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On the Homology Theory of Central Group Extensions: I-The Commutator Map and Stem Extensions

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In Memoriam Heinz Hopf (1894–1971)

1. Introduction

For any group extension

$$N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q \tag{1.1}$$

and any Q-module B, there is a five-term exact homology sequence

$$H_2(G; B) \overset{\alpha_B}{\to} H_2(Q; B) \overset{\beta_B}{\to} N_{ab} \otimes_Q B \overset{\sigma_B}{\to} H_1(G; B) \overset{\tau_B}{\to} H_1(Q; B) \to 0, \tag{1.2}$$

due to Stallings and Stammbach [8, 9]. If $B = \mathbf{Z}$, regarded as trivial Q-module, (1.2) reduces to

$$H_2G \overset{\alpha}{\to} H_2Q \overset{\beta}{\to} N/[G, N] \overset{\sigma}{\to} H_1G \overset{\tau}{\to} H_1Q \to 0. \tag{1.3}$$

For a simple proof of (1.2), including the statement of naturality, see Eckmann-Stammbach [3]; of course, α_B and τ_B in (1.2) are induced by ε; σ_B is, in a sense explained later, induced by μ (the sense is perfectly clear in the case (1.3)); and β_B will be elucidated in the next section.

In the special case where (1.1) is a *central* extension, that is, N is central in G, Ganea [5] has added a further term on the left of the exact sequence (1.3), thus,

$$G_{ab} \otimes N \overset{\chi_0}{\to} H_2G \overset{\alpha}{\to} H_2Q \to \cdots \to 0, \tag{1.4}$$

using methods of algebraic topology.

In [2], Eckmann-Hilton extended the sequence (1.4) by four further non-trivial homology terms, first replacing $G_{ab} \otimes N$ by a suitable quotient. Their method was based on a spectral sequence for the homology of a suitable fibre space. The Ganea sequence (1.4) and its extension in [2] are important for applications, beyond those of (1.3), in group theory, homology, algebraic K-theory, etc.

In the present paper we present an elementary approach to the Ganea extension (1.4) and to those parts of the extended sequence in [2] which are relevant to applications to *stem-extensions* of groups (see Section 4), and, in particular, to the study of *perfect* groups. The argument is based on a fixed, but arbitrary, free presentation of (1.1) (see Section 2), and the associated *Gruenberg resolutions* [4; Chapter VI] of $\mathbf{Z}$

over N, G and Q. The maps of (1.3) are exhibited and exactness is proved, by using these explicit resolutions.

In Section 2 we study (1.3) – and, in less detail, (1.2) – from the viewpoint of the given presentation of (1.1). In particular, we recall the relation with the *Hopf formula* for H_2G and H_2Q and obtain an explicit form for ker α. Moreover, the connection between β and the characteristic class of the central extension

$$N/[G,N] \rightarrowtail G/[G,N] \twoheadrightarrow Q, \tag{1.5}$$

associated with (1.1), is obtained.

In Section 3 the Ganea extension (1.4) is established by means of an explicit commutator map χ; and the equivalence of this map χ with Ganea's map χ_0 is demonstrated. In Section 4 we obtain an extended exact sequence

$$H_3G \to H_3Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2G \xrightarrow{\alpha} H_2Q \to \cdots \tag{1.6}$$

for stem-extensions, that is, central extensions (1.1) with $N \subseteq [G, G]$. Actually, we will obtain (1.6) for an even broader class of central extensions, which we tentatively call *weak stem-extensions.* Whereas a stem-extension (1.1) is characterized by the vanishing of the abelianization of μ, i.e. $\mu_*\colon N \to G_{ab}$ is the zeromap, for weak stem extensions we merely demand that $\mu_*\colon N \otimes N \to G_{ab} \otimes N$ is the zeromap. We give examples to show that this generalization is significant, and we also show how (1.6) may be regarded as contained in the extended sequence of [2].

Section 5 deals with perfect groups; we assume Q perfect and obtain, beyond the results of Schur [7] and Kervaire [6] on stem-extensions of Q, a description of the universal stem-extension of Q in terms of the given presentation of Q. Moreover, we use the given presentations of N, G and Q to carry further the analogy remarked by Kervaire between the theory of perfect groups and covering space theory for connected topological spaces.

Section 6 is an appendix concerning algebraic K-theory, in which we show how the exact sequence for Milnor's K_2 may be obtained from (1.3).

2. Extensions, Free Presentations, and Resolutions

Given the group extension (1.1),

$$N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q,$$

let

$$R \overset{\mu'}{\rightarrowtail} F \overset{\varepsilon'}{\twoheadrightarrow} G \tag{2.1}$$

be a (free) presentation of G. There are then presentations of Q, N,

$$S \overset{\bar{\mu}}{\rightarrowtail} F \overset{\bar{\varepsilon}}{\twoheadrightarrow} Q, \tag{2.2}$$

$$R \overset{\mu''}{\rightarrowtail} S \overset{\varepsilon''}{\twoheadrightarrow} N, \tag{2.3}$$

where

$$\bar{\mu}\mu'' = \mu', \quad \varepsilon\varepsilon' = \bar{\varepsilon}, \quad \varepsilon'\bar{\mu} = \mu\varepsilon''. \tag{2.4}$$

We sum this up in the single diagram

$$\begin{tikzcd}
 & & N \arrow[d, tail, "\mu"] \\
R \arrow[r, tail, "\mu'"] \arrow[d, tail, "\mu''"'] & F \arrow[r, two heads, "\varepsilon'"] \arrow[d, equal] & G \arrow[d, two heads, "\varepsilon"] \\
S \arrow[r, tail, "\bar{\mu}"] \arrow[d, two heads, "\varepsilon''"'] & F \arrow[r, two heads, "\bar{\varepsilon}"] & Q, \\
N & &
\end{tikzcd} \tag{2.5}$$

which we call a *presentation of the extension* (1.1).

Applying (1.3) to the two centre rows of (2.5) we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_2G \arrow[r, "\beta'"] \arrow[d, "\alpha"] & R/[F, R] \arrow[r, "\sigma'"] \arrow[d, "\gamma"] & F/[F, F] \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & H_2Q \arrow[r, "\bar{\beta}"] & S/[F, S] \arrow[r, "\bar{\sigma}"] & F/[F, F] \arrow[r] & \cdots,
\end{tikzcd} \tag{2.6}$$

where α is as in (1.3) and γ is induced by the inclusion $\mu''\colon R \rightarrowtail S$. Thus

$$\ker\gamma = (R \cap [F, S])/[F, R], \qquad \operatorname{coker}\gamma = S/R[F, S]. \tag{2.7}$$

We note that ε'' induces an isomorphism of coker γ onto $N/[G, N]$. We write η: $S/[F, S] \twoheadrightarrow N/[G, N]$ for the map induced by ε'', and thus embed (2.6) in the larger commutative diagram, with exact rows and columns,

$$\begin{tikzcd}
0 \arrow[r] & \ker\alpha \arrow[r] \arrow[d, tail] & \ker\gamma \arrow[r] \arrow[d, tail] & 0 \arrow[d] & \\
0 \arrow[r] & H_2G \arrow[r, "\beta'"] \arrow[d, "\alpha"] & R/[F, R] \arrow[r, "\sigma'"] \arrow[d, "\gamma"] & F/[F, F] \arrow[r] \arrow[d, "="] & \cdots \\
0 \arrow[r] & H_2Q \arrow[r, "\bar{\beta}"] \arrow[d, "\beta"] & S/[F, S] \arrow[r, "\bar{\sigma}"] \arrow[d, two heads, "\eta"] & F/[F, F] \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & N/[G, N] \arrow[r, "="] & N/[G, N] \arrow[r] & 0 &
\end{tikzcd} \tag{2.8}$$

