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## Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces

(Part III: Method based on Voss' Proof)

by HEINZ HOPF † (Zürich) and YOSHIE KATSURADA (Sapporo)

### Introduction

An idea that gives congruence of two hypersurfaces concerning a transformation group by a relation between the invariant of the corresponding points of these hypersurfaces was first introduced by H. Hopf and K. Voss [1], that is, in that paper congruence relations of two closed curves on a plane and of two closed surfaces in 3-dimensional euclidean space have been given by the relation of the mean curvatures.

K. Voss has generalized these theorems to hypersurfaces in an  $(m+1)$ -dimensional euclidean space ( $m+1 \geq 3$ ) and also given the congruence relations in case of Gauss curvatures or the  $r$ -th mean curvatures  $H_r$ ,  $r=1, 2, \dots, m$  [2]. A. Aepli has developed analogous statements for a central transformation group (a homothetic transformation group with the center 0) [3].

The present authors wished to generalize these theorems to Riemann spaces. In the previous papers [4], [5], we gave the generalized theorems relating to the first mean curvature.

The purpose of the present paper is to investigate a general theorem relating to the Gauss curvature or the  $r$ -th mean curvature, that is, to generalize to an orientable Riemann space  $R^{m+1}$  with constant Riemann curvature the following theorems given by K. Voss:

**THEOREM (K. Voss).** *Let  $W^m$  and  $\bar{W}^m$  be two orientable closed hypersurfaces in an  $(m+1)$ -dimensional euclidean space and let  $p$  and  $\bar{p}$  be the corresponding points of these hypersurfaces, and let  $K(p)$  and  $\bar{K}(\bar{p})$  be that Gauss curvatures at these points respectively. Assume that the second fundamental forms of  $W^m$  and  $\bar{W}^m$  are positive definite. If all straight lines  $(p\bar{p})$  are parallel to one another and if  $K(p) = \bar{K}(\bar{p})$  for all  $p \in W^m$ , then the hypersurface  $\bar{W}^m$  is produced from  $W^m$  by simple translation in the direction of  $(p\bar{p})$ . ( $W^m$  and  $\bar{W}^m$  are therefore congruent mod the translation group).*

**THEOREM (K. Voss).** *Let  $W^m$  and  $\bar{W}^m$  be two orientable closed hypersurfaces in an  $(m+1)$ -dimensional euclidean space and let  $p$  and  $\bar{p}$  be the corresponding points of these hypersurfaces, and let  $H_r(p)$  and  $\bar{H}_r(\bar{p})$  be the  $r$ -th mean curvatures at these*

points respectively, for some  $r=1, 2, \dots, m$ . Assume that the second fundamental forms of  $W^m$  and  $\bar{W}^m$  are positive definite. If all straight lines  $(p\bar{p})$  are parallel to one another and if  $H_r(p) = \bar{H}_r(\bar{p})$  for all  $p \in W^m$ , then the hypersurface  $\bar{W}^m$  is produced from  $W^m$  by simple translation in the direction of  $(p\bar{p})$ . ( $W^m$  and  $\bar{W}^m$  are congruent mod the translation group.)

## § 1. Generalized Theorems

We suppose an  $(m+1)$ -dimensional orientable Riemann space with constant curvature  $S^{m+1}$  of class  $C^v$  ( $v \geq 3$ ) which admits an infinitesimal isometric transformation

$$\hat{x}^i = x^i + \xi^i(x) \delta\tau \quad (1.1)$$

(where  $x^i$  are local coordinate in  $S^{m+1}$  and  $\xi^i$  are the components of a contravariant vector  $\xi$ ). We assume that orbits of the transformations generated by  $\xi$  cover  $S^{m+1}$  simply and that  $\xi$  is everywhere continuous and  $\neq 0$ . Let us choose a coordinate system such that the orbits of the transformations are new  $x^1$ -coordinate curves, that is, a coordinate system in which the vector  $\xi^i$  has components  $\xi^i = \delta_1^i$ , where the symbol  $\delta_j^i$  denotes Kronecker's delta; then (1.1) becomes as follows

$$\hat{x}^i = x^i + \delta_1^i \delta\tau \quad (1.2)$$

and  $S^{m+1}$  admits a one-parameter continuous group  $G$  of transformations which are 1-1-mappings of  $S^{m+1}$  onto itself and are given by the expression  $\hat{x}^i = x^i + \delta_1^i \tau$  in the new special coordinate system ([6]).

