for any $b \in B$ *there exists a family* $F_t: Q \to Q$, $t \in B$ *so that* $F_t f_b = F_t$ *for all t in B.*

Indication of proof of 6.17. The second complement to the union lemma plus the proof of Corollary 6.15 together provide a local extension as in part (I) of 6.5. Deduce global extensions by imitating the proof of part (II) of 6.5.

Foliations

For CAT = DIFF, PL or T_0 define a CAT *foliation* of a CAT object X with *model* B to the maximal family $\mathfrak{F}$ of CAT submersions $p_\alpha: U_\alpha \to B$ of open subsets U_α covering X, subject to the compatibility condition for pairs p_α, p_β in $\mathfrak{F}$: For each point x_0 in $U_\alpha \cap U_\beta$ there exists an isomorphism h of an open neighborhood of $p_\alpha(x_0)$ in B to one of $p_\beta(x_0)$ so that $h\, p_\alpha(x) = p_\beta(x)$ for all x near x_0.

Every open subset U of X clearly inherits from $\mathfrak{F}$ a foliation with model B denoted $\mathfrak{F} \mid U$.

Consider the least equivalence relation $\sim$ on X such that $x \sim y$ if, for some U_α, it is true that x, y both lie in U_α, $p_\alpha(x) = p_\alpha(y) \equiv z$ and x, y lie in the same component of $p_\alpha^{-1}(z)$. The equivalence classes are called *leaves*, and the decomposition space of leaves (with quotient topology) is called the leaf space.

The product foliation on $F \times N$, N open in B, is given by the submersions $p_\alpha: U_\alpha \to B$ compatible with $p_2: F \times N \to N \subset B$. If F is connected, the leaves are the sets $F \times \{t\}$, for t in N, and space of leaves is identified to N by p_2.

A *product chart* for $\mathfrak{F}$ is a product chart in the sense of 6.8 for one of the submersions in $\mathfrak{F}$.

A product chart for $\mathfrak{F}$ can be described as an open CAT embedding $\varphi: U \times N \to X$ carrying the product foliation to $\mathfrak{F} \mid \text{Image}(\varphi)$. Hence φ carries leaves of $U \times N$ into leaves into leaves of $\mathfrak{F}$.

LEMMA 6.18. (CAT=DIFF, PL or T_0). *If $\varphi: U\times N\to F\times B$ is a product chart for the product foliation on $F\times B$, and U, F are connected, then there exists a unique* CAT *open embedding h making commutative the square*

$$\begin{array}{ccc} U\times N & \xrightarrow{\varphi} & F\times B \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ N & \xrightarrow{h} & B \end{array}$$

Proof. Indeed h is the map of leaf spaces induced by φ. Locally, it coincides with open CAT embeddings expressing compatibility of Image$(\varphi)\xrightarrow{p_2\varphi^{-1}} N\subset B$ with $p_2: F\times B\to B$.

Let F be a closed leaf of $\mathfrak{F}$. Then F is easily seen to be a closed CAT subobject of X. Suppose one has a *retraction* $r: E\to W$ of an open set of X onto an open subset W of F. We say that r is *transverse* to $\mathfrak{F}$ near W if W is covered by open sets U such that there is product chart $\varphi: U\times N\to X$ for $\mathfrak{F}$ about U (with $\varphi(x, b)=x$ for x in U) such that $r\varphi(x, y)=\varphi(x, b)$, for all (x, y) in $U\times N$. Such a product chart φ is said to be *parallel* to r. Clearly $r: E\to W$ is a CAT microbundle with fiber germ the germ of B about b. The next proposition is a uniqueness lemma for such r.

PROPOSITION 6.19. (CAT=DIFF, PL or T_0). *Let $\mathfrak{F}$ be a* CAT *foliation with model B* of *a* CAT *object X.*

Consider $r_1, r_2: E\to F$ two CAT *retractions of an open set in X onto a closed leaf F of $\mathfrak{F}$, both transverse to $\mathfrak{F}$ near F.*

If CAT$=T_0$, *suppose that $F\in$*LOC, *and that $\mathscr{D}(F)$ holds.*

Suppose $r_1=r_2$ near a closed subset $A\subset F$. Let $C\subset F$ be a given compactum and let Ω be an open neighborhood of C in E.

Then there exists a retraction $r_2': E'\to F$ of an open set E' to F transverse to $\mathfrak{F}$ near F such that $r_2'=r_2$ near $A\cup C$, and $r_2'=r_1$ near A and outside Ω.

If B is Hausdorff and normal, then r_2' can be obtained as $h_1 r_1$ where h_t, $0\leqslant t\leqslant 1$, is a CAT *isotopy of* id $|$ *E such that h_t fixes point in F, near A, and outside Ω, and h_t respects leaves* (i.e. $h_t^*\mathfrak{F}=\mathfrak{F}$).