We use β' to induce the *Hopf formula*

$$H_2G \cong (R \cap [F, F])/[F, R]; \tag{2.9}$$

likewise $\bar{\beta}$ induces

$$H_2Q \cong (S \cap [F, F])/[F, S]. \tag{2.10}$$

We make the identifications (2.9), (2.10), so that ker α is identified with ker γ,

$$\ker\alpha = (R \cap [F, S])/[F, R],$$

and the map β,

$$\beta: (S \cap [F, F])/[F, S] \to N/[G, N],$$

is just the restriction of η to H_2Q, and thus is induced by ε''. Note that the relation $\beta = \eta\bar{\beta}$ in (2.8) simply results from the naturality of (1.3), applied to

$$\begin{array}{ccccc} S & \overset{\bar{\mu}}{\rightarrowtail} & F & \overset{\bar{\varepsilon}}{\twoheadrightarrow} & Q \\ \downarrow{\scriptstyle\varepsilon''} & & \downarrow{\scriptstyle\varepsilon'} & & \| \\ N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \end{array} .$$

We now wish to relate β to the *associated central extension* (1.5). It follows from naturality that the homomorphism $H_2Q \to N/[G, N]$ in the sequence (1.3) corresponding to (1.5) coincides with β, so that, in this part of the argument, we lose no generality in supposing that N *is itself central* in (1.1). Then

$$\beta: H_2Q \to N,$$

and we propose to relate β to the element of $H^2(Q; N)$ characterized by the *central* extension

$$N \rightarrowtail G \twoheadrightarrow Q.$$

We now have $[G, N] = 1$, so that

$$[F, S] \subseteq R, \qquad \ker\alpha = [F, S]/[F, R], \tag{2.11}$$

and $\beta: (S \cap [F, F])/[F, S] \to S/R$ is induced by the inclusions $S \cap [F, F] \subseteq S$, $[F\ S] \subseteq R$.

We begin by giving an explicit description of $H^2(Q; N)$ in terms of (2.5). We use the *Gruenberg resolution* [4; VI. 13] of $\mathbf{Z}$ over Q based on (2.2), namely,

$$\cdots \xrightarrow{\partial_3} S_{ab} \otimes \mathbf{Z}Q \xrightarrow{\partial_2} JF \otimes_F \mathbf{Z}Q \xrightarrow{\partial_1} \mathbf{Z}Q \xrightarrow{\partial_0} \mathbf{Z} \to 0. \tag{2.12}$$

Here ∂_0 is the augmentation; JF is the augmentation ideal of F; ∂_1 is given by

$$\partial_1((x - e) \otimes_F e) = \bar{\varepsilon}(x) - e, \qquad x \in F,$$

where e stands for the unity in any group; the kernel of ∂_1 is known to be isomorphic to S_{ab}, with Q operating by inner automorphisms of F, under the monomorphism

$$\sigma_1: S_{ab} \rightarrowtail JF \otimes_F \mathbf{Z}Q$$

given by

$$\sigma_1(s[S, S]) = (s - e) \otimes_F e;$$

and ∂_2 is the composite of σ_1 with the Q-module map $s[S, S]\otimes e \mapsto s[S, S]$. (See [4; VI. 6], where the argument is given in detail but for the Gruenberg resolution of $\mathbf{Z}$ as *left* Q-module.)

THEOREM 2.1. *The Gruenberg resolution* (2.12) *induces an isomorphism*

$$H^2(Q; N) \cong \mathrm{Hom}(S/[F, S], N)/\bar{\sigma}^* \, \mathrm{Hom}(F/[F, F], N)$$

for any trivial Q-module N.

Proof. For any resolution $\cdots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ of $\mathbf{Z}$ over Q, $H^2(Q; N) = \mathrm{Hom}_Q(\ker \partial_1, N)/i^* \, \mathrm{Hom}_Q(C_1, N)$, where $i\colon \ker \partial_1 \subseteq C_1$. Thus, using the Gruenberg resolution,

$$H^2(Q; N) = \mathrm{Hom}_Q(S_{ab}, N)/\sigma_1^* \, \mathrm{Hom}_Q(JF \otimes_F \mathbf{Z}Q, N).$$

Now $\mathrm{Hom}_Q(S_{ab}, N) = \mathrm{Hom}((S_{ab})_Q, N) = \mathrm{Hom}(S/[F, S], N)$, and $\mathrm{Hom}_Q(JF \otimes_F \mathbf{Z}Q, N) = \mathrm{Hom}(JF \otimes_F \mathbf{Z}, N)$. Moreover there is a natural isomorphism $\psi\colon JF \otimes_F \mathbf{Z} \cong F_{ab}$, given by $\psi((x-e)\otimes_F 1) = x[F, F]$, $x \in F$, and plainly $\psi\sigma_1$ induces $\bar{\sigma}\colon S/[F, S] \to F/[F, F]$. This proves the theorem.

Now given the central extension $N \rightarrowtail G \twoheadrightarrow Q$, the map $\eta\colon S/[F, S] \to N$ of (2.8) (recall that $[G, N] = 1$) then determines, in the light of Theorem 2.1, an element $\xi \in H^2(Q; N)$ and this is the characteristic cohomology class of the given central extension (see [4; VI. 10]). We now readily prove

THEOREM 2.2. *If* $N \rightarrowtail G \twoheadrightarrow Q$ *is a central extension with characteristic class* $\xi \in H^2(Q; N)$, *then the homomorphism* $\beta\colon H_2Q \to N$ *of* (1.3) *is the image of* ξ *under the epimorphism*

$$\Phi\colon H^2(Q; N) \twoheadrightarrow \mathrm{Hom}(H_2Q, N)$$

of the universal coefficient theorem.

Proof. For any $\zeta \in H^2(Q; N)$, $\Phi(\zeta)$ is obtained by picking a representative $\theta\colon S/[F, S] \to N$ and restricting θ to $(S \cap [F, F])/[F, S]$. Since η represents ξ and β is the restriction of η to $(S \cap [F, F])/[F, S]$, the theorem follows.

Remark. If $\mathrm{Ext}(Q_{ab}, N) = 0$, then Φ is an isomorphism, so that β characterizes the central extension. An important special case is that in which $Q_{ab} = 0$ (i.e., Q is *perfect*). The fact that β characterizes the extension when Q is perfect may be seen directly by observing that then $\bar{\sigma}\colon S/[F, S] \to F/[F, F]$ is surjective, so that two homomorphisms $S/[F, S] \to N$ determine the same element of $H^2(Q; N)$ (see Theorem 2.1) if and only if they agree on H_2Q.

We close this section by recalling from VI.8 of [4] how β_B and σ_B are defined in (1.2). In terms of the resolution (2.12),

$$H_2(Q; B) = \ker(S_{ab} \otimes_Q B \to JF \otimes_F B) \tag{2.13}$$

and then $\beta_B: H_2(Q; B) \to N_{ab} \otimes_Q B$ is given by restricting to $H_2(Q; B)$ the homomorphism $S_{ab} \otimes_Q B \to N_{ab} \otimes_Q B$ induced by ε''. As to σ_B, we exploit the short exact sequence (Theorem VI.6.3 of [4])

$$N_{ab} \rightarrowtail JG \otimes_G \mathbf{Z}Q \twoheadrightarrow JG$$

of Q-modules to obtain

$$N_{ab} \otimes_Q B \to JG \otimes_G B. \tag{2.14}$$

Moreover the image of this homomorphism obviously lies in the kernel of $JG \otimes_G B \to B$, that is, in $H_1(G; B)$, and thus (2.14) determines σ_B.

3. The Commutator Map and the Ganea Term

Given a *central* group extension $N \rightarrowtail G \twoheadrightarrow Q$ we now define a homomorphism $\chi: G_{ab} \otimes N \to H_2 G$ which will yield exactness in

$$G_{ab} \otimes N \xrightarrow{\chi} H_2 G \xrightarrow{\alpha} H_2 Q. \tag{3.1}$$

We return to the presentation (2.5) and recall (2.11) that then

$$\ker \alpha = [F, S]/[F, R].$$

Thus our objective is to define a natural surjection $G_{ab} \otimes N \to [F, S]/[F, R]$. We define a map

$$c: F \times S \to [F, S]/[F, R]$$

by

$$c(x, s) = [x, s]\,[F, R], \qquad x \in F, \quad s \in S \tag{3.2}$$

where $[x, s]$ is the commutator $xsx^{-1}s^{-1}$. We call c the *commutator map* (relative to the presentation (2.5)).

