

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 46 (1971)

**Artikel:** Semicontinuity of the Face-Function of a Convex Set  
**Autor:** Klee, Victor / Martin, Michael  
**DOI:** <https://doi.org/10.5169/seals-35502>

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# Semicontinuity of the Face-Function of a Convex Set

VICTOR KLEE and MICHAEL MARTIN<sup>1)</sup>

*Dedicated to Hugo Hadwiger on His Sixtieth Birthday*

## Introduction

Throughout this paper,  $X$  is assumed to be a closed subset of a topological linear space  $E$ ; other conditions on  $X$  and  $E$  are stated explicitly, sometimes in standing hypotheses at the beginning of a section. For each point  $x$  of  $X$ ,  $F(x)$  is the closure of the union of  $\{x\}$  with all segments in  $X$  that cross  $x$ ; that is,

$$F(x) = \text{cl} \{y: [y, y + \lambda(x - y)] \subset X \text{ for some } \lambda > 1\}.$$

The set-valued function  $F$  is called the *face-function* of  $X$ , for when  $X$  is a finite-dimensional convex body or the boundary of such a body,  $F(x)$  is the smallest face of  $X$  that includes  $x$ . (In this case the closure in the definition of  $F(x)$  is redundant.) The set of all points of  $X$  at which the face-function  $F$  is lower [resp. upper] semicontinuous is denoted by  $X_l$  [resp.  $X_u$ ].

Though some of our theorems apply to infinite-dimensional sets, this introduction describes only the finite-dimensional results. One of our two main results is the following.

**THEOREM A.** *If  $X$  is the boundary of a  $d$ -dimensional convex body then the face-function  $F$  of  $X$  is lower semicontinuous almost everywhere in the sense of category and upper semicontinuous almost everywhere in the sense of measure. However, when  $d \geq 3$  an example of Corson [4] shows that  $F$  may be lower semicontinuous almost nowhere in the sense of measure and upper semicontinuous almost nowhere in the sense of category.*

Actually, the statement that  $F$  is upper semicontinuous almost everywhere in the sense of measure is proved only for  $d \leq 3$ , so part of the above theorem remains as a conjecture.

Now let  $K$  be a compact convex set in a locally convex  $E$  and let  $C(K)$  be the set of all continuous real-valued functions on  $K$ . For each  $f \in C(K)$  let  $f_e$  denote the restriction to  $K$  of the pointwise supremum of all continuous affine functions on  $E$  that are majorized on  $K$  by  $f$ . The function  $f_e$ , which is plainly real-valued, convex, and lower semicontinuous, is here called the *envelope* of  $f$ . Let  $K_e$  denote the set of

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all points of  $K$  at which all envelopes are continuous. For a finite-dimensional  $K$ , the points of  $K_e$  were shown by Witsenhausen [17, p. 20] to be precisely the points of convergence of a certain approximate algorithm for a type of minimax stochastic control problem associated with  $K$ . Plainly  $\text{int} K \subset K_e$ , and Witsenhausen proved that  $K_e = K$  when  $K$  is a polytope as well as when  $K$  is strictly convex. Kruskal [11] reported Witsenhausen's conjecture that  $K_e = K$  for all finite-dimensional  $K$ , disproved it by a 3-dimensional example, and conjectured that  $K_e = K$  for all 2-dimensional  $K$ . Our second main result is the following.

**THEOREM B.** *If  $X$  is the boundary of a  $d$ -dimensional compact convex body  $K$ , and  $X_e = X \cap K_e$ , then  $X_e \subset X_i$  and  $X_e$  is almost all of  $X$  in the sense of category. Further,  $X_e = X$  when  $d = 2$ .*

Thus Kruskal's conjecture is proved and it is seen that Witsenhausen's algorithm has good convergence properties with respect to category, though not necessarily with respect to measure.

We are indebted to J. B. Kruskal for supplying a prepublication copy of [11], and to H. H. Corson, R. R. Phelps, and the referee for helpful comments.

### Preparatory Remarks

Semicontinuity is used in the sense of Choquet [3] and Fort [5]. If  $T$  is a topological space,  $\mathbf{S}(T)$  the set of all closed subsets of  $T$ , and  $N_\delta$  a net on a directed set  $D$  into  $\mathbf{S}(T)$ , then

$$\lim_{\delta \in D} N_\delta = \{t \in T : \text{for each neighborhood } U \text{ of } t, N_\delta \text{ eventually intersects } U\},$$

$$\overline{\lim}_{\delta \in D} N_\delta = \{t \in T : \text{for each neighborhood } U \text{ of } t, N_\delta \text{ frequently intersects } U\}.$$

If  $Z$  is a topological space and  $S$  is a function on  $Z$  into  $\mathbf{S}(T)$ , then  $S$  is said to be *lower* [resp. *upper*] *semicontinuous* at the point  $z_0$  of  $Z$  provided that  $\lim_{\delta \in D} S(z_\delta) \supset S(z_0)$  [resp.  $\overline{\lim}_{\delta \in D} S(z_\delta) \subset S(z_0)$ ] for each net  $z_\delta$  that converges to  $z_0$  in  $Z$ . When the point  $z_0$  admits a countable neighborhood base, these definitions are equivalent to the corresponding ones that involve sequences rather than nets. When  $T$  is compact they are equivalent to requiring that for each open set  $U$  which intersects  $S(z_0)$  [resp. contains  $S(z_0)$ ] the set  $\{z \in Z : U \text{ intersects } S(z)\}$  [resp.  $\{z \in Z : U \text{ contains } S(z)\}$ ] is a neighborhood of  $z_0$ .

