On the Spanier Conjecture

Autor(en): **Held, René P. / Sjerve, Denis**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **47 (1972)**

PDF erstellt am: **26.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-36352

On the Spanier Conjecture

RENÉ P. HELD and DENIS SJERVE

§ 1. Introduction

In this paper we describe how various lens spaces can be built from complex projective spaces, at least as objects in the suspension category of Spanier-Whitehead [7]. Objects in this category are finite CW complexes and morphisms (called S-maps) from X to Y are stable homotopy classes of "honest" maps from some suspension of X into some suspension of Y. We point out that the number of suspensions of X may be different from that of Y. If the "honest" map is a homotopy equivalence then the S-map is said to be an S-equivalence and X, Y are said to have the same S-type. The title of this paper refers to the following conjecture of Spanier (as communicated to the second named author):

(1.1) *Spanier Conjecture*

The mapping cone of the "canonical" map $\mathbf{R}P^q/\mathbf{R}P^r \to \mathbf{C}P^s/\mathbf{C}P^t$, where $s=[q/2]$ and $t=[r/2]$, has the S-type of some stunted complex projective space $\mathbf{C}P^u/\mathbf{C}P^v$, for some u, v. For dimensional reasons

$$u - v = \left[\frac{q-1}{2}\right] - \left[\frac{r-1}{2}\right].$$

To define the "canonical" map in the statement of the conjecture note that the Hopf map $f: S^{2n+1} \to \mathbf{C}P^n$ factors to give the map $\mathbf{R}P^{2n+1} \to \mathbf{C}P^n$, again denoted by f and also called a Hopf map. Finally, restriction gives another Hopf map $f: \mathbf{R}P^{2n} \to \mathbf{C}P^n$. The "canonical" map on the stunted real projective space $\mathbf{R}P^q/\mathbf{R}P^r$ is then the quotient of such Hopf maps.

The validity of the Spanier conjecture is suggested by the following considerations. The Puppe sequence of $f: \mathbf{R}P^{2n+1} \to \mathbf{C}P^n$ yields a map $\mu: \mathbf{C}_f \to \Sigma \mathbf{R}P^{2n+1}$. If the mapping cone C_f has the same S-type as $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ for some $d \geqslant 0$ then there would be an S-map $\lambda: \mathbf{C}P^{n+1+d}/\mathbf{C}P^d \to \Sigma^{2d+1} \mathbf{R}P^{2n+1}$ whose behaviour in ordinary cohomology is identical to that of μ. The point is that such a map λ can always be constructed by other means (see (5.1) of Adem-Gitler [3]) and this lends support to the Spanier conjecture.

The importance of this conjecture lies in the fact that if (1.1) is true then there exists an S-map $\mathbf{C}P^s/\mathbf{C}P^t \to \mathbf{C}P^u/\mathbf{C}P^v$ whose mapping cone is S-equivalent to $\mathbf{R}P^q/\mathbf{R}P^r$. In this sense $\mathbf{R}P^q/\mathbf{R}P^r$ is built from complex projective spaces.

It is obvious that one can (and should) formulate (1.1) for lens spaces associated to an arbitrary integer $p \geqslant 0$ (not necessarily a prime). Let $L^{2n+1}(p)$ denote the orbit space of the usual free action of the cyclic group $\mathbf{Z}_p$ on S^{2n+1}. Recall that if $\mathbf{Z}_p$ is presented by $\{t \mid t^p = 1\}$ then this action is given by $t \cdot (z_1, \ldots, z_{n+1}) = (\theta z_1, \ldots, \theta z_{n+1})$, where $(z_1, \ldots, z_{n+1})$ is a complex (n+1)-tuple representing a point of S^{2n+1} and $\theta = \exp(2\pi\sqrt{-1}/p)$. The $L^{2n+1}(p)$ are known as (standard) lens spaces and they carry cell structures so that the $2n$-skeleton $L^{2n}(p)$ of $L^{2n+1}(p)$ is the mapping cone C_π of the p-fold covering $\pi: S^{2n-1} \to L^{2n-1}(p)$ defined by the action. Since the lens spaces are all standard (no twisting in the action) the usual Hopf fibration $f: S^{2n+1} \to \to \mathbf{C}P^n$ factors to give another principal S^1-bundle $f: L^{2n+1}(p) \to \mathbf{C}P^n$. By restriction to the $2n$-skeleton we get another map $f: L^{2n}(p) \to \mathbf{C}P^n$. For convenience we drop the letter p from the notation and use f for all such maps $f: L^q \to \mathbf{C}P^s$, where $s = [q/2]$. Then there is a corresponding conjecture in this broader context. It reads:

(1.2) *Generalized Spanier Conjecture*

The mapping cone of the "canonical" map (quotient of two Hopf maps) $L^q/L^r \to \mathbf{C}P^s/\mathbf{C}P^t$, where $s = [q/2]$ and $t = [r/2]$, has the same S-type as $\mathbf{C}P^u/\mathbf{C}P^v$, where u, v are as in (1.1).

Our method of attack on (1.2) is to first consider the case $q = 2n+1$, $r = 0$ and determine conditions under which the mapping cone of $f: L^{2n+1} \to \mathbf{C}P^n$ is S-equivalent to $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ for some integer $d \geqslant 0$. If this is true then a cellularity argument shows that also the mapping cone of $L^{2k+1} \to \mathbf{C}P^k$ is S-equivalent to $\mathbf{C}P^{k+1+d}/\mathbf{C}P^d$ for $k \leqslant n$. Also the mapping cone of $L^{2k} \to \mathbf{C}P^k$ is homotopically equivalent to that of $L^{2k-1} \to \mathbf{C}P^{k-1}$ (see (3.1)). Thus it is sufficient to consider this case.

As remarked earlier the importance of (1.2) is that if it is true then there exists an S-map $\mathbf{C}P^s/\mathbf{C}P^t \to \mathbf{C}P^u/\mathbf{C}P^v$ whose mapping cone is S-equivalent to L^q/L^r. It is possible however that such an S-map exists without (1.2) being true. Perhaps there is always a (possibly "non-canonical") map (or maybe an S-map) $L^{2n+1} \to \mathbf{C}P^n$ whose mapping cone is a stunted complex projective space!

Finally we would like to thank E. H. Spanier and P. J. Hilton for their help and encouragement during the course of this research. We are also indebted to F. Sigrist and U. Suter for many stimulating conversations.

§ 2. Statement of Results

Let $p \geqslant 1$ be a fixed integer.

(2.1) DEFINITION. A non-negative integer d is said to satisfy *condition* $S(n, p)$ if, and only if, the coefficients of z^q, $0 \leqslant q \leqslant n$, in the power series

$$\left[\frac{\log(1+z)}{z}\right]^d \cdot \frac{(z+1)^p - 1}{pz}$$

are integers.

If $p=1$ this is the condition (C_{n+1}) of Atiyah and Todd [5]. Then d satisfies condition $S(n, 1)$ if, and only if, d is a multiple of the Atiyah-Todd number M_{n+1}.

The fact that we have made such a definition signals that there is a pertinent connection between the Spanier conjecture and $J(\mathbf{C}P^n)$.

We introduce the following notation:

(i) $T(X, \alpha)$ is the Thom complex of the vector bundle α over X

(ii) $T_n(\alpha)=T(\mathbf{C}P^n, \alpha)$

(iii) ω is the canonical complex line bundle over $\mathbf{C}P^n$

(iv) r is the realification functor from complex bundles to real ones.

Then we shall prove in §3 that the mapping cone C_f of $f: L^{2n+1} \to \mathbf{C}P^n$ is homeomorphic to the Thom complex $T_n(r(\omega^p))$. On the other hand Atiyah [4] has shown that $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ is homeomorphic to the Thom complex $T_n((d+1)\, r(\omega))$. Before stating our main theorems it is convenient to make the following definition:

(2.2) DEFINITION. The integer p is said to satisfy condition (H_n) if there exists an odd prime $q \leqslant n$ such that $p \not\equiv 0 \pmod q$.