Now we consider two orientable closed hypersurfaces  $W^m$  and  $\bar{W}^m$  of class  $C^v$  imbedded in  $S^{m+1}$  which are given as follows

$$\left. \begin{aligned} W^m: x^i &= x^i(u^\alpha) & i &= 1, \dots, m+1 & \alpha &= 1, \dots, m \\ \bar{W}^m: \bar{x}^i &= \bar{x}^i(u^\alpha) + \delta_1^i \tau(u^\alpha) \end{aligned} \right\} \quad (1.3)$$

where  $u^\alpha$  are local coordinates of  $W^m$  and  $\tau$  is a continuous function attached to each point of the hypersurface  $W^m$ . We shall henceforth confine ourselves to Latin indices running from 1 to  $m+1$  and Greek indices from 1 to  $m$ , and to two hypersurfaces  $W^m$  and  $\bar{W}^m$  which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by  $f(x^2, \dots, x^{m+1}) = 0$ .

Then we can take the family of the hypersurfaces

$$W^m(t) = (1-t) W^m + t \bar{W}^m \quad 0 \leq t \leq 1,$$

generated by  $W^m$  and  $\bar{W}^m$  whose points correspond along the orbits of the transformations where  $W^m$  and  $\bar{W}^m$  mean  $W^m(0)$  and  $W^m(1)$  respectively. Thus according to (1.3),  $W^m(t)$  is given by the expression

$$W^m(t): x^i(u^\alpha, t) = (1-t) x^i(u^\alpha) + t \bar{x}^i(u^\alpha) \quad 0 \leq t \leq 1, \quad (1.4)$$

and (1.4) may be rewritten as follows

$$W^m(t): x^i(u^\alpha, t) = x^i(u^\alpha) + \delta_1^i t \tau(u^\alpha) \quad 0 \leq t \leq 1. \quad (1.5)$$

Let us denote the normal unit vector of  $W^m(t)$  by  $n^i(t)$  and its derivative with respect to  $t$  by  $n'^i(t)$ . Then  $g_{ij}$  being the metric tensor of  $S^{m+1}$  and differentiating the following relations with respect to  $t$ ,

$$g_{ij} n^i(t) \frac{\partial x^j(u, t)}{\partial u^\alpha} = 0, \quad g_{ij} n^i(t) n^j(t) = 1,$$

since the transformation group  $G$  is isometric, that is,  $\partial g_{ij} / \partial x^1 = 0$ , we have

$$g_{ij} n'^i \frac{\partial x^j(u, t)}{\partial u^\alpha} + g_{ij} n^i(t) \frac{d}{dt} \left( \frac{\partial x^j(u, t)}{\partial u^\alpha} \right) = 0, \quad (1.6)$$

$$g_{ij} n^i(t) n'^j(t) = 0. \quad (1.7)$$

From (1.6), (1.7) and

$$\frac{d}{dt} \left( \frac{\partial x^i(u, t)}{\partial u^\alpha} \right) = \delta_1^i \frac{\partial \tau}{\partial u^\alpha},$$

we get

$$n'^i(t) = -g^{\alpha\beta}(t) \tau_\alpha \delta_1^l n_l(t) \frac{\partial x^i(u, t)}{\partial u^\beta}, \quad (1.8)$$

where  $g^{\alpha\beta}(t)$  is the contravariant metric tensor of  $W^m(t)$  and  $\tau_\alpha$  means  $\partial \tau / \partial u^\alpha$ . Throughout this paper repeated lower case Latin indices call for summation 1 to  $m+1$  and repeated lower case Greek indices for summation 1 to  $m$ . And also for its covariant differential along  $W^m(t)$  we have

$$\delta n'^i(t) = dn'^i(t) + \Gamma_{jl}^i n'^j(t) x_\gamma^l du^\gamma, \quad (1.9)$$

where  $\Gamma_{jl}^i$  is the Christoffel symbol with respect to the metric tensor  $g_{ij}$  of  $S^{m+1}$  and  $x_\gamma^l$  means  $\partial x^l(u, t) / \partial u^\gamma$ .

