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On the Homology Theory of Central Group Extensions: I-The Commutator Map and Stem Extensions

by BENO ECKMANN, PETER J. HILTON and URS STAMMBACH

In Memoriam Heinz Hopf (1894–1971)

1. Introduction

For any group extension

$$N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q \quad (1.1)$$

and any Q -module B , there is a five-term exact homology sequence

$$H_2(G; B) \xrightarrow{\alpha_B} H_2(Q; B) \xrightarrow{\beta_B} N_{ab} \otimes_Q B \xrightarrow{\sigma_B} H_1(G; B) \xrightarrow{\tau_B} H_1(Q; B) \rightarrow 0, \quad (1.2)$$

due to Stallings and Stambach [8, 9]. If $B = \mathbb{Z}$, regarded as trivial Q -module, (1.2) reduces to

$$H_2G \xrightarrow{\alpha} H_2Q \xrightarrow{\beta} N/[G, N] \xrightarrow{\sigma} H_1G \xrightarrow{\tau} H_1Q \rightarrow 0. \quad (1.3)$$

For a simple proof of (1.2), including the statement of naturality, see Eckmann-Stambach [3]; of course, α_B and τ_B in (1.2) are induced by ε ; σ_B is, in a sense explained later, induced by μ (the sense is perfectly clear in the case (1.3)); and β_B will be elucidated in the next section.

In the special case where (1.1) is a *central* extension, that is, N is central in G , Ganea [5] has added a further term on the left of the exact sequence (1.3), thus,

$$G_{ab} \otimes N \xrightarrow{\chi_0} H_2G \xrightarrow{\alpha} H_2Q \rightarrow \cdots \rightarrow 0, \quad (1.4)$$

using methods of algebraic topology.

In [2], Eckmann-Hilton extended the sequence (1.4) by four further non-trivial homology terms, first replacing $G_{ab} \otimes N$ by a suitable quotient. Their method was based on a spectral sequence for the homology of a suitable fibre space. The Ganea sequence (1.4) and its extension in [2] are important for applications, beyond those of (1.3), in group theory, homology, algebraic K -theory, etc.

In the present paper we present an elementary approach to the Ganea extension (1.4) and to those parts of the extended sequence in [2] which are relevant to applications to *stem-extensions* of groups (see Section 4), and, in particular, to the study of *perfect* groups. The argument is based on a fixed, but arbitrary, free presentation of (1.1) (see Section 2), and the associated *Gruenberg resolutions* [4; Chapter VI] of \mathbb{Z}

over N , G and Q . The maps of (1.3) are exhibited and exactness is proved, by using these explicit resolutions.

In Section 2 we study (1.3) – and, in less detail, (1.2) – from the viewpoint of the given presentation of (1.1). In particular, we recall the relation with the *Hopf formula* for H_2G and H_2Q and obtain an explicit form for $\ker \alpha$. Moreover, the connection between β and the characteristic class of the central extension

$$N/[G, N] \twoheadrightarrow G/[G, N] \rightarrow Q, \quad (1.5)$$

associated with (1.1), is obtained.

In Section 3 the Ganea extension (1.4) is established by means of an explicit commutator map χ ; and the equivalence of this map χ with Ganea's map χ_0 is demonstrated. In Section 4 we obtain an extended exact sequence

$$H_3G \rightarrow H_3Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2G \xrightarrow{\alpha} H_2Q \rightarrow \dots \quad (1.6)$$

for stem-extensions, that is, central extensions (1.1) with $N \subseteq [G, G]$. Actually, we will obtain (1.6) for an even broader class of central extensions, which we tentatively call *weak stem-extensions*. Whereas a stem-extension (1.1) is characterized by the vanishing of the abelianization of μ , i.e. $\mu_*: N \rightarrow G_{ab}$ is the zeromap, for weak stem extensions we merely demand that $\mu_*: N \otimes N \rightarrow G_{ab} \otimes N$ is the zeromap. We give examples to show that this generalization is significant, and we also show how (1.6) may be regarded as contained in the extended sequence of [2].

Section 5 deals with perfect groups; we assume Q perfect and obtain, beyond the results of Schur [7] and Kervaire [6] on stem-extensions of Q , a description of the universal stem-extension of Q in terms of the given presentation of Q . Moreover, we use the given presentations of N , G and Q to carry further the analogy remarked by Kervaire between the theory of perfect groups and covering space theory for connected topological spaces.

Section 6 is an appendix concerning algebraic K -theory, in which we show how the exact sequence for Milnor's K_2 may be obtained from (1.3).

2. Extensions, Free Presentations, and Resolutions

Given the group extension (1.1),

$$N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q,$$

let

$$R \xrightarrow{\mu'} F \xrightarrow{\varepsilon'} G \quad (2.1)$$

be a (free) presentation of G . There are then presentations of Q , N ,

$$S \xrightarrow{\bar{\mu}} F \xrightarrow{\bar{\varepsilon}} Q, \quad (2.2)$$

$$R \xrightarrow{\mu''} S \xrightarrow{\varepsilon''} N, \quad (2.3)$$

where

$$\bar{\mu}\mu'' = \mu', \quad \varepsilon\varepsilon' = \bar{\varepsilon}, \quad \varepsilon'\bar{\mu} = \mu\varepsilon''. \quad (2.4)$$

We sum this up in the single diagram

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow \mu & & \\ R & \xrightarrow{\mu'} & F & \xrightarrow{\varepsilon'} & G \\ \mu'' \downarrow & & \parallel & & \downarrow \varepsilon \\ S & \xrightarrow{\bar{\mu}} & F & \xrightarrow{\bar{\varepsilon}} & Q, \\ \varepsilon'' \downarrow & & & & \\ N & & & & \end{array} \quad (2.5)$$

which we call a *presentation of the extension* (1.1).

Applying (1.3) to the two centre rows of (2.5) we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & H_2G & \xrightarrow{\beta'} & R/[F, R] & \xrightarrow{\sigma'} & F/[F, F] \rightarrow \dots \\ & & \downarrow \alpha & & \downarrow \gamma & & \parallel \\ 0 & \rightarrow & H_2Q & \xrightarrow{\bar{\beta}} & S/[F, S] & \xrightarrow{\bar{\sigma}} & F/[F, F] \rightarrow \dots, \end{array} \quad (2.6)$$

where α is as in (1.3) and γ is induced by the inclusion $\mu'': R \rightarrow S$. Thus

$$\ker \gamma = (R \cap [F, S])/[F, R], \quad \operatorname{coker} \gamma = S/R[F, S]. \quad (2.7)$$

We note that ε'' induces an isomorphism of $\operatorname{coker} \gamma$ onto $N/[G, N]$. We write $\eta: S/[F, S] \rightarrow N/[G, N]$ for the map induced by ε'' , and thus embed (2.6) in the larger commutative diagram, with exact rows and columns,

$$\begin{array}{ccccccc} 0 & \rightarrow & \ker \alpha & \rightarrow & \ker \gamma & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H_2G & \xrightarrow{\beta'} & R/[F, R] & \xrightarrow{\sigma'} & F/[F, F] \rightarrow \dots \\ & & \downarrow \alpha & & \downarrow \gamma & & \downarrow = \\ 0 & \rightarrow & H_2Q & \xrightarrow{\bar{\beta}} & S/[F, S] & \xrightarrow{\bar{\sigma}} & F/[F, F] \rightarrow \dots \\ & & \downarrow \beta & & \downarrow \eta & & \downarrow \\ 0 & \rightarrow & N/[G, N] & \xrightarrow{=} & N/[G, N] & \rightarrow & 0 \end{array} \quad (2.8)$$