This condition is almost always satisfied! Then our main theorems are:

(2.3) THEOREM. *Let* $f: L^{2n+1} \to \mathbf{C}P^n$ *be the Hopf fibration.*

(i) *If* $J \circ r(\omega^p) = (d+1)\, J \circ r(\omega)$ *as elements of* $J(\mathbf{C}P^n)$ *then* d *satisfies condition* $S(n, p)$ *and* C_f *is* S-*equivalent to* $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$.

(ii) *If* $n>1$, $n \not\equiv 1 \pmod 4$, *and* d *satisfies condition* $S(n,p)$ *then* $J \circ r(\omega^p) = (d+1)\, J \circ r(\omega)$ *as elements of* $J(\mathbf{C}P^n)$, *and therefore* C_f *is* S-*equivalent to* $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$.

(iii) *If* C_f *is* S-*equivalent to* $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ *and* $n \geqslant 6$ *then* d *satisfies condition* $S(n-1, p)$. *Moreover, if either* n *is odd or* p *satisfies condition* (H_n), *then* d *also satisfies condition* $S(n, p)$.

The proof of this theorem is given in §3 and §4. As a corollary we have:

(2.4) COROLLARY. *Suppose* $n>1$, $n \not\equiv 1 \pmod 4$ *and* $p>n+1$ *is a prime. Then the mapping cone of* $f: L^{2n+1} \to \mathbf{C}P^n$ *is* S-*equivalent to* $\mathbf{C}P^{n+1}$.

Proof. Under the assumption $p>n+1$ is a prime it is obvious that $d=0$ satisfies condition $S(n, p)$. Now apply part (ii) of (2.3).

Some of the low dimensional cases are covered by the following theorem:

(2.5) THEOREM. (i) *For* $n=1, 2, 3$ *the mapping cone of* $f: L^{2n+1} \to \mathbf{C}P^n$ *is* S-*equivalent to* $\mathbf{C}P^{n+p^2}/\mathbf{C}P^{p^2-1}$ *and for* $n=1$ *it is also* S-*equivalent to* $\mathbf{C}P^{p+1}/\mathbf{C}P^{p-1}$.

(ii) *For* $n=4$ *the mapping cone of* $f: L^{2n+1} \to \mathbf{C}P^n$ *will have the same* S-*type as some stunted complex projective space* $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ *if, and only if,* $p \not\equiv 2 \pmod 4$.

If $p \not\equiv 2 \pmod 4$ *then a possible choice for* d *is given by:* $d=11p^4-10p^2-1$ (*resp.* $d=11p^4-10p^2+1439$) *if* $p\equiv 0, 1, 4$ *or* $7 \pmod 8$ (*resp.* $p\equiv 3$ *or* $5 \pmod 8$)). (*Note that the order of* $J\circ r(\omega)$ *in* $J(\mathbf{C}P^4)$ *is* 2880.)

Proof. We prove part (i) and leave (ii) until §6. For $n=1, 2, 3$ (but not for $n\geqslant 4$) the ring $KO(\mathbf{C}P^n)$ can be presented by one generator $\mu=r(\omega-\varepsilon)$, where ε denotes the trivial line bundle, and the relation $\mu^2=0$, with the additional relation $2\mu=0$ if $n=1$. A reference for this paragraph is Adams-Walker [2]. Now there is a unique polynomial T with integer coefficients such that $T(z-2+z^{-1})=z^p-2+z^{-p}$. Moreover $T(z)=p^2z+\text{h.o.t.}$ and the Adams operations in $KO(\mathbf{C}P^n)$ can be computed by the formula $\psi^p(\mu)=T(\mu)=p^2\mu$ (since $\mu^2=0$ for $n\leqslant 3$). Then we have $r(\omega^p)=r\cdot\psi^p(\omega)$ $=\psi^p\cdot r(\omega)=\psi^p(\mu+2\varepsilon)=p^2\mu+2\varepsilon=p^2r(\omega)+(2-2p^2)\varepsilon$. Thus, as vector bundles over $\mathbf{C}P^n$ $r(\omega^p)\oplus(2p^2-2)\varepsilon$ and $p^2r(\omega)$ are equivalent. According to Atiyah [4] $T(X, \alpha\oplus\varepsilon)$ is homeomorphic to $\Sigma T(X, \alpha)$ and $T_n(p^2r(\omega))$ is homeomorphic to $\mathbf{C}P^{n+p^2}/\mathbf{C}P^{p^2-1}$. Hence we conclude that $\Sigma^{2p^2-2}C_f$ is homeomorphic to CP^{n+p^2}/CP^{p^2-1}. In the ring $KO(\mathbf{C}P^1)$ we have the relation $r(\omega^p)=pr(\omega)+(2-2p)\varepsilon$. Then it follows that $\Sigma^{2p-2}C_f$ is homeomorphic to CP^{p+1}/CP^{p-1}. (q.e.d.)

It should be mentioned that the various conditions on n and p in (2.3) arise from the method of proof and are probably unnecessary.

§ 3. The *J*-Calculations

To establish the generalized Spanier conjecture (1.2) we must relate the mapping cone of $f: L^q\to\mathbf{C}P^s$, where $s=[q/2]$, to some stunted complex projective space.

A first step in this direction is achieved by describing C_f in terms of a Thom complex over $\mathbf{C}P^s$.

(3.1) LEMMA. (i) *The mapping cone of* $f: L^{2n+1}\to\mathbf{C}P^n$ *is homeomorphic to the Thom complex* $T_n(r(\omega^p))$.

(ii) *The mapping cone of* $f: L^{2n}\to\mathbf{C}P^n$ *is homotopically equivalent to that of* $f: L^{2n-1}\to\mathbf{C}P^{n-1}$.

Proof. The S^1-bundle $f: L^{2n+1}\to\mathbf{C}P^n$ is the associated sphere bundle of some real 2-plane bundle α over $\mathbf{C}P^n$ and therefore C_f is homeomorphic to $T_n(\alpha)$. To determine α consider the following commutative diagram of circle bundles

$$\begin{array}{ccc} S^1 & \xrightarrow{\gamma} & S^1 \\ \downarrow\cap & & \downarrow\cap \\ S^{2n+1} & \xrightarrow{\pi} & L^{2n+1} \\ \downarrow f & & \downarrow f \\ \mathbf{C}P^n & \overset{\text{id.}}{=\!=} & \mathbf{C}P^n \end{array}$$

The left hand bundle is the one associated to the 2-plane bundle $r(\omega)$. Now there is a transgression homomorphism $\tau: H^1(\mathrm{S}^1; \mathbf{Z}) \to H^2(\mathbf{C}P^n; \mathbf{Z})$ for each of the bundles and if $s \in H^1(\mathrm{S}^1; \mathbf{Z})$ is a generator then $\tau(s)$ classifies the bundle. But γ is of degree p and so, by naturality of the transgression, the cohomology class classifying α is p times the cohomology class classifying $r(\omega)$. But this is precisely the class that classifies the bundle $r(\omega^p)$.

Now let's verify part (ii) of (3.1). The $2n$-skeleton L^{2n} of L^{2n+1} is the mapping cone of the p-fold covering $\pi: \mathrm{S}^{2n-1} \to L^{2n-1}$. Denote the restriction $f \mid L^{2n-1}$ by f'. Hence by the "9 Cone-Lemma" we get the following homotopy commutative diagram:

$$\begin{array}{ccccc}
\mathrm{S}^{2n-1} & = & \mathrm{S}^{2n-1} & \longrightarrow & * \\
\pi\downarrow & & \downarrow & & \downarrow \\
L^{2n-1} & \xrightarrow{f'} & \mathbf{C}P^{n-1} & \hookrightarrow & C_{f'} \\
\cap\downarrow & & \cap\downarrow & & \downarrow \\
L^{2n} & \xrightarrow{f} & \mathbf{C}P^{n} & \hookrightarrow & C_{f}
\end{array}$$

Therefore $C_{f'} \simeq C_f$.