Proof. Because of the relative form of this result, it suffices to give a proof for all C in some base of compact neighborhoods in F. Thus we can assume that C and Ω lie in images of product charts φ_1, φ_2 for $\mathfrak{F}$ about neighborhoods of C that are parellel to r_1, r_2 respectively, and in addition satisfy Image $\varphi_1\supset$Image φ_2.

This case is equivalent to assuming (1) $\mathfrak{F}$ is the product foliation of $F\times B=X$, with F identified to $F\times B$ say, (2) r_1 is projection $F\times B\to F$, and (3) there is a product chart $\varphi_2: U\times N\to X$ about a neighborhood U of C in F so that φ_2 is parallel to r_2.

By Lemma 6.18 we can even assume that (4) φ_2 is a product chart for $p_2: F\times B\to B$.

But in this situation the proposition follows immediately from Lemma 6.12 and Addendum 6.13.

In a standard way one deduces

COROLLARY 6.20. (same data). UNIQUENESS OF TRANSVERSAL MICRO-BUNDLES. *If B is Hausdorff and normal, there exists an isotopy h_t, $0 \leqslant t \leqslant 1$ of* $\mathrm{id} \mid E$ *through homeomorphisms mapping leaves to leaves so that*

(1) $h_1 r_2 = t_1$ *near* W

(2) $h_t \mid F = \mathrm{id} \mid F$

(3) $h_t(x)$ *for x outside a prescribed neighborhood of F.*

COROLLARY 6.21. (same data). EXISTENCE[14]. *If F' is any closed leaf of $\mathfrak{F}$ there exists an open neighborhood E' of F' in X and a* CAT *retraction $r': E' \to F'$ transverse to $\mathfrak{F}$ near F' provided, for* $\mathrm{CAT} = T_0$, *that $F' \in \mathrm{LOC}$ and $\mathscr{D}(F')$ holds.*

To deduce the second corollary note that such retractions exist to open subsets forming a covering $\{W_\alpha\}$ of F'. Inductive application of 6.19 combines them to give r'. For this application note that each (open!) component of W_α is a closed leaf of $X - \delta W_\alpha$. Here δ indicates frontier.

The last corollary is exactly what is needed to complete the topological version of the classification (by holonomy) of foliated neighborhoods of a closed leaf in a foliation.

HOLONOMY THEOREM 6.22. (well known for DIFF). *Let M be a* CAT *manifold of finite dimension* (CAT = DIFF, PL or TOP) *equipped with a* CAT *foliation $\mathfrak{F}$ with model R^q. Let F be a connected leaf of $\mathfrak{F}$ that is closed in M and is a* CAT *submanifold.*

Then the germ of M about F is uniquely determined by the holonomy homorphism $\Theta: \pi_1(F) \to G$ to the germs of CAT *embeddings $(R^q, 0) \to (R^q, 0)$. Uniquely determined means that if primes indicate a similarly described situation and by chance $F = F'$ and $\Theta = \Theta'$, then there is a leaf-preserving* CAT *isomorphism of an open neighborhood of F in M to one of F' in M' that equals the identity on $F = F'$.*

Θ is defined by choosing a foliation chart about a base point $*$ in F for which $*$ projects to 0 in R^q and "sliding" a germ of it about loops in F based at $*$, always respecting leaves. (Lemma 6.17 is vital here.)

For clarifications and proof see Haefliger [16, §2.7] and [17, 298–301 and 303–304]. One could retain the generality of Proposition 6.19.

Double Foliations

There is a useful generalization of 6.19, 6.20, 6.21 that respects a second CAT folia-

[14]) There is a stronger result that applies to all leaves at once, at least if $B = R^q$. It maps Haefliger's abstract normal microbundle of $\mathfrak{F}$ into X giving an immersed normal microbundle to each leaf. One proof uses 6.23 locally.

tion $\mathfrak{F}'$ on X. Let $\mathfrak{F}'$ have model B' and suppose that $\mathfrak{F}$ and $\mathfrak{F}'$ are *mutually transverse* (form a *double foliation*) in the sense that X is covered by open sets U equipped with isomorphisms $\varphi: B'_0 \times U_0 \times B_0 \to U$ (called *double charts*) where B'_0, B_0 are open in B', B respectively, such that $\mathfrak{F}' \mid U$ is given by $p_1\varphi^{-1}: U \to B'_0 \subset B'$, and $\mathfrak{F} \mid U$ by $p_3\varphi^{-1}: U \to B_0 \subset B$.

There results a foliation $\mathfrak{F}' \cap \mathfrak{F}$ with model $B' \times B$ from the submersions (projection)$\circ\varphi^{-1}: U \to B'_0 \times B_0 \subset B' \times B$.

Consider a (normal microbundle) retraction $r: E \to W$ to a closed leaf W of $\mathfrak{F}$, which is transverse to $\mathfrak{F}$. It is said to *respect* $\mathfrak{F}'$ if W is covered by double charts $\varphi: B'_0 \times U_0 \times B_0 \to U \subset E$ such that $\varphi(B'_0 \times U_0 \times b) = U \cap W$ for some $b \in B_0$ and $\varphi^{-1} r \varphi$ is projection to $B'_0 \times U_0 \times b$.