We use β' to induce the *Hopf formula*

$$H_2G \cong (R \cap [F, F])/[F, R]; \quad (2.9)$$

likewise $\bar{\beta}$ induces

$$H_2Q \cong (S \cap [F, F])/[F, S]. \quad (2.10)$$

We make the identifications (2.9), (2.10), so that $\ker \alpha$ is identified with $\ker \gamma$,

$$\ker \alpha = (R \cap [F, S])/[F, R],$$

and the map β ,

$$\beta: (S \cap [F, F])/[F, S] \rightarrow N/[G, N],$$

is just the restriction of η to H_2Q , and thus is induced by ε'' . Note that the relation $\beta = \eta\bar{\beta}$ in (2.8) simply results from the naturality of (1.3), applied to

$$\begin{array}{ccccc} S & \xrightarrow{\bar{\mu}} & F & \xrightarrow{\bar{\varepsilon}} & Q \\ \downarrow \varepsilon'' & & \downarrow \varepsilon' & & \parallel \\ N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \end{array} .$$

We now wish to relate β to the *associated central extension* (1.5). It follows from naturality that the homomorphism $H_2Q \rightarrow N/[G, N]$ in the sequence (1.3) corresponding to (1.5) coincides with β , so that, in this part of the argument, we lose no generality in supposing that N is *itself central* in (1.1). Then

$$\beta: H_2Q \rightarrow N,$$

and we propose to relate β to the element of $H^2(Q; N)$ characterized by the *central extension*

$$N \rightarrowtail G \twoheadrightarrow Q.$$

We now have $[G, N] = 1$, so that

$$[F, S] \subseteq R, \quad \ker \alpha = [F, S]/[F, R], \quad (2.11)$$

and $\beta: (S \cap [F, F])/[F, S] \rightarrow S/R$ is induced by the inclusions $S \cap [F, F] \subseteq S$, $[F, S] \subseteq R$.

We begin by giving an explicit description of $H^2(Q; N)$ in terms of (2.5). We use the *Gruenberg resolution* [4; VI. 13] of \mathbf{Z} over Q based on (2.2), namely,

$$\cdots \xrightarrow{\partial_3} S_{ab} \otimes \mathbf{Z}Q \xrightarrow{\partial_2} JF \otimes_F \mathbf{Z}Q \xrightarrow{\partial_1} \mathbf{Z}Q \xrightarrow{\partial_0} \mathbf{Z} \rightarrow 0. \quad (2.12)$$

Here ∂_0 is the augmentation; JF is the augmentation ideal of F ; ∂_1 is given by

$$\partial_1((x - e) \otimes_F e) = \bar{\varepsilon}(x) - e, \quad x \in F,$$

where e stands for the unity in any group; the kernel of ∂_1 is known to be isomorphic to S_{ab} , with Q operating by inner automorphisms of F , under the monomorphism

$$\sigma_1: S_{ab} \rightarrowtail JF \otimes_F \mathbf{Z}Q$$

given by

$$\sigma_1(s[S, S]) = (s - e) \otimes_F e;$$

and ∂_2 is the composite of σ_1 with the Q -module map $s[S, S] \otimes e \mapsto s[S, S]$. (See [4; VI. 6], where the argument is given in detail but for the Gruenberg resolution of \mathbf{Z} as *left* Q -module.)

THEOREM 2.1. *The Gruenberg resolution (2.12) induces an isomorphism*

$$H^2(Q; N) \cong \text{Hom}(S/[F, S], N) / \bar{\sigma}^* \text{Hom}(F/[F, F], N)$$

for any trivial Q -module N .

Proof. For any resolution $\cdots \rightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ of \mathbf{Z} over Q , $H^2(Q; N) = \text{Hom}_Q(\ker \partial_1, N) / i^* \text{Hom}_Q(C_1, N)$, where $i: \ker \partial_1 \subseteq C_1$. Thus, using the Gruenberg resolution,

$$H^2(Q; N) = \text{Hom}_Q(S_{ab}, N) / \sigma_1^* \text{Hom}_Q(JF \otimes_F \mathbf{Z}Q, N).$$

Now $\text{Hom}_Q(S_{ab}, N) = \text{Hom}((S_{ab})_Q, N) = \text{Hom}(S/[F, S], N)$, and $\text{Hom}_Q(JF \otimes_F \mathbf{Z}Q, N) = \text{Hom}(JF \otimes_F \mathbf{Z}, N)$. Moreover there is a natural isomorphism $\psi: JF \otimes_F \mathbf{Z} \cong F_{ab}$, given by $\psi((x - e) \otimes_F 1) = x[F, F]$, $x \in F$, and plainly $\psi\sigma_1$ induces $\bar{\sigma}: S/[F, S] \rightarrow F/[F, F]$. This proves the theorem.

Now given the central extension $N \twoheadrightarrow G \twoheadrightarrow Q$, the map $\eta: S/[F, S] \rightarrow N$ of (2.8) (recall that $[G, N] = 1$) then determines, in the light of Theorem 2.1, an element $\xi \in H^2(Q; N)$ and this is the characteristic cohomology class of the given central extension (see [4; VI. 10]). We now readily prove

THEOREM 2.2. *If $N \twoheadrightarrow G \twoheadrightarrow Q$ is a central extension with characteristic class $\xi \in H^2(Q; N)$, then the homomorphism $\beta: H_2Q \rightarrow N$ of (1.3) is the image of ξ under the epimorphism*

$$\Phi: H^2(Q; N) \twoheadrightarrow \text{Hom}(H_2Q, N)$$

of the universal coefficient theorem.

Proof. For any $\zeta \in H^2(Q; N)$, $\Phi(\zeta)$ is obtained by picking a representative $\theta: S/[F, S] \rightarrow N$ and restricting θ to $(S \cap [F, F])/[F, S]$. Since η represents ξ and β is the restriction of η to $(S \cap [F, F])/[F, S]$, the theorem follows.

Remark. If $\text{Ext}(Q_{ab}, N) = 0$, then Φ is an isomorphism, so that β characterizes the central extension. An important special case is that in which $Q_{ab} = 0$ (i.e., Q is *perfect*). The fact that β characterizes the extension when Q is perfect may be seen directly by observing that then $\bar{\sigma}: S/[F, S] \rightarrow F/[F, F]$ is surjective, so that two homomorphisms $S/[F, S] \rightarrow N$ determine the same element of $H^2(Q; N)$ (see Theorem 2.1) if and only if they agree on H_2Q .

We close this section by recalling from VI.8 of [4] how β_B and σ_B are defined in (1.2). In terms of the resolution (2.12),

$$H_2(Q; B) = \ker(S_{ab} \otimes_Q B \rightarrow JF \otimes_F B) \quad (2.13)$$

and then $\beta_B: H_2(Q; B) \rightarrow N_{ab} \otimes_Q B$ is given by restricting to $H_2(Q; B)$ the homomorphism $S_{ab} \otimes_Q B \rightarrow N_{ab} \otimes_Q B$ induced by ε'' . As to σ_B , we exploit the short exact sequence (Theorem VI.6.3 of [4])

$$N_{ab} \twoheadrightarrow JG \otimes_G \mathbb{Z}Q \twoheadrightarrow JG$$

of Q -modules to obtain

$$N_{ab} \otimes_Q B \rightarrow JG \otimes_G B. \quad (2.14)$$

Moreover the image of this homomorphism obviously lies in the kernel of $JG \otimes_G B \rightarrow B$, that is, in $H_1(G; B)$, and thus (2.14) determines σ_B .

3. The Commutator Map and the Ganea Term

Given a *central* group extension $N \twoheadrightarrow G \twoheadrightarrow Q$ we now define a homomorphism $\chi: G_{ab} \otimes N \rightarrow H_2 G$ which will yield exactness in

$$G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\alpha} H_2 Q. \quad (3.1)$$

We return to the presentation (2.5) and recall (2.11) that then

$$\ker \alpha = [F, S]/[F, R].$$

Thus our objective is to define a natural surjection $G_{ab} \otimes N \rightarrow [F, S]/[F, R]$. We define a map

$$c: F \times S \rightarrow [F, S]/[F, R]$$

by

$$c(x, s) = [x, s] [F, R], \quad x \in F, \quad s \in S \quad (3.2)$$

where $[x, s]$ is the commutator $xsx^{-1}s^{-1}$. We call c the *commutator map* (relative to the presentation (2.5)).

