Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces

(Part III: Method based on Voss' Proof)

by Heinz Hopf † (Zürich) and Yoshie Katsurada (Sapporo)

Introduction

An idea that gives congruence of two hypersurfaces concerning a transformation group by a relation between the invariant of the corresponding points of these hyper surfaces was first introduced by H. Hopf and K. Voss [1], that is, in that paper congruence relations of two closed curves on a plane and of two closed surfaces in 3-dimensional euclidean space have been given by the relation of the mean curvatures.

K. Voss has generalized these theorems to hypersurfaces in an $(m+1)$-dimensional euclidean space $(m+1 \geq 3)$ and also given the congruence relations in case of Gauss curvatures or the r-th mean curvatures H_r $r = 1, 2, ..., m$ [2]. A. Aeppli has developed analogous statements for a central transformation group (a homothetic transformation group with the center 0) [3].

The present authors wished to generalize these theorems to Riemann spaces. In the previous papers [4], [5], we gave the generalized theorems relating to the first mean curvature.

The purpose of the present paper is to investigate a general theorem relating to the Gauss curvature or the r-th mean curvature, that is, to generalize to an orientable Riemann space R^{m+1} with constant Riemann curvature the following theorems given by K. Voss:

THEOREM (K. Voss). *Let* W^m *and* $\bar{W}^m$ *be two orientable closed hypersurfaces in an* $(m+1)$*-dimensional euclidean space and let* p *and* $\bar{p}$ *be the corresponding points of these hypersurfaces, and let* $K(p)$ *and* $\bar{K}(p)$ *be that Gauss curvatures at these points respectively. Assume that the second fundamental forms of* W^m *and* $\bar{W}^m$ *are positive definite. If all straight lines* $(p\bar{p})$ *are parallel to one another and if* $K(p) = \bar{K}(\bar{p})$ *for all* $p \in W^m$, *then the hypersurface* $\bar{W}^m$ *is produced from* W^m *by simple translation in the direction of* $(p\bar{p})$. *(*W^m *and* $\bar{W}^m$ *are therefore congruent mod the translation group).*

THEOREM (K. Voss). *Let* W^m *and* $\bar{W}^m$ *be to orientable closed hypersurfaces in an* $(m+1)$*-dimensional euclidean space and let* p *and* $\bar{p}$ *be the corresponding points of these hypersurfaces, and let* $H_r(p)$ *and* $\bar{H}_r(\bar{p})$ *be the r-th mean curvatures at these*

points respectively, for some $r = 1, 2, ..., m$. *Assume that the second fundamental forms of* W^m *and* $\bar{W}^m$ *are positive definite. If all straight lines* $(p\bar{p})$ *are parallel to one another and if* $H_r(p) = \bar{H}_r(\bar{p})$ *for all* $p \in W^m$, *then the hypersurface* $\bar{W}^m$ *is produced from* W^m *by simple translation in the direction of* $(p\bar{p})$. $(W^m$ *and* $\bar{W}^m$ *are congruent mod the translation group.)*

§ 1. Generalized Theorems

We suppose an $(m+1)$-dimensional orientable Riemann space with constant curvature S^{m+1} of class $C^\nu (\nu \geq 3)$ which admits an infinitesimal isometric transformation

$$\hat{x}^i = x^i + \xi^i(x)\,\delta\tau \tag{1.1}$$

(where x^i are local coordinate in S^{m+1} and ξ^i are the components of a contravariant vector ξ). We assume that orbits of the transformations generated by ξ cover S^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of the transformations are new x^1-coordinate curves, that is, a coordinate system in which the vector ξ^i has components $\xi^i = \delta_1^i$, where the symbol δ_j^i denotes Kronecker's delta; then (1.1) becomes as follows

$$\hat{x}^i = x^i + \delta_1^i\,\delta\tau \tag{1.2}$$

and S^{m+1} admits a one-parameter continuous group G of transformations which are $1-1$-mappings of S^{m+1} onto itself and are given by the expression $\hat{x}^i = x^i + \delta_1^i\tau$ in the new special coordinate system ([6]).

Now we consider two orientable closed hypersurfaces W^m and $\bar{W}^m$ of class C^ν imbedded in S^{m+1} which are given as follows

$$\left.\begin{array}{ll} W^m\colon x^i = x^i(u^\alpha) & i = 1, ..., m+1 \qquad \alpha = 1, ..., m \\ \bar{W}^m\colon \bar{x}^i = \bar{x}^i(u^\alpha) + \delta_1^i\tau(u^\alpha) & \end{array}\right\} \tag{1.3}$$

where u^α are local coordinates of W^m and τ is a continuous function attached to each point of the hypersurface W^m. We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to m, and to two hypersurfaces W^m and $\bar{W}^m$ which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by $f(x^2, ..., x^{m+1}) = 0$.

