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# On the Homology Theory of Central Group Extensions: I-The Commutator Map and Stem Extensions

by BENO ECKMANN, PETER J. HILTON and URS STAMMBACH

*In Memoriam Heinz Hopf (1894–1971)*

## 1. Introduction

For any group extension

$$N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q \quad (1.1)$$

and any  $Q$ -module  $B$ , there is a five-term exact homology sequence

$$H_2(G; B) \xrightarrow{\alpha_B} H_2(Q; B) \xrightarrow{\beta_B} N_{ab} \otimes_Q B \xrightarrow{\sigma_B} H_1(G; B) \xrightarrow{\tau_B} H_1(Q; B) \rightarrow 0, \quad (1.2)$$

due to Stallings and Stambach [8, 9]. If  $B = \mathbb{Z}$ , regarded as trivial  $Q$ -module, (1.2) reduces to

$$H_2G \xrightarrow{\alpha} H_2Q \xrightarrow{\beta} N/[G, N] \xrightarrow{\sigma} H_1G \xrightarrow{\tau} H_1Q \rightarrow 0. \quad (1.3)$$

For a simple proof of (1.2), including the statement of naturality, see Eckmann-Stambach [3]; of course,  $\alpha_B$  and  $\tau_B$  in (1.2) are induced by  $\varepsilon$ ;  $\sigma_B$  is, in a sense explained later, induced by  $\mu$  (the sense is perfectly clear in the case (1.3)); and  $\beta_B$  will be elucidated in the next section.

In the special case where (1.1) is a *central* extension, that is,  $N$  is central in  $G$ , Ganea [5] has added a further term on the left of the exact sequence (1.3), thus,

$$G_{ab} \otimes N \xrightarrow{\chi_0} H_2G \xrightarrow{\alpha} H_2Q \rightarrow \cdots \rightarrow 0, \quad (1.4)$$

using methods of algebraic topology.

In [2], Eckmann-Hilton extended the sequence (1.4) by four further non-trivial homology terms, first replacing  $G_{ab} \otimes N$  by a suitable quotient. Their method was based on a spectral sequence for the homology of a suitable fibre space. The Ganea sequence (1.4) and its extension in [2] are important for applications, beyond those of (1.3), in group theory, homology, algebraic  $K$ -theory, etc.

In the present paper we present an elementary approach to the Ganea extension (1.4) and to those parts of the extended sequence in [2] which are relevant to applications to *stem-extensions* of groups (see Section 4), and, in particular, to the study of *perfect* groups. The argument is based on a fixed, but arbitrary, free presentation of (1.1) (see Section 2), and the associated *Gruenberg resolutions* [4; Chapter VI] of  $\mathbb{Z}$

over  $N$ ,  $G$  and  $Q$ . The maps of (1.3) are exhibited and exactness is proved, by using these explicit resolutions.

In Section 2 we study (1.3) – and, in less detail, (1.2) – from the viewpoint of the given presentation of (1.1). In particular, we recall the relation with the *Hopf formula* for  $H_2G$  and  $H_2Q$  and obtain an explicit form for  $\ker \alpha$ . Moreover, the connection between  $\beta$  and the characteristic class of the central extension

$$N/[G, N] \twoheadrightarrow G/[G, N] \rightarrow Q, \quad (1.5)$$

associated with (1.1), is obtained.

In Section 3 the Ganea extension (1.4) is established by means of an explicit commutator map  $\chi$ ; and the equivalence of this map  $\chi$  with Ganea's map  $\chi_0$  is demonstrated. In Section 4 we obtain an extended exact sequence

$$H_3G \rightarrow H_3Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2G \xrightarrow{\alpha} H_2Q \rightarrow \dots \quad (1.6)$$

for stem-extensions, that is, central extensions (1.1) with  $N \subseteq [G, G]$ . Actually, we will obtain (1.6) for an even broader class of central extensions, which we tentatively call *weak stem-extensions*. Whereas a stem-extension (1.1) is characterized by the vanishing of the abelianization of  $\mu$ , i.e.  $\mu_*: N \rightarrow G_{ab}$  is the zeromap, for weak stem extensions we merely demand that  $\mu_*: N \otimes N \rightarrow G_{ab} \otimes N$  is the zeromap. We give examples to show that this generalization is significant, and we also show how (1.6) may be regarded as contained in the extended sequence of [2].

Section 5 deals with perfect groups; we assume  $Q$  perfect and obtain, beyond the results of Schur [7] and Kervaire [6] on stem-extensions of  $Q$ , a description of the universal stem-extension of  $Q$  in terms of the given presentation of  $Q$ . Moreover, we use the given presentations of  $N$ ,  $G$  and  $Q$  to carry further the analogy remarked by Kervaire between the theory of perfect groups and covering space theory for connected topological spaces.

Section 6 is an appendix concerning algebraic  $K$ -theory, in which we show how the exact sequence for Milnor's  $K_2$  may be obtained from (1.3).

## 2. Extensions, Free Presentations, and Resolutions

Given the group extension (1.1),

$$N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q,$$

let

$$R \xrightarrow{\mu'} F \xrightarrow{\varepsilon'} G \quad (2.1)$$

be a (free) presentation of  $G$ . There are then presentations of  $Q$ ,  $N$ ,

$$S \xrightarrow{\bar{\mu}} F \xrightarrow{\bar{\varepsilon}} Q, \quad (2.2)$$

$$R \xrightarrow{\mu''} S \xrightarrow{\varepsilon''} N, \quad (2.3)$$

where

$$\bar{\mu}\mu'' = \mu', \quad \varepsilon\varepsilon' = \bar{\varepsilon}, \quad \varepsilon'\bar{\mu} = \mu\varepsilon''. \quad (2.4)$$

We sum this up in the single diagram

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow \mu & & \\ R & \xrightarrow{\mu'} & F & \xrightarrow{\varepsilon'} & G \\ \mu'' \downarrow & & \parallel & & \downarrow \varepsilon \\ S & \xrightarrow{\bar{\mu}} & F & \xrightarrow{\bar{\varepsilon}} & Q, \\ \varepsilon'' \downarrow & & & & \\ N & & & & \end{array} \quad (2.5)$$

which we call a *presentation of the extension* (1.1).

Applying (1.3) to the two centre rows of (2.5) we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & H_2G & \xrightarrow{\beta'} & R/[F, R] & \xrightarrow{\sigma'} & F/[F, F] \rightarrow \dots \\ & & \downarrow \alpha & & \downarrow \gamma & & \parallel \\ 0 & \rightarrow & H_2Q & \xrightarrow{\bar{\beta}} & S/[F, S] & \xrightarrow{\bar{\sigma}} & F/[F, F] \rightarrow \dots, \end{array} \quad (2.6)$$

where  $\alpha$  is as in (1.3) and  $\gamma$  is induced by the inclusion  $\mu'': R \rightarrow S$ . Thus

$$\ker \gamma = (R \cap [F, S])/[F, R], \quad \operatorname{coker} \gamma = S/R[F, S]. \quad (2.7)$$

We note that  $\varepsilon''$  induces an isomorphism of  $\operatorname{coker} \gamma$  onto  $N/[G, N]$ . We write  $\eta: S/[F, S] \rightarrow N/[G, N]$  for the map induced by  $\varepsilon''$ , and thus embed (2.6) in the larger commutative diagram, with exact rows and columns,

$$\begin{array}{ccccccc} 0 & \rightarrow & \ker \alpha & \rightarrow & \ker \gamma & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H_2G & \xrightarrow{\beta'} & R/[F, R] & \xrightarrow{\sigma'} & F/[F, F] \rightarrow \dots \\ & & \downarrow \alpha & & \downarrow \gamma & & \downarrow = \\ 0 & \rightarrow & H_2Q & \xrightarrow{\bar{\beta}} & S/[F, S] & \xrightarrow{\bar{\sigma}} & F/[F, F] \rightarrow \dots \\ & & \downarrow \beta & & \downarrow \eta & & \downarrow \\ 0 & \rightarrow & N/[G, N] & \xrightarrow{=} & N/[G, N] & \rightarrow & 0 \end{array} \quad (2.8)$$