PROPOSITION 3.1. *The commutator map c is a bihomomorphism, that is,*

$$\begin{aligned} c(xx', s) &= c(x, s) c(x', s), \\ c(x, ss') &= c(x, s) c(x, s'). \end{aligned}$$

Proof. Since $[ab, c] = a[b, c]a^{-1}[a, c]$, and $[a, bc] = [a, b]b[a, c]b^{-1}$, we have

$$\begin{aligned} c(xx', s) &= xc(x', s)x^{-1}c(x, s), \\ c(x, ss') &= c(x, s)sc(x, s')s^{-1}. \end{aligned}$$

But $[F, S] \subseteq R$ and $[F, S]/[F, R]$ is commutative. Thus

$$\begin{aligned} c(xx', s) &= c(x, s) c(x', s), \\ c(x, ss') &= c(x, s) c(x, s'). \end{aligned}$$

Again, since $[F, S]/[F, R]$ is commutative, Proposition 3.1 shows that $c([F, F] \times S) = e$; plainly $c(R \times S) = e$, $c(F \times R) = e$. Thus c induces a bihomomorphism of abelian groups

$$F/[F, F] \times S/R \rightarrow [F, S]/[F, R],$$

and hence a homomorphism

$$c_1: G_{ab} \otimes N \rightarrow [F, S]/[F, R],$$

which is plainly surjective. We define χ to be the composite of c_1 and the embedding of $[F, S]/[F, R] = \ker \alpha$ in $H_2 G$. We have thus proved, for a central extension $N \rightarrowtail G \twoheadrightarrow Q$,

THEOREM 3.2. *The commutator map c defines a homomorphism $\chi: G_{ab} \otimes N \rightarrow H_2 G$ such that the sequence*

$$G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\alpha} H_2 Q \xrightarrow{\beta} N \rightarrow G_{ab} \rightarrow Q_{ab} \rightarrow 0$$

is exact.

We show later that χ is equivalent to Ganea's χ_0 . This would imply the naturality of χ , but we give now an independent proof so that our arguments may be entirely self-contained. Suppose given a map of central extensions,

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ N_0 & \xrightarrow{\mu_0} & G_0 & \xrightarrow{\varepsilon_0} & Q_0, \end{array} \quad (3.3)$$

and a presentation of $N_0 \rightarrowtail G_0 \twoheadrightarrow Q_0$, designated by (2.5) with all elements assigned the suffix 0. We lift f_2 to $\tilde{f}: F \rightarrow F_0$ such that $\varepsilon'_0 \tilde{f} = f_2 \varepsilon'$. Then $\tilde{f}(S) \subseteq S_0$, $\tilde{f}(R) \subseteq R_0$. Thus if $c_0: F_0 \times S_0 \rightarrow [F_0, S_0]/[F_0, R_0]$ is the commutator map,

$$c_0(\tilde{f}x, \tilde{f}s) = [\tilde{f}x, \tilde{f}s] [F_0, R_0] = \tilde{f}[x, s] [F_0, R_0] = \tilde{f}_* c(x, s),$$

where $\tilde{f}_*: [F, S]/[F, R] \rightarrow [F_0, S_0]/[F_0, R_0]$ is induced by \tilde{f} . It follows that the diagram

$$\begin{array}{ccc} G_{ab} \otimes N & \xrightarrow{\chi} & H_2 G \\ \downarrow f_* & & \downarrow f_* \\ G_{0ab} \otimes N_0 & \xrightarrow{\chi} & H_2 G_0 \end{array}$$

commutes, where the vertical arrows are induced by (3.3) and are independent of the choice of presentations. This shows that χ is itself independent of the choice of presentation and is natural.

Ganea's map $\chi_0: G_{ab} \otimes N \rightarrow H_2 G$ is effectively defined as follows. The multiplication in G induces a homomorphism $\mu: G \times N \rightarrow G$, inducing $\mu_*: H_2(G \times N) \rightarrow H_2 G$. According to the Künneth formula, $H_1 G \otimes H_1 N (= G_{ab} \otimes N)$ is naturally embedded in $H_2(G \times N)$.

Then

$$\chi_0 = \mu_* \mid (G_{ab} \otimes N).$$

We prove

THEOREM 3.3. $\chi_0 = -\chi$.

Proof. We first study, in general, the embedding of $G_{1ab} \otimes G_{2ab}$ in $H_2(G_1 \times G_2)$. Since we know this is natural, it suffices to choose suitable resolutions. Thus we choose the Gruenberg resolutions

$$\cdots \rightarrow R_{iab} \otimes \mathbf{Z}G_i \xrightarrow{\partial_2} JF_i \otimes_{F_i} \mathbf{Z}G_i \xrightarrow{\partial_1} \mathbf{Z}G_i \xrightarrow{\partial_0} \mathbf{Z} \rightarrow 0,$$

of \mathbf{Z} over G_i , corresponding to presentations $R_i \twoheadrightarrow F_i \twoheadrightarrow G_i$, $i=1, 2$. Then a partial resolution of \mathbf{Z} over $G_1 \times G_2$ is given by

$$\begin{aligned} \cdots \rightarrow (JF_1 \otimes_{F_1} \mathbf{Z}G_1) \otimes (JF_2 \otimes_{F_2} \mathbf{Z}G_2) &\xrightarrow{\partial \otimes 1 - 1 \otimes \partial} \{(JF_1 \otimes_{F_1} \mathbf{Z}G_1) \otimes \mathbf{Z}G_2\} \\ &\oplus \{\mathbf{Z}G_1 \otimes (JF_2 \otimes_{F_2} \mathbf{Z}G_2)\} \rightarrow \mathbf{Z}G_1 \otimes \mathbf{Z}G_2 \rightarrow \mathbf{Z} \rightarrow 0, \end{aligned} \quad (3.4)$$

where we have only written down that part of C_2 which contributes to $H_1 G_1 \otimes H_1 G_2$. Tensoring with \mathbf{Z} over $G_1 \times G_2$, we get

$$(JF_1 \otimes_{F_1} \mathbf{Z}) \otimes (JF_2 \otimes_{F_2} \mathbf{Z}) \xrightarrow{0} JF_1 \otimes \mathbf{Z} \oplus \mathbf{Z} \otimes JF_2.$$

Thus we have proved – in view of the natural isomorphisms $JF_i \otimes_{F_i} \mathbf{Z} \cong F_{iab}$, $i=1, 2$ – writing Z_k for the k th cycle group,

LEMMA 3.4. *With respect to the resolution (3.4), $Z_1 G_1 \otimes Z_2 G_2$ is embedded in $Z_2(G_1 \times G_2)$ as $F_{1ab} \otimes F_{2ab}$, and hence $G_{1ab} \otimes G_{2ab}$ is embedded in $H_2(G_1 \times G_2)$ as $F_1/[F_1, F_1] R_1 \otimes F_2/[F_2, F_2] R_2$.*

