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Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces

(Part III: Method based on Voss' Proof)

by HEINZ HOPF † (Zürich) and YOSHIE KATSURADA (Sapporo)

Introduction

An idea that gives congruence of two hypersurfaces concerning a transformation group by a relation between the invariant of the corresponding points of these hyper surfaces was first introduced by H. Hopf and K. Voss [1], that is, in that paper congruence relations of two closed curves on a plane and of two closed surfaces in 3-dimensional euclidean space have been given by the relation of the mean curvatures.

K. Voss has generalized these theorems to hypersurfaces in an $(m+1)$ -dimensional euclidean space ($m+1 \geq 3$) and also given the congruence relations in case of Gauss curvatures or the r -th mean curvatures H_r , $r=1, 2, \dots, m$ [2]. A. Aeppli has developed analogous statements for a central transformation group (a homothetic transformation group with the center 0) [3].

The present authors wished to generalize these theorems to Riemann spaces. In the previous papers [4], [5], we gave the generalized theorems relating to the first mean curvature.

The purpose of the present paper is to investigate a general theorem relating to the Gauss curvature or the r -th mean curvature, that is, to generalize to an orientable Riemann space R^{m+1} with constant Riemann curvature the following theorems given by K. Voss:

THEOREM (K. Voss). *Let W^m and \bar{W}^m be two orientable closed hypersurfaces in an $(m+1)$ -dimensional euclidean space and let p and \bar{p} be the corresponding points of these hypersurfaces, and let $K(p)$ and $\bar{K}(\bar{p})$ be that Gauss curvatures at these points respectively. Assume that the second fundamental forms of W^m and \bar{W}^m are positive definite. If all straight lines $(p\bar{p})$ are parallel to one another and if $K(p)=\bar{K}(\bar{p})$ for all $p \in W^m$, then the hypersurface \bar{W}^m is produced from W^m by simple translation in the direction of $(p\bar{p})$. (W^m and \bar{W}^m are therefore congruent mod the translation group).*

THEOREM (K. Voss). *Let W^m and \bar{W}^m be two orientable closed hypersurfaces in an $(m+1)$ -dimensional euclidean space and let p and \bar{p} be the corresponding points of these hypersurfaces, and let $H_r(p)$ and $\bar{H}_r(\bar{p})$ be the r -th mean curvatures at these*

points respectively, for some $r=1, 2, \dots, m$. Assume that the second fundamental forms of W^m and \bar{W}^m are positive definite. If all straight lines $(p\bar{p})$ are parallel to one another and if $H_r(p)=\bar{H}_r(\bar{p})$ for all $p \in W^m$, then the hypersurface \bar{W}^m is produced from W^m by simple translation in the direction of $(p\bar{p})$. (W^m and \bar{W}^m are congruent mod the translation group.)

§ 1. Generalized Theorems

We suppose an $(m+1)$ -dimensional orientable Riemann space with constant curvature S^{m+1} of class C^v ($v \geq 3$) which admits an infinitesimal isometric transformation

$$\hat{x}^i = x^i + \xi^i(x) \delta\tau \quad (1.1)$$

(where x^i are local coordinate in S^{m+1} and ξ^i are the components of a contravariant vector ξ). We assume that orbits of the transformations generated by ξ cover S^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of the transformations are new x^1 -coordinate curves, that is, a coordinate system in which the vector ξ^i has components $\xi^i = \delta_1^i$, where the symbol δ_j^i denotes Kronecker's delta; then (1.1) becomes as follows

$$\hat{x}^i = x^i + \delta_1^i \delta\tau \quad (1.2)$$

and S^{m+1} admits a one-parameter continuous group G of transformations which are 1-1-mappings of S^{m+1} onto itself and are given by the expression $\hat{x}^i = x^i + \delta_1^i \tau$ in the new special coordinate system ([6]).