We use  $\beta'$  to induce the *Hopf formula*

$$H_2G \cong (R \cap [F, F])/[F, R]; \quad (2.9)$$

likewise  $\bar{\beta}$  induces

$$H_2Q \cong (S \cap [F, F])/[F, S]. \quad (2.10)$$



We make the identifications (2.9), (2.10), so that  $\ker \alpha$  is identified with  $\ker \gamma$ ,

$$\ker \alpha = (R \cap [F, S])/[F, R],$$

and the map  $\beta$ ,

$$\beta: (S \cap [F, F])/[F, S] \rightarrow N/[G, N],$$

is just the restriction of  $\eta$  to  $H_2Q$ , and thus is induced by  $\varepsilon''$ . Note that the relation  $\beta = \eta\bar{\beta}$  in (2.8) simply results from the naturality of (1.3), applied to

$$\begin{array}{ccccc} S & \xrightarrow{\bar{\mu}} & F & \xrightarrow{\bar{\varepsilon}} & Q \\ \downarrow \varepsilon'' & & \downarrow \varepsilon' & & \parallel \\ N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \end{array} .$$

We now wish to relate  $\beta$  to the *associated central extension* (1.5). It follows from naturality that the homomorphism  $H_2Q \rightarrow N/[G, N]$  in the sequence (1.3) corresponding to (1.5) coincides with  $\beta$ , so that, in this part of the argument, we lose no generality in supposing that  $N$  is *itself central* in (1.1). Then

$$\beta: H_2Q \rightarrow N,$$

and we propose to relate  $\beta$  to the element of  $H^2(Q; N)$  characterized by the *central extension*

$$N \rightarrowtail G \twoheadrightarrow Q.$$

We now have  $[G, N] = 1$ , so that

$$[F, S] \subseteq R, \quad \ker \alpha = [F, S]/[F, R], \quad (2.11)$$

and  $\beta: (S \cap [F, F])/[F, S] \rightarrow S/R$  is induced by the inclusions  $S \cap [F, F] \subseteq S$ ,  $[F, S] \subseteq R$ .

We begin by giving an explicit description of  $H^2(Q; N)$  in terms of (2.5). We use the *Gruenberg resolution* [4; VI. 13] of  $\mathbf{Z}$  over  $Q$  based on (2.2), namely,

$$\cdots \xrightarrow{\partial_3} S_{ab} \otimes \mathbf{Z}Q \xrightarrow{\partial_2} JF \otimes_F \mathbf{Z}Q \xrightarrow{\partial_1} \mathbf{Z}Q \xrightarrow{\partial_0} \mathbf{Z} \rightarrow 0. \quad (2.12)$$

Here  $\partial_0$  is the augmentation;  $JF$  is the augmentation ideal of  $F$ ;  $\partial_1$  is given by

$$\partial_1((x - e) \otimes_F e) = \bar{\varepsilon}(x) - e, \quad x \in F,$$

where  $e$  stands for the unity in any group; the kernel of  $\partial_1$  is known to be isomorphic to  $S_{ab}$ , with  $Q$  operating by inner automorphisms of  $F$ , under the monomorphism

$$\sigma_1: S_{ab} \rightarrowtail JF \otimes_F \mathbf{Z}Q$$

given by

$$\sigma_1(s[S, S]) = (s - e) \otimes_F e;$$

and  $\partial_2$  is the composite of  $\sigma_1$  with the  $Q$ -module map  $s[S, S] \otimes e \mapsto s[S, S]$ . (See [4; VI. 6], where the argument is given in detail but for the Gruenberg resolution of  $\mathbf{Z}$  as *left*  $Q$ -module.)

**THEOREM 2.1.** *The Gruenberg resolution (2.12) induces an isomorphism*

$$H^2(Q; N) \cong \text{Hom}(S/[F, S], N) / \bar{\sigma}^* \text{Hom}(F/[F, F], N)$$

for any trivial  $Q$ -module  $N$ .

*Proof.* For any resolution  $\cdots \rightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$  of  $\mathbf{Z}$  over  $Q$ ,  $H^2(Q; N) = \text{Hom}_Q(\ker \partial_1, N) / i^* \text{Hom}_Q(C_1, N)$ , where  $i: \ker \partial_1 \subseteq C_1$ . Thus, using the Gruenberg resolution,

$$H^2(Q; N) = \text{Hom}_Q(S_{ab}, N) / \sigma_1^* \text{Hom}_Q(JF \otimes_F \mathbf{Z}Q, N).$$

Now  $\text{Hom}_Q(S_{ab}, N) = \text{Hom}((S_{ab})_Q, N) = \text{Hom}(S/[F, S], N)$ , and  $\text{Hom}_Q(JF \otimes_F \mathbf{Z}Q, N) = \text{Hom}(JF \otimes_F \mathbf{Z}, N)$ . Moreover there is a natural isomorphism  $\psi: JF \otimes_F \mathbf{Z} \cong F_{ab}$ , given by  $\psi((x - e) \otimes_F 1) = x[F, F]$ ,  $x \in F$ , and plainly  $\psi\sigma_1$  induces  $\bar{\sigma}: S/[F, S] \rightarrow F/[F, F]$ . This proves the theorem.

Now given the central extension  $N \twoheadrightarrow G \twoheadrightarrow Q$ , the map  $\eta: S/[F, S] \rightarrow N$  of (2.8) (recall that  $[G, N] = 1$ ) then determines, in the light of Theorem 2.1, an element  $\xi \in H^2(Q; N)$  and this is the characteristic cohomology class of the given central extension (see [4; VI. 10]). We now readily prove

**THEOREM 2.2.** *If  $N \twoheadrightarrow G \twoheadrightarrow Q$  is a central extension with characteristic class  $\xi \in H^2(Q; N)$ , then the homomorphism  $\beta: H_2Q \rightarrow N$  of (1.3) is the image of  $\xi$  under the epimorphism*

$$\Phi: H^2(Q; N) \twoheadrightarrow \text{Hom}(H_2Q, N)$$

of the universal coefficient theorem.

*Proof.* For any  $\zeta \in H^2(Q; N)$ ,  $\Phi(\zeta)$  is obtained by picking a representative  $\theta: S/[F, S] \rightarrow N$  and restricting  $\theta$  to  $(S \cap [F, F])/[F, S]$ . Since  $\eta$  represents  $\xi$  and  $\beta$  is the restriction of  $\eta$  to  $(S \cap [F, F])/[F, S]$ , the theorem follows.

*Remark.* If  $\text{Ext}(Q_{ab}, N) = 0$ , then  $\Phi$  is an isomorphism, so that  $\beta$  characterizes the central extension. An important special case is that in which  $Q_{ab} = 0$  (i.e.,  $Q$  is *perfect*). The fact that  $\beta$  characterizes the extension when  $Q$  is perfect may be seen directly by observing that then  $\bar{\sigma}: S/[F, S] \rightarrow F/[F, F]$  is surjective, so that two homomorphisms  $S/[F, S] \rightarrow N$  determine the same element of  $H^2(Q; N)$  (see Theorem 2.1) if and only if they agree on  $H_2Q$ .

We close this section by recalling from VI.8 of [4] how  $\beta_B$  and  $\sigma_B$  are defined in (1.2). In terms of the resolution (2.12),

$$H_2(Q; B) = \ker(S_{ab} \otimes_Q B \rightarrow JF \otimes_F B) \quad (2.13)$$

and then  $\beta_B: H_2(Q; B) \rightarrow N_{ab} \otimes_Q B$  is given by restricting to  $H_2(Q; B)$  the homomorphism  $S_{ab} \otimes_Q B \rightarrow N_{ab} \otimes_Q B$  induced by  $\varepsilon''$ . As to  $\sigma_B$ , we exploit the short exact sequence (Theorem VI.6.3 of [4])

$$N_{ab} \twoheadrightarrow JG \otimes_G \mathbb{Z}Q \twoheadrightarrow JG$$

of  $Q$ -modules to obtain

$$N_{ab} \otimes_Q B \rightarrow JG \otimes_G B. \quad (2.14)$$

Moreover the image of this homomorphism obviously lies in the kernel of  $JG \otimes_G B \rightarrow B$ , that is, in  $H_1(G; B)$ , and thus (2.14) determines  $\sigma_B$ .