According to Atiyah [4] $T_n((d+1)\, r(\omega))$ is homeomorphic to $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$. If h^* is any (reduced) cohomology theory for which there is a Thom isomorphism for both bundles $r(\omega^p)$ and $(d+1)\, r(\omega)$ then the (graded) group structures of $\tilde{h}^*(C_f)$ and $\tilde{h}^*(\mathbf{C}P^{n+1+d}/\mathbf{C}P^d)$ agree up to a shift in dimension, and this suggests that C_f and $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ are S-equivalent for some choice of d. In §5 we shall compute cohomology operations in these Thom complexes and then get conditions on d.

A reference for the remainder of this section is the Adams-Walker paper [2].

In that paper a characteristic class $\mathrm{bh}: KU(X) \to 1 + \prod_{s>0} H^{2s}(X; \mathbf{Q})$ is described. If ξ is a complex line bundle with first Chern class y then

$$\mathrm{bh}(\xi) = \frac{e^y - 1}{y}$$

bh can then be defined on arbitrary vector bundles by the splitting principle and is easily shown to be exponential (i.e., $\mathrm{bh}(\xi \oplus \xi') = \mathrm{bh}(\xi)\, \mathrm{bh}(\xi')$), hence bh can be defined on $KU(X)$.

As a first approximation to $J(X)$ a lower bound $J'(X)$ is defined by $J'(X) = KU(X)/V(X)$, where $V(X)$ is the set of $\alpha \in KU(X)$ such that $\mathrm{bh}(\alpha) = \mathrm{ch}(1+\beta)$ for some $\beta \in \widetilde{KU}(X)$. Let $J': KU(X) \to J'(X)$ be the projection. In the following theorem we have collected the results from Adams-Walker [2] that we need.

(3.2) THEOREM. (i) *There exists an epimorphism* $\theta': J(\mathbf{C}P^n) \twoheadrightarrow J'(\mathbf{C}P^n)$ *such* *that the following diagram is commutative*

$$\begin{array}{ccc} KU(CP^n) & \xrightarrow{J\circ r} & J(CP^n) \\ & {\scriptstyle J'}\searrow \quad \swarrow{\scriptstyle \theta'} & \\ & J'(CP^n) & \end{array}$$

(ii) *If* $n>1$ *and* $n\not\equiv 1 \pmod 4$ *then* $\theta' : J(CP^n) \to J'(CP^n)$ *is an isomorphism. However, if* $n>1$ *and* $n\equiv 1 \pmod 4$ *the kernel of* θ' *is* $\mathbf{Z}_2$, *generated by* $J\circ r(\omega-\varepsilon)^n$.

Thus, to decide whether $J\circ r(\omega^p)=(d+1)\,J\circ r(\omega)$ *is true or not for some* d *we first analyse what happens on the* J'*-level!*

(3.3) THEOREM. $J'(\omega^p)=(d+1)\,J'(\omega)$ *as elements of* $J'(CP^n)$ *if, and only if, the positive integer* d *satisfies condition* $S(n,p)$.

Proof. According to the definition of J' we have $J'(\omega^p)=(d+1)\,J'(\omega)$ as elements of $J'(CP^n)$ if, and only if, there exists an element $\beta\in\widetilde{KU}(CP^n)$ such that $\mathrm{bh}(\omega^p-(d+1)\,\omega)=\mathrm{ch}(1+\beta)$. Now

$$\mathrm{bh}(\omega^p-(d+1)\,\omega)=bh(\omega^p)\cdot \mathrm{bh}(\omega)^{-(d+1)}=\frac{e^{py}-1}{py}\cdot\left[\frac{e^y-1}{y}\right]^{-(d+1)},$$

where $y=c_1(\omega)$ is the first Chern class of ω. On the other hand the cohomology classes $1, \mathrm{ch}(\xi),\dots,\mathrm{ch}(\xi^n)$, where $\xi=\omega-\varepsilon$, freely generate $H^*(CP^n;\mathbf{Q})$ and so there are unique rationals $a_0,\dots,a_n$ satisfying

$$\mathrm{bh}(\omega^p-(d+1)\,\omega)=\sum_{q=0}^{n} a_q\cdot \mathrm{ch}(\xi^q).$$

Thus there exists such a β if, and only if, all the a_q are integers. Making the substitution $z=e^y-1$ we get the equation

$$\frac{(z+1)^p-1}{pz}\cdot\left[\frac{\log(1+z)}{z}\right]^d=\sum_{q=0}^{n} a_q\cdot z^q,$$

which proves (3.3). We are now in a position to prove parts (i) and (ii) of theorem (2.3).

If $J\circ r(\omega^p)=(d+1)\,J\circ r(\omega)$ then we also have $J'(\omega^p)=(d+1)\,J'(\omega)$ by (3.2). Therefore d satisfies condition $S(n,p)$. On the other hand if d satisfies condition $S(n,p)$ then $J'(\omega^p)=(d+1)\,J'(\omega)$, and if we also assume $n>1$ and $n\not\equiv 1 \pmod 4$ then θ' is an isomorphism and so $J\circ r(\omega^p)=(d+1)\,J\circ r(\omega)$. Finally, recall that if two bundles are J-equivalent then their Thom complexes are S-equivalent.

Part (iii) of (2.3) remains to be proven and this we do in the next section.

§ 4. Twisting Phenomena

In this section, together with results from §5, we essentially show that there exists an S-equivalence between $T_n(r(\omega^p))$ and $T_n((d+1)\,r(\omega))$ if, and only if, $J\circ r(\omega^p) = (d+1)\,J\circ r(\omega)$. For a precise statement see (4.5).

All through this section we shall suppose that the mapping cone C_f of $f: L^{2n+1} \to \mathbf{C}P^n$, for some fixed n, is S-equivalent to $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$. Thus we are supposing the existence of an S-map $s: \Sigma^{2d}C_f \to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ such that $\Sigma^k s$ is a homotopy equivalence for all sufficiently large k. Actually there is no loss of generality in assuming that d is chosen such that there is a homotopy equivalence $s: \Sigma^{2d}C_f \to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$. The reason for this is that for any S-equivalence $C_f \to CP^{n+1+d}/CP^d$ there are certainly homotopy equivalences $\Sigma^{2d+2e}C_f \to \Sigma^{2e}(\mathbf{C}P^{n+1+d}/\mathbf{C}P^d)$ for all sufficiently large e and so we may choose e to be some multiple of the order of the J-class $J(r(\omega))$ in the finite group $J(\mathbf{C}P^n)$. For such a choice of e we have a homotopy equivalence between $\Sigma^{2e}(CP^{n+1+d}/CP^d)$ and $CP^{n+1+d+e}/CP^{d+e}$ (see Atiyah [4]). Thus d is altered by a multiple of the order of the J-class of $r(\omega)$ in $J(\mathbf{C}P^n)$. By the results of Adams and Walker [2], if N is the order of $J\circ r(\omega)$ in $J(\mathbf{C}P^n)$, then N is the smallest positive integer satisfying the condition $\mathrm{S}(n, 1)$. Thus, altering d by a multiple of N will not affect part (iii) of (2.3).

We propose to study the homotopy equivalence $s: T_n(r(\omega^p)\oplus 2d\varepsilon) \to T_n((d+1)r(\omega))$ by studying the induced map in cohomology. The most convenient vehicle for doing this is the Thom isomorphism theorem. In order to simplify the notation put $\alpha = r(\omega^p)\oplus 2d\varepsilon$ and $\beta = (d+1)\,r(\omega)$. Then there are Thom classes $U_\alpha \in H^{2d+2}(T_n(\alpha); \mathbf{Z})$, $U_\beta \in H^{2d+2}(T_n(\beta); \mathbf{Z})$ and the Thom isomorphism theorem says that $\tilde{H}^*(T_n(\alpha); \mathbf{Z})$, $\tilde{H}^*(T_n(\beta); \mathbf{Z})$ are freely generated by the bases $\{U_\alpha, y\cup U_\alpha, \ldots, y^n\cup U_\alpha\}$, $\{U_\beta, y\cup U_\beta, \ldots, y^n\cup U_\beta\}$ respectively, where $y = c_1(\omega)$. Using these canonical bases we introduce signs $\varepsilon_1, \ldots, \varepsilon_{n+1}$ reflecting twisting properties of the homotopy equivalence s.