We now revert to our special case. We must construct a chain-map $\phi_0, \phi_1, \phi_2, \dots$

$$\begin{array}{ccc} \cdots \rightarrow (JF \otimes_F \mathbf{Z}G) \otimes (JS \otimes_S \mathbf{Z}N) & \xrightarrow{\partial_2} & \{(JF \otimes_F \mathbf{Z}G) \otimes \mathbf{Z}N\} \oplus \{\mathbf{Z}G \otimes (JS \otimes_S \mathbf{Z}N)\} \\ \downarrow \phi_2 & & \downarrow \phi_1 \\ \cdots \rightarrow R_{ab} \otimes \mathbf{Z}G & \xrightarrow{\partial_2} & JF \otimes_F \mathbf{Z}G \\ & & \downarrow \phi_0 \\ & & \begin{array}{ccc} \xrightarrow{\partial_1} & \mathbf{Z}G \otimes \mathbf{Z}N & \rightarrow \mathbf{Z} \rightarrow 0 \\ & \downarrow \phi_0 & \parallel \\ \xrightarrow{\partial_1} & \mathbf{Z}G & \rightarrow \mathbf{Z} \rightarrow 0 \end{array} \end{array}$$

compatible with $\mu: G \times N \rightarrow G$. Direct calculation shows that we may take

$$\begin{aligned}\phi_0(e \otimes e) &= e, \\ \phi_1((x - e) \otimes_F e \otimes e) &= (x - e) \otimes_F e, \\ \phi_1(e \otimes (s - e) \otimes_S e) &= (s - e) \otimes_F e, \\ \phi_2((x - e) \otimes_F e \otimes (s - e) \otimes_S e) &= [s, x] [R, R] \otimes xs, \quad x \in F, \quad s \in S.\end{aligned}$$

The last formula is justified by observing that

$$\begin{aligned}\phi_1 \partial_2((x - e) \otimes_F e \otimes (s - e) \otimes_S e) \\ &= \phi_1((\varepsilon'x - e) \otimes_F (s - e) \otimes_S e - (x - e) \otimes_F e \otimes (\varepsilon''s - e)) \\ &= (s - e) \otimes_F (\varepsilon'x - e) - (x - e) \otimes_F (\varepsilon''s - e) \\ &= (s - e)(x - e) \otimes_F e - (x - e)(s - e) \otimes_F e \\ &= (sx - xs) \otimes_F e \\ &= ([s, x] - e) \otimes_F xs \\ &= \partial_2([s, x] [R, R] \otimes xs).\end{aligned}$$

Again writing Z_k for the k th cycle group, we observe that ϕ_2 induces

$$\phi_2: Z_1 G \otimes Z_1 N \rightarrow Z_2 G$$

given by

$$\phi_2(x[F, F] \otimes s[S, S]) = [s, x] [R, R],$$

and so induces, in the light of Lemma 3.4, χ_0 given by

$$\chi_0(xR[F, F] \otimes sR) = [s, x] [F, R].$$

(Recall that $[S, S] \subseteq R$). This proves the theorem, since

$$\chi(xR[F, F] \otimes sR) = [x, s] [R, R].$$

Remark. The difference of sign is not unexpected, since Ganea is considering the *left* operation of N on G .

4. Stem-Extensions

A central extension $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ is called a *stem-extension* if $N \subseteq [G, G]$. From (1.3) we immediately deduce

PROPOSITION 4.1. *Let $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ be a central extension. Then the following statements are equivalent:*

- (i) $N \twoheadrightarrow G \twoheadrightarrow Q$ is a stem-extension;
- (ii) $\mu_*: N \rightarrow G_{ab}$ is the zeromap;
- (iii) $\tau = \varepsilon_*: G_{ab} \rightarrow Q_{ab}$ is an isomorphism;
- (iv) $\beta: H_2 Q \rightarrow N$ is an epimorphism.

Note that (iv) implies that if $\xi \in H^2(Q; N)$ is the characteristic class of the central extension, then the stem-extensions are precisely those for which $\Phi(\xi)$ is an epimorphism, where

$$\Phi: H^2(Q; N) \twoheadrightarrow \text{Hom}(H_2 Q, N)$$

is the natural epimorphism of the universal coefficient theorem.

We will apply the notion of a stem-extension in later sections. Here we wish to show that the exact sequence of Theorem 3.2 extends two further places to the left in the case of stem-extensions. However, our arguments encompass a generalization of the notion of stem-extensions, which we now give.

We say that the central extension $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ is a *weak stem-extension* if μ induces $0: N \otimes N \rightarrow G_{ab} \otimes N$. Thus we have, by applying (1.2) – compare Proposition 4.1 –

PROPOSITION 4.2. *Let $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ be a central extension. Then the following statements are equivalent:*

- (i) $N \twoheadrightarrow G \twoheadrightarrow Q$ is a weak stem-extension;
- (ii) $\tau_N = \varepsilon_*: G_{ab} \otimes N \rightarrow Q_{ab} \otimes N$ is an isomorphism;
- (iii) $\beta_N: H_2(Q; N) \rightarrow N \otimes N$ is an epimorphism.

We will abbreviate ‘weak stem-extension’ to ‘ws-extension’. We give some examples:

Examples. (a) If G is perfect then every central extension is a stem-extension.

(b) Consider $\mathbf{Z}_m \twoheadrightarrow \mathbf{Z}_{m^2} \twoheadrightarrow \mathbf{Z}_m$. Obviously this is not a stem-extension, but it is clearly a ws-extension, for $\mathbf{Z}_{m^2} \otimes \mathbf{Z}_m \rightarrow \mathbf{Z}_m \otimes \mathbf{Z}_m$ is certainly an isomorphism. It is interesting to note that, in this example, while β_N is, by Proposition 4.2, an epimorphism, $\beta \otimes 1: H_2 Q \otimes N \rightarrow N \otimes N$ is not an epimorphism.

(c) Let p be a fixed prime, let $r \geq s$ be positive integers, and let $G = G(p^r, p^s)$ be the group

$$G = \{a, b \mid a^{p^r} = b^{p^s} = a^{-1}b^{-1}ab\}.$$

Then a is of order p^{r+s} , and the center of G is generated by a^{p^r} . For any $t \geq s$, let N_t be the (central) subgroup of G generated by a^{p^t} . One may then readily verify that $N_t \twoheadrightarrow G \twoheadrightarrow Q_t$ is a stem-extension iff $t \geq r$ and a ws-extension iff $t \geq \frac{1}{2}(r+s)$.

Our main theorem, which we prove by the elementary methods of Sections 2 and 3 is the following.

THEOREM 4.3. *Let $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ be a weak stem-extension. Then there is a homo-*

morphism $\delta: H_3 Q \rightarrow G_{ab} \otimes N$ such that the sequence

$$H_3 G \xrightarrow{\varepsilon_*} H_3 Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\varepsilon_*} H_2 Q \xrightarrow{\beta} N \xrightarrow{\sigma} G_{ab} \xrightarrow{\varepsilon_*} Q_{ab} \rightarrow 0$$

is exact.

Proof. We have only to define δ and prove exactness at $H_3 Q$ and $G_{ab} \otimes N$. Now by the reduction theorem (see for example [4; Corollary VI. 6.5]), we have natural isomorphisms

$$H_3 G \cong H_1(G; R_{ab}), \quad H_3 Q \cong H_1(Q; S_{ab}),$$

so that it is sufficient to define $\delta: H_1(Q; S_{ab}) \rightarrow G_{ab} \otimes N$ and prove exactness in

$$H_1(G; R_{ab}) \xrightarrow{\varepsilon_*} H_1(Q; S_{ab}) \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2 G. \quad (4.1)$$

Now from the exact sequence of Q -modules

$$R/[S, S] \rightarrow S/[S, S] \rightarrow N$$

we get a coefficient sequence

$$H_2(Q; N) \xrightarrow{\psi} H_1(Q; R/[S, S]) \xrightarrow{\phi'} H_1(Q; S_{ab}) \xrightarrow{\delta'} H_1(Q; N) \xrightarrow{\chi'} (R/[S, S])_Q \rightarrow \dots \quad (4.2)$$