### 3. The Commutator Map and the Ganea Term

Given a *central* group extension  $N \twoheadrightarrow G \twoheadrightarrow Q$  we now define a homomorphism  $\chi: G_{ab} \otimes N \rightarrow H_2 G$  which will yield exactness in

$$G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\alpha} H_2 Q. \quad (3.1)$$

We return to the presentation (2.5) and recall (2.11) that then

$$\ker \alpha = [F, S]/[F, R].$$

Thus our objective is to define a natural surjection  $G_{ab} \otimes N \rightarrow [F, S]/[F, R]$ . We define a map

$$c: F \times S \rightarrow [F, S]/[F, R]$$

by

$$c(x, s) = [x, s] [F, R], \quad x \in F, \quad s \in S \quad (3.2)$$

where  $[x, s]$  is the commutator  $xsx^{-1}s^{-1}$ . We call  $c$  the *commutator map* (relative to the presentation (2.5)).

**PROPOSITION 3.1.** *The commutator map  $c$  is a bihomomorphism, that is,*

$$\begin{aligned} c(xx', s) &= c(x, s) c(x', s), \\ c(x, ss') &= c(x, s) c(x, s'). \end{aligned}$$

*Proof.* Since  $[ab, c] = a[b, c]a^{-1}[a, c]$ , and  $[a, bc] = [a, b]b[a, c]b^{-1}$ , we have

$$\begin{aligned} c(xx', s) &= xc(x', s)x^{-1}c(x, s), \\ c(x, ss') &= c(x, s)sc(x, s')s^{-1}. \end{aligned}$$

But  $[F, S] \subseteq R$  and  $[F, S]/[F, R]$  is commutative. Thus

$$\begin{aligned} c(xx', s) &= c(x, s) c(x', s), \\ c(x, ss') &= c(x, s) c(x, s'). \end{aligned}$$

Again, since  $[F, S]/[F, R]$  is commutative, Proposition 3.1 shows that  $c([F, F] \times S) = e$ ; plainly  $c(R \times S) = e$ ,  $c(F \times R) = e$ . Thus  $c$  induces a bihomomorphism of abelian groups

$$F/[F, F] \times S/R \rightarrow [F, S]/[F, R],$$

and hence a homomorphism

$$c_1: G_{ab} \otimes N \rightarrow [F, S]/[F, R],$$

which is plainly surjective. We define  $\chi$  to be the composite of  $c_1$  and the embedding of  $[F, S]/[F, R] = \ker \alpha$  in  $H_2 G$ . We have thus proved, for a central extension  $N \rightarrowtail G \twoheadrightarrow Q$ ,

**THEOREM 3.2.** *The commutator map  $c$  defines a homomorphism  $\chi: G_{ab} \otimes N \rightarrow H_2 G$  such that the sequence*

$$G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\alpha} H_2 Q \xrightarrow{\beta} N \rightarrow G_{ab} \rightarrow Q_{ab} \rightarrow 0$$

is exact.

We show later that  $\chi$  is equivalent to Ganea's  $\chi_0$ . This would imply the naturality of  $\chi$ , but we give now an independent proof so that our arguments may be entirely self-contained. Suppose given a map of central extensions,

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ N_0 & \xrightarrow{\mu_0} & G_0 & \xrightarrow{\varepsilon_0} & Q_0, \end{array} \quad (3.3)$$

and a presentation of  $N_0 \rightarrowtail G_0 \twoheadrightarrow Q_0$ , designated by (2.5) with all elements assigned the suffix 0. We lift  $f_2$  to  $\tilde{f}: F \rightarrow F_0$  such that  $\varepsilon'_0 \tilde{f} = f_2 \varepsilon'$ . Then  $\tilde{f}(S) \subseteq S_0$ ,  $\tilde{f}(R) \subseteq R_0$ . Thus if  $c_0: F_0 \times S_0 \rightarrow [F_0, S_0]/[F_0, R_0]$  is the commutator map,

$$c_0(\tilde{f}x, \tilde{f}s) = [\tilde{f}x, \tilde{f}s] [F_0, R_0] = \tilde{f}[x, s] [F_0, R_0] = \tilde{f}_* c(x, s),$$

where  $\tilde{f}_*: [F, S]/[F, R] \rightarrow [F_0, S_0]/[F_0, R_0]$  is induced by  $\tilde{f}$ . It follows that the diagram

$$\begin{array}{ccc} G_{ab} \otimes N & \xrightarrow{\chi} & H_2 G \\ \downarrow f_* & & \downarrow f_* \\ G_{0ab} \otimes N_0 & \xrightarrow{\chi} & H_2 G_0 \end{array}$$

commutes, where the vertical arrows are induced by (3.3) and are independent of the choice of presentations. This shows that  $\chi$  is itself independent of the choice of presentation and is natural.

Ganea's map  $\chi_0: G_{ab} \otimes N \rightarrow H_2 G$  is effectively defined as follows. The multiplication in  $G$  induces a homomorphism  $\mu: G \times N \rightarrow G$ , inducing  $\mu_*: H_2(G \times N) \rightarrow H_2 G$ . According to the Künneth formula,  $H_1 G \otimes H_1 N (= G_{ab} \otimes N)$  is naturally embedded in  $H_2(G \times N)$ .

Then

$$\chi_0 = \mu_* \mid (G_{ab} \otimes N).$$

We prove

**THEOREM 3.3.**  $\chi_0 = -\chi$ .

*Proof.* We first study, in general, the embedding of  $G_{1ab} \otimes G_{2ab}$  in  $H_2(G_1 \times G_2)$ . Since we know this is natural, it suffices to choose suitable resolutions. Thus we choose the Gruenberg resolutions

$$\cdots \rightarrow R_{iab} \otimes \mathbf{Z}G_i \xrightarrow{\partial_2} JF_i \otimes_{F_i} \mathbf{Z}G_i \xrightarrow{\partial_1} \mathbf{Z}G_i \xrightarrow{\partial_0} \mathbf{Z} \rightarrow 0,$$

of  $\mathbf{Z}$  over  $G_i$ , corresponding to presentations  $R_i \twoheadrightarrow F_i \twoheadrightarrow G_i$ ,  $i=1, 2$ . Then a partial resolution of  $\mathbf{Z}$  over  $G_1 \times G_2$  is given by

$$\begin{aligned} \cdots \rightarrow (JF_1 \otimes_{F_1} \mathbf{Z}G_1) \otimes (JF_2 \otimes_{F_2} \mathbf{Z}G_2) &\xrightarrow{\partial \otimes 1 - 1 \otimes \partial} \{(JF_1 \otimes_{F_1} \mathbf{Z}G_1) \otimes \mathbf{Z}G_2\} \\ &\oplus \{\mathbf{Z}G_1 \otimes (JF_2 \otimes_{F_2} \mathbf{Z}G_2)\} \rightarrow \mathbf{Z}G_1 \otimes \mathbf{Z}G_2 \rightarrow \mathbf{Z} \rightarrow 0, \end{aligned} \quad (3.4)$$

where we have only written down that part of  $C_2$  which contributes to  $H_1 G_1 \otimes H_1 G_2$ . Tensoring with  $\mathbf{Z}$  over  $G_1 \times G_2$ , we get

$$(JF_1 \otimes_{F_1} \mathbf{Z}) \otimes (JF_2 \otimes_{F_2} \mathbf{Z}) \xrightarrow{0} JF_1 \otimes \mathbf{Z} \oplus \mathbf{Z} \otimes JF_2.$$

Thus we have proved – in view of the natural isomorphisms  $JF_i \otimes_{F_i} \mathbf{Z} \cong F_{iab}$ ,  $i=1, 2$  – writing  $Z_k$  for the  $k$ th cycle group,

**LEMMA 3.4.** *With respect to the resolution (3.4),  $Z_1 G_1 \otimes Z_2 G_2$  is embedded in  $Z_2(G_1 \times G_2)$  as  $F_{1ab} \otimes F_{2ab}$ , and hence  $G_{1ab} \otimes G_{2ab}$  is embedded in  $H_2(G_1 \times G_2)$  as  $F_1/[F_1, F_1] R_1 \otimes F_2/[F_2, F_2] R_2$ .*