Let us give henceforth the derivative with respect to  $t$  by the dash. Calculating  $(\delta n^i)'$ , we have

$$\begin{aligned} \delta n^i &= dn^i + \Gamma_{jl}^i n^j(t) x_\gamma^l du^\gamma, \\ (\delta n^i)' &= (dn^i)' + (\Gamma_{jl}^i)' n^j(t) x_\gamma^l du^\gamma \\ &\quad + \Gamma_{jl}^i n'^j(t) x_\gamma^l du^\gamma + \Gamma_{jl}^i n^j(t) (x_\gamma^l)' du^\gamma, \end{aligned}$$

since  $G$  is isometric, that is,  $\partial g_{ij} / \partial x^1 = 0$ , we have  $\partial \Gamma_{ij}^k / \partial x^1 = 0$ . Consequently we obtain the following relation between  $\delta n'^i$  and  $(\delta n^i)'$

$$(\delta n^i)' = \delta n'^i + \Gamma_{j1}^i n^j(t) \tau_\gamma du^\gamma. \quad (1.10)$$

We claim that the following theorems hold



**THEOREM 1.1.** *Let  $K$  and  $\bar{K}$  be the Gauss curvature of  $W^m$  and  $\bar{W}^m$  respectively. Assume that the second fundamental form of  $W^m(t)$ ,  $0 \leq t \leq 1$  is positive definite. If the relation  $K = \bar{K}$  holds for each point  $p \in W^m$ , then  $W^m$  and  $\bar{W}^m$  are congruent mod  $G$ .*

*Proof.* We consider the following differential form of degree  $m-1$  attached to each point  $p$  on the hypersurface  $W^m(t)$

$$\left. \begin{aligned} ((n', \delta_1 \tau, \delta n, \dots, \delta n)) &\stackrel{\text{def.}}{=} \sqrt{g} (n', \delta_1 \tau, \delta n, \dots, \delta n) \\ &= (-1)^{m-1} \sqrt{g} (n', \delta_1 \tau, x_{\alpha_1}, \dots, x_{\alpha_{m-1}}) \\ &\quad \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}} \end{aligned} \right\} \quad (1.11)$$

where  $g$  is the determinant of the metric tensor  $g_{ij}$  of  $S^{m+1}$ , the symbol  $(\ )$  means a determinant of order  $m+1$  whose columns are the components of respective vectors,  $b_{\alpha\beta}(t)$  is the second fundamental tensor of  $W^m(t)$  and  $b_{\alpha}^{\beta}(t)$  denotes  $b_{\alpha\gamma}(t) g^{\beta\gamma}(t)$ .

Then the exterior differential of the differential form (1.11) becomes as follows

$$\left. \begin{aligned} d((n', \delta_1 \tau, \delta n, \dots, \delta n)) &= ((\delta n', \delta_1 \tau, \delta n, \dots, \delta n)) \\ &\quad + ((n', \delta(\delta_1 \tau), \delta n, \dots, \delta n)) + ((n', \delta_1 d\tau, \delta n, \dots, \delta n)), \end{aligned} \right\} \quad (1.12)$$

because since  $S^{m+1}$  is a space of constant curvature, we have

$$((n', \delta_1 \tau, \delta \delta n, \delta n, \dots, \delta n)) = 0.$$

Because  $G$  is isometric, the quantity  $n_i(t) \delta_1^i \sqrt{g^*(t)}$  is independent of  $t$ , where  $g^*(t)$  means the determinant of  $g_{\alpha\beta}(t)$ , we have

$$(((\delta n)', \delta_1 \tau, \delta n, \dots, \delta n)) = (-1)^m (m-1)! K' n_i(t) \delta_1^i \tau dA(t) \quad (1.13)$$

where  $dA(t)$  is the area element of  $W^m(t)$ , and using (1.8), we obtain

$$\left. \begin{aligned} ((n', \delta_1 d\tau, \delta n, \dots, \delta n)) &= (-1)^m (m-1)! \\ &\quad \times \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t) \tau_{\alpha} \tau_{\beta} (n_i(t) \delta_1^i)^2 \sqrt{g^*(t)} dA(t) \end{aligned} \right\} \quad (1.14)$$

where  $B^{\alpha\beta}(t)$  means the cofactor of an element  $b_{\beta\alpha}(t)$  in the determinant  $|b_{\alpha\beta}(t)|$  divided by  $g^*(t)$ .