(4.1) DEFINITION. ε_{i+1} is the sign given by $s^*(y^i\cup U_\beta) = \varepsilon_{i+1} y^i \cup U_\alpha$, $0 \leqslant i \leqslant n$.

In § 5 we derive conditions on the signs and on d from Steenrod power calculations. The crucial point is that if there exists a homotopy equivalence $s: \Sigma^{2d}C_f \to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ we can replace s by another homotopy equivalence – making some mild hypotheses – for which all the signs ε_i are equal. However, we feel it is better to isolate this argument from the main theme as it is rather ad hoc.

Let $U \in H^2(T_n(r(\omega^p)); \mathbf{Z})$ be the Thom class of the 2-plane bundle $r(\omega^p)$. Then $\tilde{H}^*(C_f; Z)$ is freely generated by the basis $\{U, y\cup U, \ldots, y^n \cup U\}$, where $y = c_1(\omega)$. Consider the commutative diagram

$$\begin{array}{ccccc} S^{2n+1} & \xrightarrow{f} & \mathbf{C}P^n & \hookrightarrow & \mathbf{C}P^{n+1} \\ \downarrow{\scriptstyle \pi} & & \|{\scriptstyle \mathrm{id.}} & & \downarrow{\scriptstyle h} \\ L^{2n+1} & \xrightarrow{f} & \mathbf{C}P^n & \xrightarrow{e} & C_f \end{array}$$

where π is the p-fold covering and h is defined by coning.

(4.2) LEMMA. *The induced homomorphism* $h^*: \tilde{H}^*(C_f; \mathbf{Z}) \to \tilde{H}^*(\mathbf{C}P^{n+1}; \mathbf{Z})$ *is given by* $h^*(y^{i-1} \cup U) = p \cdot y^i$, $1 \leqslant i \leqslant n+1$.

Proof. Under the identification of C_f with the Thom complex $T_n(r(\omega^p))$ the inclusion map e corresponds to the zero section and therefore $e^*(U) = c_1(\omega^p) = p \cdot y$. From the diagram it follows that $h^*(U) = py$. Now $U^2 = \chi(r(\omega^p)) \cup U = p \cdot y \cup U$, and in general we get $U^i = p^{i-1} \cdot y^{i-1} \cup U$ for $1 \leqslant i \leqslant n+1$. Since there is no torsion in either $H^*(C_f; \mathbf{Z})$ or $H^*(\mathbf{C}P^{n+1}; \mathbf{Z})$ we have $h^*(y^{i-1} \cup U) = 1/p^{i-1} \cdot h^*(U^i) = p \cdot y^i$.

For the proof of our main theorem we also need to know the image of the map induced by h in unitary K-theory.

(4.3) LEMMA. *The map* $h^!: \widetilde{KU}(C_f) \to \widetilde{KU}(\mathbf{C}P^{n+1})$ *is a monomorphism whose image is freely generated by the elements* $\psi^p(\xi), \xi\psi^p(\xi), \ldots, \xi^n\psi^p(\xi)$, *where* $\xi = \omega - \varepsilon \in \widetilde{KU}(\mathbf{C}P^{n+1})$.

Proof. From the Chern character it is obvious that $h^!$ is a monomorphism. For the remainder we focus our attention on the following commutative diagram:

$$\begin{array}{ccccc}
KU(\mathbf{C}P^n) & \xrightarrow[\cong]{\phi_K} & \widetilde{KU}(C_f) & \xrightarrow{h^!} & \widetilde{KU}(\mathbf{C}P^{n+1}) \\
\Big\downarrow \scriptstyle{\mathrm{bh}(\omega^p)\mathrm{ch}} & & \Big\downarrow \scriptstyle{\mathrm{ch}} & & \Big\downarrow \scriptstyle{\mathrm{ch}} \\
H^*(\mathbf{C}P^n; \mathbf{Q}) & \xrightarrow[\cong]{\phi_H} & \tilde{H}^*(C_f; \mathbf{Q}) & \xrightarrow{h^*} & \tilde{H}^*(\mathbf{C}P^{n+1}; \mathbf{Q})
\end{array}$$

We emphasize that we choose our Thom isomorphisms ϕ_K, ϕ_H in accordance with Adams [1] and therefore the deviation from commutativity with the Chern character is $\mathrm{bh}(\omega^p)$ and not as "usual" the inverse Todd genus! As $1, \xi, \ldots, \xi^n$ is a free basis for $KU(CP^n)$ we see that the image of $h^!$ is freely generated by $\xi^k\psi^p(\xi)$, $0 \leqslant k \leqslant n$, if, and only if, the image of the composite homomorphism $h^* \circ \phi_H \circ \mathrm{bh}(\omega^p) \cdot \mathrm{ch}$ is freely generated by $(e^y - 1)^k (e^{py} - 1)$, $0 \leqslant k \leqslant n$.

Writing

$$(e^y - 1)^k \left[\frac{e^{py} - 1}{py}\right] = \sum_{j \geqslant k} \gamma_j^{(k)} y^j$$

we get

$$\begin{aligned}
[h^* \circ \phi_H \circ \mathrm{bh}(\omega^p)\,\mathrm{ch}]\,(\xi^k) &= h^* \circ \phi_H \left[(e^y - 1)^k \left\{\frac{e^{py} - 1}{py}\right\}\right] \\
&= h^* \{\sum_{j \geqslant k} \gamma_j^{(k)} y^j \cup U\} \\
&= p \sum_{j \geqslant k} \gamma_j^{(k)} y^{j+1} \qquad (\text{by } (4.2)) \\
&= (e^y - 1)^k \cdot (e^{py} - 1).
\end{aligned}$$

Thus, the image of $h^*\circ\varphi_H\circ \mathrm{bh}(\omega^p)\cdot\mathrm{ch}$ is freely generated by the elements $(e^y-1)^k\,(e^{py}-1)$, $0\leqslant k\leqslant n$, and this proves (4.3).

(4.4) THEOREM. *Suppose* $s:\Sigma^{2d}C_f\to CP^{n+1+d}/CP^d$ *is a homotopy equivalence such that* $s^*(y^i\cup U_\beta)=\varepsilon y^i\cup U_\alpha$ *for* $k\leqslant i\leqslant n$, *where* ε *is a fixed sign. Then* d *satisfies condition* $\mathrm{S}(n-k,p)$.

For example, if $\varepsilon_1=\varepsilon_2=\cdots=\varepsilon_{n+1}$ then d satisfies condition $\mathrm{S}(n,p)$.