We now revert to our special case. We must construct a chain-map  $\phi_0, \phi_1, \phi_2, \dots$

$$\begin{array}{ccc} \cdots \rightarrow (JF \otimes_F \mathbf{Z}G) \otimes (JS \otimes_S \mathbf{Z}N) & \xrightarrow{\partial_2} & \{(JF \otimes_F \mathbf{Z}G) \otimes \mathbf{Z}N\} \oplus \{\mathbf{Z}G \otimes (JS \otimes_S \mathbf{Z}N)\} \\ \downarrow \phi_2 & & \downarrow \phi_1 \\ \cdots \rightarrow R_{ab} \otimes \mathbf{Z}G & \xrightarrow{\partial_2} & JF \otimes_F \mathbf{Z}G \\ & & \downarrow \phi_0 \\ & & \begin{array}{ccc} \xrightarrow{\partial_1} & \mathbf{Z}G \otimes \mathbf{Z}N & \rightarrow \mathbf{Z} \rightarrow 0 \\ & \downarrow \phi_0 & \parallel \\ \xrightarrow{\partial_1} & \mathbf{Z}G & \rightarrow \mathbf{Z} \rightarrow 0 \end{array} \end{array}$$

compatible with  $\mu: G \times N \rightarrow G$ . Direct calculation shows that we may take

$$\begin{aligned}\phi_0(e \otimes e) &= e, \\ \phi_1((x - e) \otimes_F e \otimes e) &= (x - e) \otimes_F e, \\ \phi_1(e \otimes (s - e) \otimes_S e) &= (s - e) \otimes_F e, \\ \phi_2((x - e) \otimes_F e \otimes (s - e) \otimes_S e) &= [s, x] [R, R] \otimes xs, \quad x \in F, \quad s \in S.\end{aligned}$$

The last formula is justified by observing that

$$\begin{aligned}\phi_1 \partial_2((x - e) \otimes_F e \otimes (s - e) \otimes_S e) \\ &= \phi_1((\varepsilon'x - e) \otimes_F (s - e) \otimes_S e - (x - e) \otimes_F e \otimes (\varepsilon''s - e)) \\ &= (s - e) \otimes_F (\varepsilon'x - e) - (x - e) \otimes_F (\varepsilon''s - e) \\ &= (s - e)(x - e) \otimes_F e - (x - e)(s - e) \otimes_F e \\ &= (sx - xs) \otimes_F e \\ &= ([s, x] - e) \otimes_F xs \\ &= \partial_2([s, x] [R, R] \otimes xs).\end{aligned}$$

Again writing  $Z_k$  for the  $k$ th cycle group, we observe that  $\phi_2$  induces

$$\phi_2: Z_1 G \otimes Z_1 N \rightarrow Z_2 G$$

given by

$$\phi_2(x[F, F] \otimes s[S, S]) = [s, x] [R, R],$$

and so induces, in the light of Lemma 3.4,  $\chi_0$  given by

$$\chi_0(xR[F, F] \otimes sR) = [s, x] [F, R].$$

(Recall that  $[S, S] \subseteq R$ ). This proves the theorem, since

$$\chi(xR[F, F] \otimes sR) = [x, s] [R, R].$$

*Remark.* The difference of sign is not unexpected, since Ganea is considering the *left* operation of  $N$  on  $G$ .

#### 4. Stem-Extensions

A central extension  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  is called a *stem-extension* if  $N \subseteq [G, G]$ . From (1.3) we immediately deduce

**PROPOSITION 4.1.** *Let  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  be a central extension. Then the following statements are equivalent:*

- (i)  $N \twoheadrightarrow G \twoheadrightarrow Q$  is a stem-extension;
- (ii)  $\mu_*: N \rightarrow G_{ab}$  is the zeromap;
- (iii)  $\tau = \varepsilon_*: G_{ab} \rightarrow Q_{ab}$  is an isomorphism;
- (iv)  $\beta: H_2 Q \rightarrow N$  is an epimorphism.

Note that (iv) implies that if  $\xi \in H^2(Q; N)$  is the characteristic class of the central extension, then the stem-extensions are precisely those for which  $\Phi(\xi)$  is an epimorphism, where

$$\Phi: H^2(Q; N) \twoheadrightarrow \text{Hom}(H_2 Q, N)$$

is the natural epimorphism of the universal coefficient theorem.

We will apply the notion of a stem-extension in later sections. Here we wish to show that the exact sequence of Theorem 3.2 extends two further places to the left in the case of stem-extensions. However, our arguments encompass a generalization of the notion of stem-extensions, which we now give.

We say that the central extension  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  is a *weak stem-extension* if  $\mu$  induces  $0: N \otimes N \rightarrow G_{ab} \otimes N$ . Thus we have, by applying (1.2) – compare Proposition 4.1 –

**PROPOSITION 4.2.** *Let  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  be a central extension. Then the following statements are equivalent:*

- (i)  $N \twoheadrightarrow G \twoheadrightarrow Q$  is a weak stem-extension;
- (ii)  $\tau_N = \varepsilon_*: G_{ab} \otimes N \rightarrow Q_{ab} \otimes N$  is an isomorphism;
- (iii)  $\beta_N: H_2(Q; N) \rightarrow N \otimes N$  is an epimorphism.

We will abbreviate ‘weak stem-extension’ to ‘ws-extension’. We give some examples:

*Examples.* (a) If  $G$  is perfect then every central extension is a stem-extension.

(b) Consider  $\mathbf{Z}_m \twoheadrightarrow \mathbf{Z}_{m^2} \twoheadrightarrow \mathbf{Z}_m$ . Obviously this is not a stem-extension, but it is clearly a ws-extension, for  $\mathbf{Z}_{m^2} \otimes \mathbf{Z}_m \rightarrow \mathbf{Z}_m \otimes \mathbf{Z}_m$  is certainly an isomorphism. It is interesting to note that, in this example, while  $\beta_N$  is, by Proposition 4.2, an epimorphism,  $\beta \otimes 1: H_2 Q \otimes N \rightarrow N \otimes N$  is not an epimorphism.

(c) Let  $p$  be a fixed prime, let  $r \geq s$  be positive integers, and let  $G = G(p^r, p^s)$  be the group

$$G = \{a, b \mid a^{p^r} = b^{p^s} = a^{-1}b^{-1}ab\}.$$

Then  $a$  is of order  $p^{r+s}$ , and the center of  $G$  is generated by  $a^{p^r}$ . For any  $t \geq s$ , let  $N_t$  be the (central) subgroup of  $G$  generated by  $a^{p^t}$ . One may then readily verify that  $N_t \twoheadrightarrow G \twoheadrightarrow Q_t$  is a stem-extension iff  $t \geq r$  and a ws-extension iff  $t \geq \frac{1}{2}(r+s)$ .

Our main theorem, which we prove by the elementary methods of Sections 2 and 3 is the following.

**THEOREM 4.3.** *Let  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  be a weak stem-extension. Then there is a homo-*

morphism  $\delta: H_3 Q \rightarrow G_{ab} \otimes N$  such that the sequence

$$H_3 G \xrightarrow{\varepsilon_*} H_3 Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\varepsilon_*} H_2 Q \xrightarrow{\beta} N \xrightarrow{\sigma} G_{ab} \xrightarrow{\varepsilon_*} Q_{ab} \rightarrow 0$$

is exact.

*Proof.* We have only to define  $\delta$  and prove exactness at  $H_3 Q$  and  $G_{ab} \otimes N$ . Now by the reduction theorem (see for example [4; Corollary VI. 6.5]), we have natural isomorphisms

$$H_3 G \cong H_1(G; R_{ab}), \quad H_3 Q \cong H_1(Q; S_{ab}),$$

so that it is sufficient to define  $\delta: H_1(Q; S_{ab}) \rightarrow G_{ab} \otimes N$  and prove exactness in

$$H_1(G; R_{ab}) \xrightarrow{\varepsilon_*} H_1(Q; S_{ab}) \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2 G. \quad (4.1)$$

Now from the exact sequence of  $Q$ -modules

$$R/[S, S] \rightarrow S/[S, S] \rightarrow N$$

we get a coefficient sequence

$$H_2(Q; N) \xrightarrow{\psi} H_1(Q; R/[S, S]) \xrightarrow{\phi'} H_1(Q; S_{ab}) \xrightarrow{\delta'} H_1(Q; N) \xrightarrow{\chi'} (R/[S, S])_Q \rightarrow \dots \quad (4.2)$$