By making use of (1.10), (1.12), (1.13), (1.14) and the relation

$$\delta(\delta_1^i) = \Gamma_{j1}^i x_{\gamma}^j du^{\gamma},$$

we have

$$\begin{aligned} d((n', \delta_1 \tau, \delta n, \dots, \delta n)) &= (-1)^m (m-1)! \left\{ K' n_i(t) \delta_1^i \tau dA(t) \right. \\ &\quad \left. + \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t) \tau_{\alpha} \tau_{\beta} (n_i(t) \delta_1^i)^2 \sqrt{g^*(t)} dA(t) \right\} \\ &\quad + ((n', \tau \Gamma_{j1}^i x_{\gamma}^j du^{\gamma}, \delta n, \dots, \delta n)) \\ &\quad - ((\Gamma_{j1}^i n^j(t) \tau_{\gamma} du^{\gamma}, \delta_1 \tau, \delta n, \dots, \delta n)). \end{aligned}$$

Next we shall prove that

$$((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) = 0. \quad (1.15)$$

For the first term of the left-hand member of (1.15), making use of (1.8), we can see the following

$$\left. \begin{aligned} ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) &= (-1)^{m-1} \tau n_l(t) \delta_1^l \\ &\times ((\Gamma_{j1} x_\gamma^j, g^{\alpha\beta}(t) \tau_\beta x_\alpha, x_{\alpha_1}, \dots, x_{\alpha_{m-1}})) \\ &\times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}}. \end{aligned} \right\} \quad (1.16)$$

Let  $\varepsilon_{i_1 \dots i_{m+1}}$  and  $\varepsilon_{\alpha_1 \dots \alpha_m}$  be the  $\varepsilon$ -symbol of  $S^{m+1}$  and of  $W^m(t)$  respectively,

$$\varepsilon_{i_1 \dots i_{m+1}} \stackrel{\text{def.}}{=} \sqrt{g} e_{i_1 \dots i_{m+1}}, \quad \varepsilon_{\alpha_1 \dots \alpha_m} \stackrel{\text{def.}}{=} \sqrt{g^*(t)} e_{\alpha_1 \dots \alpha_m},$$

the symbol  $e_{i_1 \dots i_{m+1}}$  meaning plus one or minus one, depending on whether the indices  $i_1, \dots, i_{m+1}$  denote an even permutation of  $1, 2, \dots, m+1$  or odd permutation, and zero when at least any two indices have the same value, and also the symbol  $e_{\alpha_1 \dots \alpha_m}$  meaning similarly for the indices  $\alpha_1, \dots, \alpha_m$  running from 1 to  $m$ .

Making use of the relation

$$n_i(t) \varepsilon_{\alpha\alpha_1 \dots \alpha_{m-1}} = \varepsilon_{ii_2 \dots i_{m+1}} x_{\alpha}^{i_2} x_{\alpha_1}^{i_3} \dots x_{\alpha_{m-1}}^{i_{m+1}}$$

we have

$$\begin{aligned} ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) &= (-1)^{m-1} \tau n_l(t) \delta_1^l \Gamma_{j1}^i n_i(t) x_\gamma^j \tau_\beta g^{\beta d}(t) \\ &\times \varepsilon_{\alpha\alpha_1 \dots \alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}} \\ &= (-1)^{m-1} \tau n_l(t) \delta_1^l \Gamma_{j1}^i n_i(t) x_\gamma^j \tau_\beta \varepsilon_{\alpha_1 \dots \alpha_{m-1}}^{\beta} \varepsilon^{\gamma\beta_1 \dots \beta_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) dA(t) \end{aligned}$$

and we can see easily the following relation

$$\begin{aligned} \varepsilon_{\alpha_1 \dots \alpha_{m-1}}^{\beta} \varepsilon^{\gamma\beta_1 \dots \beta_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) &= \varepsilon^{\beta\gamma_1 \dots \gamma_{m-1}} \varepsilon^{\gamma\beta_1 \dots \beta_{m-1}} b_{\gamma_1\beta_1}(t) \dots b_{\gamma_{m-1}\beta_{m-1}}(t) \\ &= (m-1)! B^{\beta\gamma}(t). \end{aligned}$$

Since  $B^{\beta\gamma}(t)$  is the symmetric tensor, we have

$$\left. \begin{aligned} ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) &= (-1)^{m-1} (m-1)! \tau n_l(t) \\ &\times \delta_1^l \Gamma_{ji1} n^i(t) x_{(\gamma}^j \tau_{\beta)} B^{\beta\gamma}(t) dA(t) \end{aligned} \right\} \quad (1.17)$$

where  $\Gamma_{ji1}$  means  $g_{il} \Gamma_{j1}^l$  and the symbol  $(\gamma\beta)$  denotes the symmetric part for the indices  $\gamma$  and  $\beta$ .