PROPOSITION 3.1. *The commutator map c is a bihomomorphism, that is,*

$$c(xx', s) = c(x, s)\, c(x', s),$$
$$c(x, ss') = c(x, s)\, c(x, s').$$

Proof. Since $[ab, c] = a[b, c]\,a^{-1}[a, c]$, and $[a, bc] = [a, b]\,b[a, c]\,b^{-1}$, we have

$$c(xx', s) = x\,c(x', s)\,x^{-1}\,c(x, s),$$
$$c(x, ss') = c(x, s)\,s\,c(x, s')\,s^{-1}.$$

But $[F, S] \subseteq R$ and $[F, S]/[F, R]$ is commutative. Thus

$$c(xx', s) = c(x, s)\, c(x', s),$$
$$c(x, ss') = c(x, s)\, c(x, s').$$

Again, since $[F, S]/[F, R]$ is commutative, Proposition 3.1 shows that $c([F, F] \times S) = e$; plainly $c(R \times S) = e$, $c(F \times R) = e$. Thus c induces a bihomomorphism of abelian groups

$$F/[F, F]\,R \times S/R \to [F, S]/[F, R],$$

and hence a homomorphism

$$c_1 : G_{ab} \otimes N \to [F, S]/[F, R],$$

which is plainly surjective. We define χ to be the composite of c_1 and the embedding of $[F, S]/[F, R] = \ker\alpha$ in H_2G. We have thus proved, for a central extension $N \rightarrowtail G \twoheadrightarrow Q$,

THEOREM 3.2. *The commutator map c defines a homomorphism* $\chi : G_{ab} \otimes N \to H_2G$ *such that the sequence*

$$G_{ab} \otimes N \xrightarrow{\chi} H_2G \xrightarrow{\alpha} H_2Q \xrightarrow{\beta} N \to G_{ab} \to Q_{ab} \to 0$$

is exact.

We show later that χ is equivalent to Ganea's χ_0. This would imply the naturality of χ, but we give now an independent proof so that our arguments may be entirely self-contained. Suppose given a map of central extensions,

$$\begin{array}{ccccc} N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ N_0 & \overset{\mu_0}{\rightarrowtail} & G_0 & \overset{\varepsilon_0}{\twoheadrightarrow} & Q_0, \end{array} \tag{3.3}$$

and a presentation of $N_0 \rightarrowtail G_0 \twoheadrightarrow Q_0$, designated by (2.5) with all elements assigned the suffix 0. We lift f_2 to $f : F \to F_0$ such that $\varepsilon_0' f = f_2 \varepsilon'$. Then $f(S) \subseteq S_0$, $f(R) \subseteq R_0$. Thus if $c_0 : F_0 \times S_0 \to [F_0, S_0]/[F_0, R_0]$ is the commutator map,

$$c_0(fx, fs) = [fx, fs]\,[F_0, R_0] = f[x, s]\,[F_0, R_0] = f_* c(x, s),$$

where $f_* : [F, S]/[F, R] \to [F_0, S_0]/[F_0, R_0]$ is induced by f. It follows that the diagram

$$\begin{array}{ccc} G_{ab} \otimes N & \xrightarrow{\chi} & H_2G \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ G_{0ab} \otimes N_0 & \xrightarrow{\chi} & H_2G_0 \end{array}$$

commutes, where the vertical arrows are induced by (3.3) and are independent of the choice of presentations. This shows that χ is itself independent of the choice of presentation and is natural.

Ganea's map $\chi_0 : G_{ab} \otimes N \to H_2G$ is effectively defined as follows. The multiplication in G induces a homomorphism $\mu : G \times N \to G$, inducing $\mu_* : H_2(G \times N) \to H_2G$. According to the Künneth formula, $H_1G \otimes H_1N (= G_{ab} \otimes N)$ is naturally embedded in $H_2(G \times N)$.

Then

$$\chi_0 = \mu_* \,|\, (G_{ab} \otimes N).$$

We prove

THEOREM 3.3. $\chi_0 = -\chi$.

Proof. We first study, in general, the embedding of $G_{1ab} \otimes G_{2ab}$ in $H_2(G_1 \times G_2)$. Since we know this is natural, it suffices to choose suitable resolutions. Thus we choose the Gruenberg resolutions

$$\cdots \to R_{iab} \otimes \mathbf{Z}G_i \xrightarrow{\partial_2} JF_i \otimes_{F_i} \mathbf{Z}G_i \xrightarrow{\partial_1} \mathbf{Z}G_i \xrightarrow{\partial_0} \mathbf{Z} \to 0,$$

of $\mathbf{Z}$ over G_i, corresponding to presentations $R_i \rightarrowtail F_i \twoheadrightarrow G_i$, $i=1, 2$. Then a partial resolution of $\mathbf{Z}$ over $G_1 \times G_2$ is given by

$$\begin{aligned} \cdots \to (JF_1 \otimes_{F_1} \mathbf{Z}G_1) \otimes (JF_2 \otimes_{F_2} \mathbf{Z}G_2) &\xrightarrow{\partial \otimes 1 - 1 \otimes \partial} \{(JF_1 \otimes_{F_1} \mathbf{Z}G_1) \otimes \mathbf{Z}G_2\} \\ &\oplus \{\mathbf{Z}G_1 \otimes (JF_2 \otimes_{F_2} \mathbf{Z}G_2)\} \to \mathbf{Z}G_1 \otimes \mathbf{Z}G_2 \to \mathbf{Z} \to 0, \end{aligned} \tag{3.4}$$

where we have only written down that part of C_2 which contributes to $H_1G_1 \otimes H_1G_2$. Tensoring with $\mathbf{Z}$ over $G_1 \times G_2$, we get

$$(JF_1 \otimes_{F_1} \mathbf{Z}) \otimes (JF_2 \otimes_{F_2} \mathbf{Z}) \xrightarrow{0} JF_1 \otimes \mathbf{Z} \oplus \mathbf{Z} \otimes JF_2.$$

Thus we have proved – in view of the natural isomorphisms $JF_i \otimes_{F_i} \mathbf{Z} \cong F_{iab}$, $i=1, 2$ – writing Z_k for the kth cycle group,

LEMMA 3.4. *With respect to the resolution* (3.4), $Z_1G_1 \otimes Z_2G_2$ *is embedded in* $Z_2(G_1 \times G_2)$ *as* $F_{1ab} \otimes F_{2ab}$, *and hence* $G_{1ab} \otimes G_{2ab}$ *is embedded in* $H_2(G_1 \times G_2)$ *as* $F_1/[F_1, F_1]\, R_1 \otimes F_2/[F_2, F_2]\, R_2$.

