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Autor:	Gilmer, Robert
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# **On Factorization into Prime Ideals**

ROBERT GILMER<sup>1</sup>)

Let r be a regular element of the commutative ring R. It is well known that if r can be written as a finite product of prime elements of R, then this representation is unique. We consider here the corresponding question for ideals:

If A is a regular ideal of R such that A can be represented as a finite product of prime ideals of R, is this representation unique?

We begin by listing some observations concerning this question.

(1) Without the assumption that A is regular, the answer to the question is negative, even if R is Noetherian with identity. For example, (0) is prime in R for any integral domain R, and yet  $(0) = [(0)]^n$  for each positive integer n. If R is the direct sum of two fields  $F_1$  and  $F_2$ , then  $P = F_1 \oplus (0)$  is maximal in R, and  $P = P^n$  for each positive integer n.

(2) Even with the assumption that A is regular, the answer to the question is negative, even if R is an integral domain with identity. For instance,  $P_1 = P_1P_2$  for any prime ideals  $P_1$ ,  $P_2$  of a valuation ring R with  $P_1 \subset P_2$ ; more generally the equality  $P_1 = P_1P_2$  holds for any prime ideals  $P_1$ ,  $P_2$  of a Prüfer domain with  $P_1 \subset P_2$  [1, Theorem 19.3].

(3) The following result appears as Theorem 30.13 of [1]:

Let A be a nonzero ideal of a Noetherian domain D such that A can be expressed as a finite product of prime ideals of D. Then this representation is unique if D contains no identity, and is unique to within factors of D if D contains an identity.

(4) By examining the proof of Theorem 30.13, we can see that the following result, which we label as (\*), is valid.

(\*) Let A be a regular ideal of a commutative ring R such that A can be expressed as a finite product of finitely generated prime ideals of R. Then this representation is unique if R contains no identity, and is unique to within factors of R of R contains an identity.

In this paper we exted (\*) to the case where A is finitely generated, but the prime factors of A need not be finitely generated (Theorems 1 and 2). In Proposition 1, we prove that our results are stronger than (\*) by proving that for any positive integers

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k and n, there is an integral domain  $D_k$  with identity containing prime ideals  $P_1, \ldots, P_n$ such that  $P_1^{e_1} P_2^{e_2} \ldots P_n^{e_n}$  is finitely generated if and only if  $e_1 + \cdots + e_n \ge k$ . Our proofs of Theorems 1 and 2 are independent of (\*) and the result cited in (3). Moreover, our proofs are more elementary than the proof of (30.13) in [1]; these proofs rest on the following facts.

OBSERVATION 1. If  $\{A_i\}_{1}^{n}$  is a finite family of ideals of the commutative ring R, then  $A_1A_2...A_n$  is regular if and only if each  $A_i$  is regular.

OBSERVATION 2. If A is a regular ideal of the commutative ring R and if N is a multiplicative system in R, then the extension of A to the ring of quotients  $R_N$  is regular in  $R_N$ .

RESULT 1. [1, Corollary 5.2] If A and B are ideals of the commutative ring R such that AB=B, where B is finitely generated, then there is an element x of A such that xb=b for each b in B: if B is regular, then R has an identity element and A=R.

LEMMA 1. Assume that A and B are ideals of the commutative ring R, where B is proper, finitely generated, and regular. Moreover, assume that  $\{P_1, ..., P_m\}$  and  $\{Q_1, ..., Q_n\}$  are two families of proper prime ideals of R such that  $B = AP_1^{s_1} ... P_m^{s_m} = AQ_1^{t_1} ... Q_n^{t_n}$ , where each  $s_i$  and each  $t_i$  is positive. Then each minimal element of the set  $\{P_1, ..., P_m, Q_1, ..., Q_n\}$  occurs both as a  $P_i$  and as a  $Q_j$ , and the corresponding exponents  $s_i$  and  $t_j$  are equal.

*Proof.* We assume that the labeling is such that  $P_1$  is a minimal element of  $\{P_i\}_1^m \cup \{Q_j\}_1^n$ . If 'e' denotes extension of ideals with respect to the quotient ring  $R_{P_1}$ , then

$$B^{e} = A^{e} (P_{1}^{e})^{s_{1}} = A^{e} (Q_{j}^{e})^{wt_{j}},$$

where w=0 if  $P_1 \notin \{Q_j\}_1$ , while w=1 and  $P_1=Q_j$  otherwise. The assumption w=0 would lead to the equation

$$B^e = B^e (P_1^e)^{s_1},$$

where  $B^e$  is finitely generated, regular, and proper, and  $(P_1^e)^{s_1}$  is proper in  $R_{P_1}$ , in contradiction to Result 1. Hence w=1 and

$$B^e = A^e (P_1^e)^{s_1} = A^e (P_1^e)^{t_j}.$$

Again, if  $t_j > s_1$ , we obtain a contradiction, as above, from the equation

$$B^e = B^e (P_1^e)^{t_j - s_1}$$

Therefore  $s_1 = t_j$ , and our proof is complete.