Now we consider two orientable closed hypersurfaces W^m and \bar{W}^m of class C^v imbedded in S^{m+1} which are given as follows

$$\left. \begin{array}{l} W^m: x^i = x^i(u^\alpha) \quad i = 1, \dots, m+1 \quad \alpha = 1, \dots, m \\ \bar{W}^m: \bar{x}^i = \bar{x}^i(u^\alpha) + \delta_1^i \tau(u^\alpha) \end{array} \right\} \quad (1.3)$$

where u^α are local coordinates of W^m and τ is a continuous function attached to each point of the hypersurface W^m . We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to m , and to two hypersurfaces W^m and \bar{W}^m which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by $f(x^2, \dots, x^{m+1}) = 0$.

Then we can take the family of the hypersurfaces

$$W^m(t) = (1-t)W^m + t\bar{W}^m \quad 0 \leq t \leq 1,$$

generated by W^m and \bar{W}^m whose points correspond along the orbits of the transformations where W^m and \bar{W}^m mean $W^m(0)$ and $W^m(1)$ respectively. Thus according to (1.3), $W^m(t)$ is given by the expression

$$W^m(t): x^i(u^\alpha, t) = (1-t)x^i(u^\alpha) + t\bar{x}^i(u^\alpha) \quad 0 \leq t \leq 1, \quad (1.4)$$

and (1.4) may be rewritten as follows

$$W^m(t): x^i(u^\alpha, t) = x^i(u^\alpha) + \delta_1^i t \tau(u^\alpha) \quad 0 \leq t \leq 1. \quad (1.5)$$

Let us denote the normal unit vector of $W^m(t)$ by $n^i(t)$ and its derivative with respect to t by $n'^i(t)$. Then g_{ij} being the metric tensor of S^{m+1} and differentiating the following relations with respect to t ,

$$g_{ij} n^i(t) \frac{\partial x^j(u, t)}{\partial u^\alpha} = 0, \quad g_{ij} n^i(t) n^j(t) = 1,$$

since the transformation group G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have

$$g_{ij} n'^i(t) \frac{\partial x^j(u, t)}{\partial u^\alpha} + g_{ij} n^i(t) \frac{d}{dt} \left(\frac{\partial x^j(u, t)}{\partial u^\alpha} \right) = 0, \quad (1.6)$$

$$g_{ij} n^i(t) n'^j(t) = 0. \quad (1.7)$$

From (1.6), (1.7) and

$$\frac{d}{dt} \left(\frac{\partial x^i(u, t)}{\partial u^\alpha} \right) = \delta_1^i \frac{\partial \tau}{\partial u^\alpha},$$

we get

$$n'^i(t) = -g^{\alpha\beta}(t) \tau_\alpha \delta_1^i n_l(t) \frac{\partial x^l(u, t)}{\partial u^\beta}, \quad (1.8)$$

where $g^{\alpha\beta}(t)$ is the contravariant metric tensor of $W^m(t)$ and τ_α means $\partial \tau / \partial u^\alpha$. Throughout this paper repeated lower case Latin indices call for summation 1 to $m+1$ and repeated lower case Greek indices for summation 1 to m . And also for its covariant differential along $W^m(t)$ we have

$$\delta n'^i(t) = d n'^i(t) + \Gamma_{jl}^i n'^j(t) x_\gamma^l du^\gamma, \quad (1.9)$$

where Γ_{jl}^i is the Christoffel symbol with respect to the metric tensor g_{ij} of S^{m+1} and x_γ^l means $\partial x^l(u, t) / \partial u^\gamma$.