Since  $\varepsilon_*: G_{ab} \otimes N \cong Q_{ab} \otimes N = H_1(Q; N)$ , we may define  $\delta$  by  $\delta = \varepsilon_*^{-1} \delta'$ . We now show that the diagram

$$\begin{array}{ccc} G_{ab} \otimes N & \xrightarrow{\chi} & R/[F, R] \\ \downarrow \varepsilon_* & & \downarrow \zeta \\ Q_{ab} \otimes N & \xrightarrow{\chi'} & R/[F, R][S, S] \end{array} \quad (4.3)$$

commutes, where  $\zeta$  is the natural projection; here no use is made of the assumption that  $N \rightarrow G \rightarrow Q$  is weak stem. Using the Gruenberg resolution of  $\mathbb{Z}$  over  $Q$ , we have  $C_1 \otimes_Q N = JF \otimes_F N$ , consisting exclusively of cycles. Thus, to compute  $\chi'$  on  $[(x-e) \otimes_F sR]$  we pass to  $(x-e) \otimes_F s[S, S]$  and apply the boundary  $\partial_1$ , obtaining  $\chi s x^{-1} s^{-1} [S, S]$ . We have shown that  $\chi'[(x-e) \otimes_F sR] = [x, s][F, R][S, S]$ . Identifying  $JF \otimes_F N$  with  $F_{ab} \otimes N$ , we find

$$\chi'(xS[F, F] \otimes sR) = [x, s][F, R][S, S],$$

proving (4.3). Now enlarge (4.3) to the diagram, with exact columns,

$$\begin{array}{ccc} N \otimes N & \xrightarrow{\tilde{\chi}} & [S, S]/[S, R] \\ \downarrow \mu_* & \chi & \downarrow \\ G_{ab} \otimes N & \rightarrow & R/[F, R] \\ \downarrow \varepsilon_* & & \downarrow \zeta \\ Q_{ab} \otimes N & \xrightarrow{\chi'} & R/[F, R][S, S] \end{array}, \quad (4.4)$$



where  $\bar{\chi}$  is defined by the commutator map for the extension  $N \twoheadrightarrow N \twoheadrightarrow 1$ . It follows from (4.4) that, in the case of a ws-extension,  $\varepsilon_*$  induces an isomorphism of  $\ker \chi$  onto  $\ker \chi'$ . Thus the exactness of (4.2) at  $H_1(Q; N)$  implies the exactness of the sequence of Theorem 4.3 at  $G_{ab} \otimes N$ .

Before proceeding with the argument, we remark that it follows from (4.4) that, for a ws-extension,

$$[S, S] \subseteq [F, R], \quad (4.5)$$

so that  $\zeta$  in (4.3) is the identity, and  $(R/[S, S])_Q = R/[F, R]$ .

It remains to prove the exactness of (4.1) at  $H_1(Q; S_{ab})$ . Our first step is to factorize  $\varepsilon_*: H_1(G; R_{ab}) \rightarrow H_1(Q; S_{ab})$  as  $\varepsilon_* = \phi' \phi'' \phi'''$ ,

$$H_1(G; R_{ab}) \xrightarrow{\phi'''} H_1(Q; (R_{ab})_N) \xrightarrow{\phi''} H_1(Q; R/[S, S]) \xrightarrow{\phi'} H_1(Q; S_{ab}). \quad (4.6)$$

Here  $\phi'''$  is the 'change-of-rings' homomorphism;  $\phi''$  is induced by the sequence of  $Q$ -modules,

$$[S, S]/[S, R] \twoheadrightarrow R/[S, R] \twoheadrightarrow R/[S, S]$$

(note that  $(R_{ab})_N = R/[S, R]$ ); and  $\phi'$  is as in (4.2). We will prove

$$\phi''' \text{ is surjective;} \quad (4.7)$$

$$\text{im } \phi' \phi'' = \text{im } \phi'. \quad (4.8)$$

These two facts together establish the exactness of (4.1) at  $H_1(Q; S_{ab})$ —in view of (4.2)—and hence Theorem 4.3. To prove (4.7), we demonstrate the following more general lemma.

**LEMMA 4.4.** *Given any extension  $N \twoheadrightarrow G \xrightarrow{e} Q$  and any  $G$ -module  $B$ ,*

$$\varepsilon_*: H_1(G; B) \rightarrow H_1(Q; B_N)$$

*is surjective.*

*Proof of Lemma.* This is well-known (see for example [9]), but we give an elementary proof here in the framework of our paper. Using the Gruenberg resolutions, we readily obtain a commutative diagram

$$\begin{array}{ccccc} H_1(G; B) & \xrightarrow{\varepsilon_*} & H_1(Q; B_N) & & \\ \downarrow & & \downarrow & & \\ JN \otimes_N B & \xrightarrow{\mu_*} & JG \otimes_G B & \xrightarrow{\varepsilon_*} & JQ \otimes_Q B_N \\ \parallel & & \downarrow & & \downarrow \\ JN \otimes_N B & \longrightarrow & B & \longrightarrow & B_N \end{array}$$

in which the columns are exact, the bottom row is exact, and the middle row is differential. It is now trivial that  $\varepsilon_*: H_1(G; B) \rightarrow H_1(Q; B_N)$  is surjective.

The proof of (4.8) is based on the following lemma. As usual, we refer to the presentation (2.5).

LEMMA 4.5. *Let  $N \twoheadrightarrow G \twoheadrightarrow Q$  be a central extension. Let  $\delta'': H_1(Q; R/[S, S]) \rightarrow ([S, S]/[S, R])_Q$  be the connecting homomorphism associated with the sequence of  $Q$ -modules*

$$[S, S]/[S, R] \twoheadrightarrow R/[S, R] \twoheadrightarrow R/[S, S]$$

*and let  $\bar{\chi}: N \otimes N \rightarrow [S, S]/[S, R]$  be the commutator map (4.4) for the extension  $N \twoheadrightarrow N \twoheadrightarrow 1$ . Then  $([S, S]/[S, R])_Q = [S, S]/[S, R]$  and the square*

$$\begin{array}{ccc} H_2(Q; N) & \xrightarrow{\psi} & H_1(Q; R/[S, S]) \\ \downarrow \beta_N & & \downarrow \delta'' \\ N \otimes N & \xrightarrow{\bar{\chi}} & [S, S]/[S, R] \end{array} \quad (4.9)$$

*commutes.*

*Proof of Lemma.* Since  $N$  is central,  $Q$  operates trivially on  $H_2 N = [S, S]/[S, R]$ . We now prove the commutativity of (4.9) by again appealing to the Gruenberg resolution of  $\mathbf{Z}$  over  $Q$ . Then a 2-chain of  $C_2(Q; N)$  has the form  $\sum_i s_i [S, S] \otimes s'_i R$ . Since  $\psi$  is the connecting homomorphism associated with the sequence of  $Q$ -modules

$$R/[S, S] \twoheadrightarrow S/[S, S] \twoheadrightarrow N,$$

it follows that the value of  $\psi$  on the class of the cycle  $w = \sum_i s_i [S, S] \otimes s'_i R$  is the class, in  $H_1(Q; R/[S, S])$ , of  $\sum_i (s_i - e) \otimes_F s'_i [S, S]$ , i.e., (using  $\{ \}$  for homology classes)

$$\psi \{w\} = \left\{ \sum_i (s_i - e) \otimes_F s'_i [S, S] \right\}. \quad (4.10)$$

Now a 1-chain of  $C_1(Q; R/[S, S]) = JF \otimes_F R/[S, S]$  has the form  $\sum_j (x_j - e) \otimes_F r_j [S, S]$ , and the value of  $\delta''$  on the class of the cycle  $z = \sum_j (x_j - e) \otimes_F r_j [S, S]$  is the class, in  $H_0(Q; [S, S]/[S, R])$ , of  $\sum_j [x_j, r_j] [S, R]$ . But since  $Q$  operates trivially on  $[S, S]/[S, R]$ , we may write

$$\delta'' \{z\} = \sum_j [x_j, r_j] [S, R]. \quad (4.11)$$

We now define a 'commutator map'  $\theta: JF \otimes_F S/[S, S] \rightarrow [F, S]/[S, R]$  by the rule

$$\theta((x - e) \otimes_F s [S, S]) = [x, s] [S, R]. \quad (4.12)$$

To check that  $\theta$  is well-defined we must verify that

$$[x, s_1 s_2] [S, R] = [x, s_1] [x, s_2] [S, R] \quad (4.13)$$

and that  $\theta((x - e) y \otimes_F s [S, S]) = \theta((x - e) \otimes_F y s y^{-1} [S, S])$ , or

$$[xy, s] [y, s]^{-1} [S, R] = [x, y s y^{-1}] [S, R]. \quad (4.14)$$

Now (4.13) follows exactly as in the proof of Proposition 3.1, since  $[S, [F, S]] \subseteq [S, R]$ , and (4.14) holds since, in fact,  $[xy, s] = [x, ysy^{-1}][y, s]$ . Thus  $\theta$  is well-defined; and we observe that  $JF \otimes_F R/[S, S]$  is a subgroup of  $JF \otimes_F S/[S, S]$  and that, from (4.11),

$$\theta(z) = \delta''\{z\}. \quad (4.15)$$

It thus follows from (4.10), (4.12) and (4.15) that, if  $w = \sum s_i [S, S] \otimes s'_i R$ ,

$$\delta''\psi\{w\} = \sum_i [s_i, s'_i] [S, R]. \quad (4.16)$$

Now, as shown in Section 2,  $\beta_N: H_2(Q; N) \rightarrow N \otimes N = N \otimes_Q N$  is given by restricting to  $H_2(Q; N)$  the homomorphism  $S_{ab} \otimes_Q N \rightarrow N \otimes_Q N$  induced by  $\varepsilon''$ . Thus, with  $w = \sum_i s_i [S, S] \otimes s'_i R$ ,

$$\beta_N\{w\} = \sum_i s_i R \otimes s'_i R,$$

so that

$$\bar{\chi}\beta_N\{w\} = \sum_i [s_i, s'_i] [S, R],$$

and the lemma is proved.

