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## On the Absolute Continuity of a Surface Representation

by HANS MARTIN REIMANN

This note contains an example of a 2-dimensional surface in 3-space, which is represented by an absolutely continuous (in the sense of Tonelli) homeomorphism  $f$ . Although the surface has finite Lebesgue area and  $f$  is a mapping “of bounded distortion” with  $L^2$  – integrable partial derivatives, there exists a 2-dimensional zero set which is mapped onto a set of positive 2-dimensional Hausdorff measure.

A real valued continuous function  $f$  defined in a bounded domain  $G^k$  in  $k$ -dimensional Euclidean space  $E^k$  is absolutely continuous in the sense of Tonelli if:

(i) Given any closed interval  $I^k \subset G^k$ ,  $I^k = \{(x_1, \dots, x_k) \mid a_i \leq x_i \leq b_i, i=1, \dots, k\}$   $f$  is absolutely continuous as a function of  $x_i$  on a.e. line parallel to the  $x_i$  axis;  $i=1, \dots, k$ ;

(ii) The partial derivatives which exist a.e. are integrable in  $G^k$ . For mappings  $f=(f_1, \dots, f_n): G^k \rightarrow E^n$  we write  $f \in ACL^p$  ( $p > 1$ ), if all coordinate functions  $f_i$ ,  $i=1, \dots, n$ , are absolutely continuous in the sense of Tonelli and furthermore the partial derivatives are integrable to the power  $p$ .

Cesari [1952] proved that mapping  $s f \in ACL^p$ ,  $p > 2$ ,  $f: G^2 \rightarrow E^2$  have the following property: Every subset of  $G^2$  of zero (2-dim.) measure is mapped onto a set of zero measure. We will refer to this property by saying that  $f$  satisfies condition  $N$  with respect to 2-dimensional Lebesgue measure  $m_2: N(m_2)$ . In the same paper Cesari presented examples of mappings  $f \in ACL^2$ ,  $f: G^2 \rightarrow E^2$ , which do not satisfy condition  $N(m_2)$  and give rise to further phenomena. Some of Cesari’s examples are based on conformal representations as the one below.

Cesari’s result carries over to higher dimensions: Calderon [1951] has shown that mappings  $f \in ACL^p$ ,  $p > k$ ,  $f: G^k \rightarrow E^k$  are generalized Lipschitzian in the sense of Rado-Reichelderfer [1955]. From their results it then follows that  $f$  satisfies condition  $N(m_k)$ . This result still holds if  $f \in ACL^p$ ,  $p > k$ , is a mapping  $f: G^k \rightarrow E^n$ ,  $n > k$ . Condition  $N$  is then satisfied with respect to  $k$ -dimensional Hausdorff measure  $H_k$ .

If  $f \in ACL^k$  is a homeomorphism,  $f: G^k \rightarrow E^k$ , one can also conclude that  $f$  satisfies  $N(m_k)$ . This is well known for  $k=2$  (for a proof see e.g. Lehto-Virtanen [1965] p.158). A proof for the case  $k > 2$  has been given by Reshetnjak [1966].

A mapping  $f \in ACL^k$ ,  $f: G^k \rightarrow E^k$  is said to be of bounded distortion if there exists a constant  $C \geq 1$  such that

$$|df|^k \leq C Jf$$

holds a.e. in  $G^k$ . Here  $Jf(x)$  is the (signed) Jacobian and  $|df(x)|$  is the norm of the linear transformation  $df(x)$ , which is given by the partial derivatives of  $f$  at  $x$ . For mappings  $f \in ACL^k$ ,  $f: G^k \rightarrow E^n$ ,  $n > k$ , we interpret this condition as  $|df|^k \leq C \|Jf\|$

a.e. in  $G^k$  with

$$\|Jf\| = \left( \frac{1}{k!} \sum \left[ \frac{\partial(f_{\alpha_1}, \dots, f_{\alpha_k})}{\partial(x_1, \dots, x_k)} \right]^2 \right)^{1/2},$$

where the sum in this expression extends over all multiindices  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $1 \leq \alpha_i \leq n$ . (Intuitively  $\|Jf\|$  denotes the ‘‘surface element’’.) To guarantee that  $f: G^2 \rightarrow E^3$ ,  $f \in ACL^2$ , is of bounded distortion it is sufficient to verify that a.e. in  $G^2$

$$\sum_{i,j} \left( \frac{\partial f_i}{\partial x_j} \right)^2 \leq C' \|Jf\|$$

for some constant  $C'$ .

From Reshetnjak’s work [1967] it is known that mappings  $f \in ACL^k$ ,  $f: G^k \rightarrow E^k$ , which are of bounded distortion, satisfy  $N(m_k)$ . The homeomorphisms of bounded distortion are the quasiconformal mappings (see e.g. Gehring [1962]). The investigation of extremal length properties of quasiconformal mappings leads to the following question: Do homeomorphisms  $f: G^k \rightarrow E^n$ ,  $n > k$ ,  $f \in ACL^k$ , which are of bounded distortion, satisfy condition  $N(H_k)$ ? The following example provides a negative answer to this question.