Proof. The theorem will be proved by chasing from $\widetilde{KU}(CP^{n+1+d}/CP^d)$ to $\tilde{H}^*(\Sigma^{2d}CP^{n+1})$ in the commutative diagram

$$\begin{array}{ccccc}
\widetilde{KU}(\Sigma^{2d}CP^{n+1}) & \xleftarrow{(\Sigma^{2d}h)^!} & \widetilde{KU}(\Sigma^{2d}C_f) & \xleftarrow{s^!} & \widetilde{KU}(CP^{n+1+d}/CP^d)\\
\uparrow{\scriptstyle \cong b^d} & & \uparrow{\scriptstyle \cong b^d} & & \\
\widetilde{KU}(CP^{n+1}) & \xleftarrow{h^!} & \widetilde{KU}(C_f) & & \downarrow{\scriptstyle \mathrm{ch}}\\
\downarrow{\scriptstyle \sigma^{2d}\mathrm{ch}} & & \downarrow{\scriptstyle \sigma^{2d}\mathrm{ch}} & & \\
\tilde{H}^*(\Sigma^{2d}CP^{n+1}) & \xleftarrow{(\Sigma^{2d}h)^*} & \tilde{H}^*(\Sigma^{2d}C_f) & \xleftarrow{s^*} & \tilde{H}^*(CP^{n+1+d}/CP^d)
\end{array}$$

where b is the Bott isomorphism and σ is the suspension isomorphism. If V_β is the $\widetilde{KU}$ Thom class of $\beta=(d+1)\,r(\omega)$ then the Thom isomorphism theorem implies that $\widetilde{KU}(CP^{n+1+d}/CP^d)$ is freely generated by the elements $\xi^i\cup V_\beta$, $0\leqslant i\leqslant n$, where $\xi=\omega-\varepsilon$. Then

$$\mathrm{ch}(\xi^i\cup V_\beta)=\mathrm{bh}((d+1)\,\omega)\,\mathrm{ch}(\xi^i)\cup U_\beta=\left\{\frac{e^y-1}{y}\right\}^{d+1}\cdot(e^y-1)^i\cup U_\beta,$$

where U_β is the Thom class in ordinary cohomology. Recall that $U\in H^2(T_n(r(\omega^p);\mathbf{Z}))$ is the Thom class of $r(\omega^p)$ and that $U_\alpha\in H^{2d+2}(T_n(r(\omega^p)\oplus 2d\varepsilon);\mathbf{Z})$ is the Thom class of $\alpha=r(\omega^p)\oplus 2d\varepsilon$. Then the bases $\{U, y\cup U,\dots, y^n\cup U\}$ and $\{U_\alpha, y\cup U_\alpha,\dots, y^n\cup U_\alpha\}$ of $\tilde{H}^*(C_f;\mathbf{Z})$ and $\tilde{H}^*(\Sigma^{2d}C_f;\mathbf{Z})$ resp. correspond under the suspension isomorphism σ^{2d}. If $i\geqslant k$ then we have

$$\begin{aligned}
(\Sigma^{2d}h)^*s^*\mathrm{ch}(\xi^i\cup V_\beta)&=(\Sigma^{2d}h)^*s^*\left\{\left[\frac{e^y-1}{y}\right]^{d+1}\cdot(e^y-1)^i\cup U_\beta\right\}\\
&=\varepsilon(\Sigma^{2d}h)^*\left\{\left[\frac{e^y-1}{y}\right]^{d+1}\cdot(e^y-1)^i\cup U_\alpha\right\}\\
&=\varepsilon\sigma^{2d}h^*\left\{\left[\frac{e^y-1}{y}\right]^{d+1}\cdot(e^y-1)^i\cup U\right\}\\
&=p\,\varepsilon\,\sigma^{2d}\left\{y\left[\frac{e^y-1}{y}\right]^{d+1}\cdot(e^y-1)^i\right\}\quad\text{by (4.2)}.
\end{aligned}$$

On the other hand (4.3) guarantees the existence of integers $\lambda_j^{(i)}$ such that

$$(\Sigma^{2d}h)^!s^!(\xi^i\cup V_\beta)=b^d\sum_{j=i}^{n}\lambda_j^{(i)}\,\xi^j\psi^p(\xi).$$

The following calculation shows that the sum starts at i and that $\lambda_i^{(i)}=\varepsilon$ for $i\geqslant k$:

$$[\sigma^{2d}\circ \mathrm{ch}\circ b^{-d}\circ (\Sigma^{2d}h)^! s^!]\,(\xi^i\cup V_\beta)=\sigma^{2d}\circ \mathrm{ch}\sum_{j=i}^{n}\lambda_j^{(i)}\,\xi^j\psi^p(\xi)$$
$$=\sigma^{2d}\sum_{j=i}^{n}\lambda_j^{(i)}\,(e^y-1)^i\,(e^{py}-1).$$

Hence in $H^*(\mathbf{C}P^{n+1};\mathbf{Q})$ we have the equations

$$p\,\varepsilon\, y\left[\frac{e^y-1}{y}\right]^{d+1}\cdot(e^y-1)^i=\sum_{j=i}^{n}\lambda_j^{(i)}\,(e^y-1)^j\,(e^{py}-1)$$

for $k\leqslant i\leqslant n$.

Making the substitution $z=e^y-1$ gives

$$\frac{pz}{(z+1)^p-1}\cdot\left[\frac{z}{\log(1+z)}\right]^d=\varepsilon\sum_{j=i}^{n}\lambda_j^{(i)}\,z^{j-i}\ (\mathrm{mod}\ z^{n+1-i}).$$

But both

$$\frac{pz}{(z+1)^p-1}\quad\text{and}\quad\left[\frac{z}{\log(1+z)}\right]^d$$

may be inverted without losing integrality. Putting $i=k$ then implies that d satisfies condition $\mathrm{S}(n-k,p)$.

The statement (2.3) part (iii) is the following corollary of (4.4).

(4.5) COROLLARY. *Suppose C_f is* S-*equivalent to* $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ *and* $n\geqslant 6$. *Then d satisfies condition* $\mathrm{S}(n-1,p)$. *Moreover, if either n is odd or p satisfies condition (H_n), then d also satisfies condition* $\mathrm{S}(n,p)$.

Proof. As observed in the introduction of this section we may assume there is a homotopy equivalence $s:\Sigma^{2d}C_f\to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$, where d is possibly altered by adding on a multiple of the order of $J\circ r(\omega)$ in $J(\mathbf{C}P^n)$. But this order is the Atiyah-Todd number M_{n+1} and M_{n+1} satisfies the condition $\mathrm{S}(n,1)$. Therefore the conclusion of (4.5) is not altered by adding a multiple of M_{n+1} to d. Then (5.1) together with (4.4) now proves most of (4.5). If n is odd and $n\geqslant 3$ then the inclusion $\mathbf{C}P^{n-1}\subset \mathbf{C}P^n$ induces an isomorphism $J'(\mathbf{C}P^n)\cong J'(\mathbf{C}P^{n-1})$; (see Adams-Walker [2]). From (3.3) it now follows that d satisfies condition $\mathrm{S}(n,p)$ if, and only if, d satisfies condition $\mathrm{S}(n-1,p)$.

§5. Characteristic Class Computations

In this section we resume the analysis of the twisting phenomenon of a homotopy equivalence $s:T_n(r(\omega^p)\oplus 2d\varepsilon)\to T_n((d+1)\,r(\omega))$ and derive congruences on d from characteristic class computations.

One of the main theorems we shall prove is the following "untwisting" theorem:

(5.1) THEOREM. *Suppose $n\geqslant 6$ and $s:\Sigma^{2d}C_f\to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ is a homotopy* *equivalence. Then there exists a sign* ε *and a homotopy equivalence* $t\colon \Sigma^{2d} C_f \to CP^{n+1+d}/CP^d$ *such that* $t^*(y^i \cup U_\beta) = \varepsilon y^i \cup U_\alpha$ *for* $1 \leqslant i \leqslant n$. *Moreover* $t^*(U_\beta) = \varepsilon U_\alpha$ *if* p *satisfies condition* (H_n).

It is probable that such a homotopy equivalence always exists but our method of proof does not show this.

The proof of (5.1) is preceded by a sequence of lemmas.

(5.2) LEMMA. *If* q *is an odd prime then we have the following congruences*

$$\varepsilon_i \left[\binom{i-1}{j} + p^{q-1} \binom{i-1}{j-1} \right] \equiv \varepsilon_{i+j(q-1)} \binom{d+i}{j} \pmod q$$

for all $i, j \geqslant 1$ *such that* $i + j(q-1) \leqslant n+1$.