THEOREM 1. Let B be a proper, finitely generated regular ideal of the commutative ring R such that B is a product of proper prime ideals of R. Then the representation of B as a finite product of proper prime ideals is unique.

Proof. Let

 $B = P_1^{e_1} \dots P_m^{s_m}$  and  $B = Q_1^{t_1} \dots Q_n^{t_n}$ 

be two representations of *B* as a finite product of proper prime ideals. From the proof of Lemma 1, it follows that the set *S* of primes  $P_i$  such that  $P_i = Q_j$  for some *j*, and  $s_i = t_j$ , is nonempty. We assume that  $S = \{P_1, ..., P_r\}$ , where  $r \le m$  and where  $P_i = Q_i$  for  $1 \le i \le r$ . Setting  $A = P_1^{s_1} \dots P_r^{s_r}$ , we have

$$B = AP_{r+1}^{s_{r+1}} \dots P_m^{s_m} = AQ_{r+1}^{t_{r+1}} \dots Q_n^{t_n},$$

and the assumption r < m or r < n would lead to a contradiction of Lemma 1. Hence r=m=n, and this completes the proof of Theorem 1.

THEOREM 2. Suppose that R is a commutative ring without identity and that B is a finitely generated regular ideal of R that is representable as a finite product of prime ideals of R. Then this representation is unique.

*Proof.* We consider first the case when  $B = R^n$  is a power of R. It is clear that R is the only prime factor of B. Hence we need only prove in this case that  $R^n = R^m$  implies that m = n. Since R is a ring without identity, this follows immediately from Result 1.

If B is not a power of R, then we write

$$B = R^s P_1^{s_1} \dots P_m^{s_m} = R^t Q_1^{t_1} \dots Q_n^{t_n},$$

where  $\{P_i\}_1^m$  and  $\{Q_i\}_1^n$  are sets of *m* and *n* proper prime ideals of *R*, where  $s_i$  and  $t_j$  are positive, and *s* and *t* are nonnegative  $(R^0U$ , for *U* an ideal of *R*, is defined to be *U*). If  $N = R - [(\bigcup_{i=1}^{m} P_i) \cup (\bigcup_{i=1}^{n} Q_i)]$ , and if 'e' denotes extension of ideals of *R* to the quotient ring  $R_N$ , then

$$B^{e} = (P_{1}^{e})^{s_{1}} \dots (P_{m}^{e})^{s_{m}} = (Q_{1}^{e})^{t_{1}} \dots (Q_{n}^{e})^{t_{n}}$$

in  $R_N$ . By Theorem 1, m=n and, by proper labeling,  $P_i^e = Q_i^e$  and  $s_i = t_i$  for  $1 \le i \le m$ . It follows that  $P_i = Q_i$  for  $1 \le i \le n$ , and we have

$$B = R^s P_1^{s_1} \dots P_m^{s_m} = R^t P_1^{s_1} \dots P_m^{s_m}.$$

As before, the assumption s > t would lead to the equation  $R^{s-t} B = B$ , and to a contradiction of the assumption that R does not contain an identity.

It is clear that Theorems 1 and 2 imply (\*). It is conceivable, however, that Theorems 1 and 2 are not actually stronger than (\*). That is, if the following statement (\*\*) were true, then (\*) would imply Theorems 1 and 2. (\*\*) If  $\{P_i\}_1^m$  is a finite family of regular prime ideals of the commutative ring R, and if  $e_1, \ldots, e_m$  are positive integers such that  $P_1^e \ldots P_m^{e_m}$  is finitely generated, then each  $P_i$  is finitely generated.

We proceed to show that a very strong negation of (\*\*) is, in fact, true.

**PROPOSITION 1.** Let k and n be positive integers, where  $k \ge 1$ . There is an integral domain with identity containing prime ideals  $P_1, P_2, ..., P_n$  such that the product  $P_1^{e_1} P_2^{e_2} ... P_n^{e_n}$  is finitely generated if and only if  $\sum_{i=1}^n e_i \ge k$ .