Since $\varepsilon_*: G_{ab} \otimes N \cong Q_{ab} \otimes N = H_1(Q; N)$, we may define δ by $\delta = \varepsilon_*^{-1} \delta'$. We now show that the diagram

$$\begin{array}{ccc} G_{ab} \otimes N & \xrightarrow{\chi} & R/[F, R] \\ \downarrow \varepsilon_* & & \downarrow \zeta \\ Q_{ab} \otimes N & \xrightarrow{\chi'} & R/[F, R][S, S] \end{array} \quad (4.3)$$

commutes, where ζ is the natural projection; here no use is made of the assumption that $N \rightarrow G \rightarrow Q$ is weak stem. Using the Gruenberg resolution of \mathbb{Z} over Q , we have $C_1 \otimes_Q N = JF \otimes_F N$, consisting exclusively of cycles. Thus, to compute χ' on $[(x-e) \otimes_F sR]$ we pass to $(x-e) \otimes_F s[S, S]$ and apply the boundary ∂_1 , obtaining $\chi s x^{-1} s^{-1} [S, S]$. We have shown that $\chi'[(x-e) \otimes_F sR] = [x, s][F, R][S, S]$. Identifying $JF \otimes_F N$ with $F_{ab} \otimes N$, we find

$$\chi'(xS[F, F] \otimes sR) = [x, s][F, R][S, S],$$

proving (4.3). Now enlarge (4.3) to the diagram, with exact columns,

$$\begin{array}{ccc} N \otimes N & \xrightarrow{\tilde{\chi}} & [S, S]/[S, R] \\ \downarrow \mu_* & \chi & \downarrow \\ G_{ab} \otimes N & \rightarrow & R/[F, R] \\ \downarrow \varepsilon_* & & \downarrow \zeta \\ Q_{ab} \otimes N & \xrightarrow{\chi'} & R/[F, R][S, S] \end{array}, \quad (4.4)$$

where $\bar{\chi}$ is defined by the commutator map for the extension $N \twoheadrightarrow N \twoheadrightarrow 1$. It follows from (4.4) that, in the case of a ws-extension, ε_* induces an isomorphism of $\ker \chi$ onto $\ker \chi'$. Thus the exactness of (4.2) at $H_1(Q; N)$ implies the exactness of the sequence of Theorem 4.3 at $G_{ab} \otimes N$.

Before proceeding with the argument, we remark that it follows from (4.4) that, for a ws-extension,

$$[S, S] \subseteq [F, R], \quad (4.5)$$

so that ζ in (4.3) is the identity, and $(R/[S, S])_Q = R/[F, R]$.

It remains to prove the exactness of (4.1) at $H_1(Q; S_{ab})$. Our first step is to factorize $\varepsilon_*: H_1(G; R_{ab}) \rightarrow H_1(Q; S_{ab})$ as $\varepsilon_* = \phi' \phi'' \phi'''$,

$$H_1(G; R_{ab}) \xrightarrow{\phi'''} H_1(Q; (R_{ab})_N) \xrightarrow{\phi''} H_1(Q; R/[S, S]) \xrightarrow{\phi'} H_1(Q; S_{ab}). \quad (4.6)$$

Here ϕ''' is the 'change-of-rings' homomorphism; ϕ'' is induced by the sequence of Q -modules,

$$[S, S]/[S, R] \twoheadrightarrow R/[S, R] \twoheadrightarrow R/[S, S]$$

(note that $(R_{ab})_N = R/[S, R]$); and ϕ' is as in (4.2). We will prove

$$\phi''' \text{ is surjective; } \quad (4.7)$$

$$\text{im } \phi' \phi'' = \text{im } \phi'. \quad (4.8)$$

These two facts together establish the exactness of (4.1) at $H_1(Q; S_{ab})$ —in view of (4.2)—and hence Theorem 4.3. To prove (4.7), we demonstrate the following more general lemma.

LEMMA 4.4. *Given any extension $N \twoheadrightarrow G \xrightarrow{e} Q$ and any G -module B ,*

$$\varepsilon_*: H_1(G; B) \rightarrow H_1(Q; B_N)$$

is surjective.

Proof of Lemma. This is well-known (see for example [9]), but we give an elementary proof here in the framework of our paper. Using the Gruenberg resolutions, we readily obtain a commutative diagram

$$\begin{array}{ccccc} H_1(G; B) & \xrightarrow{\varepsilon_*} & H_1(Q; B_N) & & \\ \downarrow & & \downarrow & & \\ JN \otimes_N B & \xrightarrow{\mu_*} & JG \otimes_G B & \xrightarrow{\varepsilon_*} & JQ \otimes_Q B_N \\ \parallel & & \downarrow & & \downarrow \\ JN \otimes_N B & \longrightarrow & B & \longrightarrow & B_N \end{array}$$

in which the columns are exact, the bottom row is exact, and the middle row is differential. It is now trivial that $\varepsilon_*: H_1(G; B) \rightarrow H_1(Q; B_N)$ is surjective.

The proof of (4.8) is based on the following lemma. As usual, we refer to the presentation (2.5).

LEMMA 4.5. *Let $N \twoheadrightarrow G \twoheadrightarrow Q$ be a central extension. Let $\delta'': H_1(Q; R/[S, S]) \rightarrow ([S, S]/[S, R])_Q$ be the connecting homomorphism associated with the sequence of Q -modules*

$$[S, S]/[S, R] \twoheadrightarrow R/[S, R] \twoheadrightarrow R/[S, S]$$

and let $\bar{\chi}: N \otimes N \rightarrow [S, S]/[S, R]$ be the commutator map (4.4) for the extension $N \twoheadrightarrow N \twoheadrightarrow 1$. Then $([S, S]/[S, R])_Q = [S, S]/[S, R]$ and the square

$$\begin{array}{ccc} H_2(Q; N) & \xrightarrow{\psi} & H_1(Q; R/[S, S]) \\ \downarrow \beta_N & & \downarrow \delta'' \\ N \otimes N & \xrightarrow{\bar{\chi}} & [S, S]/[S, R] \end{array} \quad (4.9)$$

commutes.

Proof of Lemma. Since N is central, Q operates trivially on $H_2 N = [S, S]/[S, R]$. We now prove the commutativity of (4.9) by again appealing to the Gruenberg resolution of \mathbf{Z} over Q . Then a 2-chain of $C_2(Q; N)$ has the form $\sum_i s_i [S, S] \otimes s'_i R$. Since ψ is the connecting homomorphism associated with the sequence of Q -modules

$$R/[S, S] \twoheadrightarrow S/[S, S] \twoheadrightarrow N,$$

it follows that the value of ψ on the class of the cycle $w = \sum_i s_i [S, S] \otimes s'_i R$ is the class, in $H_1(Q; R/[S, S])$, of $\sum_i (s_i - e) \otimes_F s'_i [S, S]$, i.e., (using $\{ \}$ for homology classes)

$$\psi \{w\} = \left\{ \sum_i (s_i - e) \otimes_F s'_i [S, S] \right\}. \quad (4.10)$$

Now a 1-chain of $C_1(Q; R/[S, S]) = JF \otimes_F R/[S, S]$ has the form $\sum_j (x_j - e) \otimes_F r_j [S, S]$, and the value of δ'' on the class of the cycle $z = \sum_j (x_j - e) \otimes_F r_j [S, S]$ is the class, in $H_0(Q; [S, S]/[S, R])$, of $\sum_j [x_j, r_j] [S, R]$. But since Q operates trivially on $[S, S]/[S, R]$, we may write

$$\delta'' \{z\} = \sum_j [x_j, r_j] [S, R]. \quad (4.11)$$

We now define a 'commutator map' $\theta: JF \otimes_F S/[S, S] \rightarrow [F, S]/[S, R]$ by the rule

$$\theta((x - e) \otimes_F s [S, S]) = [x, s] [S, R]. \quad (4.12)$$

To check that θ is well-defined we must verify that

$$[x, s_1 s_2] [S, R] = [x, s_1] [x, s_2] [S, R] \quad (4.13)$$

and that $\theta((x - e) y \otimes_F s [S, S]) = \theta((x - e) \otimes_F y s y^{-1} [S, S])$, or