As the term is used here, a *body* is a set that has nonempty interior. Hausdorff  $k$ -dimensional measure is denoted  $\mu_k$ .

Now suppose that  $X$  is a closed convex set or is the boundary of such a set. Then a *face* of  $X$  is defined as a convex subset  $C$  of  $X$  such that  $C$  contains  $[p, q]$  whenever  $p, q \in X$  and  $C$  intersects  $]p, q[$ . For  $x \in X$ , the smallest face of  $X$  that includes  $x$  is the set

$$F_a(x) = \{y : [y, y + \lambda(x - y)] \subset X \text{ for some } \lambda > 1\}.$$

If  $X$  is finite-dimensional, then  $F(x) = F_a(x)$  and  $F$  deserves its name of *face-function*. However, the reader should be warned that when  $X$  is infinite-dimensional the set  $F(x)$  need not be a face of  $X$ . Suppose, for example, that  $X$  is a compact Choquet simplex. Then it may happen that the set  $\text{ext } X$  of all extreme points of  $X$  is dense in  $X$  (Thue Poulsen [16]). On the other hand, it follows from results of Alfsen [1, p. 101] and Størmer [15, p. 257] that  $\text{ext } X$  is closed if and only if  $F(x)$  is a face of  $X$  for all  $x \in X$ .

### Lower Semicontinuity of the Face-Function

*Recall the standing hypotheses that  $X$  is a closed subset of a topological linear space  $E$ ,  $F$  is the face-function of  $X$ , and  $X_l$  is the set of all points of  $X$  at which  $F$  is lower semicontinuous.*

**1.1 THEOREM.** *If  $X$  is locally compact, separable, and metrizable, then  $X_l$  is a dense  $G_\delta$  subset of  $X$ .*

*Proof.* Let  $X_1, X_2, \dots$  be a sequence of compact sets whose union is  $X$ . For each pair of positive integers  $i$  and  $j$  let  $W_{ij}$  denote the set of all points  $x$  of  $X$  for which there exists a point  $y \in X_j$  and there exists a sequence  $x_\delta$  in  $X$  converging to  $x$  such that

- (a)  $[y, y + \lambda(x - y)] \subset X$  for some  $\lambda \in [1 + i^{-1}, 1 + i]$ , and
- (b) the distance from the point  $y$  to the set  $\overline{\lim} F(x_\delta)$  is at least  $i^{-1}$ .

It follows by routine arguments that  $X \sim X_l = \bigcup_{i,j=1}^{\infty} W_{ij}$  and each set  $W_{ij}$  is closed. Hence  $X_l$  is a  $G_\delta$  set.

For each positive integer  $i$  and each point  $x$  of  $X$ , let  $F_i(x) = \text{cl} \{y : [y, y + \lambda(x - y)] \subset X \text{ for some } \lambda \in [1 + i^{-1}, 1 + i]\}$ . As each  $F_i$  is everywhere upper semicontinuous, it follows from Theorem 12 of Fort [5] and the Baire category theorem that there is a dense  $G_\delta$  subset  $V$  of  $X$  such that each function  $F_i$  is lower semicontinuous at each point of  $V$ . Consider an arbitrary point  $v$  of  $V$  and sequence  $x_\delta$  in  $X$  converging to  $v$ . For each  $i$ ,

$$F_i(v) \subset \underline{\lim} F_i(x_\delta) \subset \underline{\lim} F(x_\delta),$$

and as the last set is closed it follows that

$$F(v) = \text{cl} \bigcup_1^{\infty} F_i(v) \subset \underline{\lim} F(x_\delta). \quad \square$$

For the  $X$  of Theorem A, 1.1 implies  $X_l$  is a dense  $G_\delta$  subset of  $X$ . If  $d \leq 2$  the set  $\text{ext } X$  is closed and consequently  $X_l = X$ . We now describe a simple example showing that if  $d \geq 3$   $X_l$  may be a small subset of  $X$  in the sense of measure. Let  $C$  be the unit circle  $\{(\alpha, \beta, 0) : \alpha^2 + \beta^2 = 1\}$ , let  $p = (0, 0, 1)$ , and for each  $c \in C$  and  $\lambda \in [0, 1]$  let

$$A(c, \lambda) = [c, c + \lambda(p - c)] \cup [c, c + \lambda(-p - c)].$$

For each  $\eta \in [0, \frac{1}{2}]$  let  $C^\eta$  be a Cantor set in  $C$  such that  $\mu_1(C \sim C^\eta) < \eta$ , let  $K^\eta$  be the

convex hull of the set

$$\left(\bigcup_{c \in C^n} A(c, 1 - \eta)\right) \cup \left(\bigcup_{c \in C \sim C^n} A(c, 1 - 2\eta)\right),$$

and let  $X^n$  denote the boundary of  $K^n$ . Then

$$\bigcup_{c \in C^n} A(c, 1 - \eta) \subset X^n \sim X_1^n,$$

whence it follows that  $\lim_{\eta \rightarrow 0} \mu_2(X_1^n)/\mu_2(X^n) = 0$  even though  $X^n$  converges to the set  $\bigcup_{c \in C} A(c, 1)$  as  $\varepsilon \rightarrow 0$ .