PROPOSITION 6.23. (CAT=DIFF, PL or T_0). *The propositions* 6.19, 6.20, 6.21 *remain valid when a foliation $\mathfrak{F}'$ transverse to $\mathfrak{F}$ is given and we alter both hypotheses and conclusions* 1) *by insisting that all retractions and isotopies mentioned respect $\mathfrak{F}'$, and* 2) (when CAT$=T_0$) *by supposing $\mathscr{D}(Y)$ for the leaves*[15]) *Y of $\mathfrak{F} \cap \mathfrak{F}'$ rather than for leaves of $\mathfrak{F}$.*

Proof of 6.23. Recall that the proof of 6.19, 6.20, 6.21 boils down to local applications of 6.12 and 6.13. But as we have observed under 6.11, the proposition 6.12 and 6.13 have an appropriate respectful version, namely the version respecting the collection $\mathscr{S}$ of fibers of a projection $p': F \times B \to B'$ to a cartesian factor of F i.e. $F = B' \times F_0$. This version comes from the respectful version of the isotopy extension principle denoted $\mathscr{I}(F;\ \mathscr{S}_0)$, $\mathscr{S}_0 = \{a \times F_0 \mid a \in B'\}$ in 6.7 (cf. 6.6), or from $\mathscr{D}(F,\ \mathscr{S}_0)$ if CAT $= T_0$.

Caution. In the proof of 6.23, when we work in a double chart $U \cong B'_0 \times U_0 \times B_0$ with $U \cap F = B'_0 \times U_0 \times b$ (F being a closed leaf of $\mathfrak{F}$ for 6.19 generalized), we have to deal with certain open embeddings $f: V \to U$, where $V \subset U$, which are known a priori to (thoroughly) respect the leaves of $\mathfrak{F}'$. Yet we need to know that f respects the (more numerous) leaves $b' \times (U_0 \times B_0)$ of $\mathfrak{F}' \mid U$. A similar difficulty explains the counterexample 6.2. Fortunately we also know that f fixes points of $U \cap F \cong B'_0 \times U_0 \times b$. And we can arrange that each leaf of $\mathfrak{F}' \mid V$ is connected and meets $B'_0 \times U_0 \times b$. Then f must respect the leaves of $\mathfrak{F}' \mid U$, as desired.

To show the way toward applications of 6.23 we give a corollary of generalized 6.20 which can be regarded as a version of 6.15 (submersions are bundles) respecting a foliation.

COROLLARY 6.24. (CAT=DIFF, PL or T_0). *Let $\mathfrak{F}$, $\mathfrak{F}'$ be a pair of mutually*

---

[15]) The leaves are CAT objects with the leaf topology obtained by allowing as open each fiber of each submersion defining the foliation.

transverse CAT *foliations on* X, *with models* B, B' *respectively. Suppose that* $\mathfrak{F}$ *is given by a submersion* $p: X \to B$. *In case* $\mathrm{CAT} = T_0$, *suppose* $F \in \mathrm{LOC}$ *for each leaf of* $\mathfrak{F}$ *and* $\mathscr{D}(Y)$ *holds for each leaf of* $\mathfrak{F} \cap \mathfrak{F}'$. *Then, for any sigma-compact leaf* F *of* $\mathfrak{F}$, *there exists a neighborhood* U *of* $F \times b$ *in* $F \times B$, *where* $b = p(F)$, *and an open embedding* $\varphi: U \to X$ *such that* 1) $p\varphi = p$, *and* 2) $\varphi^* \mathfrak{F}' = \{(\mathfrak{F}' \mid F) \times B\} \mid U$, *where* $\varphi^* \mathfrak{F}'$ *is the pullback of* $\mathfrak{F}'$ *by* φ *and* $\mathfrak{F}' \mid F$ *is the foliation on* F *induced by* $\mathfrak{F}'$ (*or by* $\mathfrak{F} \cap \mathfrak{F}'$).

Proof of 6.24. By 6.21 and 6.23 there exists an open neighborhood E of F in X and a (microbundle) retraction $r: E \to F$ transverse to $\mathfrak{F}$ and respecting $\mathfrak{F}'$.

The map $\psi: E \to F \times B = X$ given by $\psi(x) = (r(x), p(x))$ is an immersion near F, hence an embedding on a smaller neighborhood of F. If E' is a sufficiently small neighborhood of F we can define φ as ψ^{-1}

$$\varphi: U = \psi E' \xrightarrow{\psi^{-1}} E \subset X.$$

This completes 6.24.

CONSEQUENCES OF 6.24. (a). In case $X = F \times B$ and p is projection to B, 6.24 asserts a local triviality of a family $\mathfrak{F}'_t$, $t \in B$, of foliations on F. And if F is compact and $B = [0, 1]$, we quickly deduce an isotopy φ_t, $0 \leqslant t \leqslant 1$, of id $\mid F$ so that $\varphi_t^* \mathfrak{F}_t = \mathfrak{F}_0$ for *all* t.