Let us give henceforth the derivative with respect to t by the dash. Calculating $(\delta n^i)'$, we have

$$\begin{aligned} \delta n^i &= d n^i + \Gamma_{jl}^i n^j(t) x_\gamma^l d u^\gamma, \\ (\delta n^i)' &= (d n^i)' + (\Gamma_{jl}^i)' n^j(t) x_\gamma^l du^\gamma \\ &\quad + \Gamma_{jl}^i n'^j(t) x_\gamma^l du^\gamma + \Gamma_{jl}^i n^j(t) (x_\gamma^l)' du^\gamma, \end{aligned}$$

since G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have $\partial \Gamma_{ij}^l / \partial x^1 = 0$. Consequently we obtain the following relation between $\delta n'^i$ and $(\delta n^i)'$

$$(\delta n^i)' = \delta n'^i + \Gamma_{j1}^i n^j(t) \tau_\gamma du^\gamma. \quad (1.10)$$

We claim that the following theorems hold

THEOREM 1.1. *Let K and \bar{K} be the Gauss curvature of W^m and \bar{W}^m respectively. Assume that the second fundamental form of $W^m(t)$, $0 \leq t \leq 1$ is positive definite. If the relation $K = \bar{K}$ holds for each point $p \in W^m$, then W^m and \bar{W}^m are congruent mod G .*

Proof. We consider the following differential form of degree $m-1$ attached to each point p on the hypersurface $W^m(t)$

$$\left. \begin{aligned} ((n', \delta_1 \tau, \delta n, \dots, \delta n)) &\stackrel{\text{def.}}{=} \sqrt{g}(n', \delta_1 \tau, \delta n, \dots, \delta n) \\ &= (-1)^{m-1} \sqrt{g}(n', \delta_1 \tau, x_{\alpha_1}, \dots, x_{\alpha_{m-1}}) \\ &\quad \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}} \end{aligned} \right\} \quad (1.11)$$

where g is the determinant of the metric tensor g_{ij} of S^{m+1} , the symbol $(\)$ means a determinant of order $m+1$ whose columns are the components of respective vectors, $b_{\alpha\beta}(t)$ is the second fundamental tensor of $W^m(t)$ and $b_{\alpha}^{\beta}(t)$ denotes $b_{\alpha\gamma}(t) g^{\beta\gamma}(t)$.

Then the exterior differential of the differential form (1.11) becomes as follows

$$\left. \begin{aligned} d((n', \delta_1 \tau, \delta n, \dots, \delta n)) &= ((\delta n', \delta_1 \tau, \delta n, \dots, \delta n)) \\ &+ ((n', \delta(\delta_1) \tau, \delta n, \dots, \delta n)) + ((n', \delta_1 d\tau, \delta n, \dots, \delta n)), \end{aligned} \right\} \quad (1.12)$$

because since S^{m+1} is a space of constant curvature, we have

$$((n', \delta_1 \tau, \delta \delta n, \delta n, \dots, \delta n)) = 0.$$

Because G is isometric, the quantity $n_i(t) \delta_1^i \sqrt{g^*(t)}$ is independent of t , where $g^*(t)$ means the determinant of $g_{\alpha\beta}(t)$, we have

$$((\delta n)', \delta_1 \tau, \delta n, \dots, \delta n) = (-1)^m (m-1)! K' n_i(t) \delta_1^i \tau dA(t) \quad (1.13)$$

where $dA(t)$ is the area element of $W^m(t)$, and using (1.8), we obtain

$$\left. \begin{aligned} ((n', \delta_1 d\tau, \delta n, \dots, \delta n)) &= (-1)^m (m-1)! \\ &\times \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t) \tau_{\alpha} \tau_{\beta} (n_i(t) \delta_1^i)^2 \sqrt{g^*(t)} dA(t) \end{aligned} \right\} \quad (1.14)$$

where $B^{\alpha\beta}(t)$ means the cofactor of an element $b_{\beta\alpha}(t)$ in the determinant $|b_{\alpha\beta}(t)|$ divided by $g^*(t)$.