Up to this point, our discussion has been aimed at the case in which  $X$  is the boundary of a finite-dimensional convex body. When  $X$  is the boundary of an infinite-dimensional convex body (or, in particular, is the unit sphere of an infinite-dimensional separable Banach space), we do not know whether  $X_1$  is necessarily nonempty or dense in  $X$ . Now suppose, on the other hand, that  $X$  is an entire convex set.  $X_1$  is dense in  $X$  when  $X$  is a body. But what happens when  $X$  is an infinite-dimensional compact convex set? In the metrizable case, 1.1 implies  $X_1$  is almost all of  $X$  in the sense of category. In the locally convex case,  $\text{cl con } X_1 = X$  by the Krein-Milman theorem (for plainly  $X_1 \supset \text{ext } X$ ), but we do not know whether  $X_1$  must be dense in  $X$ . If  $X$  is compact and convex but nonmetrizable and  $E$  is not locally convex, must  $X_1$  be nonempty?

### Continuity of Envelope Functions

*Throughout this section,  $X$  is the boundary of a compact convex set  $K$  in a locally convex space. The set  $K_e$  is as defined in the introduction, and  $X_e = X \cap K_e$ .*

**2.1 PROPOSITION.** *For any  $f \in C(K)$  and any  $p \in K$ ,  $f_e(p)$  is the infimum of all numbers of the form  $\sum_1^n \alpha_i f(k_i)$ , where  $p = \sum_1^n \alpha_i k_i$  is an expression of  $p$  as a convex combination of points  $k_i$  of  $K$ . For any closed face  $L$  of  $K$ , the envelope of  $f$ 's restriction to  $L$  is equal to the restriction to  $L$  of  $f$ 's envelope.*

*Proof.* The first assertion follows from Lemma 9.6 and Proposition 4.5 of Phelps [14]. The second assertion is an immediate consequence of the first one and the definition of a face.  $\square$

The following formalizes the idea behind Kruskal's example [11] of a 3-dimensional  $K$  for which  $K_e \neq K$ .

**2.2 THEOREM.** *If  $F$  is the face-function of  $K$  and  $F(k)$  is a face for all  $k \in K$ , then  $K_e \subset K_1$ ; that is,  $F$  is lower semicontinuous at each point of  $K_e$ .*

*Proof.* Consider an arbitrary point  $p$  of  $K \sim K_1$ , and let  $k_\delta$  be a net converging to  $p$  in  $K$  such that  $\underline{\lim} F(k_\delta) \not\supset F(p)$ . By passing to a subnet if necessary, we can find a

point  $q$  of  $F(p)$  and a neighborhood  $V$  of  $q$  that is disjoint from  $C$ , the closure of the union of the sets  $F(k_\delta)$ . There exist points  $v \in V \cap F(p)$  and  $w \in K$  and a number  $\lambda \in [0, 1]$  such that  $p = \lambda v + (1 - \lambda) w$ . Now let  $f(c) = 1$  for all  $c \in C \cup \{w\}$ , let  $f(v) = 0$ , and extend  $f$  to a member of  $C(K)$ . As  $f$  is identically 1 on each of the faces  $F(k_\delta)$ , it follows from the second part of 2.1 that  $f_e(k_\delta) = 1$  for all  $\delta$ . However, any affine function  $g$  majorized on  $K$  by  $f$  must have

$$g(p) = \lambda g(v) + (1 - \lambda) g(w) \leq 1 - \lambda,$$

so that  $f_e(p) < 1$ ,  $f_e$  is discontinuous at  $p$ , and  $p \notin K_e$ .  $\square$

**2.3 THEOREM.** *If  $K$  is metrizable then the set of points at which the restriction to  $X$  of every envelope function on  $K$  is continuous forms a dense  $G_\delta$  subset of  $X$ .*

*Proof.* Since  $K$  is metrizable there is a sequence  $f_\delta$  of functions uniformly dense in  $C(K)$ . It follows from [14, p. 19(c) and 4.5] that the sequence  $f_{\delta_e}$  of envelopes is uniformly dense in the collection of all envelope functions on  $K$ . Therefore the set in question is the intersection of the sets of points of continuity of the restrictions of the  $f_{\delta_e}$ , each of which is well known to be a dense  $G_\delta$  subset of  $X$ . The desired conclusion is then an immediate consequence of the Baire category theorem.  $\square$

**2.4 COROLLARY.** *If  $K$  is a metrizable compact convex subset of a locally convex space then  $K_e$  is a dense  $G_\delta$  subset of  $K$ .*

**2.5 COROLLARY.** *If  $X$  is the boundary of a  $d$ -dimensional compact convex body  $K$  then  $X_e \subset X_l$  and  $X_e$  is a dense  $G_\delta$  subset of  $X$ .*

*Proof.* The first statement of 2.5 follows from 2.2. In view of 2.3 it suffices for the second to show that a real-valued, lower semicontinuous, convex function  $f$  on  $K$  is continuous at every point of continuity of its restriction to  $X$ . This is left to the reader.  $\square$

We do not know, in general, whether  $X_e = X_l$ .