The relation (4.8) quickly follows from Lemma 4.5. For we have the diagram (with exact row and column)

$$\begin{array}{ccccc} & & H_1(Q; R/[S, R]) & & \\ & & \downarrow \phi'' & & \\ H_2(Q; N) & \xrightarrow{\psi} & H_1(Q; R/[S, S]) & \xrightarrow{\phi'} & H(Q; S/[S, S]) \\ & & \downarrow \delta'' & & \\ & & [S, S]/[S, R] & & \end{array}$$

and, by Lemma 4.5,  $\delta''\psi$  is surjective in the case of a ws-extension; for, in that case,  $\beta_N$  is surjective (Proposition 4.2), and  $\bar{\chi}$  is always surjective. It follows immediately that  $H_1(Q; R/[S, S]) = \text{im } \psi + \text{im } \phi''$ , so that  $\text{im } \phi' = \text{im } \phi' \phi''$ . Thus Theorem 4.3 is completely proved.

*Remark.* The exact sequence of [2] yields Theorem 4.3 as a special case. For the homomorphism  $\sigma: H_4(N, 2) \rightarrow G_{ab} \otimes N$  of [2] factors through  $\mu_*: N \otimes N \rightarrow G_{ab} \otimes N$  and is thus zero for a ws-extension. Indeed, the exact sequence of [2] shows that the conclusion of Theorem 4.3 holds if and only if  $\mu_* \mid \ker \bar{\chi} = 0$ . We will revert to this point in a subsequent paper [11].

In the case of a stem-extension, the exact sequence of Theorem 4.3 becomes

$$H_3 G \xrightarrow{\varepsilon_*} H_3 Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\varepsilon_*} H_2 Q \xrightarrow{\beta} N \rightarrow 0 \quad (4.17)$$

Now let  $Q$  be a given group and let  $U$  be any subgroup of  $H_2 Q$ . Set  $N = H_2 Q/U$  and let  $\beta: H_2 Q \rightarrow N$  be the canonical projection. Let  $\xi$  be any element of  $H^2(Q; N)$  such that

$\Phi(\xi) = \beta$ , and let  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  be a central extension with characteristic class  $\xi$ . Then it follows from Theorem 2.2 and Proposition 4.1 that  $N \twoheadrightarrow G \twoheadrightarrow Q$  is a stem-extension and (4.17) shows that

$$U = \operatorname{im} \varepsilon_* \cong \operatorname{coker} \chi.$$

The stem-extensions yielding the given epimorphism  $\beta$  are in one-to-one correspondence with the elements of  $\operatorname{Ext}(H_1 Q, N)$ . The stem-extensions for which  $U=0$ ,  $\beta=1$  are called *stem-covers*. For a stem-cover

$$H_2 Q \twoheadrightarrow G \xrightarrow{\varepsilon} Q$$

we have an exact sequence

$$H_3 G \xrightarrow{\varepsilon_*} H_3 Q \xrightarrow{\delta} G_{ab} \otimes H_2 Q \xrightarrow{\chi} H_2 G \rightarrow 0; \quad (4.18)$$

and the stem-covers of  $Q$  are in one-to-one correspondence with elements of  $\operatorname{Ext}(H_1 Q, H_2 Q)$ .

## 5. Perfect Groups

Let

$$N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q \quad (5.1)$$

be a central extension classified by  $\xi \in H^2(Q; N)$  and let  $\varrho: N \rightarrow N_1$  be a homomorphism of commutative groups. We then recall that if  $\xi_1 = \varrho_*(\xi) \in H^2(Q; N_1)$  and

$$N_1 \twoheadrightarrow G_1 \twoheadrightarrow Q$$

is the central extension classified by  $\xi_1$ , there is a map of extensions

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ e \downarrow & & \downarrow \tau & & \parallel \\ N_1 & \xrightarrow{\mu_1} & G_1 & \xrightarrow{\varepsilon_1} & Q \end{array} . \quad (5.2)$$

We study this situation when  $Q$  is perfect, that is,  $Q_{ab} = 0$ . In that case (see the Remark following Theorem 2.2) the central extension (5.1) is characterized by  $\beta = \Phi(\xi): H_2 Q \rightarrow N$ , which appears in the exact sequence (1.3)

$$H_2 G \rightarrow H_2 Q \xrightarrow{\beta} \mathcal{N} \rightarrow G_{ab} \rightarrow 0;$$

and, if  $\xi_1 = \varrho_*(\xi)$ ,

$$\Phi(\xi_1) = \varrho \beta. \quad (5.3)$$

Moreover,

PROPOSITION 5.1. *Let*

$$\begin{array}{ccccc} N & \xrightarrow{\mu} & G & \xrightarrow{\varepsilon} & Q \\ \varrho \downarrow & & \tau \downarrow & & \downarrow \psi \\ N_1 & \xrightarrow[\mu_1]{} & G_1 & \xrightarrow[\varepsilon_1]{} & Q_1 \end{array}$$

*be a map of extensions in which  $Q$  is perfect and  $N_1$  is central. Then  $\tau$  is uniquely determined by  $\varrho$  and  $\psi$ .*

*Proof.* Let  $\tau, \tau': G \rightarrow G_1$  be two homomorphisms each yielding commutativity in relation to  $\varrho$  and  $\psi$ . Consider the function  $f: G \rightarrow G_1$  given by  $f(x) = \tau(x) \tau'(x)^{-1}$ ,  $x \in G$ . Since  $\varepsilon_1 \tau = \varepsilon_1 \tau'$ ,  $f$  maps  $G$  into  $N_1$ . Since  $N_1$  is central, it is clear that  $f: G \rightarrow N_1$  is a homomorphism. Since  $\tau\mu = \tau'\mu$ ,  $f$  is trivial on  $N$ , and thus induces a homomorphism  $g: Q \rightarrow N_1$ . Since  $Q$  is perfect and  $N_1$  is commutative,  $g = 0$ . Thus  $f = 0$ , so that  $\tau = \tau'$ .

COROLLARY 5.2. *In the diagram (5.2), with  $Q$  perfect,  $\tau$  is uniquely determined by  $\varrho$ .*

We now, temporarily, restrict attention to *stem*-extensions (5.1) with  $Q$  perfect. We note that,  $Q$  being perfect, (5.1) is a stem extension if and only if  $G$  is also perfect. Theorem 3.2 then yields the short exact sequence, for stem extensions (5.1),

$$0 \rightarrow H_2G \xrightarrow{\alpha} H_2Q \xrightarrow{\beta} N \rightarrow 0. \quad (5.4)$$

Thus, with every stem extension (5.1) of the perfect group  $Q$ , we may associate a subgroup  $U = H_2G$  of  $H_2Q$ , such that the stem extension is characterized by the projection  $H_2Q \rightarrow H_2Q/U = N$ . Conversely, given  $U \subseteq H_2Q$ , set  $N = H_2Q/U$ ,  $\beta: H_2Q \rightarrow N$  the projection, and let (5.1) be the central extension characterized by  $\beta$ . Then, plainly, (5.1) is a stem extension and

$$H_2G = U.$$

Thus we have proved

THEOREM 5.3. *There is a one-one correspondence between stem extensions of the perfect group  $Q$  and subgroups of  $H_2Q$ , given by associating with (5.1) the group  $H_2G$ .*