By making use of (1.10), (1.12), (1.13), (1.14) and the relation

$$\delta(\delta_1^i) = \Gamma_{j1}^i x_{\gamma}^j du^{\gamma},$$

we have

$$\begin{aligned} d((n', \delta_1 \tau, \delta n, \dots, \delta n)) &= (-1)^m (m-1)! \left\{ K' n_i(t) \delta_1^i \tau dA(t) \right. \\ &+ \left. \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t) \tau_{\alpha} \tau_{\beta} (n_i(t) \delta_1^i)^2 \sqrt{g^*(t)} dA(t) \right\} \\ &+ ((n', \tau \Gamma_{j1}^i x_{\gamma}^j du^{\gamma}, \delta n, \dots, \delta n)) \\ &- ((\Gamma_{j1}^i n^j(t) \tau_{\gamma} du^{\gamma}, \delta_1 \tau, \delta n, \dots, \delta n)). \end{aligned}$$

Next we shall prove that

$$((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) = 0. \quad (1.15)$$

For the first term of the left-hand member of (1.15), making use of (1.8), we can see the following

$$\left. \begin{aligned} ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) &= (-1)^{m-1} \tau n_l(t) \delta_1^l \\ &\times ((\Gamma_{j1} x_\gamma^j, g^{\alpha\beta}(t) \tau_\beta x_\alpha, x_{\alpha_1}, \dots, x_{\alpha_{m-1}})) \\ &\times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}}. \end{aligned} \right\} \quad (1.16)$$

Let $\varepsilon_{i_1 \dots i_{m+1}}$ and $\varepsilon_{\alpha_1 \dots \alpha_m}$ be the ε -symbol of S^{m+1} and of $W^m(t)$ respectively,

$$\varepsilon_{i_1 \dots i_{m+1}} \stackrel{\text{def.}}{=} \sqrt{g} e_{i_1 \dots i_{m+1}}, \quad \varepsilon_{\alpha_1 \dots \alpha_m} \stackrel{\text{def.}}{=} \sqrt{g^*(t)} e_{\alpha_1 \dots \alpha_m},$$

the symbol $e_{i_1 \dots i_{m+1}}$ meaning plus one or minus one, depending on whether the indices i_1, \dots, i_{m+1} denote an even permutation of $1, 2, \dots, m+1$ or odd permutation, and zero when at least any two indices have the same value, and also the symbol $e_{\alpha_1 \dots \alpha_m}$ meaning similarly for the indices $\alpha_1, \dots, \alpha_m$ running from 1 to m .

Making use of the relation

$$n_i(t) \varepsilon_{\alpha \alpha_1 \dots \alpha_{m-1}} = \varepsilon_{i i_2 \dots i_{m+1}} x_\alpha^{i_2} x_{\alpha_1}^{i_3} \dots x_{\alpha_{m-1}}^{i_{m+1}}$$

we have

$$\begin{aligned} ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) &= (-1)^{m-1} \tau n_l(t) \delta_1^l \Gamma_{j1}^i n_i(t) x_\gamma^j \tau_\beta g^{\beta d}(t) \\ &\times \varepsilon_{\alpha \alpha_1 \dots \alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}} \\ &= (-1)^{m-1} \tau n_l(t) \delta_1^l \Gamma_{j1}^i n_i(t) x_\gamma^j \tau_\beta \varepsilon_{\alpha_1 \dots \alpha_{m-1}}^{\gamma \beta_1 \dots \beta_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) dA(t) \end{aligned}$$

and we can see easily the following relation

$$\begin{aligned} \varepsilon_{\alpha_1 \dots \alpha_{m-1}}^{\beta} \varepsilon^{\gamma \beta_1 \dots \beta_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) &= \varepsilon^{\beta_1 \dots \beta_{m-1}} \varepsilon^{\gamma \beta_1 \dots \beta_{m-1}} b_{\gamma_1 \beta_1}(t) \dots b_{\gamma_{m-1} \beta_{m-1}}(t) \\ &= (m-1)! B^{\beta \gamma}(t). \end{aligned}$$

Since $B^{\beta \gamma}(t)$ is the symmetric tensor, we have

$$\left. \begin{aligned} ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) &= (-1)^{m-1} (m-1)! \tau n_l(t) \\ &\times \delta_1^l \Gamma_{j1}^i n_i(t) x_\gamma^j \tau_\beta B^{\beta \gamma}(t) dA(t) \end{aligned} \right\} \quad (1.17)$$

where Γ_{j1} means $g_{il} \Gamma_{j1}^l$ and the symbol $(\gamma \beta)$ denotes the symmetric part for the indices γ and β .