**2.6 PROPOSITION.**  $X_e = X$  if  $d \leq 2$ .

*Proof.* If  $F(x)$  is a segment it follows from the proof of 2.5 that  $f_e$  is continuous at  $x$ . If  $x$  is an extreme point then  $f_e(x) = f(x)$  as a result of 2.1. It follows easily that  $f_e$  is upper semicontinuous at  $x$ .  $\square$

## Upper Semicontinuity of the Face-Function

*The standing hypotheses for this section are the same as for the preceding section. In addition, for any set  $Y$  with face-function  $G$ , the inside  $I(Y)$  is defined as the set of all points  $y \in Y$  such that  $G(y) = Y$ . (This notion is due to Michael [12].)*

**3.1 THEOREM.** *If  $X$  is the boundary of a convex body and  $Y$  is a maximal convex subset of  $X$  then  $I(Y) \subset X_u$ .*

*Proof.* Let  $F$  and  $G$  be the face-functions of  $X$  and  $Y$  respectively, and consider an arbitrary point  $y \in I(Y)$ . Then  $G(y) = Y$  by definition, and from the maximality of  $Y$  it follows that  $F(y) = Y$ . We want to show that for an arbitrary net  $x_\delta$  converging to  $y$  in  $X$ , and an arbitrary point  $p \in \overline{\lim} F(x_\delta)$ , it is true that  $p \in F(y)$ . As  $[p, y] \subset X$ , there is a supporting hyperplane  $H$  of  $X$  such that  $[p, y] \subset H$ . Any segment in  $X$  that has  $y$  as an inner point must lie in  $H$ , whence  $Y \subset H$  and it follows from the maximality of  $Y$  that

$$Y = X \cap H \ni p. \quad \square$$

A subset  $U$  of a set  $X$  is said to be *ubiquitous* in  $X$  provided that every point of  $X$  belongs to  $U$  or is an endpoint of a segment in  $U$ .

**3.2 THEOREM.** *If  $E$  is complete, metrizable, and separable, and  $X$  is the boundary of a convex body in  $E$ , then  $X_u$  is the union of all sets of the form  $I(Y)$ , where  $Y$  is a maximal convex subset of  $X$ . The set  $X_u$  is ubiquitous in  $X$ .*

*Proof.* With  $U$  denoting the union in question, it follows from 3.1 that  $U \subset X_u$ . Any point  $x$  of  $X$  lies in a maximal convex subset  $Y$  of  $X$ , and as  $Y$  is closed a construction of Michael [12, 5.1] and Klee [7, 2.6] produces a point  $y \in I(Y)$ . It is easily verified that  $]x, y[ \subset I(Y)$  and thus  $U$  is ubiquitous. If  $x \notin U$ , then  $F(x)$  does not contain  $Y$  even though  $F(v) = Y$  for all  $v \in ]x, y]$ , and consequently  $x \notin X_u$ . Thus  $X_u = U$ .  $\square$

If  $X$  is as described in Theorem *A* of the introduction, it follows from 3.2 that  $X_u$  is dense in  $X$ . We conjecture, moreover, that  $\mu_{d-1}(X \sim X_u) = 0$ , but are able to prove this only for  $d \leq 3$ . The proof is based on the following result concerning upper semicontinuous collections of convex sets.

**3.3 THEOREM.** *Let  $\mathbf{C}$  be an upper semicontinuous collection of compact convex subsets of Euclidean  $n$ -space  $E^n$  such that  $\bigcup \mathbf{C}$  is a Borel set, and let  $W = \bigcup_{C \in \mathbf{C}} (C \sim I(C))$ . Then  $\mu_n(W) = 0$  when  $n \leq 2$ .*

The theorem is trivial when  $n = 1$ , for then  $W$  is countable. Suppose that  $n = 2$  and let  $\mathbf{J}$ ,  $\mathbf{K}$ , and  $\mathbf{L}$  denote respectively the collections of all 0-, 1-, and 2-dimensional members of  $\mathbf{C}$ . Then  $\bigcup \mathbf{J}$  is a  $G_\delta$  set relative to  $\bigcup \mathbf{C}$  and contributes nothing to  $W$ , while  $\mathbf{L}$  is an  $F_\sigma$  set and contributes nothing to  $\mu_2(W)$ . Thus we may (and will) assume without loss of generality  $\mathbf{C} = \mathbf{K}$ .