(b) Suppose $\mathfrak{F}'$ comes from a family of submersions $f_t: F \to B'$, $t \in B$, giving a submersion $f: F \times B \to B' \times B$ by $f(x, t) = (f_t(x), t)$. Then for any bounded open set $U \subset F$, 6.24 offers a CAT isotopy through open embeddings $\varphi_t: U \to F$, t near b in B, such that $\varphi_t^* \mathfrak{F}'_t = \mathfrak{F}'_b$ on U, and hence $f_t \varphi_t = f_b$ for all t near b. This means that every reasonable family of submersions arises (locally) from pushing the source over itself.[16)]

(c) A relative form of 6.25 (coming from 6.19 and 6.23) implies the CAT isotopy extension principle $\mathscr{I}(X; \mathfrak{F})$ for embeddings thoroughly respecting the leaves of a CAT foliation $\mathfrak{F}$ of X. For CAT=LOC one assumes $\mathscr{D}(Y)$, for each leaf Y of $\mathfrak{F}$ (with leaf topology).

Line Fields Normal to a Codimension One Foliation

A foliation with model R^q is regularly called a codimension q foliation. Consider a codimension q CAT foliation $\mathfrak{F}$, CAT=(DIFF, PL or T_0), on a connected CAT object X. If $\mathrm{CAT} = T_0$ we assume X is Hausdorff, $F \in \mathrm{LOC}$, and $\mathscr{D}(F)$ holds for each leaf F of $\mathfrak{F}$ (with leaf topology). We enquire whether there exists a foliation $\mathfrak{F}'$ transverse to $\mathfrak{F}$ so that the leaves of $\mathfrak{F}'$ are q-manifolds (with leaf topology). Then $\mathfrak{F}'$ clearly can have as model F where F is any leaf of $\mathfrak{F}$, and the double foliation $\mathfrak{F}$, $\mathfrak{F}'$ looks locally like that on $F \times R^q$.

The answer is no in general. The Hopf fibration $S^3 \to S^2$, regarded as a foliation of S^3 by circles, is a simple counterexample (suggested by Haefliger). If there were a

16) This fact was first proved by Gauld [15]. An open question: When does a merely continuous family f_t, $t \in B$, of submersions $X \to Y$ in LOC provide a submersion $f: X \times B \to Y \times B$ via the rule $f(x, t) = (f_t(x), t)$?

transverse foliation each leaf would be a covering of S^2, hence a copy of S^2 giving a section of the fibration – which does not exist.

However we show $\mathfrak{F}'$ exists if $q=1$. This is obvious (but useful) if CAT=DIFF; just find a line field normal to $\mathfrak{F}$ and integrate. The proof for PL and T_0 rests on the following relative uniqueness theorem for $\mathfrak{F}'$. I thank Harold Rosenberg for encouraging me to give a proof.

THEOREM 6.25. (CAT=PL or LOC). *Given X and the codimension* 1 *foliation $\mathfrak{F}$ as above, consider $\mathfrak{F}_1$, $\mathfrak{F}_2$ two foliations by* 1*-manifolds, transverse to $\mathfrak{F}$.*

Suppose $\mathfrak{F}_1=\mathfrak{F}_2$ near a closed subset $A\subset X$. Let $B\subset X$ be a compactum and $U\subset X$ an open neighborhood of N. Then there exists a third foliation $\mathfrak{F}_2'$ by 1*-manifolds, transverse to $\mathfrak{F}$ such that $\mathfrak{F}_2'$ equals $\mathfrak{F}_1$ near $A\cup B$ and equals $\mathfrak{F}_2$ outside U.*

Proof of 6.25. In view of the strongly relative form of this proposition it suffices to prove it in case U belongs to a given base of neighborhoods of X. Thus we can and do assume that we have $U=F_0\times(-1, 1)$ relatively compact in $X=F\times R$, and that $\mathfrak{F}$, $\mathfrak{F}_1$ come respectively from projection to R and to F.

Notice that the hypotheses give a foliation $\mathfrak{F}_2''$ defined (only) on some neighborhood V of $A\cup B\cup(X-U)$ satisfying all other conditions on $\mathfrak{F}_2'$.

Consider any leaf $F_t=F\times t$ of $\mathfrak{F}$. Now find a neighborhood N_t of t and a foliation by 1-manifolds $\mathfrak{F}_2^t$ of $F\times N_t$ with model F so that $\mathfrak{F}_2^t$ is transverse to $\mathfrak{F}$, and equals $\mathfrak{F}_2''$ near $A\cup B\cup(X-V)$. If t is not in $(-1, 1)$ we can and do choose $\mathfrak{F}_2^t=\mathfrak{F}_2 \mid F\times N_t$. If $t\in(-1, 1)$, note that one some open neighborhood E of F_t the foliations $\mathfrak{F}_1$, $\mathfrak{F}_2$ are given by retractions $r_1, r_2: E\to F_t$ transverse to $\mathfrak{F}$. Then 6.19 offers another $r_2': E'\to F_t$ whose foliation $\mathfrak{F}(r_2')$ would serve as $\mathfrak{F}_2'$ if E' contains some $F\times N_t$. If not, for N_t small, we can define $\mathfrak{F}_2' \mid W\times N_t=\mathfrak{F}_2(r_2')$ for an open neighborhood W of U in F with compact closure, and define $\mathfrak{F}_2'=\mathfrak{F}_2''$ elsewhere in $F\times N_t$.