On the other hand, we calculate the second term of the left-hand member of (1.15). Since  $G$  is isometric, that is,  $\partial g_{ij}/\partial x^1 = 0$ , we have

$$\left. \begin{aligned} \Gamma_{j1}^l n^j(t) n_l(t) &= \frac{1}{2} g^{lk} \left( \frac{\partial g_{kj}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^j} - \frac{\partial g_{j1}}{\partial x^k} \right) n^j(t) n_l(t) \\ &= \frac{1}{2} \frac{\partial g_{kj}}{\partial x^1} n^j(t) n^k(t) = 0, \end{aligned} \right\} \quad (1.18)$$

and we can give the vector  $\delta_1^i$  by the expression

$$\delta_1^i = n_l(t) \delta_1^l n^i(t) + \varphi^\beta x_\beta^i. \quad (1.19)$$

Substituting (1.19) in the second term of the left-hand member of (1.15) and making use of (1.18), we have

$$\left. \begin{aligned} & - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) \\ & = - (-1)^{m-1} \tau n_l(t) \delta_1^l ((\Gamma_{j1} n^j(t) \tau_\gamma, n, x_{\alpha_1}, \dots, x_{\alpha_{m-1}})) \\ & \quad \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{m-1}}. \end{aligned} \right\} \quad (1.20)$$

Let us take the relation

$$\varepsilon_{\alpha\alpha_1 \dots \alpha_{m-1}} g^{\alpha\beta}(t) x_\beta^j g_{ij} = (-1)^m \varepsilon_{ii_2 \dots i_{m+1}} x_{\alpha_1}^{i_2} \dots x_{\alpha_{m-1}}^{i_m} n^{i_{m+1}}.$$

Then we have

$$\left. \begin{aligned} & - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) \\ & = (-1)^{m-1} (m-1)! \tau n_l(t) \delta_1^l \Gamma_{ij1} n^i(t) x_{(\beta}^j \tau_{\gamma)} B^{\beta\gamma}(t) dA(t). \end{aligned} \right\} \quad (1.21)$$

Thus from (1.17), (1.21) and  $\Gamma_{ij1} + \Gamma_{ji1} = \partial g_{ij} / \partial x^1 = 0$ , we can arrive at (1.15) as follows

$$\begin{aligned} & ((n', \tau \Gamma_{j1} x_\gamma^j du^\gamma, \delta n, \dots, \delta n)) - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n)) \\ & = (-1)^{m-1} (m-1)! \tau n_l(t) \delta_1^l (\Gamma_{ij1} + \Gamma_{ji1}) n^i(t) x_{(\gamma}^j \tau_{\beta)} B^{\beta\gamma}(t) dA(t) = 0. \end{aligned}$$

Finally we have

$$\left. \begin{aligned} & \frac{(-1)^m}{(m-1)!} d((n', \delta_1 \tau, \delta n, \dots, \delta n)) = K' n_i(t) \delta_1^i \tau dA(t) \\ & + \frac{1}{\sqrt{g^*(t)}} B^{\alpha\beta}(t) \tau_\alpha \tau_\beta (n_i(t) \delta_1^i)^2 \sqrt{g^*(t)} dA(t). \end{aligned} \right\} \quad (1.22)$$

Integrating both members of (1.22) over the interval  $0 \leq t \leq 1$ , we get

$$\left. \begin{aligned} & \frac{(-1)^m}{(m-1)!} d \int_0^1 ((n', \delta_1 \tau, \delta n, \dots, \delta n)) dt = (K' - K) n_i(0) \delta_1^i \tau dA(0) \\ & + \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0). \end{aligned} \right\} \quad (1.23)$$