$$[xy, s] [y, s]^{-1} [S, R] = [x, y s y^{-1}] [S, R]. \quad (4.14)$$

Now (4.13) follows exactly as in the proof of Proposition 3.1, since $[S, [F, S]] \subseteq [S, R]$, and (4.14) holds since, in fact, $[xy, s] = [x, ysy^{-1}] [y, s]$. Thus θ is well-defined; and we observe that $JF \otimes_F R/[S, S]$ is a subgroup of $JF \otimes_F S/[S, S]$ and that, from (4.11),

$$\theta(z) = \delta'' \{z\}. \quad (4.15)$$

It thus follows from (4.10), (4.12) and (4.15) that, if $w = \sum s_i [S, S] \otimes s'_i R$,

$$\delta'' \psi \{w\} = \sum_i [s_i, s'_i] [S, R]. \quad (4.16)$$

Now, as shown in Section 2, $\beta_N: H_2(Q; N) \rightarrow N \otimes N = N \otimes_Q N$ is given by restricting to $H_2(Q; N)$ the homomorphism $S_{ab} \otimes_Q N \rightarrow N \otimes_Q N$ induced by ε'' . Thus, with $w = \sum_i s_i [S, S] \otimes s'_i R$,

$$\beta_N \{w\} = \sum_i s_i R \otimes s'_i R,$$

so that

$$\bar{\chi} \beta_N \{w\} = \sum_i [s_i, s'_i] [S, R],$$

and the lemma is proved.

The relation (4.8) quickly follows from Lemma 4.5. For we have the diagram (with exact row and column)

$$\begin{array}{ccccc} & & H_1(Q; R/[S, R]) & & \\ & & \downarrow \phi'' & & \\ H_2(Q; N) & \xrightarrow{\psi} & H_1(Q; R/[S, S]) & \xrightarrow{\phi'} & H(Q; S/[S, S]) \\ & & \downarrow \delta'' & & \\ & & [S, S]/[S, R] & & \end{array}$$

and, by Lemma 4.5, $\delta'' \psi$ is surjective in the case of a ws-extension; for, in that case, β_N is surjective (Proposition 4.2), and $\bar{\chi}$ is always surjective. It follows immediately that $H_1(Q; R/[S, S]) = \text{im } \psi + \text{im } \phi''$, so that $\text{im } \phi' = \text{im } \phi' \phi''$. Thus Theorem 4.3 is completely proved.

Remark. The exact sequence of [2] yields Theorem 4.3 as a special case. For the homomorphism $\sigma: H_4(N, 2) \rightarrow G_{ab} \otimes N$ of [2] factors through $\mu_*: N \otimes N \rightarrow G_{ab} \otimes N$ and is thus zero for a ws-extension. Indeed, the exact sequence of [2] shows that the conclusion of Theorem 4.3 holds if and only if $\mu_* \mid \ker \bar{\chi} = 0$. We will revert to this point in a subsequent paper [11].

In the case of a stem-extension, the exact sequence of Theorem 4.3 becomes

$$H_3 G \xrightarrow{\varepsilon_*} H_3 Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\varepsilon_*} H_2 Q \xrightarrow{\beta} N \rightarrow 0 \quad (4.17)$$

Now let Q be a given group and let U be any subgroup of $H_2 Q$. Set $N = H_2 Q/U$ and let $\beta: H_2 Q \rightarrow N$ be the canonical projection. Let ξ be any element of $H^2(Q; N)$ such that

$\Phi(\xi) = \beta$, and let $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ be a central extension with characteristic class ξ . Then it follows from Theorem 2.2 and Proposition 4.1 that $N \twoheadrightarrow G \rightarrow Q$ is a stem-extension and (4.17) shows that

$$U = \operatorname{im} \varepsilon_* \cong \operatorname{coker} \chi.$$

The stem-extensions yielding the given epimorphism β are in one-to-one correspondence with the elements of $\operatorname{Ext}(H_1 Q, N)$. The stem-extensions for which $U=0$, $\beta=1$ are called *stem-covers*. For a stem-cover

$$H_2 Q \twoheadrightarrow G \xrightarrow{\varepsilon} Q$$

we have an exact sequence

$$H_3 G \xrightarrow{\varepsilon_*} H_3 Q \xrightarrow{\delta} G_{ab} \otimes H_2 Q \xrightarrow{\chi} H_2 G \rightarrow 0; \quad (4.18)$$

and the stem-covers of Q are in one-to-one correspondence with elements of $\operatorname{Ext}(H_1 Q, H_2 Q)$.

5. Perfect Groups

Let

$$N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q \quad (5.1)$$

be a central extension classified by $\xi \in H^2(Q; N)$ and let $\varrho: N \rightarrow N_1$ be a homomorphism of commutative groups. We then recall that if $\xi_1 = \varrho_*(\xi) \in H^2(Q; N_1)$ and

$$N_1 \twoheadrightarrow G_1 \rightarrow Q$$

is the central extension classified by ξ_1 , there is a map of extensions

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ e \downarrow & & \downarrow \tau & & \parallel \\ N_1 & \xrightarrow{\mu_1} & G_1 & \xrightarrow{\varepsilon_1} & Q \end{array} . \quad (5.2)$$

We study this situation when Q is perfect, that is, $Q_{ab}=0$. In that case (see the Remark following Theorem 2.2) the central extension (5.1) is characterized by $\beta = \Phi(\xi): H_2 Q \rightarrow N$, which appears in the exact sequence (1.3)

$$H_2 G \rightarrow H_2 Q \xrightarrow{\beta} \mathcal{N} \rightarrow G_{ab} \rightarrow 0;$$

and, if $\xi_1 = \varrho_*(\xi)$,

$$\Phi(\xi_1) = \varrho\beta. \quad (5.3)$$

Moreover,

PROPOSITION 5.1. *Let*

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ \varrho \downarrow & & \tau \downarrow & & \downarrow \psi \\ N_1 & \xrightarrow[\mu_1]{} & G_1 & \xrightarrow[\varepsilon_1]{} & Q_1 \end{array}$$

be a map of extensions in which Q is perfect and N_1 is central. Then τ is uniquely determined by ϱ and ψ .

Proof. Let $\tau, \tau': G \rightarrow G_1$ be two homomorphisms each yielding commutativity in relation to ϱ and ψ . Consider the function $f: G \rightarrow G_1$ given by $f(x) = \tau(x) \tau'(x)^{-1}$, $x \in G$. Since $\varepsilon_1 \tau = \varepsilon_1 \tau'$, f maps G into N_1 . Since N_1 is central, it is clear that $f: G \rightarrow N_1$ is a homomorphism. Since $\tau\mu = \tau'\mu$, f is trivial on N , and thus induces a homomorphism $g: Q \rightarrow N_1$. Since Q is perfect and N_1 is commutative, $g = 0$. Thus $f = 0$, so that $\tau = \tau'$.