Since $[-1, 1]$ is compact there exists a subdivision $-1=t_0<t_1<t_2<\cdots<t_s=1$ such that each interval $[t_{k-1}, t_k]$ is contained in some N_t, say $N_{t(k)}$. Write $\mathfrak{F}_2(k)$ for $\mathfrak{F}_2^{t(k)} \mid F\times[t_{k-1}, t_k]$. The foliations

$$\mathfrak{F}_2 \mid F\times(-\infty, -1],\ \mathfrak{F}_2(1),\ \mathfrak{F}_2(2), \ldots, \mathfrak{F}_2(s),\ \mathfrak{F}_2 \mid F\times[1, \infty)$$

agree on the interfaces $F\times t_0, \ldots, F\times t_s$ and together define a CAT foliation $\mathfrak{F}_2'$ on X as required by 6.25.

Remarks (a) Only the last paragraph of this proof breaks down for codimensions $q>1$.

(b) This proof does not quite work for DIFF. Our definition of $\mathfrak{F}_2'$ allows kinks in the leaves at the interfaces $F\times t_0, \ldots, F\times t_k$. Fortunately, for CAT=PL or LOC there is no trouble, because of $X=X_1\cup X_2$ and f_1, f_2 are maps of the category defined on closed subobjects X_1, X_2 that agree on $X_1\cap X_2$, then there is a unique map of the category defined on X and extending f_1, f_2.

THEOREM 6.26. (CAT=PL or LOC). *Given X and a codimension 1 foliation $\mathfrak{F}$ as for 6.25, one can always find $\mathfrak{F}'$ a foliation by 1-manifolds, transverse to $\mathfrak{F}$, provided X is sigma-compact. If $\mathfrak{F}''$ is a given foliation by 1-manifolds transverse to $\mathfrak{F}$ and defined near a closed set $C \subset X$, then one can choose $\mathfrak{F}'$ equal $\mathfrak{F}''$ near C.*

Complement. The same result for foliations F with model $R_+ = \{x \in R \mid x \geqslant 0\}$, follows by the device of doubling (Unite two copies of X by identifying to its duplicate each leaf of $\mathfrak{F}$ that must project to $0 \in R_+$).

Proof of 6.26. Since $\mathfrak{F}'$ exists for $\mathfrak{F} \mid U$, where U is any chart for $\mathfrak{F}$, this follows from the uniqueness theorem 6.25 by a routine argument.

Theorem 6.25, 6.26 permit one to establish topological versions of many theorems about codimension 1 DIFF foliations – simply by inspecting existing proofs. The generalizations of 6.25, 6.26 respecting a foliation transverse to $\mathfrak{F}$ (see 6.23) may also prove useful.

Counting Compact CS Sets

COUNTING THEOREM 6.27. *There are only countably many ($=\aleph_0$) homeomorphism classes of (Hausdorff) compacta X such that each point of X has an open neighborhood that is homeomorphic to a CS set (of* § 1*).*

Notice that this class of compacta includes compact manifolds, locally triangulable compacta, and compact CS sets.

Cheeger and Kister counted compact manifolds in [10].

Proof of 6.27 *by induction on depth.* Suppose the result has been proved for the smaller class $\mathscr{C}_{d-1}$ of compacta covered by open sets that are (homeomorphic to) CS sets of depth $\leqslant d-1$. Any compactum $X \in \mathscr{C}_d$ is covered by finitely many open CS sets of the form $R^m \times C$ where C is a stratified open cone of depth $\leqslant d$.

ASSERTION 6.28. *Such a CS set $R^m \times C$ is homeomorphic to one of $\aleph_0$ model CS sets $S_1, S_2, S_3, \ldots$.*

Proof. By our induction on depth there are up to homeomorphism only $\aleph_0$ CS sets of the form $T^{m+1} \times L$. In this proof L stands for a compact stratified set of depth $\leqslant d-1$, which may not be a CS set. Hence there are only $\aleph_0$ such up to homeomorphism respecting projection of fundamental group to $Z^{m+1} = \pi_1 T^{m+1}$.

Now, passing to coverings with group Z^{m+1} (as under 5.13 or in [34a, Theorem A]) and adding the m-sphere S^m at infinity we find that there are up to homeomorphism respecting S^m, only $\aleph_0$ CS sets of the form $S^m * L$. Then, by the open star uniqueness theorem 4.13, there are, up to homeomorphism, only $\aleph_0$ CS sets of the form $R^m \times cL$. This proves the assertion.