On the other hand, we calculate the second term of the left-hand member of (1.15). Since G is isometric, that is, $\partial g_{ij} / \partial x^1 = 0$, we have

$$\left. \begin{aligned} \Gamma_{j1}^l n^j(t) n_l(t) &= \frac{1}{2} g^{lk} \left(\frac{\partial g_{kj}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^j} - \frac{\partial g_{j1}}{\partial x^k} \right) n^j(t) n_l(t) \\ &= \frac{1}{2} \frac{\partial g_{kj}}{\partial x^1} n^j(t) n^k(t) = 0, \end{aligned} \right\} \quad (1.18)$$

and we can give the vector δ_1^i by the expression

$$\delta_1^i = n_l(t) \delta_1^l n^i(t) + \varphi^\beta x_\beta^i. \quad (1.19)$$

Substituting (1.19) in the second term of the left-hand member of (1.15) and making use of (1.18), we have

$$\left. \begin{aligned} & -((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) \\ & = -(-1)^{m-1} \tau n_l(t) \delta_1^l ((\Gamma_{j1} n^j(t) \tau_\gamma, n, x_{\alpha_1}, \dots, x_{\alpha_{m-1}})) \\ & \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}}. \end{aligned} \right\} \quad (1.20)$$

Let us take the relation

$$\varepsilon_{\alpha\alpha_1\dots\alpha_{m-1}} g^{\alpha\beta}(t) x_\beta^j g_{ij} = (-1)^m \varepsilon_{i_1\dots i_{m+1}} x_{\alpha_1}^{i_2} \dots x_{\alpha_{m-1}}^{i_m} n^{i_{m+1}}.$$

Then we have

$$\left. \begin{aligned} & -((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) \\ & = (-1)^{m-1} (m-1)! \tau n_l(t) \delta_1^l \Gamma_{ij1} n^i(t) x_{(\beta}^j \tau_{\beta)} B^{\beta\gamma}(t) dA(t). \end{aligned} \right\} \quad (1.21)$$

Thus from (1.17), (1.21) and $\Gamma_{ij1} + \Gamma_{ji1} = \partial g_{ij} / \partial x^1 = 0$, we can arrive at (1.15) as follows

$$\begin{aligned} & ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) \\ & = (-1)^{m-1} (m-1)! \tau n_l(t) \delta_1^l (\Gamma_{ij1} + \Gamma_{ji1}) n^i(t) x_{(\gamma}^j \tau_{\beta)} B^{\beta\gamma}(t) dA(t) = 0. \end{aligned}$$

Finally we have

$$\left. \begin{aligned} & \frac{(-1)^m}{(m-1)!} d((n', \delta_1 \tau, \delta n, \dots, \delta n)) = K' n_i(t) \delta_1^i \tau dA(t) \\ & + \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t) \tau_\alpha \tau_\beta (n_i(t) \delta_1^i)^2 \sqrt{g^*(t)} dA(t). \end{aligned} \right\} \quad (1.22)$$

Integrating both members of (1.22) over the interval $0 \leq t \leq 1$, we get

$$\left. \begin{aligned} & \frac{(-1)^m}{(m-1)!} d \int_0^1 ((n', \delta_1 \tau, \delta n, \dots, \delta n)) dt = (K - K) n_i(0) \delta_1^i \tau dA(0) \\ & + \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0). \end{aligned} \right\} \quad (1.23)$$