Now let us take the stem extension

$$H_2Q \rightarrow Q_0 \rightarrow Q \quad (5.5)$$

of the perfect group  $Q$  with  $H_2Q_0 = 0$ . It is *universal* in the following sense. Given any central extension (5.1), characterized by  $\beta: H_2Q \rightarrow N$ , there exists, according to Corollary 5.2, a unique homomorphism  $\tau: Q_0 \rightarrow G$  such that the diagram

$$\begin{array}{ccccc} H_2Q & \rightarrow & Q_0 & \rightarrow & Q \\ \downarrow \beta & & \downarrow \tau & & \parallel \\ N & \rightarrow & G & \rightarrow & Q \end{array}$$

commutes (note that (5.5) is characterized by the identity map of  $H_2Q$ ). Notice that, if (5.1) is also a stem extension, then, in (5.6),  $\beta$  is surjective, so that  $\tau$  is surjective. Moreover,  $\ker \tau = \ker \beta = H_2G$ , which is central in  $Q_0$ . Thus if we describe  $G$  as a *cover* of  $Q$  if there exists a stem extension (5.1), then Theorem 5.3 establishes a one-one correspondence between covers of  $Q$  and subgroups of  $H_2Q$  [6], and our subsequent argument shows that  $Q_0$  is the *universal cover* of  $Q$  in that it (uniquely) covers any cover of  $Q$ .

**PROPOSITION 5.4.** *The central extension (5.1) is the universal cover of the perfect group  $Q$  if and only if  $H_1G=0$ ,  $H_2G=0$ .*

*Proof.* This is immediate, since, if  $H_1G=0$ ,  $H_2G=0$ , then  $\beta$  and  $\tau$  are isomorphisms in (5.6).

We next describe  $Q_0$  and  $\tau$  in (5.6) by means of the presentation (2.5). We have the evident

**PROPOSITION 5.5.** *Let (5.1) be a stem extension of the perfect group  $Q$ . Then*

$$[F, S] = [F, R].$$

*Proof.* We showed in Section 3 that for any central extension, the kernel of  $\alpha: H_2G \rightarrow H_2Q$  is  $[F, S]/[F, R]$ . But for a stem extension of the perfect group  $Q$ ,  $\ker \alpha = 0$  (5.4); indeed, it is plain that  $[F, S] = [F, R]$  for a ws-extension of the perfect group  $Q$ .

Now if (5.1) is the universal cover, then  $H_1G=0$ ,  $H_2G=0$ ,  $N=H_2Q$ . Thus

$$R \cap [F, F] = [F, R] = [F, S],$$

so that

$$G = F/R = [F, F]/(R \cap [F, F]) = [F, F]/[F, S].$$

Conversely, let  $S \twoheadrightarrow F \twoheadrightarrow Q$  be a free presentation of the perfect group  $Q$ ; then  $Q = [F, F]/(S \cap [F, F])$  and if we set  $G = [F, F]/[F, S]$  we obtain the extension

$$H_2Q \twoheadrightarrow G \twoheadrightarrow Q. \quad (5.7)$$

This extension is plainly central and is characterized by the identity on  $H_2Q$ . It follows that (5.7) is a stem extension and  $H_2G=0$ .

If  $N \twoheadrightarrow G \twoheadrightarrow Q$  is an arbitrary central extension of the perfect group  $Q$ , characterized by  $\beta: H_2Q \rightarrow N$ , we induce a homomorphism  $\tau: [F, F]/[F, S] \rightarrow F/R$  from the inclusions  $[F, F] \subseteq F$ ,  $[F, S] \subseteq R$ . Plainly,  $\tau$  restricts to  $\beta$  on  $H_2Q = (S \cap [F, F])/[F, S]$  and induces a commutative diagram (5.6). By uniqueness, it is therefore the homomorphism there described. We sum up:

**THEOREM 5.6.** *Given the presentation  $S \twoheadrightarrow F \twoheadrightarrow Q$  of the perfect group  $Q$ , the*

universal central (stem) extension  $H_2Q \xrightarrow{\mu_0} Q_0 \xrightarrow{\varepsilon_0} Q$  can be given by  $Q_0 = [F, F]/[F, S]$  with  $\varepsilon_0$  induced by the inclusions  $[F, F] \subseteq F$ ,  $[F, S] \subseteq S$ . For any central extension  $N \rightarrowtail G \twoheadrightarrow Q$  with presentation (2.5), the universal homomorphism  $\tau: Q_0 \rightarrow G = F/R$  is induced by the inclusions  $[F, F] \subseteq F$ ,  $[F, S] \subseteq R$ .

*Remarks.* 1) Theorem 5.6 may readily be used to prove, without any spectral sequence techniques, the result cited in [6] that, given a short exact sequence of perfect groups  $K \rightarrowtail G \twoheadrightarrow Q$ , the lifted sequence  $K_0 \rightarrow G_0 \twoheadrightarrow Q_0$  is exact.

2) It is also clear how to present the cover of  $Q$  corresponding to any subgroup  $U$  of  $H_2Q$ . With  $H_2Q = (S \cap [F, F])/[F, S]$  we have  $U = V/[F, S]$ ,  $[F, S] \subseteq V \subseteq S \cap [F, F]$ , and  $V \rightarrowtail [F, F] \twoheadrightarrow G$  is then a presentation of the required cover  $G$ .

We now prove a theorem motivated by the analogy with covering space theory.

**THEOREM 5.7** *Let  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  be a central extension, let  $X$  be a perfect group and let  $\psi: X \rightarrow Q$  be a homomorphism. Then  $\psi$  lifts, uniquely, to  $\phi: X \rightarrow G$  with  $\varepsilon\phi = \psi$ , if and only if*

$$\psi_* H_2 X \subseteq \varepsilon_* H_2 G. \quad (5.8)$$

*Proof.* Plainly, if  $\psi$  lifts, (5.8) holds. Also Proposition 5.1 affirms that if  $\phi$  exists, it is unique. Thus it suffices to prove that (5.8) implies the existence of  $\phi$ .

We use the presentation of  $X$ ,

$$U \rightarrowtail V \twoheadrightarrow X.$$

Let  $\psi$  be lifted to  $\eta: V \rightarrow F$  with  $\eta(U) \subseteq S$ . Then

$$\begin{aligned} \varepsilon_* H_2 G &= (R \cap [F, F]) [F, S] / [F, S] \\ &= R \cap [F, F] / [F, S], \end{aligned}$$

since  $N$  is central in  $G$ . Thus the hypothesis (5.8) may be translated into the condition

$$\eta(U \cap [V, V]) \subseteq R \cap [F, F]. \quad (5.9)$$

Since  $X$  is perfect we may present it by

$$U \cap [V, V] \rightarrowtail [V, V] \twoheadrightarrow X.$$

We have the diagram

$$\begin{array}{ccccc} U \cap [V, V] & \xrightarrow{\eta} & R \cap [F, F] & \rightarrowtail & S \cap [F, F] \\ \downarrow & & \downarrow & & \downarrow \\ [V, V] & \xrightarrow{\eta} & [F, F] & \longrightarrow & [F, F] \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & G & \xrightarrow{\varepsilon} & Q \end{array}, \quad (5.10)$$

determining a map  $\phi: X \rightarrow G$  such that  $\varepsilon\phi = \psi$ .

We may apply this theorem in the following situation. Consider the diagram

$$\begin{array}{ccc} N & \xrightarrow{\mu} & G \xrightarrow{\varepsilon} Q \\ & & \downarrow \psi \\ N_1 & \xrightarrow{\mu_1} & G_1 \xrightarrow{\varepsilon_1} Q_1 \end{array}, \quad (5.11)$$

where the top row is a stem extension of the perfect group  $Q$ , and the bottom row is a central extension. Then we know that we may complete (5.11) to the commutative diagram (and uniquely)

$$\begin{array}{ccc} N & \xrightarrow{\mu} & G \xrightarrow{\varepsilon} Q \\ \downarrow \varrho & & \downarrow \tau \quad \downarrow \psi \\ N_1 & \xrightarrow{\mu_1} & G_1 \xrightarrow{\varepsilon_1} Q_1 \end{array} \quad (5.12)$$

if and only if  $\psi_* \varepsilon_* H_2 G \subseteq \varepsilon_{1*} H_2 G_1$ . However this latter is precisely the condition that we may find  $\varrho': N \rightarrow N_1$  such that the diagram

$$\begin{array}{ccc} H_2 Q & \xrightarrow{\beta} & N \\ \downarrow \psi_* & & \downarrow \varrho' \\ H_2 Q_1 & \xrightarrow{\beta_1} & N_1 \end{array} \quad (5.13)$$

commutes, as follows immediately from (1.3). Moreover, since (5.12) induces

$$\begin{array}{ccc} H_2 Q & \xrightarrow{\beta} & N \\ \downarrow \psi_* & & \downarrow \varrho \\ H_2 Q_1 & \xrightarrow{\beta_1} & N_1 \end{array} \quad (5.14)$$

and  $\beta$  is surjective, it follows that  $\varrho = \varrho'$ . We thus have the corollary of Theorem 5.7.