Proof. We establish this lemma by computing the Steenrod power $\mathscr{P}^j$ in both $H^*(T_n(\alpha); \mathbf{Z}_q)$ and $H^*(T_n(\beta); \mathbf{Z}_q)$ and then applying the induced map s^*. To compute $\mathscr{P}^j$ in $H^*(T(X, \zeta); \mathbf{Z}_q)$ we use the Cartan formula and a result of Wu (see Milnor [6]) saying that $\mathscr{P}^j(U_\zeta) = Q_j(\zeta) \cup U_\zeta$, where the total characteristic class $Q(\zeta) = 1 + Q_1(\zeta) + \cdots$ corresponds to the power series $1 + z^{1/2(q-1)}$. For example, if ζ is the real 2-plane bundle underlying a complex bundle ξ then $Q(\zeta) = 1 + P_1(\zeta)^{(q-1)/2} = 1 + c_1(\xi)^{q-1}$, where $P_1(\zeta)$ is the first Pontrjagin class and $c_1(\xi)$ is the first Chern class. Since Q is exponential (i.e., $Q(\zeta \oplus \zeta') = Q(\zeta) \cup Q(\zeta')$) we get $Q(\alpha) = Q(r(\omega^p) \oplus \oplus 2d\varepsilon) = 1 + p^{q-1} y^{q-1}$, where $y = c_1(\omega)$. Likewise $Q(\beta) = \{1 + y^{q-1}\}^{d+1}$. Thus $\mathscr{P}^j$ evaluated on the Thom classes gives

$$\mathscr{P}^j(U_\beta) = \binom{d+1}{j} y^{(q-1)j} \cup U_\beta \quad \text{for all} \quad j \geqslant 0,$$

and

$$\mathscr{P}^j(U_\alpha) = \begin{cases} U_\alpha & \text{for} \quad j = 0 \\ p^{q-1}\, y^{q-1} \cup U_\alpha & \text{for} \quad j = 1 \\ 0 \quad \text{for} \quad j > 1 \end{cases}$$

By the Cartan formula we derive

$$\begin{aligned} \mathscr{P}^j(y^{i-1} \cup U_\alpha) &= \mathscr{P}^j(y^{i-1}) \cup U_\alpha + \mathscr{P}^{j-1}(y^{i-1}) \cup \mathscr{P}^1(U_\alpha) \\ &= \left[\binom{i-1}{j} + p^{q-1} \binom{i-1}{j-1} \right] y^{i-1+j(q-1)} \cup U_\alpha \end{aligned}$$

and

$$\begin{aligned} \mathscr{P}^j(y^{i-1} \cup U_\beta) &= \sum_{k+m=j} \mathscr{P}^k(y^{i-1}) \cup \mathscr{P}^m(U_\beta) \\ &= \sum_{k+m=j} \binom{i-1}{k} y^{i-1+k(q-1)} \cup \binom{d+1}{m} y^{m(q-1)} \cup U_\beta \\ &= \binom{d+i}{j} y^{i-1+j(q-1)} \cup U_\beta . \end{aligned}$$

The result now follows from $\mathscr{P}^j s^*(y^{i-1} \cup U_\beta) = s^* \mathscr{P}^j (y^{i-1} \cup U_\beta)$.

In passing one should note that no information on the signs ε_i comes from cohomology with coefficients $\mathbf{Z}_2$. However the mod 2 Steenrod algebra gives conditions on d.

(5.3) LEMMA. *Suppose* $s: \Sigma^{2d} C_f \to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ *is an* S-*equivalence. Then* $d+1 \equiv p \pmod 2$ *and* $\binom{d+1}{i} \equiv 0 \pmod 2$ *for* $2 \leqslant i \leqslant n$.

Proof. If the Thom complexes $T_n(r(\omega^p))$, $T_n((d+1)\, r(\omega))$ have the same S-type then we have equality of total Stiefel-Whitney classes $w(r(\omega^p)) = w(r(\omega))^{d+1}$, since the total Steenrod square Sq is the same for both spaces (the group $\mathbf{Z}_2$ has a unique automorphism). But $w(r(\omega^p)) = 1 + p\bar{y}$ and $w(r(\omega))^{d+1} = (1+\bar{y})^{d+1}$, where $\bar{y}$ is the mod 2 reduction of $y = c_1(\omega)$. Thus $(d+1)\,\bar{y} = p \cdot \bar{y}$ and $\binom{d+1}{i} \bar{y}^i = 0$ if $i \geqslant 2$.

(5.4) COROLLARY. *Suppose* $s: \Sigma^{2d} C_f \to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ *is an* S-*equivalence. If* $p \equiv 0 \pmod 2$ (*resp.* $p \not\equiv 0 \pmod 2$), *then* $d+1$ (*resp.* d) $\equiv 0 \pmod{2^\nu}$ *where* $2^\nu > n \geqslant 2^{\nu-1}$.

Proof. If p is even then $\binom{d+1}{i} \equiv 0 \pmod 2$ for $1 \leqslant i \leqslant n$. Writing $d+1 = 2^\mu + \text{h.o.t.}$, where $\mu \geqslant 1$, we have $\binom{d+1}{2^\mu} \equiv 1 \pmod 2$ and therefore $2^\mu > n$. Now suppose p is odd. Then d is even and we write $d+1 = 1 + 2^\mu + \text{h.o.t.}$, where $\mu \geqslant 1$, and proceed as before, thus completing the proof.

For the next lemma suppose that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$ is a sequence of signs chosen to satisfy the congruences of (5.2).

(5.5) LEMMA (i) *If* $p \equiv 0 \pmod q$ (*resp.* $p \not\equiv 0 \pmod q$), *where* q *is an odd prime such that* $q \leqslant n+1$ (*resp.* $q \leqslant n$), *then* $d+1$ (*resp.* d) $\equiv 0 \pmod{q^\nu}$, *where* $q^\nu > (n/q - 1) \geqslant q^{\nu-1}$.

(ii) *If* $n \geqslant 6$ *then* $\varepsilon_k = \varepsilon_{k+2}$ *for* $1 < k < n$. *Moreover, if* p *satisfies condition* (H_n) *then* $\varepsilon_k = \varepsilon_{k+2}$ *for* $1 \leqslant k < n$.

Proof. We first verify (i). There are two cases to consider: $p \equiv 0 \pmod q$ and $p \not\equiv 0 \pmod q$. If $p \equiv 0 \pmod q$ then (5.2) becomes $\varepsilon_i \binom{i-1}{j} \equiv \varepsilon_{i+j(q-1)} \binom{d+i}{j} \pmod q$ for all $i, j \geqslant 1$ satisfying $i + j(q-1) \leqslant n+1$. Putting $i = j = 1$ gives $d+1 \equiv 0 \pmod q$. Writing $d+1 = a_\mu q^\mu + \text{h.o.t.}$, where $\mu \geqslant 1$ and $0 > a_\mu > q$, we now argue as in the proof of (5.4).

If $p \not\equiv 0 \pmod q$, then (5.2) becomes $\varepsilon_i \binom{i}{j} \equiv \varepsilon_{i+j(q-1)} \binom{d+i}{j} \pmod q$ for all $i, j \geqslant 1$ such that $i + j(q-1) \leqslant n+1$. Putting $i = j = 1$ gives $d+1 \equiv 1$ or $-1 \pmod q$ and putting $i = 2, j = 1$ (here is there we need $q \leqslant n$) gives $d+2 \equiv 2$ or $-2 \pmod q$. Together these congruences imply $d \equiv 0 \pmod q$. Now proceed as in the first case.

