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On the Absolute Continuity of a Surface Representation

by HANS MARTIN REIMANN

This note contains an example of a 2-dimensional surface in 3-space, which is represented by an absolutely continuous (in the sense of Tonelli) homeomorphism f . Although the surface has finite Lebesgue area and f is a mapping “of bounded distortion” with L^2 – integrable partial derivatives, there exists a 2-dimensional zero set which is mapped onto a set of positive 2-dimensional Hausdorff measure.

A real valued continuous function f defined in a bounded domain G^k in k -dimensional Euclidean space E^k is absolutely continuous in the sense of Tonelli if:

(i) Given any closed interval $I^k \subset G^k$, $I^k = \{(x_1, \dots, x_k) | a_i \leq x_i \leq b_i, i=1, \dots, k\}$ f is absolutely continuous as a function of x_i on a.e. line parallel to the x_i axis; $i=1, \dots, k$;

(ii) The partial derivatives which exist a.e. are integrable in G^k . For mappings $f = (f_1, \dots, f_n) : G^k \rightarrow E^n$ we write $f \in ACL^p$ ($p > 1$), if all coordinate functions f_i , $i = 1, \dots, n$, are absolutely continuous in the sense of Tonelli and furthermore the partial derivatives are integrable to the power p .

Cesari [1952] proved that mapping $s f \in ACL^p$, $p > 2$, $f : G^2 \rightarrow E^2$ have the following property: Every subset of G^2 of zero (2-dim.) measure is mapped onto a set of zero measure. We will refer to this property by saying that f satisfies condition N with respect to 2-dimensional Lebesgue measure $m_2 : N(m_2)$. In the same paper Cesari presented examples of mappings $f \in ACL^2$, $f : G^2 \rightarrow E^2$, which do not satisfy condition $N(m_2)$ and give rise to further phenomena. Some of Cesari’s examples are based on conformal representations as the one below.

Cesari’s result carries over to higher dimensions: Calderon [1951] has shown that mappings $f \in ACL^p$, $p > k$, $f : G^k \rightarrow E^k$ are generalized Lipschitzian in the sense of Rado-Reichelderfer [1955]. From their results it then follows that f satisfies condition $N(m_k)$. This result still holds if $f \in ACL^p$, $p > k$, is a mapping $f : G^k \rightarrow E^n$, $n > k$. Condition N is then satisfied with respect to k -dimensional Hausdorff measure H_k .

If $f \in ACL^k$ is a homeomorphism, $f : G^k \rightarrow E^k$, one can also conclude that f satisfies $N(m_k)$. This is well known for $k = 2$ (for a proof see e.g. Lehto-Virtanen [1965] p.158). A proof for the case $k > 2$ has been given by Reshetnjak [1966].

A mapping $f \in ACL^k$, $f : G^k \rightarrow E^k$ is said to be of bounded distortion if there exists a constant $C \geq 1$ such that

$$|df|^k \leq C Jf$$

holds a.e. in G^k . Here $Jf(x)$ is the (signed) Jacobian and $|df(x)|$ is the norm of the linear transformation $df(x)$, which is given by the partial derivatives of f at x . For mappings $f \in ACL^k$, $f : G^k \rightarrow E^n$, $n > k$, we interpret this condition as $|df|^k \leq C \|Jf\|$

a.e. in G^k with

$$\|Jf\| = \left(\frac{1}{k!} \sum \left[\frac{\partial(f_{\alpha_1}, \dots, f_{\alpha_k})}{\partial(x_1, \dots, x_k)} \right]^2 \right)^{1/2},$$

where the sum in this expression extends over all multiindices $\alpha = (\alpha_1, \dots, \alpha_k)$, $1 \leq \alpha_i \leq n$. (Intuitively $\|Jf\|$ denotes the “surface element”.) To guarantee that $f: G^2 \rightarrow E^3$, $f \in ACL^2$, is of bounded distortion it is sufficient to verify that a.e. in G^2

$$\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j} \right)^2 \leq C' \|Jf\|$$

for some constant C' .

From Reshetnjak's work [1967] it is known that mappings $f \in ACL^k$, $f: G^k \rightarrow E^k$, which are of bounded distortion, satisfy $N(m_k)$. The homeomorphisms of bounded distortion are the quasiconformal mappings (see e.g. Gehring [1962]). The investigation of extremal length properties of quasiconformal mappings leads to the following question: Do homeomorphisms $f: G^k \rightarrow E^n$, $n > k$, $f \in ACL^k$, which are of bounded distortion, satisfy condition $N(H_k)$? The following example provides a negative answer to this question.