COROLLARY 5.2. *In the diagram (5.2), with Q perfect, τ is uniquely determined by ϱ .*

We now, temporarily, restrict attention to *stem*-extensions (5.1) with Q perfect. We note that, Q being perfect, (5.1) is a stem extension if and only if G is also perfect. Theorem 3.2 then yields the short exact sequence, for stem extensions (5.1),

$$0 \rightarrow H_2G \xrightarrow{\alpha} H_2Q \xrightarrow{\beta} N \rightarrow 0. \quad (5.4)$$

Thus, with every stem extension (5.1) of the perfect group Q , we may associate a subgroup $U = H_2G$ of H_2Q , such that the stem extension is characterized by the projection $H_2Q \rightarrow H_2Q/U = N$. Conversely, given $U \subseteq H_2Q$, set $N = H_2Q/U$, $\beta: H_2Q \rightarrow N$ the projection, and let (5.1) be the central extension characterized by β . Then, plainly, (5.1) is a stem extension and

$$H_2G = U.$$

Thus we have proved

THEOREM 5.3. *There is a one-one correspondence between stem extensions of the perfect group Q and subgroups of H_2Q , given by associating with (5.1) the group H_2G .*

Now let us take the stem extension

$$H_2Q \rightarrow Q_0 \rightarrow Q \quad (5.5)$$

of the perfect group Q with $H_2Q_0 = 0$. It is *universal* in the following sense. Given any central extension (5.1), characterized by $\beta: H_2Q \rightarrow N$, there exists, according to Corollary 5.2, a unique homomorphism $\tau: Q_0 \rightarrow G$ such that the diagram

$$\begin{array}{ccccc} H_2Q & \rightarrow & Q_0 & \rightarrow & Q \\ \downarrow \beta & & \downarrow \tau & & \parallel \\ N & \rightarrow & G & \rightarrow & Q \end{array}$$

commutes (note that (5.5) is characterized by the identity map of H_2Q). Notice that, if (5.1) is also a stem extension, then, in (5.6), β is surjective, so that τ is surjective. Moreover, $\ker \tau = \ker \beta = H_2G$, which is central in Q_0 . Thus if we describe G as a *cover* of Q if there exists a stem extension (5.1), then Theorem 5.3 establishes a one-one correspondence between covers of Q and subgroups of H_2Q [6], and our subsequent argument shows that Q_0 is the *universal cover* of Q in that it (uniquely) covers any cover of Q .

PROPOSITION 5.4. *The central extension (5.1) is the universal cover of the perfect group Q if and only if $H_1G=0$, $H_2G=0$.*

Proof. This is immediate, since, if $H_1G=0$, $H_2G=0$, then β and τ are isomorphisms in (5.6).

We next describe Q_0 and τ in (5.6) by means of the presentation (2.5). We have the evident

PROPOSITION 5.5. *Let (5.1) be a stem extension of the perfect group Q . Then*

$$[F, S] = [F, R].$$

Proof. We showed in Section 3 that for any central extension, the kernel of $\alpha: H_2G \rightarrow H_2Q$ is $[F, S]/[F, R]$. But for a stem extension of the perfect group Q , $\ker \alpha = 0$ (5.4); indeed, it is plain that $[F, S] = [F, R]$ for a ws-extension of the perfect group Q .

Now if (5.1) is the universal cover, then $H_1G=0$, $H_2G=0$, $N=H_2Q$. Thus

$$R \cap [F, F] = [F, R] = [F, S],$$

so that

$$G = F/R = [F, F]/(R \cap [F, F]) = [F, F]/[F, S].$$

Conversely, let $S \twoheadrightarrow F \twoheadrightarrow Q$ be a free presentation of the perfect group Q ; then $Q = [F, F]/(S \cap [F, F])$ and if we set $G = [F, F]/[F, S]$ we obtain the extension

$$H_2Q \twoheadrightarrow G \twoheadrightarrow Q. \quad (5.7)$$

This extension is plainly central and is characterized by the identity on H_2Q . It follows that (5.7) is a stem extension and $H_2G=0$.

If $N \twoheadrightarrow G \twoheadrightarrow Q$ is an arbitrary central extension of the perfect group Q , characterized by $\beta: H_2Q \rightarrow N$, we induce a homomorphism $\tau: [F, F]/[F, S] \rightarrow F/R$ from the inclusions $[F, F] \subseteq F$, $[F, S] \subseteq R$. Plainly, τ restricts to β on $H_2Q = (S \cap [F, F])/[F, S]$ and induces a commutative diagram (5.6). By uniqueness, it is therefore the homomorphism there described. We sum up:

THEOREM 5.6. *Given the presentation $S \twoheadrightarrow F \twoheadrightarrow Q$ of the perfect group Q , the*

universal central (stem) extension $H_2Q \xrightarrow{\mu_0} Q_0 \xrightarrow{\varepsilon_0} Q$ can be given by $Q_0 = [F, F]/[F, S]$ with ε_0 induced by the inclusions $[F, F] \subseteq F$, $[F, S] \subseteq S$. For any central extension $N \rightarrowtail G \twoheadrightarrow Q$ with presentation (2.5), the universal homomorphism $\tau: Q_0 \rightarrow G = F/R$ is induced by the inclusions $[F, F] \subseteq F$, $[F, S] \subseteq R$.

Remarks. 1) Theorem 5.6 may readily be used to prove, without any spectral sequence techniques, the result cited in [6] that, given a short exact sequence of perfect groups $K \rightarrowtail G \twoheadrightarrow Q$, the lifted sequence $K_0 \rightarrow G_0 \twoheadrightarrow Q_0$ is exact.

2) It is also clear how to present the cover of Q corresponding to any subgroup U of H_2Q . With $H_2Q = (S \cap [F, F])/[F, S]$ we have $U = V/[F, S]$, $[F, S] \subseteq V \subseteq S \cap [F, F]$, and $V \rightarrowtail [F, F] \twoheadrightarrow G$ is then a presentation of the required cover G .

We now prove a theorem motivated by the analogy with covering space theory.

THEOREM 5.7 *Let $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ be a central extension, let X be a perfect group and let $\psi: X \rightarrow Q$ be a homomorphism. Then ψ lifts, uniquely, to $\phi: X \rightarrow G$ with $\varepsilon\phi = \psi$, if and only if*

$$\psi_* H_2 X \subseteq \varepsilon_* H_2 G. \quad (5.8)$$

Proof. Plainly, if ψ lifts, (5.8) holds. Also Proposition 5.1 affirms that if ϕ exists, it is unique. Thus it suffices to prove that (5.8) implies the existence of ϕ .

We use the presentation of X ,

$$U \rightarrowtail V \twoheadrightarrow X.$$

Let ψ be lifted to $\eta: V \rightarrow F$ with $\eta(U) \subseteq S$. Then

$$\begin{aligned} \varepsilon_* H_2 G &= (R \cap [F, F]) [F, S] / [F, S] \\ &= R \cap [F, F] / [F, S], \end{aligned}$$

since N is central in G . Thus the hypothesis (5.8) may be translated into the condition

$$\eta(U \cap [V, V]) \subseteq R \cap [F, F]. \quad (5.9)$$

Since X is perfect we may present it by

$$U \cap [V, V] \rightarrowtail [V, V] \twoheadrightarrow X.$$

We have the diagram

$$\begin{array}{ccccc} U \cap [V, V] & \xrightarrow{\eta} & R \cap [F, F] & \rightarrowtail & S \cap [F, F] \\ \downarrow & & \downarrow & & \downarrow \\ [V, V] & \xrightarrow{\eta} & [F, F] & \longrightarrow & [F, F] \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & G & \xrightarrow{\varepsilon} & Q \end{array}, \quad (5.10)$$

determining a map $\phi: X \rightarrow G$ such that $\varepsilon\phi = \psi$.

We may apply this theorem in the following situation. Consider the diagram

$$\begin{array}{ccc} N & \xrightarrow{\mu} & G \twoheadrightarrow Q \\ & & \downarrow \psi \\ N_1 & \xrightarrow{\mu_1} & G_1 \twoheadrightarrow Q_1 \end{array}, \quad (5.11)$$

where the top row is a stem extension of the perfect group Q , and the bottom row is a central extension. Then we know that we may complete (5.11) to the commutative diagram (and uniquely)

$$\begin{array}{ccc} N & \xrightarrow{\mu} & G \twoheadrightarrow Q \\ \downarrow \varrho & & \downarrow \tau \quad \downarrow \psi \\ N_1 & \xrightarrow{\mu_1} & G_1 \twoheadrightarrow Q_1 \end{array} \quad (5.12)$$

if and only if $\psi_* \varepsilon_* H_2 G \subseteq \varepsilon_{1*} H_2 G_1$. However this latter is precisely the condition that we may find $\varrho': N \rightarrow N_1$ such that the diagram

$$\begin{array}{ccc} H_2 Q & \xrightarrow{\beta} & N \\ \downarrow \psi_* & & \downarrow \varrho' \\ H_2 Q_1 & \xrightarrow{\beta_1} & N_1 \end{array} \quad (5.13)$$

commutes, as follows immediately from (1.3). Moreover, since (5.12) induces

$$\begin{array}{ccc} H_2 Q & \xrightarrow{\beta} & N \\ \downarrow \psi_* & & \downarrow \varrho \\ H_2 Q_1 & \xrightarrow{\beta_1} & N_1 \end{array} \quad (5.14)$$

and β is surjective, it follows that $\varrho = \varrho'$. We thus have the corollary of Theorem 5.7.