Furthermore integrating both members of (1.23) over  $W^m$  and applying Stokes' theorem, since  $W^m$  is closed, we have

$$\begin{aligned} & \int \int_{W^m} (K' - K) n_i(0) \delta_1^i \tau dA(0) \\ & + \int \int_{W^m} \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0) = 0, \end{aligned}$$

making use of the hypothesis  $\bar{K} = K$ , we obtain

$$\int \int_{W^m} \sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0) = 0.$$

On the other hand, from that the second fundamental form of  $W^m(t)$  is positive definite everywhere in  $W^m(t)$ ,  $0 \leq t \leq 1$ , the quantity

$$\sqrt{g^*(0)} \int_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) dt v_\alpha v_\beta$$

becomes positive definite. From that two hypersurfaces  $W^m$  and  $\bar{W}^m$  do not contain a piece of a hypersurface covered by the orbits of transformations, a point on  $W^m$  such that  $n_i(0)\delta_1^i = 0$  must be an isolate point. Moreover since  $\tau$  is a continuous function of  $W^m$ , we have

$$\tau = \text{constant}$$

for all points of  $W^m$ . Consequently we can arrive at the following result

$$W^m \equiv \bar{W}^m \pmod{G}.$$

**THEOREM 1.2.** *Let  $H_r$  and  $\bar{H}_r$  be the  $r$ -th mean curvature of  $W^m$  and  $\bar{W}^m$  respectively. Assume that the second fundamental form of  $W^m(t)$ ,  $0 \leq t \leq 1$ , is positive definite. If the relation*

$$H_r = \bar{H}_r$$

*holds for each point  $p \in W^m$ , then  $W^m$  and  $\bar{W}^m$  are congruent mod  $G$ .*

*Proof.* We consider the following differential form of degree  $m-1$  attached to each point  $p$  on the hypersurface  $W^m(t)$

$$\left. \begin{aligned} ((n', \underbrace{\delta_1 \tau, \delta n, \dots, \delta n}_{r-1}, dx, \dots, dx)) &\stackrel{\text{def.}}{=} \sqrt{g}(n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx) \\ &= (-1)^{r-1} \sqrt{g}(n', \delta_1 \tau, x_{\alpha_1}, \dots, x_{\alpha_{r-1}}, x_{\beta_r} \dots x_{\beta_{m-1}}) \\ &\quad \times b_{\beta_r}^{\alpha_r}(t) \dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t) du^{\beta_1} \wedge \dots \wedge du^{\beta_{r-1}} \wedge du^{\beta_r} \wedge \dots \wedge du^{\beta_{m-1}} \end{aligned} \right\} \quad (1.24)$$

The exterior differential of the differential form (1.24) becomes as follows

$$\begin{aligned} d((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) &= ((\delta n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &\quad + ((n', \delta(\delta_1 \tau), \delta n, \dots, \delta n, dx, \dots, dx)) + ((n', \delta_1 d\tau, \delta n, \dots, \delta n, dx, \dots, dx)) \end{aligned}$$

because since  $S^{m+1}$  is a space of constant curvature, it follows that

$$((n', \delta_1 \tau, \delta \delta n, \dots, \delta n, dx, \dots, dx)) = 0,$$

and also we have

$$((n', \delta_1 \tau, \delta n, \dots, \delta n, \delta dx, \dots, dx)) = 0.$$

Making use of (1.8), we have

$$\left. \begin{aligned} & ((n', \delta_1 d\tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &= (-1)^{r-1} g^{\alpha\beta}(t) \tau_\alpha n_i(t) \delta_1^i ((\delta_1 \tau_\gamma, x_\beta, x_{\alpha_1}, \dots, x_{\alpha_{r-1}}, x_{\alpha_r}, \dots, x_{\alpha_{m-1}})) \\ &\quad \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{r-1}}^{\alpha_{r-1}}(t) du^\gamma \wedge du^{\beta_1} \wedge \dots \wedge du^{\beta_{r-1}} \wedge du^{\sigma_r} \\ &\quad \wedge \dots \wedge du^{\alpha_{m-1}} \\ &= (-1)^{r-1} g^{\alpha\beta}(t) \varepsilon_{\beta\alpha_1 \dots \alpha_{r-1} \alpha_r \dots \alpha_{m-1}} \varepsilon^{\gamma\beta_1 \dots \beta_{r-1} \alpha_r \dots \alpha_{m-1}} \\ &\quad \times b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{r-1}}^{\alpha_{r-1}}(t) (n_i(t) \delta_1^i)^2 \tau_\alpha \tau_\gamma dA(t). \end{aligned} \right\} \quad (1.25)$$