To prove (ii) we use induction on n. Given a sequence of signs $\varepsilon_1,\dots,\varepsilon_{n+1}$ satisfying (5.2) let P_n be the statement that $\varepsilon_k=\varepsilon_{k+2}$ for $1<k<n$. Then one checks directly that P_2, P_3, P_4 and P_5 are false, whereas P_6 is true. Assume that we have proved P_n is true for $6\leqslant n<N$. To prove that P_N is true we need only find an integer i such that $1<i<N+1$, $i\equiv N+1 \pmod 2$, and $\varepsilon_{N+1}=\varepsilon_i$. Putting $j=1$ into (5.2) gives $\varepsilon_i[i-1+p^{q-1}]\equiv\varepsilon_{i+q-1}(d+i) \pmod q$, where $1\leqslant i\leqslant N+2-q$. Thus we need only find an odd prime $q\leqslant N$ such that $i-1+p^{q-1}\equiv d+i\not\equiv 0 \pmod q$, where $i=N+2-q$. But part (i) of this lemma implies that $d+1\equiv p^{q-1} \pmod q$ for all odd primes $q\leqslant N$ and so we need only find an odd prime $q\leqslant N$ so that $N+1+p^{q-1}\not\equiv 0 \pmod q$. Using Bertrand's postulate it is now a simple matter to show that such an odd prime always exists if $N\geqslant 5$. This completes the induction.

The only possible way to get information on ε_1 from (5.2) is to pick $i=1, j=1$, in which case we get $p^{q-1}\varepsilon_1\equiv(d+1)\varepsilon_q \pmod q$ for all odd primes $q\leqslant n+1$. Since $p^{q-1}\equiv d+1 \pmod q$ for all odd primes $q\leqslant n$ we get $\varepsilon_1=\varepsilon_q$ if p is relatively prime to q. Thus we get $\varepsilon_k=\varepsilon_{k+2}$ for $1\leqslant k<n$ if p satisfies condition (H_N).

(5.6) *Remark.* If we pick $n=5$ then the above argument yields $\varepsilon_2=\varepsilon_4=\varepsilon_6$ (for all p) and $\varepsilon_1=\varepsilon_3=\varepsilon_5$ (for $p\not\equiv 0 \pmod 5$). Similar statements hold for $n=2, 3, 4$.

Proof of (5.1). Either s^* already satisfies the stated condition or there is a sign δ such that $s^*(y^i\cup U_\beta)=(-1)^i\delta y^i\cup U_\alpha$ for $1\leqslant i\leqslant n$. Moreover, this equation holds for $i=0$ as well if p satisfies condition (H_n). By altering s by a self equivalence of $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ we can change every second sign. This proves (5.1).

§ 6. Special Results and Closing Remarks

In this section we prove part (ii) of (2.5) together with a few isolated results. To facilitate the computations we work with the characteristic class sh rather than bh. The characteristic class $\mathrm{sh}: KO(X)\to 1+\prod_{s>0}H^{4s}(X;\mathbf{Q})$ is the one associated to the power series $(e^{1/2y}-e^{-1/2y})/y$. It is multiplicative and if ξ is a complex line bundle over X then $\mathrm{sh}(r(\xi))=(e^{1/2y}-e^{-1/2y})/y$, where $y=c_1(\xi)$, (see [2]). In Adams-Walker [2] a lower bound $J'_{\mathbb{R}}(X)$ of $J(X)$ is defined by $J'_{\mathbb{R}}(X)=KO(X)/V_{\mathbb{R}}(X)$, where $V_{\mathbb{R}}^{(X)}$ is the set of $\alpha\in KO(X)$ such that $\mathrm{sh}(\alpha)=\mathrm{ch}\circ c(1+\beta)$ for some $\beta\in\widetilde{KO}(X)$. Here c denotes the complexification ringhomomorphism $c: KO(X)\to KU(X)$. The advantage of sh over bh is that $(e^{1/2y}-e^{-1/2y})/y$ is a power series in y^2 whereas $(e^y-1)/y$ is a power series in y only. Let $J'_{\mathbb{R}}: KO(X)\to J'_{\mathbb{R}}(X)$ denote the projection and let α_t, $t\geqslant 1$, be defined by

$$\log\left[\frac{e^z-1}{z}\right]=\sum_{t\geqslant 1}\alpha_t\frac{z^t}{t!}.$$

In the following theorem we collect the pertinent facts from Adams-Walker [2] that we shall need.

(6.1) THEOREM. (i) $\log \operatorname{sh}(\zeta)=\frac{1}{2}\cdot\Sigma_{s\geq 1}\,\alpha_{2s}\mathrm{ch}_{2s}\circ c(\zeta)$, *where* ch_{2s} *denotes the component of the Chern character in dimension* $4s$.

(ii) *there exists an epimorphism* $\theta'_{\mathbb{R}}: J(\mathbf{C}P^n)\to J'_{\mathbb{R}}(\mathbf{C}P^n)$ *making the following diagram commutative*

$$\begin{array}{ccc} KO(\mathbf{C}P^n) & \xrightarrow{J} & J(\mathbf{C}P^n) \\ & {\scriptstyle J'_{\mathbb{R}}}\searrow \quad \swarrow{\scriptstyle \theta'_{R}} & \\ & J'_R(\mathbf{C}P^n) & \end{array}$$

(iii) $\theta'_{\mathbb{R}}$ *is an isomorphism if* $n>1$ *and* $n\not\equiv 1 \pmod 4$.

It should be remarked that if $n>1$ then $J'(\mathbf{C}P^n)\cong J'_{\mathbb{R}}(\mathbf{C}P^n)$ and under this isomorphism θ' and $\theta'_{\mathbb{R}}$ correspond.

(6.2) THEOREM. *There exists an integer* $d\geq 0$ *such that* $J\circ r(\omega^p)=(d+1)\,J\circ r(\omega)$ *as elements of* $J(\mathbf{C}P^4)$ *if, and only if,* $p\not\equiv 2 \pmod 4$.

If $p\equiv 0, 1, 4$ *or* $7 \pmod 8$ (*resp.* $p\equiv 3$ *or* $5 \pmod 8$) *then a possible choice for* $d+1$ *is* $d+1=11p^4-10p^2$ (*resp.* $11p^4-10p^2+1440$).

Proof. By (6.1) we know that $J\circ r(\omega^p)=(d+1)\,J\circ r(\omega)$ as elements of $J(\mathbf{C}P^4)$ if and only if, $J'_{\mathbb{R}}\circ r(\omega^p)=(d+1)\,J'_{\mathbb{R}}\circ r(\omega)$. According to the definition of $J'_{\mathbb{R}}$ this happens if, and only if, there exists an element $\beta\in\widetilde{KO}(\mathbf{C}P^4)$ such that $\operatorname{sh}(r(\omega^p)-(d+1)\,r(\omega))=\mathrm{ch}\circ c(1+\beta)$. Now $\widetilde{KO}(\mathbf{C}P^4)$ can be presented by one generator $\mu=r(\omega-\varepsilon)$ and one relation $\mu^3=0$ (see [2]). Writing $\beta=a_1\mu+a_2\mu^2$ for some integers a_1, a_2 and carrying out the calculation we get

$$\log\left\{\mathrm{ch}\circ c\,(1+\beta)\right\}=a_1 y^2+\left\{\frac{a_1+12a_2-6a_1^2}{12}\right\}\cdot y^4 .$$

Using part (i) of (6.1) and $\alpha_2=\frac{1}{12}, \alpha_4=-\frac{1}{120}$ we have

$$\log\left\{\operatorname{sh}\left(r(\omega^p)-(d+1)\,r(\omega)\right)\right\}=\frac{1}{24}(p^2-d-1)\,y^2-\frac{1}{120.24}(p^4-d-1)\,y^4 .$$

Thus $J\circ r(\omega^p)=(d+1)\,J\circ r(\omega)$ as elements of $J(\mathbf{C}P^4)$ if, and only if, there are integers a_1, a_2 such that (*) $p^2-1-d=24a_1$, $p^4-d-1=240(6a_1^2-a_1-12a_2)$. Thus a necessary condition is that $p^2-p^4\equiv 0 \pmod{24}$ and this happens if, and only if, $p\not\equiv 2 \pmod 4$. Now assume $p^2-p^4=24q$ for some integer q. Then q is even (resp. odd) if, and only if, $p\equiv 0, 1, 4$ or $7 \pmod 8$ (resp. $p\equiv 3, 5 \pmod 8$). Then it is a simple matter to show that (*) is satisfied for the values of d given in (6.2). Hence the "numerical" theorem (6.2) is established. Therefore also part of (2.5) (ii) is verified.