The following proposition now shows that up to homeomorphism there are $\leqslant \aleph_0$ sets in $\mathscr{C}_d$ and thus completes the induction to prove the counting theorem. Fix a finite collection $A_1, A_2, \ldots, A_k$ of spaces in LOC so that $\mathscr{D}(A_i)$ hold $1 \leqslant i \leqslant k$.

PROPOSITION 6.29. *Up to homeomorphism there exist only countably many metric compacta X such that X is expressible as a union of open subsets $A_1^*, A_2^*, \ldots, A_k^*$ with $A_i^* \approx A_i$, and $\dim X < \infty$.*

Proof of 6.29. In each A_i consider a compactum B_i. In this proof variable subscripts are understood to range through, 1, 2,... k. Form a metric space $\mathcal{M}$ depending on A_i, B_i, $i = 1, \ldots, k$, as follows. A point of $\mathcal{M}$ consists of a compactum X in separable Hilbert space H and embeddings $f_i : B_i \to H$, such that $X = \cup f_i \mathring{B}_i$ and each set $f_i \mathring{B}_i$ is open in X. We write $\{f_i\}$ for this point, X being determined as $\cup f_i \mathring{B}_i$.

We install a rather fine metric[17] on $\mathcal{M}$. Let $\hat{B}_i$ the Alexandroff one point compactification of $\mathring{B}_i$ and fix a metric d_i on $\hat{B}_i$. Define f_{ij} to be the composed map $B_i \to X \to \hat{B}_j$, where the second map collapses $X - f_j(\mathring{B}_j)$ to a point mapping the quotient to $\hat{B}_j$ under f_j^{-1}. Define

$$d(\{f_i\}, \{f_i'\}) = \sum_i \sup_x \{|f_i(x) - f_i'(x)|;\, x \in B_i\} + \sum_{ij} \sup_x \{d_j(f_{ij}(x), f_{ij}'(x));\, x \in B_i\}$$

Thus $\mathcal{M}$ has a metric inherited as a subspace of $\prod_i \operatorname{Map}(B_i, H \times \hat{B}_1 \times \cdots \times \hat{B}_k)$. Now $\mathcal{M}$ is separable since any subset of a separable metric space is separable and $\operatorname{Map}(B, H)$ is clearly separable for any compactum B in an euclidean space E. (There is a countable dense subset consisting of certain maps that extend to simplicial maps of a compact neighborhoods of B in E to finite dimensional subspaces of H).

Clearly the proposition will follow when we show that, for each such space $\mathcal{M}$, $\leqslant \aleph_0$ compacta X up to homeomorphism occur as image sets $X = \cup f_i B_i$.

Consider the subset $\mathscr{E} \subset \mathcal{M} \times H$ consisting of points $(\{f_i\}, x)$ with $x \in X = \cup_i f_i B_i$.

ASSERTION 6.30. *The (first factor) projection map $\pi : \mathscr{E} \to \mathcal{M}$ is a (locally trivial) bundle map.*

Since $\mathcal{M}$ is separable it has a countable base of open sets, hence also a countable base over which π is trivial. Thus π has up to homeomorphism $\leqslant \aleph_0$ fibers which are just the image sets $\cup f_i B_i$ for $\{f_i\} \in \mathcal{M}$. Thus the assertion implies the theorem.

Proof of Assertion 6.30. According to 6.14, which characterises bundle maps with compact fiber, it suffices to show (a) that π maps closed sets to closed sets and (b) that π is a submersion.

To prove (a) let $C \subset \mathscr{E}$ be closed and consider a point $\{f_i\} \in \mathcal{M} - \pi C$. Then the compactum $\pi^{-1}\{f_i\} = \{f_i\} \times \cup_i f_i(B_i)$ lies in the open set $\mathscr{E} - C$ of $\mathscr{E}$. As $X = \cup_i f_i(B_i)$ is compact, $\pi^{-1}\{f_i\}$ has a neighborhood in $\mathscr{E} - C$ of the form $\mathscr{E} \cap \{U \times W\}$ where $U \subset \mathcal{M}$ is a neighborhood of $\{f_i\}$ in $\mathcal{M}$ and W is a neighborhood of X in H. But if U is small, the metric on $\mathcal{M}$ dictates that, for every $\{f_i'\} \in U$, one has $\cup_i f_i' B_i \subset W$. Thus $\pi^{-1} U \subset \mathscr{E} \cap \{U \times W\} \subset \mathscr{E} - C$. Hence $U \subset \mathcal{M} - \pi C$ and we conclude that $\mathcal{M} - \pi C$ is open proving (a).

---

17) cf. A. Shilepsky's remark in [10].