We now revert to our special case. We must construct a chain-map $\phi_0, \phi_1, \phi_2, \cdots$

$$\begin{array}{ccccccccc} \cdots \to (JF \otimes_F \mathbf{Z}G) \otimes (JS \otimes_S \mathbf{Z}N) & \xrightarrow{\partial_2} & \{(JF \otimes_F \mathbf{Z}G) \otimes \mathbf{Z}N\} \oplus \{\mathbf{Z}G \otimes (JS \otimes_S \mathbf{Z}N)\} & \xrightarrow{\partial_1} & \mathbf{Z}G \otimes \mathbf{Z}N & \to & \mathbf{Z} & \to & 0 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \| & & \\ \cdots \to R_{ab} \otimes \mathbf{Z}G & \xrightarrow{\partial_2} & JF \otimes_F \mathbf{Z}G & \xrightarrow{\partial_1} & \mathbf{Z}G & \longrightarrow & \mathbf{Z} & \to & 0 \end{array}$$

compatible with $\mu: G\times N \to G$. Direct calculation shows that we may take

$$\begin{aligned}
&\phi_0(e\otimes e) = e,\\
&\phi_1((x-e)\otimes_F e\otimes e) = (x-e)\otimes_F e,\\
&\phi_1(e\otimes(s-e)\otimes_S e) = (s-e)\otimes_F e,\\
&\phi_2((x-e)\otimes_F e\otimes(s-e)\otimes_S e) = [s,x]\,[R,R]\otimes xs, \quad x\in F, \quad s\in S.
\end{aligned}$$

The last formula is justified by observing that

$$\begin{aligned}
&\phi_1\partial_2((x-e)\otimes_F e\otimes(s-e)\otimes_S e)\\
&\quad = \phi_1((\varepsilon' x-e)\otimes_F(s-e)\otimes_S e-(x-e)\otimes_F e\otimes(\varepsilon'' s-e))\\
&\quad = (s-e)\otimes_F(\varepsilon' x-e)-(x-e)\otimes_F(\varepsilon'' s-e)\\
&\quad = (s-e)(x-e)\otimes_F e-(x-e)(s-e)\otimes_F e\\
&\quad = (sx-xs)\otimes_F e\\
&\quad = ([s,x]-e)\otimes_F xs\\
&\quad = \partial_2([s,x]\,[R,R]\otimes xs).
\end{aligned}$$

Again writing Z_k for the kth cycle group, we observe that ϕ_2 induces

$$\phi_2: Z_1G\otimes Z_1N\to Z_2G$$

given by

$$\phi_2(x[F,F]\otimes s[S,S]) = [s,x]\,[R,R],$$

and so induces, in the light of Lemma 3.4, χ_0 given by

$$\chi_0(xR[F,F]\otimes sR) = [s,x]\,[F,R].$$

(Recall that $[S,S]\subseteq R$). This proves the theorem, since

$$\chi(xR[F,F]\otimes sR) = [x,s]\,[R,R].$$

Remark. The difference of sign is not unexpected, since Ganea is considering the *left* operation of N on G.

4. Stem-Extensions

A central extension $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ is called a *stem-extension* if $N\subseteq[G,G]$. From (1.3) we immediately deduce

PROPOSITION 4.1. *Let $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a central extension. Then the following statements are equivalent:*

(i) $N \rightarrowtail G \twoheadrightarrow Q$ *is a stem-extension;*

(ii) $\mu_*: N \to G_{ab}$ *is the zeromap;*

(iii) $\tau = \varepsilon_*: G_{ab} \to Q_{ab}$ *is an isomorphism;*

(iv) $\beta: H_2Q \to N$ *is an epimorphism.*

Note that (iv) implies that if $\xi \in H^2(Q; N)$ is the characteristic class of the central extension, then the stem-extensions are precisely those for which $\Phi(\xi)$ is an epimorphism, where

$$\Phi: H^2(Q; N) \twoheadrightarrow \mathrm{Hom}(H_2Q, N)$$

is the natural epimorphism of the universal coefficient theorem.

We will apply the notion of a stem-extension in later sections. Here we wish to show that the exact sequence of Theorem 3.2 extends two further places to the left in the case of stem-extensions. However, our arguments encompass a generalization of the notion of stem-extensions, which we now give.

We say that the central extension $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ is a *weak stem-extension* if μ induces $0: N \otimes N \to G_{ab} \otimes N$. Thus we have, by applying (1.2) – compare Proposition 4.1 –

PROPOSITION 4.2. *Let* $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ *be a central extension. Then the following statements are equivalent:*

(i) $N \rightarrowtail G \twoheadrightarrow Q$ *is a weak stem-extension;*

(ii) $\tau_N = \varepsilon_*: G_{ab} \otimes N \to Q_{ab} \otimes N$ *is an isomorphism;*

(iii) $\beta_N: H_2(Q; N) \to N \otimes N$ *is an epimorphism.*

We will abbreviate 'weak stem-extension' to 'ws-extension'. We give some examples:

Examples. (a) If G is perfect then every central extension is a stem-extension.

(b) Consider $\mathbf{Z}_m \rightarrowtail \mathbf{Z}_{m^2} \twoheadrightarrow \mathbf{Z}_m$. Obviously this is not a stem-extension, but it is clearly a ws-extension, for $\mathbf{Z}_{m^2} \otimes \mathbf{Z}_m \to \mathbf{Z}_m \otimes \mathbf{Z}_m$ is certainly an isomorphism. It is interesting to note that, in this example, while β_N is, by Proposition 4.2, an epimorphism, $\beta \otimes 1: H_2Q \otimes N \to N \otimes N$ is not an epimorphism.

(c) Let p be a fixed prime, let $r \geqslant s$ be positive integers, and let $G = G(p^r, p^s)$ be the group

$$G = \{a, b \mid a^{p^r} = b^{p^s} = a^{-1}b^{-1}ab\}.$$

Then a is of order p^{r+s}, and the center of G is generated by a^{p^s}. For any $t \geqslant s$, let N_t be the (central) subgroup of G generated by a^{p^t}. One may then readily verify that $N_t \rightarrowtail G \twoheadrightarrow Q_t$ is a stem-extension iff $t \geqslant r$ and a ws-extension iff $t \geqslant \frac{1}{2}(r+s)$.

Our main theorem, which we prove by the elementary methods of Sections 2 and 3 is the following.

THEOREM 4.3. *Let* $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ *be a weak stem-extension. Then there is a homo-*

morphism $\delta : H_3Q \to G_{ab} \otimes N$ such that the sequence

$$H_3G \xrightarrow{\varepsilon_*} H_3Q \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2G \xrightarrow{\varepsilon_*} H_2Q \xrightarrow{\beta} N \xrightarrow{\sigma} G_{ab} \xrightarrow{\varepsilon_*} Q_{ab} \to 0$$

is exact.

Proof. We have only to define δ and prove exactness at H_3Q and $G_{ab} \otimes N$. Now by the reduction theorem (see for example [4; Corollary VI. 6.5]), we have natural isomorphisms

$$H_3G \cong H_1(G; R_{ab}), \qquad H_3Q \cong H_1(Q; S_{ab}),$$

so that it is sufficient to define $\delta : H_1(Q; S_{ab}) \to G_{ab} \otimes N$ and prove exactness in

$$H_1(G; R_{ab}) \xrightarrow{\varepsilon_*} H_1(Q; S_{ab}) \xrightarrow{\delta} G_{ab} \otimes N \xrightarrow{\chi} H_2G. \tag{4.1}$$

Now from the exact sequence of Q-modules

$$R/[S, S] \rightarrowtail S/[S, S] \twoheadrightarrow N$$

we get a coefficient sequence

$$H_2(Q; N) \xrightarrow{\psi} H_1(Q; R/[S, S]) \xrightarrow{\phi'} H_1(Q; S_{ab}) \xrightarrow{\delta'} H_1(Q; N) \xrightarrow{\chi'} (R/[S, S])_Q \to \cdots. \tag{4.2}$$