*Proof.* Let D be a non-Noetherian domain with identity containing distinct ideals  $A_1, A_2, ..., A_n$  such that  $A_{i_1} + A_{i_2} + \cdots + A_{i_r}$  is not finitely generated for any nonempty subset  $\{i_1, i_2, ..., i_r\}$  of  $\{1, 2, ..., n\}^2$ ). Let t be an indeterminate over D, and let  $E_k$  be the subring  $D[t^{k+1}, t^{k+2}, ..., t^{2k+1}]$  of D[t];  $E_k$  is a graded ring with gradation D,  $Dt^{k+1}, Dt^{k+2}, ...$  We set

$$A = (t^{k+1}, t^{k+2}, ..., t^{2k+1}), \quad B = (t^{k+1}, t^{k+2}).$$

It is straightforward to verify that

$$A^{n} = \left(t^{n(k+1)}, t^{n(k+1)+1}, \dots, t^{n(k+1)+k}\right)$$

for any positive integer n,

 $A^n = B^n$  for  $n \ge k$ , and  $B^r = (t^{r(k+1)}, t^{r(k+1)+1}, \dots, t^{r(k+1)+r})$  for r < k.

We set  $C_i = B + A_i t^{k+3}$  for  $1 \le i \le n$ . Each  $C_i$  is a homogeneous ideal of  $E_k$ , and  $B \subseteq C_i \subseteq A$  for each *i*. Hence

 $C_1^{e_1}C_2^{e_2}\ldots C_n^{e_n} = A^{e_1+\ldots+e_n} = B^{e_1+\ldots+e_n}$ 

if  $e_1 + \dots + e_n \ge k$ , and  $C_1^{e_1} C_2^{e_2} \dots C_n^{e_n}$  is finitely generated.

If  $e = e_1 + \dots + e_n < k$ , then  $C_1^{e_1} C_2^{e_2} \dots C_n^{e_n}$  is a homogeneous ideal of  $E_k$ , and its homogeneous component in  $Dt^{e(k+1)+(e+1)}$  is

 $(A_{i_1} + A_{i_2} + \dots + A_{i_r}) t^{e(k+1)+(e+1)},$ 

where  $\{i_1, i_2, ..., i_r\}$  is the set of integers *j* such that  $e_j \neq 0$ . Because  $E_k$  is a graded ring and  $A_{i_1} + A_{i_2} + \cdots + A_{i_r}$  is not finitely generated as an ideal of *D*, it follows that  $C_1^{e_1} C_2^{e_2} \dots C_n^{e_n}$  is not finitely generated as an ideal of  $E_k$ .

The ideals  $C_i$  of  $E_k$  are not prime in  $E_k$ . To obtain our desired example, we let  $D_k$  be the subring of  $E = E_k[X_1, ..., X_n]$  consisting of all polynomials f such that the

<sup>2)</sup> Take, for example,  $D = Z[\{X_j\}_{j=1}^{\infty}]$ , and for  $1 \le i \le n$ , take  $A_i = (\{X_j \mid j \in S_i\})$ , where  $S_1, \ldots, S_n$  are distinct infinite subsets of N, the set of positive integers.

coefficient of  $X_1^{e_1} X_2^{e_2} \dots X_n^{e_n}$  in f is in  $A_1^{e_1} A_2^{e_2} \dots A_n^{e_n}$  for each  $e_1, e_2, \dots, e_n \ge 0$ . Again  $D_k$  is a graded ring with gradation

 $E_k, \sum A_i X_i, \sum A_i^{e_i} A_j^{2-e_i} X_i^{e_i} X_j^{2-e_i}, \dots;$ 

in fact,  $D_k$  is a graded subring of E, where E has the usual gradation by degree. The ideal  $X_i E \cap D_k = P_i$  is a homogeneous prime ideal of  $D_k$ : in fact,

$$P_i = A_i X_i + A_i A_j X_i X_j + \dots = A_i X_i D_k.$$

Hence

 $P_1^{e_1}P_2^{e_2}\dots P_n^{e_n} = A_1^{e_1}A_2^{e_2}\dots A_n^{e_n}X_1^{e_1}X_2^{e_2}\dots X_n^{e_n}D_k$ 

is the set of polynomials f in  $D_k$  such that the coefficient of  $X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$  in f is zero if  $i_j < e_j$  for some j, and is in  $A_1^{i_1} A_2^{i_2} \dots A_n^{i_n}$  otherwise. Hence  $P_1^{e_1} P_2^{e_2} \dots P_n^{e_n}$  is finitely generated if and only if  $e_1 + \dots + e_n \ge k$ .

We remark that the prime ideals  $P_i$  of Proposition 1 extend to maximal ideals of the quotient ring  $(D_k)_S$ , where  $S = D_k - (\bigcup_{i=1}^{n} P_i)$ . But  $(D_k)_S$  is a quotient ring of  $L[X_1, ..., X_n]$ , where L is the quotient field of  $E_k$ , and hence  $(D_k)_S$  is Noetherian.

We have no counterexample to (\*\*) in the case where the ideals  $P_i$  are maximal in R. In particular, we know of no example of a regular maximal ideal M of a commutative ring S with identity such that M is not finitely generated, but some power of M is finitely generated. If such M and S exist, then they also exist with M maximal in a quasi-local ring S.

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Florida State University, Tallahassee

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