Choose a small positive  $\eta$ — $\eta < 1/100$  will surely suffice. For each pair  $p, q$  of distinct points of  $E^2$  whose coordinates are all rational, let  $\mathbf{K}(p, q)$  denote the collection of all members of  $\mathbf{K}$  that have one endpoint within  $\eta \|p - q\|$  of  $p$  and the other within  $\eta \|p - q\|$  of  $q$ ; the set of former endpoints is denoted by  $W(p, q)$ . Then  $\mathbf{K} = \bigcup_{p, q} \mathbf{K}(p, q)$  and  $W = \bigcup_{p, q} W(p, q)$ , and with the aid of upper semicontinuity

it follows that for each  $p, q, \bigcup \mathbf{K}(p, q)$  is a  $G_\delta$  set relative to  $\bigcup \mathbf{K}$ . To complete the proof it suffices to show  $\mu_2 W(p, q) = 0$ .

For each point  $x \in W(p, q)$ , and each  $\varepsilon$  with  $0 < \varepsilon < \eta$ , let  $S(x)$  be the member of  $\mathbf{K}(p, q)$  that has  $x$  as one of its endpoints and let  $x_\varepsilon$  be the point of  $S(x)$  whose distance from  $x$  is  $\varepsilon$ . Let  $W_\varepsilon(p, q) = \{x_\varepsilon : x \in W(p, q)\}$ . It follows with the aid of upper semi-continuity that each set  $W_\varepsilon(p, q)$  is an  $F_{\sigma\delta}$  set relative to  $\bigcup \mathbf{K}(p, q)$  and hence is  $\mu_2$ -measurable. To complete the proof it suffices to show that

$$(*) \quad \|x_\varepsilon - y_\varepsilon\| \geq \|x - y\|/2$$

for all  $x, y \in W(p, q)$  and  $0 < \varepsilon < \eta$ , for it then follows that  $\mu_2 W_\varepsilon(p, q) \geq \mu_2 W(p, q)/4$ , and, as the various sets  $W_\varepsilon(p, q)$  are pairwise disjoint, a contradiction would ensue if  $\mu_2 W(p, q) > 0$ .

Let  $L(x)$  denote the line containing  $S(x)$ ,  $L'(x)$  the part of  $L(x) \sim S(x)$  adjoining  $x$ , and  $L''(x)$  the rest of  $L(x)$ . If  $L(x)$  and  $L(y)$  are parallel the inequality (\*) is trivial, so we assume that  $L(x)$  and  $L(y)$  intersect and, for notational simplicity, that their point of intersection is the origin 0. There are four possibilities: (a)  $0 \in L'(x) \cap L'(y)$ ; (b)  $0 \in L''(x) \cap L''(y)$ ; (c)  $0 \in L'(x) \cap L''(y)$ ; (d)  $0 \in L''(x) \cap L'(y)$ . Denoting  $\|x\|$  and  $\|y\|$  by  $\alpha$  and  $\beta$  respectively, in case (a),  $x_\varepsilon = x + \varepsilon x \alpha^{-1}$  and  $y_\varepsilon = y + \varepsilon y \beta^{-1}$ , while in case (b)  $x_\varepsilon = x - \varepsilon x \alpha^{-1}$  and  $y_\varepsilon = y - \varepsilon y \beta^{-1}$ . The two cases will be treated together. We have

$$\begin{aligned} \|x_\varepsilon - y_\varepsilon\| &= \|x \pm \varepsilon x \alpha^{-1} - (y \pm \varepsilon y \beta^{-1})\| \\ &= \|x(1 \pm \varepsilon \alpha^{-1}) - y(1 \pm \varepsilon \alpha^{-1}) \pm \varepsilon y(\alpha^{-1} - \beta^{-1})\| \\ &= \|(1 \pm \varepsilon \alpha^{-1})(x - y) \pm \varepsilon y(\beta - \alpha) \alpha^{-1} \beta^{-1}\| \\ &\geq |1 \pm \varepsilon \alpha^{-1}| \cdot \|x - y\| - \varepsilon \alpha^{-1} \|x - y\|. \end{aligned}$$

It follows that  $\|x_\varepsilon - y_\varepsilon\| \geq \|x - y\|$  in case (a) and

$$\|x_\varepsilon - y_\varepsilon\| \geq (1 - 2\varepsilon \alpha^{-1}) \|x - y\| \geq \|x - y\|/2$$

in case (b) because  $\varepsilon \leq \alpha/4$  in this case. Finally, in case (c), let  $\theta$  denote the acute angle between  $L$  and  $M$  and suppose first that 0 comes between  $y$  and  $y_\varepsilon$ . Then by the law of cosines

$$\begin{aligned} \|x_\varepsilon - y_\varepsilon\|^2 &= (\varepsilon - \beta)^2 + (\varepsilon + \alpha)^2 - 2(\varepsilon - \beta)(\varepsilon + \alpha) \cos \theta \\ &= 2\varepsilon(\varepsilon - \beta + \alpha)(1 - \cos \theta) + \alpha^2 + \beta^2 + 2\alpha\beta \cos \theta \\ &> \alpha^2 + \beta^2 - 2\alpha\beta \cos(\pi - \theta) = \|x - y\|^2. \end{aligned}$$