**COROLLARY 5.8.** *Let  $N \rightarrowtail G \twoheadrightarrow Q$  be a stem extension of the perfect group  $Q$ , let  $N_1 \rightarrowtail G_1 \twoheadrightarrow Q_1$  be a central extension, and let  $\varrho: N \rightarrow N_1$ ,  $\psi: Q \rightarrow Q_1$  be homomorphisms. Then there exists  $\tau: G \rightarrow G_1$  rendering the diagram (5.12) commutative if and only if (5.14) commutes. If  $\tau$  exists, it is unique.*

*Remark.* We note, in particular, that if (5.14) commutes, then there exists a canonical homomorphism  $\theta: H_2 G \rightarrow H_2 G_1$ , namely  $\tau_*$ , such that

$$\begin{array}{ccc} H_2 G & \xrightarrow{\varepsilon} & H_2 Q \\ \downarrow \theta & & \downarrow \psi_* \\ H_2 G_1 & \xrightarrow{\varepsilon_{1*}} & H_2 Q_1 \end{array}$$

commutes.

We close this section by observing that Theorem 4.3 leads to an immediate proof of the following result of Kervaire [6] (see also [2]):



**THEOREM 5.9.** *Let  $N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$  be a stem extension of the perfect group  $Q$ . Then  $\varepsilon_*: H_3 G \rightarrow H_3 Q$  is an epimorphism.*

## 6. Appendix: Remarks on Algebraic $K$ -Theory

Certain exact sequences of algebraic  $K$ -theory can easily be obtained within the framework of this paper. We first recall some known facts and definitions.

We consider  $G = GL(A)$ ,  $A$  being a ring with unity, and the subgroup  $E = E(A)$  generated by elementary matrices  $1 + \lambda E_{ij}$ ,  $i \neq j$ ,  $\lambda \in A$ . Given an ideal  $\mathfrak{a} \subset A$ , we write  $\bar{G} = G(A/\mathfrak{a})$ ,  $\bar{E} = E(A/\mathfrak{a})$  and denote by  $\pi: G \rightarrow \bar{G}$ ,  $\pi': E \rightarrow \bar{E}$  the canonical maps, by  $N$  the kernel of  $\pi$  (i.e., the *congruence subgroup*  $G(A, \mathfrak{a})$ ). The group  $E(A, \mathfrak{a})$ , the normal hull in  $E$  of all elementary matrices  $1 + \alpha E_{ij}$ ,  $\alpha \in \mathfrak{a}$ , is contained in  $^1) \ker \pi' = E \cap N$ . The following facts are easily proved:  $E = [E, E]$ ,  $\bar{E} = [\bar{E}, \bar{E}]$ ,  $E(A, \mathfrak{a}) = [E, E(A, \mathfrak{a})]$ , and  $\pi': E \rightarrow \bar{E}$  is an epimorphism.

The generalized *Whitehead Lemma* [10, Theorem 15.1]

$$[G, N] \subseteq E(A, \mathfrak{a})$$

yields the further relations

$$E = [G, G] \tag{6.1}$$

$$E(A, \mathfrak{a}) = [G, N] \tag{6.2}$$

$$[E, E \cap N] = [G, N]. \tag{6.3}$$

The proof of (6.3) is as follows: the inclusion  $[E, E \cap N] \subseteq [G, N]$  is obvious, and, on the other hand, we have

$$[E, E \cap N] \supseteq [E, E(A, \mathfrak{a})] = E(A, \mathfrak{a}) = [G, N].$$

We now reflect this situation by the following more general set-up. Let  $G$  and  $\bar{G}$  be groups with perfect commutator subgroups, let  $E = [G, G]$  and  $\bar{E} = [\bar{G}, \bar{G}]$ ; let  $\pi$  be a homomorphism  $G \rightarrow \bar{G}$  which maps  $E$  onto  $\bar{E}$ , and suppose that  $[G, N] = [E, E \cap N]$ , where  $N = \ker \pi$ .

We write  $Q = G/N$  and map the extension  $E \cap N \rightarrow E \rightarrow \bar{E}$  into  $N \rightarrow G \xrightarrow{\varepsilon} Q$  by inclusions. The induced map of the exact sequences (1.3) is given by the commutative diagram

$$\begin{array}{ccccccc} H_2 E & \xrightarrow{\alpha'} & H_2 \bar{E} & \xrightarrow{\beta'} & E \cap N / [E, E \cap N] & \longrightarrow & 0 \longrightarrow 0 \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow e & & \downarrow & \downarrow \\ H_2 G & \xrightarrow{\alpha} & H_2 Q & \xrightarrow{\beta} & N / [G, N] & \xrightarrow{\delta} & H_1 G \xrightarrow{\varepsilon_*} H_1 Q \rightarrow 0. \end{array} \tag{6.4}$$

<sup>1)</sup> On p. 211 of [10], Swan inadvertently defines  $E(A, \mathfrak{a})$  to be  $\ker \pi'$ . The statement on p. 212 is the correct one – and the only one which Swan uses.

We note that  $\text{Im } \varrho = (E \cap N)/[G, N] = \ker \delta$ ; this is also  $\text{im } \varrho\beta'$ . Since  $[E, E \cap N] = [G, N]$ ,  $\varrho$  is a monomorphism, so  $\ker \varrho\beta' = \ker \beta' = \text{im } \alpha'$ . We further remark that  $\pi$  induces a monomorphism  $Q \rightarrow \tilde{G}$  which maps  $[Q, Q] = \varepsilon(E)$  isomorphically onto  $\tilde{E}$ . It follows that the kernel of  $\pi_*: H_1 G = G/E \rightarrow H_1 \tilde{G} = \tilde{G}/\tilde{E}$  is equal to the kernel of  $\varepsilon_*: G/E \rightarrow Q/[Q, Q]$ . Writing  $\sigma = \varrho\beta'$ , we obtain from (6.4) the exact sequence

$$H_2 E \xrightarrow{\alpha'} H_2 \tilde{E} \xrightarrow{\sigma} N/[G, N] \xrightarrow{\delta} H_1 G \xrightarrow{\pi_*} H_1 \tilde{G}. \quad (6.5)$$

In the case of  $K$ -theory, the groups can all be identified with familiar  $K$ -groups:  $H_1 G = G/E = K_1(\Lambda)$ ;  $H_1 \tilde{G} = K_1(\Lambda/\alpha)$ ;  $N/[G, N] = GL(\Lambda, \alpha)/E(\Lambda, \alpha) = K_1(\Lambda, \alpha)$ ; and finally one defines  $K_2(\Lambda)$  by  $H_2 E(\Lambda)$ . Then (6.5) becomes the exact sequence

$$K_2(\Lambda) \rightarrow K_2(\Lambda/\alpha) \rightarrow K_1(\Lambda, \alpha) \rightarrow K_1(\Lambda) \rightarrow K_1(\Lambda/\alpha). \quad (6.6)$$

The above definition of  $K_2(\Lambda)$  coincides with that of Milnor, where  $K_2(\Lambda)$  is the kernel of  $St(\Lambda) \rightarrow E(\Lambda)$ . The group  $St(\Lambda)$ , the Steinberg group, is defined by an explicit free presentation  $S \rightarrow F \rightarrow St(\Lambda)$ ; it is a stem-extension of  $E(\Lambda)$ , and in fact the universal one – which implies that the kernel is  $H_2 E(\Lambda)$ . To prove that  $St(\Lambda)$  is universal, it suffices to show that it is its own universal stem-extension; i.e., that the homomorphism  $[F, F]/[F, S] \rightarrow F/S$  induced by the inclusions  $[F, F] \subseteq F$ ,  $[F, S] \subseteq S$ , is an isomorphism. This can be done by following Kervaire's procedure [6], using various commutator relations.

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