Let  $J$  be an Osgood curve, i.e. a closed Jordan curve in the plane with positive 2-dimensional measure.  $J$  separates the plane into a bounded and an unbounded component. We map the unit square  $Q = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$  conformally onto the bounded component  $J^0$ . By the Carathéodory extension theorem this mapping  $h$  can be extended continuously and one to one to a mapping  $h_c$  of the closed square  $\bar{Q}$  onto  $J^0 \cup J$ . Furthermore we can choose  $h$  in such a way as to have  $A = \{(x, y) \mid x = 0, 0 < y < 1\}$  mapped onto a set of positive 2-dimensional measure.

We define now the continuous mapping  $g = (u, v) : R \rightarrow J^0 \cup J$  by setting  $R = \{(x, y) \mid 0 \leq |x| < 1, 0 < y < 1\}$  and

$$g(x, y) = \begin{cases} h_c(x, y) & \text{for } (x, y) \in Q \cup A \\ h(-x, y) & \text{otherwise} \end{cases}$$

Next we construct an auxiliary function  $w: R \rightarrow E^1$  in terms of the bounded positive function  $a(x, y) = \min\{1, |h'(x, y)|\}: Q \rightarrow E^1$ , where  $h = (u, v)$ ,  $|h'|^2 = |u_x v_y - u_y v_x|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 > 0$ . We define

$$w(x, y) = \begin{cases} \inf_{\gamma} \int_{\gamma} a(x, y) ds & \text{for } (x, y) \in Q \\ 0 & \text{for } (x, y) \in R \setminus Q \end{cases}$$

where the infimum is taken over all rectifiable curves  $\gamma \subset Q$  connecting  $(x, y)$  with  $A$ .  $w(x, y)$  is positive for all  $(x, y) \in Q$  since  $a(x, y)$  is positive and continuous in  $Q$ .

**THEOREM.** The mapping  $f=(u, v, w):R\rightarrow E^3$  constructed above has all the properties:

- a)  $f\in ACL^2$
- b)  $f$  is a homeomorphism
- c)  $f$  is of bounded distortion
- d)  $f$  maps the set  $A$  (with  $H_2(A)=0$ ) onto a set  $B$  with  $H_2(B)>0$ .

a)  $w$  satisfies a uniform Lipschitz condition with constant 1, hence  $w\in ACL^2$ .  $g=(u, v)$  is conformal in  $Q$  and maps  $Q$  onto a bounded domain. Therefore

$$\int_R |g'|^2 dx dy = 2 \int_Q |g'|^2 dx dy < \infty,$$

which means that the partial derivatives of  $u$  and  $v$  are square integrable. In order to show that  $g\in ACL^2$  it is sufficient to prove that for a.e.  $y$ ,  $0<y<1$ ,  $g(x, y)$  is absolutely continuous as a function of  $x$ . We choose  $y$  in such a way that

$V(y)=\int_{-1}^1 |g'(x, y)| dx < \infty$ . For these values the function  $g(x, y)$  is absolutely continuous in  $x$ , since it has an integral representation

$$g(x, y) = g(x, 0) + \int_0^x g'(t, y) dt$$

and the total variation  $V(y)$  is finite.

b) Because  $w(x, y)\neq 0$  for  $(x, y)\in Q$ ,  $f$  is a homeomorphism.

c)  $F$  satisfies the distortion condition  $|df|^2 \leq C \|Jf\|$  a.e. in  $R$ . For  $(x, y)\in R\setminus\bar{Q}$  this is clearly true for any constant  $C\geq 1$ . In the case  $(x, y)\in Q$  we obtain the following estimates:

$$\|Jf\| \geq |u_x v_y - u_y v_x| = |g'|^2$$

and

$$|w_x| \leq \left| \lim_{h\rightarrow 0} h^{-1} \int_x^{x+h} a(t, y) dt \right| \leq a(x, y) \leq |g'(x, y)|$$

From this we conclude

$$|(u_x, v_x, w_x)|^2 \leq |g'|^2 + a^2 \leq 2 |g'|^2.$$

An analogous relation holds for the derivatives with respect to  $y$  and therefore  $|df|^2 \leq C \|Jf\|$  for any  $C\geq 4$ . This clearly is not the best estimate. We remark that by replacing the function  $a(x, y)$  in the definition for  $w(x, y)$  by  $c\cdot a(x, y)$ ,  $c$  constant, we obtain  $C\rightarrow 1$  for  $c\rightarrow 0$ .

d)  $f$  does not satisfy condition  $N(H_2)$

The set  $A = \{(x, y) \mid x=0, 0 < y < 1\}$  has zero 2-dimensional measure ( $H_2(A) = 0$ ) and  $f$  maps  $A$  onto a set  $B$  with  $H_2(B) > 0$ . (Observe that  $H_2(B) = m_2(B)$ , since  $B$  lies in the plane  $w=0$ .)

We add a few remarks:

1)  $f$  does not satisfy condition  $N$  with respect to 2-dimensional integralgeometric measure  $I_2$ . Using the characterization of  $I_2$  given by Federer [1947] p. 145, this statement can easily be verified.

2) Since  $f \in ACL^2$ , the Lebesgue area of  $f$  is given by  $L(f) = \int_R \|Jf\| dx dy$ .  $f$  therefore is an example of a homeomorphism with the property that  $L(f) \neq H_2(f(R))$ . A similar example of such a mapping has been constructed by Breckenridge [1970].

3)  $g: R \rightarrow E^2$  is another example of a mapping of the type described by Cesari:  $g \in ACL^2$  does not satisfy condition  $N(m_2)$ .

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