Now let us complete the proof of (2.5) (ii)! For that assume that there exists a homotopy equivalence $s:\Sigma^{2d}C_f \to \mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ for some positive integer d.

We shall show that such a homotopy equivalence does not exist in case $p\equiv 2 \pmod 4$ and $n\geqslant 4$. This result is obtained by comparing the Adams operations under the induced map $s^!$ in complex K-theory.

Recall that $\alpha=r(\omega^p)\oplus 2d\varepsilon$ and $\beta=(d+1)\,r(\omega)$. Then the spaces $\Sigma^{2d}C_f$, $\mathbf{C}P^{n+1+d}/\mathbf{C}P^d$ are the respective Thom complexes $T_n(\alpha)$, $T_n(\beta)$. If V_α, V_β are the KU Thom classes then the Thom isomorphism theorem implies that $\widetilde{KU}(T_n(\alpha))$, $\widetilde{KU}(T_n(\beta))$ are freely generated by the bases $\{V_\alpha, \xi\cup V_\alpha, \ldots, \xi^n\cup V_\alpha\}$, $\{V_\beta, \xi\cup V_\beta, \ldots, \xi^n\cup V_\beta\}$ respectively, where $\xi=\omega-\varepsilon\in\widetilde{KU}(\mathbf{C}P^n)$. In §4 we introduced the Thom classes U_α, U_β for integral cohomology and then defined a sequence of signs $\varepsilon_1, \ldots, \varepsilon_{n+1}$ by $s^*(y^i\cup U_\beta)=\varepsilon_{i+1}y^i\cup U_\alpha$, $0\leqslant i\leqslant n$, where $y=c_1(\omega)$.

Now

$$\mathrm{ch}(\xi^i\cup V_\alpha)=\mathrm{ch}(\xi^i)\cup\mathrm{bh}(\omega^p)\cup U_\alpha=(e^y-1)^i\left\{\frac{e^{py}-1}{py}\right\}\cup U_\alpha=y^i\cup U_\alpha+\text{h.o.t.}$$

Likewise

$$\mathrm{ch}(\xi^i\cup V_\beta)=(e^y-1)^i\left\{\frac{e^y-1}{y}\right\}^{d+1}\cup U_\beta=y^i\cup U_\beta+\text{h.o.t.}$$

Thus $s^!:\widetilde{KU}(T_n(\beta))\to\widetilde{KU}(T_n(\alpha))$ satisfies

$$s^!(\xi^i\cup V_\beta)=\sum_{j\geqslant i} x_j^{(i)}\,\xi^j\cup V_\alpha,\qquad 0\leqslant i\leqslant n,$$

where the $x_j^{(i)}$ are integers and $x_i^{(i)}=\varepsilon_{i+1}$.

The Adams operations ψ^k are computed by means of the cannibalistic characteristic classes ϱ^k. If ζ is a complex bundle with KU Thom class V_ζ then $\psi^k(x\cup V_\zeta)=\varrho^k(\zeta)\cdot\psi^k(x)\cup V_\zeta$ for any $x\in KU(X)$, (see Adams [1]). ϱ^k is exponential, $\varrho^k(\varepsilon)=k$, and $\varrho^k(\zeta)=1+\zeta^{-1}+\cdots+\zeta^{-(k-1)}$ if ζ is a complex line bundle.

Thus we have

$$\Psi^k(\xi^i\cup V_\alpha)=\varrho^k(\omega^p\oplus d\varepsilon)\,(\omega^k-1)^i\cup V_\alpha=k^d\left(\frac{\omega^{-pk}-1}{\omega^{-p}-1}\right)(\omega^k-1)^i\cup V_\alpha$$

and

$$\begin{aligned}\Psi^k(\xi^i\cup V_\beta)&=\varrho^k((d+1)\,\omega)\,(\omega^k-1)^i\cup V_\beta\\&=\left(\frac{\omega^{-k}-1}{\omega^{-1}-1}\right)^{d+1}\cdot(\omega^k-1)^i\cup V_\beta.\end{aligned}$$

Now we are in a position to deduce the result we are looking for. By equating the coefficients of $\xi\cup V_\alpha$ and $\xi^2\cup V_\alpha$ respectively in the equation $\psi^2 s^!(V_\beta)=s^!\psi^2(V_\beta)$

one gets

$$x_1^{(0)} = -\binom{d+1}{2}\varepsilon_2 + \frac{p}{2}\varepsilon_1$$

and

$$2^d\left\{x_0^{(0)}\binom{-p}{2} + 2x_1^{(0)}(-p+1) + 8x_2^{(0)}\right\} =$$
$$2^{d+1}x_2^{(0)} - 2^d(d+1)x_2^{(1)} + 2^{d-2}(d+1)(d+4)x_2^{(2)}$$

Furthermore, by equating the coefficients of $\xi^2 \cup V_\alpha$ in $\psi^2 s^1(\xi \cup V_\beta) = s^1\psi^2(\xi \cup V_\beta)$ we get

$$x_2^{(1)} = -\frac{d}{2}\varepsilon_3 + \binom{p-1}{2}\varepsilon_2 .$$

Combining these three equations we end up with the following congruence

$$2p^2\varepsilon_1 - 6p(\varepsilon_1 + \varepsilon_2) + 6\varepsilon_2 + 4\varepsilon_3 + 6d\varepsilon_2(1-p) + 3d^2\varepsilon_3 + 7d\varepsilon_3 \equiv 0 \pmod{24}$$

Let p be even, $n \geqslant 4$ and hence by (5.4) $d+1 \equiv 0 \pmod 8$. Then it is easy to see that all $p \equiv 2 \pmod 4$ do not satisfy this congruence relation, which is the required result.

In conclusion we mention that we have made number theoretic calculations to decide when there exists an integer d satisfying condition $S(n, p)$ for a fixed n and p. For example we have

(6.3) *Remark.* If $n \geqslant 32$ and $p \equiv 4 \pmod 8$ satisfies condition (H_n), then there is no integer d satisfying condition $S(n, p)$. The same is true for $n \geqslant 64$ and $p = 9, 21, 33$. In all these cases it then follows that the mapping cone of $f : L^{2n+1}(p) \to CP^n$ does not have the S-type of a stunted complex projective space. We understand that F. Sigrist has also done similar calculations.

In a subsequent paper we intend to study the problem of finding number theoretic conditions on n and p that guarantee the existence of an integer d satisfying $S(n, p)$ and then determine the congruence class of d modulo the Atiyah-Todd number M_{n+1}. We suspect that solving these number theory problems will remove the hypothesis $n \not\equiv 1 \pmod 4$ from (2.3).

BIBLIOGRAPHY

[1] ADAMS, J. F., *On the groups J(X)-II*, Topology *3*, (1965), 137–171.
[2] ADAMS, J. F. and WALKER, G., *On complex Stiefel manifolds*, Proc. Camb. Phil. Soc. *61* (1965), 81–103.
[3] ADEM, J. and GITLER, S., *Non-immersion theorems for real projective spaces*, Bol. Soc. Mat. Mexicana *9* (2) (1964), 47–50.
[4] ATIYAH, M., *Thom complexes*, Proc. London Math. Soc. *11* (3) (1961), 291–310.

[5] Atiyah, M. and Todd, J. A., *On complex Stiefel manifolds*, Proc. Camb. Phil. Soc. *56* (1960), 342–353.
[6] Milnor, J., *Lecture notes on characteristic classes* (mimeographed notes). (Princeton University, N.J., 1957).
[7] Spanier, E. H. and Whitehead, J. H. C., *The theory of carriers and S-theory*. A symposium in honor of S. Lefschetz. (Princeton University Press, Princeton, N.J., 1957), pp. 330–360.
["On the Spanier Conjecture", by R. P. Held, D. Sjerve]

University of British Columbia
Department of Mathematics
Vancouver 8, B.C. (Canada)

Received October 14, 1971