If, instead,  $y_\varepsilon$  comes between  $y$  and 0, then

$$\|x_\varepsilon - y_\varepsilon\|^2 = (\varepsilon + \alpha)^2 + (\beta - \varepsilon)^2 + 2(\varepsilon + \alpha)(\beta - \varepsilon) \cos \theta$$

and

$$\|x - y\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta \cos \theta.$$

Hence  $\|x_\varepsilon - y_\varepsilon\| \geq 2^{-\frac{1}{2}}\|x - y\|$  if and only if  $\varepsilon^2 + \alpha^2 + (\beta - \varepsilon)^2 + 2\alpha(\beta - \varepsilon)\cos\theta + 4\varepsilon\alpha + 4\varepsilon(\beta - \varepsilon)\cos\theta \geq 2\varepsilon(\beta - \varepsilon) + 2\varepsilon\alpha\cos\theta$ . As  $\beta - \varepsilon \geq 0$ ,  $\varepsilon^2 + (\beta - \varepsilon)^2 \geq 2\varepsilon(\beta - \varepsilon)$ , and  $4\varepsilon\alpha \geq 2\varepsilon\alpha\cos\theta$ , the desired conclusion follows. The same argument handles case (d).  $\square$

The assumption of upper semicontinuity was used only to get measurability, but simple examples show this assumption cannot be completely discarded. In addition, some geometric condition related to the straightness of the segments is necessary. For consider a Jordan region  $J$  in the plane whose boundary curve has positive 2-dimensional measure, let  $\mathbf{C}$  be a continuous collection of parallel segments filling all but two antipodal points of a circular disk  $D$ , and let  $\mathbf{K}$  be the collection of arcs obtained from the members of  $\mathbf{C}$  under a homeomorphism of  $D$  onto  $J$ . Then  $\mathbf{K}$  is a continuous collection of pairwise disjoint arcs whose endpoints form a set of positive 2-dimensional measure.

For any collection  $\mathbf{S}$  of line segments in  $E^d$  let  $P(\mathbf{S})$  denote the set of all endpoints of the members of  $\mathbf{S}$ . A subset  $M$  of  $E^d$  is called an *endset* provided that  $M = P(\mathbf{S})$  for some collection  $\mathbf{S}$  of pairwise disjoint segments. It follows from 3.3 that when  $d \leq 2$  the  $d$ -dimensional Lebesgue measure of any compact (and hence any measurable) endset in  $E^d$  is zero. However, for  $d=4$  (and hence for all  $d \geq 4$ ) Bruckner and Ceder [2] have produced in  $E^d$  a compact endset of positive  $d$ -measure. Their construction is based on Nikodym's example [13] of a Cantor set  $X$  of positive measure in  $E^2$  such that for each point  $x$  of  $X$  there is a line in  $E^2$  intersecting  $X$  only at  $x$ . It is unknown whether a compact endset in  $E^3$  must be of measure zero [10].

**3.4 THEOREM.** *If  $X$  is the boundary of a  $d$ -dimensional convex body then  $\mu_{d-1}(X \sim X_u) = 0$  when  $d \leq 3$ .*

*Proof.* No restriction on  $d$  is required until the end of the argument, when 3.3 is used. Assuming without loss of generality that the convex body in question is compact and is situated in a  $d$ -dimensional *Euclidean* space  $E^d$ , we show first that the body may also be assumed smooth.

Suppose  $X'$  is the boundary of a compact convex body  $Y'$  in  $E^d$ , and let  $X'_u$  denote the set of all points at which the face-function of  $X'$  is upper semicontinuous. Let  $B$  denote the unit ball of  $E^d$  and let  $Y = Y' + B$ , a smooth convex body. Let  $X$  denote the boundary of  $Y$ , and for each point  $x \in X$  let  $x'$  denote the unique point of  $X'$  nearest to  $x$ . It is well-known that  $\|x'_1 - x'_2\| \leq \|x_1 - x_2\|$  for all  $x_1, x_2 \in X$ , and hence the mapping  $'$  carries sets of zero  $(d-1)$ -dimensional Hausdorff measure onto such sets. Thus the reduction to the case of smooth convex bodies is justified if we can show that the set  $X \sim X_u$  is carried onto the set  $X' \sim X'_u$  by the mapping  $'$ . For this it suffices, in view of 3.2, to show that for any maximal convex subset  $M'$  of  $X'$  there is a maximal convex  $M \subset X$  such that the restriction to  $M$  of the mapping  $'$  simply translates  $M$  onto  $M'$ . Indeed, choose  $x' \in I(M')$ , let  $J$  be a closed halfspace supporting  $X'$  at  $x'$ , and let  $b$  be a point at which  $B$  is supported by a translate of  $J$ . Then

the set  $M = M' + b$  is a maximal convex subset of  $X$ , with  $m' = m - b$  for all  $m \in M$ .