Let J be an Osgood curve, i.e. a closed Jordan curve in the plane with positive 2-dimensional measure. J separates the plane into a bounded and an unbounded component. We map the unit square $Q = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$ conformally onto the bounded component J^0 . By the Carathéodory extension theorem this mapping h can be extended continuously and one to one to a mapping h_c of the closed square \bar{Q} onto $J^0 \cup J$. Furthermore we can choose h in such a way as to have $A = \{(x, y) \mid x = 0, 0 < y < 1\}$ mapped onto a set of positive 2-dimensional measure.

We define now the continuous mapping $g = (u, v) : R \rightarrow J^0 \cup J$ by setting $R = \{(x, y) \mid 0 \leq |x| < 1, 0 < y < 1\}$ and

$$g(x, y) = \begin{cases} h_c(x, y) & \text{for } (x, y) \in Q \cup A \\ h(-x, y) & \text{otherwise} \end{cases}$$

Next we construct an auxiliary function $w: R \rightarrow E^1$ in terms of the bounded positive function $a(x, y) = \min\{1, |h'(x, y)|\} : Q \rightarrow E^1$, where $h = (u, v)$, $|h'|^2 = |u_x v_y - u_y v_x| = u_x^2 + v_x^2 = u_y^2 + v_y^2 > 0$. We define

$$w(x, y) = \begin{cases} \inf_{\gamma} \int_{\gamma} a(x, y) ds & \text{for } (x, y) \in Q \\ 0 & \text{for } (x, y) \in R \setminus Q \end{cases}$$

where the infimum is taken over all rectifiable curves $\gamma \subset Q$ connecting (x, y) with A . $w(x, y)$ is positive for all $(x, y) \in Q$ since $a(x, y)$ is positive and continuous in Q .

THEOREM. The mapping $f=(u, v, w): R \rightarrow E^3$ constructed above has all the properties:

- a) $f \in ACL^2$
- b) f is a homeomorphism
- c) f is of bounded distortion
- d) f maps the set A (with $H_2(A)=0$) onto a set B with $H_2(B)>0$.

a) w satisfies a uniform Lipschitz condition with constant 1, hence $w \in ACL^2$. $g=(u, v)$ is conformal in Q and maps Q onto a bounded domain. Therefore

$$\int_R |g'|^2 dx dy = 2 \int_Q |g'|^2 dx dy < \infty,$$

which means that the partial derivatives of u and v are square integrable. In order to show that $g \in ACL^2$ it is sufficient to prove that for a.e. y , $0 < y < 1$, $g(x, y)$ is absolutely continuous as a function of x . We choose y in such a way that

$V(y) = \int_{-1}^1 |g'(x, y)| dx < \infty$. For these values the function $g(x, y)$ is absolutely continuous in x , since it has an integral representation

$$g(x, y) = g(x, 0) + \int_0^x g'(t, y) dt$$

and the total variation $V(y)$ is finite.

- b) Because $w(x, y) \neq 0$ for $(x, y) \in Q$, f is a homeomorphism.
- c) F satisfies the distortion condition $|df|^2 \leq C \|Jf\|$ a.e. in R . For $(x, y) \in R \setminus \bar{Q}$ this is clearly true for any constant $C \geq 1$. In the case $(x, y) \in Q$ we obtain the following estimates:

$$\|Jf\| \geq |u_x v_y - u_y v_x| = |g'|^2$$

and

$$|w_x| \leq \left| \lim_{h \rightarrow 0} h^{-1} \int_x^{x+h} a(t, y) dt \right| \leq a(x, y) \leq |g'(x, y)|$$

From this we conclude

$$|(u_x, v_x, w_x)|^2 \leq |g'|^2 + a^2 \leq 2 |g'|^2.$$

An analogous relation holds for the derivatives with respect to y and therefore $|df|^2 \leq C \|Jf\|$ for any $C \geq 4$. This clearly is not the best estimate. We remark that by replacing the function $a(x, y)$ in the definition for $w(x, y)$ by $c \cdot a(x, y)$, c constant, we obtain $C \rightarrow 1$ for $c \rightarrow 0$.

- d) f does not satisfy condition $N(H_2)$

The set $A = \{(x, y) \mid x=0, 0 < y < 1\}$ has zero 2-dimensional measure ($H_2(A)=0$) and f maps A onto a set B with $H_2(B) > 0$. (Observe that $H_2(B) = m_2(B)$, since B lies in the plane $w=0$.)

We add a few remarks:

1) f does not satisfy condition N with respect to 2-dimensional integralgeometric measure I_2 . Using the characterization of I_2 given by Federer [1947] p. 145, this statement can easily be verified.

2) Since $f \in ACL^2$, the Lebesgue area of f is given by $L(f) = \int_R \|Jf\| dx dy$. f therefore is an example of a homeomorphism with the property that $L(f) \neq H_2(f(R))$. A similar example of such a mapping has been constructed by Breckenridge [1970].

3) $g: R \rightarrow E^2$ is another example of a mapping of the type described by Cesari: $g \in ACL^2$ does not satisfy condition $N(m_2)$.

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