Since $\varepsilon_* : G_{ab} \otimes N \cong Q_{ab} \otimes N = H_1(Q; N)$, we may define δ by $\delta = \varepsilon_*^{-1}\delta'$. We now show that the diagram

$$\begin{array}{ccc} G_{ab} \otimes N & \xrightarrow{\chi} & R/[F, R] \\ \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \zeta} \\ Q_{ab} \otimes N & \xrightarrow{\chi'} & R/[F, R][S, S] \end{array} \tag{4.3}$$

commutes, where ζ is the natural projection; here no use is made of the assumption that $N \rightarrowtail G \twoheadrightarrow Q$ is weak stem. Using the Gruenberg resolution of $\mathbf{Z}$ over Q, we have $C_1 \otimes_Q N = JF \otimes_F N$, consisting exclusively of cycles. Thus, to compute χ' on $[(x-e) \otimes_F sR]$ we pass to $(x-e) \otimes_F s[S, S]$ and apply the boundary ∂_1, obtaining $xsx^{-1}s^{-1}[S, S]$. We have shown that $\chi'[(x-e) \otimes_F sR] = [x, s][F, R][S, S]$. Identifying $JF \otimes_F N$ with $F_{ab} \otimes N$, we find

$$\chi'(xS[F, F] \otimes sR) = [x, s][F, R][S, S],$$

proving (4.3). Now enlarge (4.3) to the diagram, with exact columns,

$$\begin{array}{ccc} N \otimes N & \xrightarrow{\bar{\chi}} & [S, S]/[S, R] \\ \downarrow{\scriptstyle \mu_*} & & \downarrow \\ G_{ab} \otimes N & \xrightarrow{\chi} & R/[F, R] \\ \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \zeta} \\ Q_{ab} \otimes N & \xrightarrow{\chi'} & R/[F, R][S, S] \end{array}, \tag{4.4}$$

where $\bar{\chi}$ is defined by the commutator map for the extension $N \rightarrowtail N \twoheadrightarrow 1$. It follows from (4.4) that, in the case of a ws-extension, ε_* induces an isomorphism of $\ker\chi$ onto $\ker\chi'$. Thus the exactness of (4.2) at $H_1(Q; N)$ implies the exactness of the sequence of Theorem 4.3 at $G_{ab}\otimes N$.

Before proceeding with the argument, we remark that it follows from (4.4) that, for a ws-extension,

$$[S, S] \subseteq [F, R], \tag{4.5}$$

so that ζ in (4.3) is the identity, and $(R/[S, S])_Q = R/[F, R]$.

It remains to prove the exactness of (4.1) at $H_1(Q; S_{ab})$. Our first step is to factorize $\varepsilon_*: H_1(G; R_{ab}) \to H_1(Q; S_{ab})$ as $\varepsilon_* = \phi'\phi''\phi'''$,

$$H_1(G; R_{ab}) \xrightarrow{\phi'''} H_1(Q; (R_{ab})_N) \xrightarrow{\phi''} H_1(Q; R/[S, S]) \xrightarrow{\phi'} H_1(Q; S_{ab}). \tag{4.6}$$

Here ϕ''' is the 'change-of-rings' homomorphism; ϕ'' is induced by the sequence of Q-modules,

$$[S, S]/[S, R] \rightarrowtail R/[S, R] \twoheadrightarrow R/[S, S]$$

(note that $(R_{ab})_N = R/[S, R]$); and ϕ' is as in (4.2). We will prove

$$\phi''' \text{ is surjective}; \tag{4.7}$$

$$\operatorname{im}\phi'\phi'' = \operatorname{im}\phi'. \tag{4.8}$$

These two facts together establish the exactness of (4.1) at $H_1(Q; S_{ab})$ – in view of (4.2) – and hence Theorem 4.3. To prove (4.7), we demonstrate the following more general lemma.

LEMMA 4.4. *Given any extension* $N \rightarrowtail G \overset{\varepsilon}{\twoheadrightarrow} Q$ *and any* G*-module* B,

$$\varepsilon_*: H_1(G; B) \to H_1(Q; B_N)$$

is surjective.

Proof of Lemma. This is well-known (see for example [9]), but we give an elementary proof here in the framework of our paper. Using the Gruenberg resolutions, we readily obtain a commutative diagram

$$\begin{tikzcd}
 & H_1(G; B) \arrow[r, "\varepsilon_*"] \arrow[d, tail] & H_1(Q; B_N) \arrow[d, tail] \\
JN \otimes_N B \arrow[r, "\mu_*"] \arrow[d, equal] & JG \otimes_G B \arrow[r, two heads, "\varepsilon_*"] \arrow[d] & JQ \otimes_Q B_N \arrow[d] \\
JN \otimes_N B \arrow[r] & B \arrow[r] & B_N
\end{tikzcd}$$

in which the columns are exact, the bottom row is exact, and the middle row is differential. It is now trivial that $\varepsilon_*: H_1(G; B) \to H_1(Q; B_N)$ is surjective.

The proof of (4.8) is based on the following lemma. As usual, we refer to the presentation (2.5).

LEMMA 4.5. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a central extension. Let $\delta'' : H_1(Q; R/[S, S]) \to \to ([S, S]/[S, R])_Q$ be the connecting homomorphism associated with the sequence of Q-modules*

$$[S, S]/[S, R] \rightarrowtail R/[S, R] \twoheadrightarrow R/[S, S]$$

and let $\bar{\chi} : N \otimes N \to [S, S]/[S, R]$ be the commutator map (4.4) *for the extension $N \rightarrowtail N \twoheadrightarrow \twoheadrightarrow 1$. Then $([S, S]/[S, R])_Q = [S, S]/[S, R]$ and the square*

$$\begin{array}{ccc} H_2(Q; N) & \xrightarrow{\psi} & H_1(Q; R/[S, S]) \\ \downarrow{\scriptstyle \beta_N} & & \downarrow{\scriptstyle \delta''} \\ N \otimes N & \xrightarrow{\bar{\chi}} & [S, S]/[S, R] \end{array} \tag{4.9}$$

commutes.

Proof of Lemma. Since N is central, Q operates trivially on $H_2 N = [S, S]/[S, R]$. We now prove the commutativity of (4.9) by again appealing to the Gruenberg resolution of $\mathbf{Z}$ over Q. Then a 2-chain of $C_2(Q; N)$ has the form $\sum_i s_i [S, S] \otimes s'_i R$. Since ψ is the connecting homomorphism associated with the sequence of Q-modules

$$R/[S, S] \rightarrowtail S/[S, S] \twoheadrightarrow N,$$

it follows that the value of ψ on the class of the cycle $w = \sum_i s_i [S, S] \otimes s'_i R$ is the class, in $H_1(Q; R/[S, S])$, of $\sum_i (s_i - e) \otimes_F s'_i [S, S]$, i.e., (using $\{ \}$ for homology classes)

$$\psi \{w\} = \{\sum_i (s_i - e) \otimes_F s'_i [S, S]\}. \tag{4.10}$$

Now a 1-chain of $C_1(Q; R/[S, S]) = JF \otimes_F R/[S, S]$ has the form $\sum_j (x_j - e) \otimes_F r_j [S, S]$, and the value of δ'' on the class of the cycle $z = \sum_j (x_j - e) \otimes_F r_j \, [S, S]$ is the class, in $H_0(Q; [S, S]/[S, R])$, of $\sum_j [x_j, r_j] \, [S, R]$. But since Q operates trivially on $[S, S]/[S, R]$, we may write

$$\delta'' \{z\} = \sum_j [x_j, r_j] \, [S, R]. \tag{4.11}$$

We now define a 'commutator map' $\theta : JF \otimes_F S/[S, S] \to [F, S]/[S, R]$ by the rule

$$\theta((x - e) \otimes_F s [S, S]) = [x, s] \, [S, R]. \tag{4.12}$$

To check that θ is well-defined we must verify that

$$[x, s_1 s_2] \, [S, R] = [x, s_1] \, [x, s_2] \, [S, R] \tag{4.13}$$

and that $\theta((x - e) \, y \otimes_F s [S, S]) = \theta((x - e) \otimes_F y s y^{-1} [S, S])$, or

$$[xy, s] \, [y, s]^{-1} \, [S, R] = [x, y s y^{-1}] \, [S, R]. \tag{4.14}$$

Now (4.13) follows exactly as in the proof of Proposition 3.1, since $[S,[F,S]]\subseteq[S,R]$, and (4.14) holds since, in fact, $[xy,s]=[x,ysy^{-1}][y,s]$. Thus θ is well-defined; and we observe that $JF\otimes_F R/[S,S]$ is a subgroup of $JF\otimes_F S/[S,S]$ and that, from (4.11),

$$\theta(z)=\delta''\{z\}. \tag{4.15}$$

It thus follows from (4.10), (4.12) and (4.15) that, if $w=\sum s_i[S,S]\otimes s_i'R$,

$$\delta''\psi\{w\}=\sum_i[s_i,s_i'][S,R]. \tag{4.16}$$

Now, as shown in Section 2, $\beta_N: H_2(Q;N)\to N\otimes N=N\otimes_Q N$ is given by restricting to $H_2(Q;N)$ the homomorphism $S_{ab}\otimes_Q N\to N\otimes_Q N$ induced by ε''. Thus, with $w=\sum_i s_i[S,S]\otimes s_i'R$,

$$\beta_N\{w\}=\sum_i s_iR\otimes s_i'R,$$

so that

$$\bar{\chi}\beta_N\{w\}=\sum_i[s_i,s_i'][S,R],$$

and the lemma is proved.