Now suppose  $X$  is the boundary of a smooth compact convex body  $Y$  in  $E^d$ , and let  $\mathbf{M}$  denote the collection of all maximal convex subsets of  $X$ . For each  $M \in \mathbf{M}$  there is a supporting hyperplane  $H$  of  $X$  such that  $M = X \cap H$ , whence it follows from smoothness that the members of  $\mathbf{M}$  are pairwise disjoint and by a routine argument that  $\mathbf{M}$  is upper semicontinuous. Let  $S$  denote the boundary of a  $d$ -dimensional simplex that contains  $Y$  and let  $\varrho$  be a radial mapping of  $S$  onto  $X$ . Let  $T_0, \dots, T_d$  be the  $(d-1)$ -dimensional faces of  $S$ , and for  $0 \leq i \leq d$  let

$$\mathbf{M}_i = \{(\varrho^{-1}M) \cap T_i : M \in \mathbf{M}\}, \quad W_i = \bigcup_{C \in \mathbf{M}_i} (C \sim I(C)).$$

As  $\mathbf{M}_i$  is an upper semicontinuous collection of compact convex sets, it follows from 3.3 with  $n = d-1$  that  $\mu_{d-1}(W_i) = 0$ . But  $X \sim X_u \subset \bigcup_0^d \varrho W_i$  and the mapping  $\varrho$  is known to be lipschitzian ([6]), so the desired conclusion follows when  $d \leq 3$ .  $\square$

**3.5 PROPOSITION.** *If  $X$  is the boundary of a  $d$ -dimensional convex body then  $X_u$  is an  $F_{\sigma\delta}$  set.*

*Proof.* Let  $A$  denote the set of all points  $x$  of  $X$  such that  $x$  lies in two or more maximal convex subsets of  $X$ . For each  $i$ , let  $A_i$  denote the set of all  $x \in X$  such that there are two hyperplanes  $H'$  and  $H''$  supporting  $X$  at  $x$  and forming an angle of at least  $1/i$  and there are points  $x' \in X \cap H'$  and  $x'' \in X \cap H''$  such that

$$\|x' - x\| \geq 1/i \leq \|x'' - x\|.$$

Then  $A = \bigcup_1^\infty A_i$  and a routine compactness argument shows each set  $A_i$  is closed. Thus  $A$  is an  $F_\sigma$  set. Let  $S = X \sim A$ , and for each  $p \in S$  let  $M_p$  denote the unique maximal convex subset of  $X$  containing  $p$ . Then  $S$  is a  $G_\delta$  set.

For  $1 \leq j \leq d-1$ , the  $j$ -interior  $\text{int}_j X$  is defined as the set of all points  $x$  of  $X$  for which there exists a  $j$ -dimensional flat  $J$  such that  $x$  is interior to  $X \cap J$  relative to  $J$ . Plainly  $\text{int}_{d-1} X$  is open relative to  $X$ , and a routine compactness argument shows each set  $\text{int}_j X$  is an  $F_\sigma$  set.

For each pair of positive integers  $i$  and  $j$  with  $j < d$  let

$$S_{ij} = \{p \in S : M_p \text{ contains a } j\text{-dimensional ball of radius } \geq 1/i\}.$$

Then  $S_{ij}$  is closed relative to  $S$  and it follows from 3.2 that

$$X_u = S \sim \bigcup_{j=1}^{d-1} \bigcup_{i=1}^\infty [(X \sim \text{int}_j X) \cap S_{ij}],$$

the difference of a  $G_\delta$  and a  $G_{\delta\sigma}$ .  $\square$

If  $X$  is as described in the conjecture stated in the introduction, it follows from 3.5 that  $X_u$  is an  $F_{\sigma\delta}$  set, and from 3.4 that  $X_u$  is almost all of  $X$  in the sense of measure (at least when  $d \leq 3$ ). Of course  $X \sim X_u$  is countable when  $d = 2$ . We now

describe some examples with  $d=3$  showing  $X_u$  may fail to be either an  $F_\sigma$  set or a  $G_\delta$  set, and it may be of the first category in  $X$ .

Suppose  $X$  is the boundary of a  $d$ -dimensional convex body and all the maximal convex subsets of  $X$  are 0- or 1-dimensional. By 3.2,  $X \sim X_u$  is the set  $Z$  of all endpoints of 1-dimensional maximal convex subsets of  $X$ . Now suppose, in addition, that both  $Z$  and  $(\text{ext } X) \sim Z$  are dense in  $X$ . As  $(\text{ext } X) \sim Z$  is a  $G_\delta$  set, it follows from the Baire category theorem that  $Z$  is not a  $G_\delta$  set and hence  $X_u$  is not an  $F_\sigma$  set. With  $d=3$ , an  $X$  in which the sets  $Z$  and  $(\text{ext } X) \sim Z$  are both dense can be constructed by a procedure used by Klee [8, pp. 99–103] for constructing an  $X$  in which the set  $\text{exp } X$  of all exposed points of  $x$  is not a  $G_\delta$  set. (A point  $x$  of  $X$  is *exposed* provided that there is a supporting hyperplane  $H$  of  $X$  such that  $X \cap H = \{x\}$ .)