Furthermore integrating both members of (1.23) over W^m and applying Stokes' theorem, since W^m is closed, we have

$$\begin{aligned} & \iint_{W^m} (K - K) n_i(0) \delta_1^i \tau dA(0) \\ & + \iint_{W^m} \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0) = 0, \end{aligned}$$

making use of the hypothesis $K = K$, we obtain

$$\iint_{W^m} \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0) = 0.$$

On the other hand, from that the second fundamental form of $W^m(t)$ is positive definite everywhere in $W^m(t)$, $0 \leq t \leq 1$, the quantity

$$\sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt v_\alpha v_\beta$$

becomes positive definite. From that two hypersurfaces W^m and \bar{W}^m do not contain a piece of a hypersurface covered by the orbits of transformations, a point on W^m such that $n_i(0) \delta_1^i = 0$ must be an isolate point. Moreover since τ is a continuous function of W^m , we have

$$\tau = \text{constant}$$

for all points of W^m . Consequently we can arrive at the following result

$$W^m \equiv \bar{W}^m \pmod{G}.$$

THEOREM 1.2. *Let H_r and \bar{H}_r be the r -th mean curvature of W^m and \bar{W}^m respectively. Assume that the second fundamental form of $W^m(t)$, $0 \leq t \leq 1$, is positive definite. If the relation*

$$H_r = \bar{H}_r$$

holds for each point $p \in W^m$, then W^m and \bar{W}^m are congruent mod G .

Proof. We consider the following differential form of degree $m-1$ attached to each point p on the hypersurface $W^m(t)$

$$\left. \begin{aligned} ((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) &\stackrel{\text{def.}}{=} \sqrt{g}(n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx) \\ &= (-1)^{r-1} \sqrt{g}(n', \delta_1 \tau, x_{\alpha_1}, \dots, x_{\alpha_{r-1}}, x_{\beta_r} \dots x_{\beta_{m-1}}) \\ &\quad \times b_{\beta_r}^{\alpha_r}(t) \dots b_{\beta_{m-1}}^{\alpha_{r-1}}(t) du^{\beta_1} \wedge \dots \wedge du^{\beta_{r-1}} \wedge du^{\beta_r} \wedge \dots \wedge du^{\beta_{m-1}} \end{aligned} \right\} \quad (1.24)$$

The exterior differential of the differential form (1.24) becomes as follows

$$\begin{aligned} d((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) &= ((\delta n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &\quad + ((n', \delta(\delta_1) \tau, \delta n, \dots, \delta n, dx, \dots, dx)) + ((n', \delta_1 d\tau, \delta n, \dots, \delta n, dx, \dots, dx)) \end{aligned}$$

because since S^{m+1} is a space of constant curvature, it follows that

$$((n', \delta_1 \tau, \delta \delta n, \dots, \delta n, dx, \dots, dx)) = 0,$$

and also we have

$$((n', \delta_1 \tau, \delta n, \dots, \delta n, \delta dx, \dots, dx)) = 0.$$

Making use of (1.8), we have

$$\left. \begin{aligned} & ((n', \delta_1 d\tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ & = (-1)^{r-1} g^{\alpha\beta}(t) \tau_\alpha n_i(t) \delta_1^i ((\delta_1 \tau_\gamma, x_\beta, x_{\alpha_1}, \dots, x_{\alpha_{r-1}}, x_{\alpha_r}, \dots, x_{\alpha_{m-1}})) \\ & \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{r-1}}^{\alpha_{r-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{r-1}} \wedge du^{\alpha_r} \\ & \wedge \dots \wedge du^{\alpha_{m-1}} \\ & = (-1)^{r-1} g^{\alpha\beta}(t) \varepsilon_{\beta\alpha_1\dots\alpha_{r-1}\alpha_r\dots\alpha_{m-1}} \varepsilon^{\gamma\beta_1\dots\beta_{r-1}\alpha_r\dots\alpha_{m-1}} \\ & \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{r-1}}^{\alpha_{r-1}}(t) (n_i(t) \delta_1^i)^2 \tau_\alpha \tau_\gamma dA(t). \end{aligned} \right\} \quad (1.25)$$