The relation (4.8) quickly follows from Lemma 4.5. For we have the diagram (with exact row and column)

$$\begin{array}{ccccc} & & H_1(Q;R/[S,R]) & & \\ & & \downarrow{\scriptstyle\phi''} & & \\ H_2(Q;N) & \xrightarrow{\psi} & H_1(Q;R/[S,S]) & \xrightarrow{\phi'} & H(Q;S/[S,S]) \\ & & \downarrow{\scriptstyle\delta''} & & \\ & & [S,S]/[S,R] & & \end{array}$$

and, by Lemma 4.5, $\delta''\psi$ is surjective in the case of a ws-extension; for, in that case, β_N is surjective (Proposition 4.2), and $\bar{\chi}$ is always surjective. It follows immediately that $H_1(Q;R/[S,S])=\operatorname{im}\psi+\operatorname{im}\phi''$, so that $\operatorname{im}\phi'=\operatorname{im}\phi'\phi''$. Thus Theorem 4.3 is completely proved.

Remark. The exact sequence of [2] yields Theorem 4.3 as a special case. For the homomorphism $\sigma: H_4(N,2)\to G_{ab}\otimes N$ of [2] factors through $\mu_*: N\otimes N\to G_{ab}\otimes N$ and is thus zero for a ws-extension. Indeed, the exact sequence of [2] shows that the conclusion of Theorem 4.3 holds if and only if $\mu_*\mid\ker\bar{\chi}=0$. We will revert to this point in a subsequent paper [11].

In the case of a stem-extension, the exact sequence of Theorem 4.3 becomes

$$H_3G\xrightarrow{\varepsilon_*}H_3Q\xrightarrow{\delta}G_{ab}\otimes N\xrightarrow{\chi}H_2G\xrightarrow{\varepsilon_*}HQ\xrightarrow{\beta}N\to 0 \tag{4.17}$$

Now let Q be a given group and let U be any subgroup of H_2Q. Set $N=H_2Q/U$ and let $\beta: H_2Q\twoheadrightarrow N$ be the canonical projection. Let ξ be any element of $H^2(Q;N)$ such that

$\Phi(\xi)=\beta$, and let $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a central extension with characteristic class ξ. Then it follows from Theorem 2.2 and Proposition 4.1 that $N \rightarrowtail G \twoheadrightarrow Q$ is a stem-extension and (4.17) shows that

$$U = \operatorname{im} \varepsilon_* \cong \operatorname{coker} \chi .$$

The stem-extensions yielding the given epimorphism β are in one-to-one correspondence with the elements of $\operatorname{Ext}(H_1Q, N)$. The stem-extensions for which $U=0$, $\beta=1$ are called *stem-covers.* For a stem-cover

$$H_2Q \rightarrowtail G \overset{\varepsilon}{\twoheadrightarrow} Q$$

we have an exact sequence

$$H_3G \xrightarrow{\varepsilon_*} H_3Q \xrightarrow{\delta} G_{ab} \otimes H_2Q \xrightarrow{\chi} H_2G \to 0; \tag{4.18}$$

and the stem-covers of Q are in one-to-one correspondence with elements of $\operatorname{Ext}(H_1Q, H_2Q)$.

5. Perfect Groups

Let

$$N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q \tag{5.1}$$

be a central extension classified by $\xi \in H^2(Q; N)$ and let $\varrho: N \to N_1$ be a homomorphism of commutative groups. We then recall that if $\xi_1 = \varrho_*(\xi) \in H^2(Q; N_1)$ and

$$N_1 \rightarrowtail G_1 \twoheadrightarrow Q$$

is the central extension classified by ξ_1, there is a map of extensions

$$\begin{array}{ccccc} N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ \varrho\downarrow & & \downarrow\tau & & \| \\ N_1 & \overset{\mu_1}{\rightarrowtail} & G_1 & \overset{\varepsilon_1}{\twoheadrightarrow} & Q \end{array} . \tag{5.2}$$

We study this situation when Q is perfect, that is, $Q_{ab}=0$. In that case (see the Remark following Theorem 2.2) the central extension (5.1) is characterized by $\beta=\Phi(\xi): H_2Q \to N$, which appears in the exact sequence (1.3)

$$H_2G \to H_2Q \xrightarrow{\beta} \mathcal{N} \to G_{ab} \to 0;$$

and, if $\xi_1 = \varrho_*(\xi)$,

$$\Phi(\xi_1) = \varrho\beta . \tag{5.3}$$

Moreover,

PROPOSITION 5.1. *Let*

$$\begin{array}{ccccc} N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \\ \varrho\downarrow & & \tau\downarrow & & \downarrow\psi \\ N_1 & \underset{\mu_1}{\rightarrowtail} & G_1 & \underset{\varepsilon_1}{\twoheadrightarrow} & Q_1 \end{array}$$

be a map of extensions in which Q is perfect and N_1 is central. Then τ is uniquely determined by ϱ and ψ.

Proof. Let τ, $\tau': G \to G_1$ be two homomorphisms each yielding commutativity in relation to ϱ and ψ. Consider the function $f: G \to G_1$ given by $f(x) = \tau(x)\,\tau'(x)^{-1}$, $x \in G$. Since $\varepsilon_1\tau = \varepsilon_1\tau'$, f maps G into N_1. Since N_1 is central, it is clear that $f: G \to N_1$ is a homomorphism. Since $\tau\mu = \tau'\mu$, f is trivial on N, and thus induces a homomorphism $g: Q \to N_1$. Since Q is perfect and N_1 is commutative, $g = 0$. Thus $f = 0$, so that $\tau = \tau'$.

COROLLARY 5.2. *In the diagram* (5.2), *with Q perfect, τ is uniquely determined by ϱ.*

We now, temporarily, restrict attention to *stem*-extensions (5.1) with *Q perfect.* We note that, Q being perfect, (5.1) is a stem extension if and only if G is also perfect. Theorem 3.2 then yields the short exact sequence, for stem extensions (5.1),

$$0 \to H_2G \overset{\alpha}{\to} H_2Q \overset{\beta}{\to} N \to 0. \tag{5.4}$$

Thus, with every stem extension (5.1) of the perfect group Q, we may associate a subgroup $U = H_2G$ of H_2Q, such that the stem extension is characterized by the projection $H_2Q \twoheadrightarrow H_2Q/U = N$. Conversely, given $U \subseteq H_2Q$, set $N = H_2Q/U$, $\beta: H_2Q \to N$ the projection, and let (5.1) be the central extension characterized by β. Then, plainly, (5.1) is a stem extension and

$$H_2G = U.$$

Thus we have proved

THEOREM 5.3. *There is a one-one correspondence between stem extensions of the perfect group Q and subgroups of H_2Q, given by associating with* (5.1) *the group H_2G.*

Now let us take the stem extension

$$H_2Q \rightarrowtail Q_0 \twoheadrightarrow Q \tag{5.5}$$

of the perfect group Q with $H_2Q_0 = 0$. It is *universal* in the following sense. Given any central extension (5.1), characterized by $\beta: H_2Q \to N$, there exists, according to Corollary 5.2, a unique homomorphism $\tau: Q_0 \to G$ such that the diagram

$$\begin{array}{ccccc} H_2Q & \rightarrowtail & Q_0 & \twoheadrightarrow & Q \\ \downarrow\beta & & \downarrow\tau & & \| \\ N & \rightarrowtail & G & \twoheadrightarrow & Q \end{array}$$

commutes (note that (5.5) is characterized by the identity map of H_2Q). Notice that, if (5.1) is also a stem extension, then, in (5.6), β is surjective, so that τ is surjective. Moreover, $\ker \tau = \ker \beta = H_2G$, which is central in Q_0. Thus if we describe G as a *cover* of Q if there exists a stem extension (5.1), then Theorem 5.3 establishes a one-one correspondence between covers of Q and subgroups of H_2Q [6], and our subsequent argument shows that Q_0 is the *universal cover* of Q in that it (uniquely) covers any cover of Q.