To prove (b) consider the maps $\Theta_j: \mathcal{M} \times B_j \to \mathcal{E}$ given by $\Theta_j(\{f_i\}, x) = (\{f_i\}, f_j(x)) \in \mathcal{E}$. As $\pi\Theta_j$ is projection to $\mathcal{M}$ it is easily seen that Θ_j is a homeomorphism onto its image. Also $\cup_j \Theta_j(\mathcal{M} \times \mathring{B}_j) = \mathcal{E}$. To verify that π is a submersion with charts $\Theta_j \mid \mathcal{M} \times \mathring{B}_j$. we now check that each $\Theta_j \mid \mathcal{M} \times \mathring{B}_j$ is an open mapping into $\mathcal{E}$. This amounts to showing that $\Theta_\alpha(\mathcal{M} \times \mathring{B}_\alpha) \cap \Theta_\beta(\mathcal{M} \times \mathring{B}_\beta)$ is open in $\Theta_\alpha(\mathcal{M} \times \mathring{B}_\alpha)$ for any index values α, β among $1, \ldots, k$. For this it suffices to show that if $\{f_i\}$ is fixed and $f_\alpha(x) = f_\beta(y)$ for some $x \in \mathring{B}_\alpha$, $y \in \mathring{B}_\beta$, then for any compact neighborhood K of x in $\mathring{B}_\alpha \cap f_\alpha^{-1} f_\beta \mathring{B}_\beta$ there exists a neighborhood U of $\{f_i\}$ in $\mathcal{M}$ so that $\{f_i'\} \in U$ implies $f_\beta'(\mathring{B}_\beta) \supset f_\alpha'(K)$ i.e. $\mathring{B}_\beta \supset f_{\alpha\beta}'(K)$. But as $\{f_i'\}$ approaches $\{f_i\}$ our choice of metric on $\mathcal{M}$ makes $f_{\alpha\beta}': B_\alpha \to \hat{B}_\beta$ approach $f_{\alpha\beta}$, so U exists. This establishes the submersion.

Assertion 6.30 is now proved and with it Proposition 6.29 and Theorem 6.27.

Historical Remark 6.32. Cheeger and Kister observe in [10] [1, p.3] that their counting argument also gives the submersion characterisation (6.14) of bundle maps with compact fibers, a result that had already been proved (as in 6.14) to show that a proper (topological) Morse function on a manifold yields a handle decomposition. We prefer to deduce the counting from 6.14.

Next we indicate two standard generalizations for the counting theorem 6.27.

DEFINITION 6.33. If in the definition of WCS set §5, we insist that each stratified mock open cone C be *regular* in the sense that it

(i) be orderly in the sense of 5.8,

(ii) be an isotopy regular neighborhood [32] of its vertex v (in the category of stratified sets),[18, 19])

then we have the definition of a so-called *regular* WCS set.

COMPLEMENT 6.34 to 6.27. *Theorem* 6.27 *remains valid if regular WCS sets replace CS sets.*

The only part of the proof of 6.27 requiring adjustment is Assertion 6.28. The CS sets $T^{m+1} \times L$ are replaced by regular WCS sets $T^m \times M$ where M is obtained from $C - v$ by gluing its ends in the manner of [30, §5] respecting strata. Here C is a stratified regular mock open cone. To complete the adjustment of 6.28 one needs an isomorphism of $C - v$ with the standard infinite cyclic covering $\bar{M}$ of M. To get this follow the proof of [30, §7.8] and refer to [32].

COMPLEMENT 6.35 to 6.27. *There are* $\leqslant \aleph_0$ *homeomorphism classes of compact*

---

[18]) Definition of an isotopy regular neighborhood as a nest of open sets each compressible towards v in the next larger [32] shows that (ii) implies (i) (always).

[19]) Does (i) imply (ii)? If so, then *regular* would mean no more than *orderly* (see 5.8). A sufficiently general isotopy extension principle would prove this.

(Hausdorff) $(n+1)$*-ads* $(X; X_1, \ldots, X_n)$ *(a 2-ad is a pair) in which each point has an open neighborhood homeomorphic (as* $(n+1)$*-ad) to a regular WCS* $(n+1)$*-ad.*

Proof of 6.35. The adjustments to the proof of 6.27 and 6.34 required are strictly routine. Theorem 6.14 is replaced by its respectful version (see 6.16).

Note that the group H of automorphisms of such an $(n+1)$-ad fixing X_1 has countable homotopy groups $\pi_i H$ since the maps $S^i \to H$ sufficiently near a given map are homotopic by 5.10, and H is separable. Hence H is weakly homotopy equivalent to a countable complex.

REFERENCES

[1] *Topology of Manifolds*, editors J. C. Cantrell and C. H. Edwards (Markham, Chicago, 1970).

[2] Akin, E., *Manifold phenomena in the theory of polyhedra*, Trans. Amer. Math. Soc. *143* (1969), 413–473.

[3] Alexander, J. W., *On the deformation of an n-cell*, Proc. Nat. Acad. Sc. U.S.A. *9* (1923), 406–407.

[4] Andrews, J. and Curtis, M., *n-Space modulo an arc*. Ann. of Math. *75* (1962), 1–7.