On the other hand, from (1.10) we get

$$\left. \begin{aligned} & ((\delta n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ & = ((\delta n)', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx) \\ & - ((\Gamma_{j1} n^j(t) \tau_\gamma) du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx). \end{aligned} \right\} \quad (1.26)$$

And after some calculations, we have

$$(-1)^r m! H_r' n_i(t) \delta_1^i dA(t) = r((\delta_1, (\delta n)', \delta n, \dots, \delta n, dx, \dots, dx)), \quad (1.27)$$

because $n_i(t) \delta_1^i dA(t)$ is independent of t and $dx'^i = \delta_1^i d\tau$, that is, the same direction to δ_1 . Moreover we can prove similarly the following relation as the proof of (1.15)

$$\left. \begin{aligned} & ((n', \delta(\delta_1) \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ & - ((\Gamma_{j1} n^j(t) \tau_\gamma) du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) = 0. \end{aligned} \right\} \quad (1.28)$$

Then putting

$$(m-1)! c_{(r)}^{\alpha\beta} = \varepsilon_{\alpha_1\dots\alpha_{r-1}\alpha_r\dots\alpha_{m-1}} \varepsilon^{\beta\beta_1\dots\beta_{r-1}\alpha_r\dots\alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{r-1}}^{\alpha_{r-1}}(t)$$

and using (1.25), (1.26), (1.27) and (1.28), we have

$$\left. \begin{aligned} & d((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ & = \frac{(-1)^{r-1}}{r} m! H_r' n_i(t) \delta_1^i \tau dA(t) \\ & + (-1)^{r-1} (m-1)! c_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta (n_i(t) \delta_1^i)^2 dA(t). \end{aligned} \right\} \quad (1.29)$$

Integrating both members of (1.29) over the interval $0 \leq t \leq 1$, and putting

$$C_{(r)}^{\alpha\beta} = g^*(0)^{1/2} \int_0^1 g^*(t)^{-1/2} c_{(r)}^{\alpha\beta} dt,$$

we have

$$\left. \begin{aligned} & m(\bar{H}_r - H_r) n_i(0) \delta_1^i \tau dA(0) + r C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0) \\ & = \frac{r(-1)^{r-1}}{(m-1)!} d \int_0^1 ((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) dt. \end{aligned} \right\} \quad (1.30)$$

Furthermore integrating both members of (1.30) over W^m and applying Stokes' theorem

$$\begin{aligned} & \frac{m}{r} \iint_{W^m} (\bar{H}_r - H_r) n_i(0) \delta_1^i \tau \, dA(0) + \iint_{W^m} (n_i(0) \delta_1^i)^2 C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta \, dA(0) \\ &= \frac{(-1)^{r-1}}{(m-1)!} \int_{\partial W^m} \int_0^1 ((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \, dt. \end{aligned}$$

Since W^m is closed, we have

$$\frac{m}{r} \iint_{W^m} (\bar{H}_r - H_r) n_i(0) \delta_1^i \tau \, dA(0) + \iint_{W^m} (n_i(0) \delta_1^i)^2 C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta \, dA(0) = 0,$$

using the hypotheses $H_r = \bar{H}_r$, and that the second fundamental form of $W^m(t)$, $0 \leq t \leq 1$, is positive definite, and from that two hypersurfaces W^m and \bar{W}^m do not contain a piece of a hypersurface covered by the orbits of transformations, we can arrive at

$$\tau_\alpha = 0$$

for all points of W^m , consequently we have

$$\tau = \text{constant}$$

for all points of W^m . Accordingly we can see the following result

$$W^m = \bar{W}^m \bmod G.$$

This proof follows to the method of that due to K. Voss [2].

Remark. In an euclidean space, if G is translation group, that is, a special isometric transformation group, Theorem 1.1 and Theorem 1.2 just coincide with theorems of K. Voss given in the introduction.

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