PROPOSITION 5.4. *The central extension* (5.1) *is the universal cover of the perfect group Q if and only if $H_1G=0$, $H_2G=0$.*

Proof. This is immediate, since, if $H_1G=0, H_2G=0$, then β and τ are isomorphisms in (5.6).

We next describe Q_0 and τ in (5.6) by means of the presentation (2.5). We have the evident

PROPOSITION 5.5. *Let* (5.1) *be a stem extension of the perfect group Q. Then*

$$[F, S] = [F, R].$$

Proof. We showed in Section 3 that for any central extension, the kernel of $\alpha: H_2G \to H_2Q$ is $[F, S]/[F, R]$. But for a stem extension of the perfect group Q, $\ker\alpha = 0$ (5.4); indeed, it is plain that $[F, S] = [F, R]$ for a ws-extension of the perfect group Q.

Now if (5.1) is the universal cover, then $H_1G=0$, $H_2G=0$, $N=H_2Q$. Thus

$$R \cap [F, F] = [F, R] = [F, S],$$

so that

$$G = F/R = [F, F]/(R \cap [F, F]) = [F, F]/[F, S].$$

Conversely, let $S \rightarrowtail F \twoheadrightarrow Q$ be a free presentation of the perfect group Q; then $Q = [F, F]/(S \cap [F, F])$ and if we set $G = [F, F]/[F, S]$ we obtain the extension

$$H_2Q \rightarrowtail G \twoheadrightarrow Q. \tag{5.7}$$

This extension is plainly central and is characterized by the identity on H_2Q. It follows that (5.7) is a stem extension and $H_2G=0$.

If $N \rightarrowtail G \twoheadrightarrow Q$ is an arbitrary central extension of the perfect group Q, characterized by $\beta: H_2Q \to N$, we induce a homomorphism $\tau: [F, F]/[F, S] \to F/R$ from the inclusions $[F, F] \subseteq F$, $[F, S] \subseteq R$. Plainly, τ restricts to β on $H_2Q = (S \cap [F, F])/[F, S]$ and induces a commutative diagram (5.6). By uniqueness, it is therefore the homomorphism there described. We sum up:

THEOREM 5.6. *Given the presentation $S \rightarrowtail F \twoheadrightarrow Q$ of the perfect group Q, the* *universal central (stem) extension* $H_2Q \overset{\mu_0}{\rightarrowtail} Q_0 \overset{\varepsilon_0}{\twoheadrightarrow} Q$ *can be given by* $Q_0 = [F, F]/[F, S]$ *with* ε_0 *induced by the inclusions* $[F, F] \subseteq F$, $[F, S] \subseteq S$. *For any central extension* $N \rightarrowtail G \twoheadrightarrow Q$ *with presentation* (2.5), *the universal homomorphism* $\tau: Q_0 \to G = F/R$ *is induced by the inclusions* $[F, F] \subseteq F$, $[F, S] \subseteq R$.

Remarks. 1) Theorem 5.6 may readily be used to prove, without any spectral sequence techniques, the result cited in [6] that, given a *short exact sequence of perfect groups* $K \rightarrowtail G \twoheadrightarrow Q$, the lifted sequence $K_0 \to G_0 \twoheadrightarrow Q_0$ is exact.

2) It is also clear how to present the cover of Q corresponding to any subgroup U of H_2Q. With $H_2Q = (S \cap [F, F])/[F, S]$ we have $U = V/[F, S]$, $[F, S] \subseteq V \subseteq S \cap [F, F]$, and $V \rightarrowtail [F, F] \twoheadrightarrow G$ is then a presentation of the required cover G.

We now prove a theorem motivated by the analogy with covering space theory.

THEOREM 5.7 *Let* $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ *be a central extension, let* X *be a perfect group and let* $\psi: X \to Q$ *be a homomorphism. Then* ψ *lifts, uniquely, to* $\phi: X \to G$ *with* $\varepsilon\phi = \psi$, *if and only if*

$$\psi_* H_2 X \subseteq \varepsilon_* H_2 G. \tag{5.8}$$

Proof. Plainly, if ψ lifts, (5.8) holds. Also Proposition 5.1 affirms that if ϕ exists, it is unique. Thus it suffices to prove that (5.8) implies the existence of ϕ.

We use the presentation of X,

$$U \rightarrowtail V \twoheadrightarrow X.$$

Let ψ be lifted to $\eta: V \to F$ with $\eta(U) \subseteq S$. Then

$$\begin{aligned} \varepsilon_* H_2 G &= (R \cap [F, F])\,[F, S]/[F, S] \\ &= R \cap [F, F]/[F, S], \end{aligned}$$

since N is central in G. Thus the hypothesis (5.8) may be translated into the condition

$$\eta(U \cap [V, V]) \subseteq R \cap [F, F]. \tag{5.9}$$

Since X is perfect we may present it by

$$U \cap [V, V] \rightarrowtail [V, V] \twoheadrightarrow X.$$

We have the diagram

$$\begin{tikzcd} U \cap [V, V] \arrow[r, "\eta"] \arrow[d, tail] & R \cap [F, F] \arrow[r, tail] \arrow[d, tail] & S \cap [F, F] \arrow[d, tail] \\ {[V, V]} \arrow[r, "\eta"] \arrow[d, two heads] & {[F, F]} \arrow[r] \arrow[d] & {[F, F]} \arrow[d] \\ X \arrow[r, dashed, "\phi"] & G \arrow[r, "\varepsilon"] & Q \end{tikzcd} \quad , \tag{5.10}$$

determining a map $\phi: X \to G$ such that $\varepsilon\phi = \psi$.

We may apply this theorem in the following situation. Consider the diagram

$$\begin{tikzcd} N \arrow[r, tail, "\mu"] & G \arrow[r, two heads, "\varepsilon"] & Q \arrow[d, "\psi"] \\ N_1 \arrow[r, tail, "\mu_1"] & G_1 \arrow[r, two heads, "\varepsilon_1"] & Q_1 \end{tikzcd}, \tag{5.11}$$

where the top row is a stem extension of the perfect group Q, and the bottom row is a central extension. Then we know that we may complete (5.11) to the commutative diagram (and uniquely)

$$\begin{tikzcd} N \arrow[r, tail, "\mu"] \arrow[d, "\varrho"] & G \arrow[r, two heads, "\varepsilon"] \arrow[d, "\tau"] & Q \arrow[d, "\psi"] \\ N_1 \arrow[r, tail, "\mu_1"] & G_1 \arrow[r, two heads, "\varepsilon_1"] & Q_1 \end{tikzcd} \tag{5.12}$$

if and only if $\psi_* \varepsilon_* H_2 G \subseteq \varepsilon_{1*} H_2 G_1$. However this latter is precisely the condition that we may find $\varrho' : N \to N_1$ such that the diagram

$$\begin{tikzcd} H_2 Q \arrow[r, "\beta"] \arrow[d, "\psi_*"] & N \arrow[d, "\varrho'"] \\ H_2 Q_1 \arrow[r, "\beta_1"] & N_1 \end{tikzcd} \tag{5.13}$$

commutes, as follows immediately from (1.3). Moreover, since (5.12) induces

$$\begin{tikzcd} H_2 Q \arrow[r, "\beta"] \arrow[d, "\psi_*"] & N \arrow[d, "\varrho"] \\ H_2 Q_1 \arrow[r, "\beta_1"] & N_1 \end{tikzcd} \tag{5.14}$$

and β is surjective, it follows that $\varrho = \varrho'$. We thus have the corollary of Theorem 5.7.