**3.6 EXAMPLE.** *For  $d \geq 3$  there is a  $d$ -dimensional compact convex body with boundary  $X$  such that  $\mu_{d-1}(X_l) = 0$  and  $X_u$  is of the first category in  $X$ .*

*Proof.* Let  $Y'$  denote the union of all segments joining the point  $(0, 0, 1)$  to a point of the semicircle  $\{(\alpha, \beta, 0) : \alpha \geq 0, \alpha^2 + \beta^2 = 1\}$ , and let  $Z$  denote the union of the semidisk  $\{(\alpha, \beta, 0) : \alpha \geq 0, \alpha^2 + \beta^2 \leq 1\}$  with the triangle  $\text{con}\{(0, 1, 0), (0, -1, 0), (0, 0, 1)\}$ . Let  $K' = \text{con } Y' = \text{con } Z$ , a 3-dimensional convex body (half of a truncated circular cone) whose boundary is  $Y' \cup Z$ . By a slight perturbation of  $Y'$ , moving  $Y'$  to a nearby position  $Y$  while leaving invariant the points of  $Y' \cap Z$ , Corson [4] constructs a convex body  $K = \text{con } Y$  such that the boundary of  $K$  is  $Y \cup Z$  and the following conditions are satisfied:  $\text{ext } Y$  is dense in  $Y$ ,  $\text{exp } Y$  is an  $F_\sigma$  set, the set  $W = \text{ext } Y \sim \text{exp } Y$  is dense in  $\text{ext } Y$ , and every exposed point of  $Y \sim Z$  is an endpoint of a segment in  $Y \sim Z$ . It follows that  $Y_u \subset Y \sim W$ ,  $Y_l = \text{ext } Y$ , and  $Y_l \cap Y_u \subset Y \cap Z$ . As  $W$  is a dense  $G_\delta$  set in  $Y$ , it follows from the Baire category theorem that  $Y_u$  is not a  $G_\delta$  set in  $Y$  and is of the first category in  $Y$ . Further, it follows from 3.4 that  $\mu_2(Y_l) = 0$ .

Four copies of the above example  $Y$  (or, rather, of its intersection with a suitable halfspace) can be fitted together to form a slight perturbation of a double cone resulting in a 3-dimensional compact convex body with boundary  $X$  such that  $X_u$  is not a  $G_\delta$  set in  $X$  and is of the first category in  $X$ , and  $\mu_2(X_l) = 0$ . Successive double cones over this set  $X$  result in similar examples in higher dimensions.  $\square$

Our discussion has thus far been aimed at the case in which  $X$  is the boundary of a finite-dimensional convex body. When  $X$  is the boundary of a convex body in a metrizable linear space  $E$  that is complete and separable, it follows from 3.2 that  $X_u$  is dense in  $X$ . To see that the completeness assumption cannot be abandoned, let  $X$  be the unit sphere of the subspace of  $l(\aleph_0)$  consisting of all points having only finitely many nonzero coordinates. To see that the separability assumption cannot be abandoned, let  $X$  be the unit sphere of the space  $l(\aleph_1)$ . In each case the set  $X_u$  is empty. Now suppose, on the other hand, that  $X$  is an entire convex set. Then  $X_u \supset I(X)$  and hence  $X_u$  is dense in  $X$  if  $I(X) \neq \emptyset$  (in particular, if  $X$  is centrally symmetric or

is complete, metrizable, and separable). If  $X$  is a body then  $X_u = \text{int } X$ . If  $X$  is the unit ball of  $l^2(\aleph_0)$  in the weak topology, then  $X$  is compact and metrizable and  $X_u$  is an  $F_\sigma$  set of the first category in  $X$ . If  $X$  is the intersection of the unit ball of  $l^2(\aleph_0)$  with the positive cone of  $l^2(\aleph_0)$ , in the weak topology, then  $X$  is compact and  $X_u$  is empty.

### Continuity of the Face-Function

With  $X$  denoting a closed subset of a topological linear space  $E$ , let  $X_c = X_l \cap X_u$ , the set of all points at which the face-function of  $X$  is continuous. Then  $X_c = \text{int } X$  when  $X$  is a convex body. Suppose, on the other hand, that  $E$  is an infinite-dimensional separable Banach space, and let  $Z_E$  denote the space of all compact convex subsets of  $E$ , metrized by the Hausdorff metric. Then  $Z_E$  is a complete metric space and those  $X \in Z_E$  for which  $X_c$  is nonempty form a first category subset of  $Z_E$ . To see this, note that  $X_c = \emptyset$  whenever  $\text{ext } X$  is dense in  $X$ , and then apply two results of Klee [9, 2.1 and 2.2].

Note that  $\mu_2(X_c) = 0$  in the example of 3.6; indeed,  $X_c$  is the union of a finite number of rectifiable arcs. To obtain an unbounded 3-dimensional convex body with boundary  $X$  such that  $X_c$  is empty, fit together two copies of the  $Y$  of 3.6 so as to obtain a perturbation of a circular cone that has the same base as the cone; then send the base to infinity by a projective transformation.

We do not know whether  $X_c$  may be empty when  $X$  is the boundary of a bounded convex body in a complete metrizable separable space  $E$  of dimension  $d \geq 3$ . However, it seems probable that  $X_c$  may be empty when  $d > 3$  but not when  $d = 3$ .

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Received February 4, 1970