On the other hand, from (1.10) we get

$$\left. \begin{aligned} & ((\delta n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &= (((\delta n)', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &\quad - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)). \end{aligned} \right\} \quad (1.26)$$

And after some calculations, we have

$$(-1)^r m! H'_r n_i(t) \delta_1^i dA(t) = r((\delta_1, (\delta n)', \delta n, \dots, \delta n, dx, \dots, dx)), \quad (1.27)$$

because  $n_i(t) \delta_1^i dA(t)$  is independent of  $t$  and  $dx'^i = \delta_1^i d\tau$ , that is, the same direction to  $\delta_1$ . Moreover we can prove similarly the following relation as the proof of (1.15)

$$\left. \begin{aligned} & ((n', \delta(\delta_1) \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &\quad - ((\Gamma_{j1} n^j(t) \tau_\gamma du^\gamma, \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) = 0. \end{aligned} \right\} \quad (1.28)$$

Then putting

$$(m-1)! c_{(r)}^{\alpha\beta} = \varepsilon_{\alpha_1 \dots \alpha_{r-1} \alpha_r \dots \alpha_{m-1}} \varepsilon^{\beta\beta_1 \dots \beta_{r-1} \alpha_r \dots \alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) \dots b_{\beta_{r-1}}^{\alpha_{r-1}}(t)$$

and using (1.25), (1.26), (1.27) and (1.28), we have

$$\left. \begin{aligned} & d((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &= \frac{(-1)^{r-1}}{r} m! H'_r n_i(t) \delta_1^i \tau dA(t) \\ &\quad + (-1)^{r-1} (m-1)! c_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta (n_i(t) \delta_1^i)^2 dA(t). \end{aligned} \right\} \quad (1.29)$$

Integrating both members of (1.29) over the interval  $0 \leq t \leq 1$ , and putting

$$C_{(r)}^{\alpha\beta} = g^*(0)^{1/2} \int_0^1 g^*(t)^{-1/2} c_{(r)}^{\alpha\beta} dt,$$

we have

$$\left. \begin{aligned} & m(\bar{H}_r - H_r) n_i(0) \delta_1^i \tau dA(0) + r C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta (n_i(0) \delta_1^i)^2 dA(0) \\ &= \frac{r(-1)^{r-1}}{(m-1)!} d \int_0^1 ((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) dt. \end{aligned} \right\} \quad (1.30)$$

Furthermore integrating both members of (1.30) over  $W^m$  and applying Stokes' theorem

$$\begin{aligned} & \frac{m}{r} \iint_{W^m} (\bar{H}_r - H_r) n_i(0) \delta_1^i \tau dA(0) + \iint_{W^m} (n_i(0) \delta_1^i)^2 C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta dA(0) \\ &= \frac{(-1)^{r-1}}{(m-1)!} \int_0^1 \int_{\partial W^m} ((n', \delta_1 \tau, \delta n, \dots, \delta n, dx, \dots, dx)) dt. \end{aligned}$$

Since  $W^m$  is closed, we have

$$\frac{m}{r} \iint_{W^m} (\bar{H}_r - H_r) n_i(0) \delta_1^i \tau dA(0) + \iint_{W^m} (n_i(0) \delta_1^i)^2 C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta dA(0) = 0,$$

using the hypotheses  $H_r = \bar{H}_r$  and that the second fundamental form of  $W^m(t)$ ,  $0 \leq t \leq 1$ , is positive definite, and from that two hypersurfaces  $W^m$  and  $\bar{W}^m$  do not contain a piece of a hypersurface covered by the orbits of transformations, we can arrive at

$$\tau_\alpha = 0$$

for all points of  $W^m$ , consequently we have

$$\tau = \text{constant}$$

for all points of  $W^m$ . Accordingly we can see the following result

$$W^m = \bar{W}^m \bmod G.$$

This proof follows to the method of that due to K. Voss [2].

*Remark.* In an euclidean space, if  $G$  is translation group, that is, a special isometric transformation group, Theorem 1.1 and Theorem 1.2 just coincide with theorems of K. Voss given in the introduction.

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