[5] Anderson, R. D. and Bing, R. H., *A complete elementary proof that Hilbert space is homeomorphic to the countable product of lines* Bull. Amer. Math. Soc. *74* (1968), 771–792.

[6] Armstrong, A. and Zeeman, E. C., *Transversality for polyhedra* Ann. of Math, *86* (1967), 172–191

[7] Bourbaki, N., *Topologie Générale*, 4ième édition (Hermann, Paris, 1971).

[8] Černavskii, A. V., *Local contractibility of the homeomorphism group of a manifold* Soviet Math. Doklady *9* (1968), 1171–1174 Dokl. Akad. Nauk SSSR, Tom *182*, No. 3 (1968).

[9] Černavskii, A. V. *Local contractibility of the homeomorphisms group of a manifold* (Russian) Math. Sbornik Vol. 79 (121) 1969, 307–356.

[10] Cheeger, J. and Kister, J., *Counting topological manifolds*, Topology *9* (1970), 149–152.

[11] Crowell, R. H., *Invertible isotopies*, Proc. Am. Math. Soc. *14* (1963), 658–664.

[12] Edwards, R. D., *Wrapping small homeomorphism around tori*, p. 4 in [1].

[13] Edwards, R. D., *93* (1971), 63–88. (article in preparation).

[14] Edwards, R. D. and Kirby, R. C., *Deformations of spaces of imbeddings*, Ann. of Math. *93* (1971), 63–88.

[15] Gauld, D., *Mersions of topological manifolds*, Trans. Amer. Math. Soc. *149* (1970), 539–560.

[16] Haefliger, A., *Structures feuilletées…*, Comm. Math. Helv. *32* (1958), 248–329.

[17] Haefliger, A., *Variétés feuilletées*, Annali Scu. norm. sup. Pisa *CVI* (1962), 367–397.

[18] J. G. Hocking and Young, G. S., *Topology*, Addison-Wesley, Reading Mass. 1960.

[19] Hudson, J. F. P. *Piecewise-linear topology*, W. A. Benjamin, Inc., New-York, 1969.

[20] Kirby, R. C., *Stable homeomorphisms and the annulus conjecture*, Ann. of Math. *89* (1969), 575–582.

[21] Kirby, R. C. and Siebenmann, L. C., *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. *75* (1969), 742–749.

[22] Kirby, R. C. and Siebenmann, L. C., *Deformation of smooth and piecewise linear manifold structures* (to appear).

[23] Kwun, K. W., *Uniqueness of the open cone neighborhood*, Proc. Amer. Math. Soc. *15* (1964), 476–479.

[23A] Lees, J., *Immersions and surgeries of topological manifolds*, Bull. Amer. Math. Soc.

[24] Mather, J., *Notes on topological stability*, mimeographed, Harvard U. 1970.

[25] Milnor, J., *Two complexes that are homeomorphic but combinatorially distinct*, Ann. of Math. *74* (1961), 575–590.

[26] Rourke, C. P., *Covering the track of an isotopy*, Proc. Amer. Math. Soc. *18* (1967), 320–324.

[26A] Rourke, C. P. and Sanderson, B. J., *An embedding without a normal microbundle*, Invent. Math. *3* (1967), 293–399.

[27] SIEBENMANN, L. C., *The obstruction to finding a boundary for an open manifold of dimension greater than five, doctoral dissertation*, Princeton Univ. 1965, Order number 66–5012, University Microfilms Ltd., 300N. Zeeb Rd., Ann. Arbor, Mich. 48106 U.S.A. (to appear in revised form).
[28] SIEBENMANN, L. C., *On the homotopy type of compact topological manifolds*, Bull. Amer. Math. Soc. *74* (1968), 738–742.; Zentralbatt.
[29] SIEBENMANN, L. C., *Pseudo-annuli and invertible cobordisms*, Archiv der Math. *19* (1968), 28–35.
[30] SIEBENMANN, L. C., *A total Whitehead torsion obstruction*, Comment. Math. Helv. *45* (1970) 1–48.
[31] SIEBENMANN, L. C., *Approximating cellular maps by homeomorphisms*, Topology 1972.
[32] SIEBENMANN, L. C., *Open regular neighborhoods*, General Topology.
[33] SIEBENMANN, L. C., *Some locally triangulable compact metric spaces that are not simplicial complexes.*
[33A] SIEBENMANN, L. C., *Are non-triangulable manifolds triangulable?*, pages 77–84 in [1].
[34] STEENROD, N., *A convenient category of topological spaces*, Mich. Math. J. *14* (1967), 133–152.
[35] THOM, R., *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. *75* (1969), 133–152.
[36] KOBAYASHI, S. and NOMIZU, K., *Foundations of Differential Geometry* (Interscience, New York, 1963).
[37] SULLIVAN, D., *Combinatorial invariants of analytic spaces*, in Springer Lecture Notes, No. 192 (Liverpool Symposium I).

Faculté des Sciences
91-Orsay, France

Received December 1970