COROLLARY 5.8. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a stem extension of the perfect group Q, let $N_1 \rightarrowtail G_1 \twoheadrightarrow Q_1$ be a central extension, and let $\varrho : N \to N_1$, $\psi : Q \to Q_1$ be homomorphisms. Then there exists $\tau : G \to G_1$ rendering the diagram* (5.12) *commutative if and only if* (5.14) *commutes. If τ exists, it is unique.*

Remark. We note, in particular, that if (5.14) commutes, then there exists a canonical homomorphism $\theta : H_2 G \to H_2 G_1$, namely τ_*, such that

$$\begin{tikzcd} H_2 G \arrow[r, "\varepsilon"] \arrow[d, "\theta"] & H_2 Q \arrow[d, "\psi_*"] \\ H_2 G_1 \arrow[r, "\varepsilon_{1*}"] & H_2 Q_1 \end{tikzcd}$$

commutes.

We close this section by observing that Theorem 4.3 leads to an immediate proof of the following result of Kervaire [6] (see also [2]):

THEOREM 5.9. *Let $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a stem extension of the perfect group Q. Then $\varepsilon_*: H_3G \to H_3Q$ is an epimorphism.*

6. Appendix: Remarks on Algebraic K-Theory

Certain exact sequences of algebraic K-theory can easily be obtained within the framework of this paper. We first recall some known facts and definitions.

We consider $G=GL(\Lambda)$, Λ being a ring with unity, and the subgroup $E=E(\Lambda)$ generated by elementary matrices $1+\lambda E_{ij}$, $i\neq j$, $\lambda\in\Lambda$. Given an ideal $\mathfrak{a}\subset\Lambda$, we write $\bar{G}=G(\Lambda/\mathfrak{a})$, $\bar{E}=E(\Lambda/\mathfrak{a})$ and denote by $\pi: G\to\bar{G}$, $\pi': E\to\bar{E}$ the canonical maps, by N the kernel of π (i.e., the *congruence subgroup* $G(\Lambda, \mathfrak{a})$). The group $E(\Lambda, \mathfrak{a})$, the normal hull in E of all elementary matrices $1+\alpha E_{ij}$, $\alpha\in\mathfrak{a}$, is contained in [1]) $\ker\pi' = E\cap N$. The following facts are easily proved: $E=[E, E]$, $\bar{E}=[\bar{E}, \bar{E}]$, $E(\Lambda, \mathfrak{a}) =[E, E(\Lambda, \mathfrak{a})$, and $\pi': E\to\bar{E}$ is an epimorphism.

The generalized *Whitehead Lemma* [10, Theorem 15.1]

$$[G, N] \subseteq E(\Lambda, \mathfrak{a})$$

yields the further relations

$$E = [G, G] \tag{6.1}$$

$$E(\Lambda, \mathfrak{a}) = [G, N] \tag{6.2}$$

$$[E, E\cap N] = [G, N]. \tag{6.3}$$

The proof of (6.3) is as follows: the inclusion $[E, E\cap N]\subseteq[G, N]$ is obvious, and, on the other hand, we have

$$[E, E\cap N] \supseteq [E, E(\Lambda, \mathfrak{a})] = E(\Lambda, \mathfrak{a}) = [G, N].$$

We now reflect this situation by the following more general set-up. Let G and $\bar{G}$ be *groups with perfect commutator subgroups*, let $E=[G, G]$ and $\bar{E}=[\bar{G}, \bar{G}]$; let π be a homorphism $G\to\bar{G}$ which maps E onto $\bar{E}$, and suppose that $[G, N]=[E, E\cap N]$, where $N=\ker\pi$.

We write $Q=G/N$ and map the extension $E\cap N\rightarrowtail E\twoheadrightarrow\bar{E}$ into $N\rightarrowtail G\overset{\varepsilon}{\twoheadrightarrow} Q$ by inclusions. The induced map of the exact sequences (1.3) is given by the commutative diagram

$$\begin{array}{ccccccccc}
H_2E & \xrightarrow{\alpha'} & H_2\bar{E} & \xrightarrow{\beta'} & E\cap N/[E, E\cap N] & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\varrho} & & \downarrow & & \downarrow & \\
H_2G & \xrightarrow{\alpha} & H_2Q & \xrightarrow{\beta} & N/[G,N] & \xrightarrow{\delta} & H_1G & \xrightarrow{\varepsilon_*} & H_1Q & \to 0\,.
\end{array} \tag{6.4}$$

[1]) On p. 211 of [10], Swan inadvertently defines $E(\Lambda, \mathfrak{a})$ to be $\ker\pi'$. The statement on p. 212 is the correct one – and the only one which Swan uses.

We note that $\operatorname{Im}\varrho = (E \cap N)/[G, N] = \ker\delta$; this is also $\operatorname{im}\varrho\beta'$. Since $[E, E \cap N] = [G, N]$, ϱ is a monomorphism, so $\ker\varrho\beta' = \ker\beta' = \operatorname{im}\alpha'$. We further remark that π induces a monomorphism $Q \rightarrowtail \bar{G}$ which maps $[Q, Q] = \varepsilon(E)$ isomorphically onto $\bar{E}$. It follows that the kernel of $\pi_*: H_1G = G/E \to H_1\bar{G} = \bar{G}/\bar{E}$ is equal to the kernel of $\varepsilon_*: G/E \to Q/[Q, Q]$. Writing $\sigma = \varrho\beta'$, we obtain from (6.4) the exact sequence

$$H_2E \xrightarrow{\alpha'} H_2\bar{E} \xrightarrow{\sigma} N/[G, N] \xrightarrow{\delta} H_1G \xrightarrow{\pi_*} H_1\bar{G}. \tag{6.5}$$

In the case of K-theory, the groups can all be identified with familiar K-groups: $H_1G = G/E = K_1(\Lambda)$; $H_1\bar{G} = K_1(\Lambda/\mathfrak{a})$; $N/[G, N] = GL(\Lambda, \mathfrak{a})/E(\Lambda, \mathfrak{a}) = K_1(\Lambda, \mathfrak{a})$; and finally one *defines* $K_2(\Lambda)$ by $H_2E(\Lambda)$. Then (6.5) becomes the exact sequence

$$K_2(\Lambda) \to K_2(\Lambda/\mathfrak{a}) \to K_1(\Lambda, \mathfrak{a}) \to K_1(\Lambda) \to K_1(\Lambda/\mathfrak{a}). \tag{6.6}$$

The above definition of $K_2(\Lambda)$ coincides with that of Milnor, where $K_2(\Lambda)$ is the kernel of $St(\Lambda) \twoheadrightarrow E(\Lambda)$. The group $St(\Lambda)$, the Steinberg group, is defined by an explicit free presentation $S \rightarrowtail F \twoheadrightarrow St(\Lambda)$; it is a stem-extension of $E(\Lambda)$, and in fact the universal one – which implies that the kernel is $H_2E(\Lambda)$. To prove that $St(\Lambda)$ is universal, it suffices to show that it is its own universal stem-extension; i.e., that the homomorphism $[F, F]/[F, S] \to F/S$ induced by the inclusions $[F, F] \subseteq F$, $[F, S] \subseteq S$, is an isomorphism. This can be done by following Kervaire's procedure [6], using various commutator relations.

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