COROLLARY 5.8. *Let $N \twoheadrightarrow G \twoheadrightarrow Q$ be a stem extension of the perfect group Q , let $N_1 \twoheadrightarrow G_1 \twoheadrightarrow Q_1$ be a central extension, and let $\varrho: N \rightarrow N_1$, $\psi: Q \rightarrow Q_1$ be homomorphisms. Then there exists $\tau: G \rightarrow G_1$ rendering the diagram (5.12) commutative if and only if (5.14) commutes. If τ exists, it is unique.*

Remark. We note, in particular, that if (5.14) commutes, then there exists a canonical homomorphism $\theta: H_2 G \rightarrow H_2 G_1$, namely τ_* , such that

$$\begin{array}{ccc} H_2 G & \xrightarrow{\varepsilon} & H_2 Q \\ \downarrow \theta & & \downarrow \psi_* \\ H_2 G_1 & \xrightarrow{\varepsilon_{1*}} & H_2 Q_1 \end{array}$$

commutes.

We close this section by observing that Theorem 4.3 leads to an immediate proof of the following result of Kervaire [6] (see also [2]):

THEOREM 5.9. *Let $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$ be a stem extension of the perfect group Q . Then $\varepsilon_*: H_3 G \rightarrow H_3 Q$ is an epimorphism.*

6. Appendix: Remarks on Algebraic K -Theory

Certain exact sequences of algebraic K -theory can easily be obtained within the framework of this paper. We first recall some known facts and definitions.

We consider $G = GL(A)$, A being a ring with unity, and the subgroup $E = E(A)$ generated by elementary matrices $1 + \lambda E_{ij}$, $i \neq j$, $\lambda \in A$. Given an ideal $\mathfrak{a} \subset A$, we write $\bar{G} = G(A/\mathfrak{a})$, $\bar{E} = E(A/\mathfrak{a})$ and denote by $\pi: G \rightarrow \bar{G}$, $\pi': E \rightarrow \bar{E}$ the canonical maps, by N the kernel of π (i.e., the *congruence subgroup* $G(A, \mathfrak{a})$). The group $E(A, \mathfrak{a})$, the normal hull in E of all elementary matrices $1 + \alpha E_{ij}$, $\alpha \in \mathfrak{a}$, is contained in $^1) \ker \pi' = E \cap N$. The following facts are easily proved: $E = [E, E]$, $\bar{E} = [\bar{E}, \bar{E}]$, $E(A, \mathfrak{a}) = [E, E(A, \mathfrak{a})]$, and $\pi': E \rightarrow \bar{E}$ is an epimorphism.

The generalized *Whitehead Lemma* [10, Theorem 15.1]

$$[G, N] \subseteq E(A, \mathfrak{a})$$

yields the further relations

$$E = [G, G] \tag{6.1}$$

$$E(A, \mathfrak{a}) = [G, N] \tag{6.2}$$

$$[E, E \cap N] = [G, N]. \tag{6.3}$$

The proof of (6.3) is as follows: the inclusion $[E, E \cap N] \subseteq [G, N]$ is obvious, and, on the other hand, we have

$$[E, E \cap N] \supseteq [E, E(A, \mathfrak{a})] = E(A, \mathfrak{a}) = [G, N].$$

We now reflect this situation by the following more general set-up. Let G and \bar{G} be groups with perfect commutator subgroups, let $E = [G, G]$ and $\bar{E} = [\bar{G}, \bar{G}]$; let π be a homomorphism $G \rightarrow \bar{G}$ which maps E onto \bar{E} , and suppose that $[G, N] = [E, E \cap N]$, where $N = \ker \pi$.

We write $Q = G/N$ and map the extension $E \cap N \rightarrow E \rightarrow \bar{E}$ into $N \rightarrow G \xrightarrow{\varepsilon} Q$ by inclusions. The induced map of the exact sequences (1.3) is given by the commutative diagram

$$\begin{array}{ccccccc} H_2 E & \xrightarrow{\alpha'} & H_2 \bar{E} & \xrightarrow{\beta'} & E \cap N / [E, E \cap N] & \longrightarrow & 0 \longrightarrow 0 \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow e & & \downarrow & \downarrow \\ H_2 G & \xrightarrow{\alpha} & H_2 Q & \xrightarrow{\beta} & N / [G, N] & \xrightarrow{\delta} & H_1 G \xrightarrow{\varepsilon_*} H_1 Q \rightarrow 0. \end{array} \tag{6.4}$$

¹⁾ On p. 211 of [10], Swan inadvertently defines $E(A, \mathfrak{a})$ to be $\ker \pi'$. The statement on p. 212 is the correct one – and the only one which Swan uses.

We note that $\text{Im } \varrho = (E \cap N)/[G, N] = \ker \delta$; this is also $\text{im } \varrho\beta'$. Since $[E, E \cap N] = [G, N]$, ϱ is a monomorphism, so $\ker \varrho\beta' = \ker \beta' = \text{im } \alpha'$. We further remark that π induces a monomorphism $Q \rightarrow \tilde{G}$ which maps $[Q, Q] = \varepsilon(E)$ isomorphically onto \tilde{E} . It follows that the kernel of $\pi_*: H_1 G = G/E \rightarrow H_1 \tilde{G} = \tilde{G}/\tilde{E}$ is equal to the kernel of $\varepsilon_*: G/E \rightarrow Q/[Q, Q]$. Writing $\sigma = \varrho\beta'$, we obtain from (6.4) the exact sequence

$$H_2 E \xrightarrow{\alpha'} H_2 \tilde{E} \xrightarrow{\sigma} N/[G, N] \xrightarrow{\delta} H_1 G \xrightarrow{\pi_*} H_1 \tilde{G}. \quad (6.5)$$

In the case of K -theory, the groups can all be identified with familiar K -groups: $H_1 G = G/E = K_1(\Lambda)$; $H_1 \tilde{G} = K_1(\Lambda/\alpha)$; $N/[G, N] = GL(\Lambda, \alpha)/E(\Lambda, \alpha) = K_1(\Lambda, \alpha)$; and finally one defines $K_2(\Lambda)$ by $H_2 E(\Lambda)$. Then (6.5) becomes the exact sequence

$$K_2(\Lambda) \rightarrow K_2(\Lambda/\alpha) \rightarrow K_1(\Lambda, \alpha) \rightarrow K_1(\Lambda) \rightarrow K_1(\Lambda/\alpha). \quad (6.6)$$

The above definition of $K_2(\Lambda)$ coincides with that of Milnor, where $K_2(\Lambda)$ is the kernel of $St(\Lambda) \rightarrow E(\Lambda)$. The group $St(\Lambda)$, the Steinberg group, is defined by an explicit free presentation $S \rightarrow F \rightarrow St(\Lambda)$; it is a stem-extension of $E(\Lambda)$, and in fact the universal one – which implies that the kernel is $H_2 E(\Lambda)$. To prove that $St(\Lambda)$ is universal, it suffices to show that it is its own universal stem-extension; i.e., that the homomorphism $[F, F]/[F, S] \rightarrow F/S$ induced by the inclusions $[F, F] \subseteq F$, $[F, S] \subseteq S$, is an isomorphism. This can be done by following Kervaire's procedure [6], using various commutator relations.

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