Then we can take the family of the hypersurfaces

$$W^m(t) = (1 - t)W^m + t\bar{W}^m \qquad 0 \leq t \leq 1,$$

generated by W^m and $\bar{W}^m$ whose points correspond along the orbits of the transformations where W^m and $\bar{W}^m$ mean $W^m(0)$ and $W^m(1)$ respectively. Thus according to (1.3), $W^m(t)$ is given by the expression

$$W^m(t)\colon x^i(u^\alpha, t) = (1 - t)x^i(u^\alpha) + t\bar{x}^i(u^\alpha) \qquad 0 \leq t \leq 1, \tag{1.4}$$

and (1.4) may be rewritten as follows

$$W^m(t): x^i(u^\alpha, t) = x^i(u^\alpha) + \delta_1^i t\tau(u^\alpha) \qquad 0 \leq t \leq 1. \tag{1.5}$$

Let us denote the normal unit vector of $W^m(t)$ by $n^i(t)$ and its derivative with respect to t by $n'^i(t)$. Then g_{ij} being the metric tensor of S^{m+1} and differentiating the following relations with respect to t,

$$g_{ij}n^i(t)\frac{\partial x^j(u, t)}{\partial u^\alpha} = 0, \qquad g_{ij}n^i(t) n^j(t) = 1,$$

since the transformation group G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have

$$g_{ij}n'^i\frac{\partial x^j(u, t)}{\partial u^\alpha} + g_{ij}n^i(t)\frac{d}{dt}\left(\frac{\partial x^j(u, t)}{\partial u^\alpha}\right) = 0, \tag{1.6}$$

$$g_{ij}n^i(t) n'^j(t) = 0. \tag{1.7}$$

From (1.6), (1.7) and

$$\frac{d}{dt}\left(\frac{\partial x^i(u, t)}{\partial u^\alpha}\right) = \delta_1^i\frac{\partial \tau}{\partial u^\alpha}, \qquad \qquad \cdot$$

we get

$$n'^i(t) = -g^{\alpha\beta}(t) \tau_\alpha\delta_1^l n_l(t)\frac{\partial x^i(u, t)}{\partial u^\beta}, \tag{1.8}$$

where $g^{\alpha\beta}(t)$ is the contravariant metric tensor of $W^m(t)$ and τ_α means $\partial\tau/\partial u^\alpha$. Throughout this paper repeated lower case Latin indices call for summation 1 to $m+1$ and repeated lower case Greek indices for summation 1 to m. And also for its co-variant differential along $W^m(t)$ we have

$$\delta n'^i(t) = dn'^i(t) + \Gamma_{jl}^i n'^j(t) x_\gamma^l du^\gamma, \tag{1.9}$$

where Γ_{jl}^i is the Christoffel symbol with respect to the metric tensor g_{ij} of S^{m+1} and x_γ^l means $\partial x^l(u, t)/\partial u^\gamma$.

Let us give henceforth the derivative with respect to t by the dash. Calculating $(\delta n^i)'$, we have

$$\delta n^i = dn^i + \Gamma_{jl}^i n^j(t) x_\gamma^l d u^\gamma,$$

$$(\delta n^i)' = (dn^i)' + (\Gamma_{jl}^i)' n^j(t) x_\gamma^l du^\gamma$$

$$+ \Gamma_{jl}^i n'^j(t) x_\gamma^l du^\gamma + \Gamma_{jl}^i n^j(t) (x_\gamma^l)' du^\gamma,$$

since G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have $\partial\Gamma_{ij}^i/\partial x^1 = 0$. Consequently we obtain the following relation between $\delta n'^i$ and $(\delta n^i)'$

$$(\delta n^i)' = \delta n'^i + \Gamma_{j1}^i n^j(t) \tau_\gamma du^\gamma. \tag{1.10}$$

We claim that the following theorems hold

THEOREM 1.1. *Let K and $\bar{K}$ be the Gauss curvature of W^m and $\bar{W}^m$ respectively. Assume that the second fundamental form of $W^m(t)$, $0 \leq t \leq 1$ is positive definite. If the relation $K = \bar{K}$ holds for each point $p \in W^m$, then W^m and $\bar{W}^m$ are congruent mod G.*

Proof. We consider the following differential form of degree $m-1$ attached to each point p on the hypersurface $W^m(t)$

$$
\begin{aligned}
\left((n', \delta_1\tau, \delta n, ..., \delta n)\right) &\overset{\text{def.}}{\equiv} \sqrt{g}\,(n', \delta_1\tau, \delta n, ..., \delta n) \\
&= (-1)^{m-1}\sqrt{g}\,(n', \delta_1\tau, x_{\alpha_1}, ..., x_{\alpha_{m-1}}) \\
&\quad \times b^{\alpha_1}_{\beta_1}(t) ... b^{\alpha_{m-1}}_{\beta_{m-1}}(t)\, du^{\beta_1} \wedge ... \wedge du^{\beta_{m-1}}
\end{aligned}
\tag{1.11}
$$

where g is the determinant of the metric tensor g_{ij} of S^{m+1}, the symbol $(\ )$ means a determinant of order $m+1$ whose columns are the components of respective vectors, $b_{\alpha\beta}(t)$ is the second fundamental tensor of $W^m(t)$ and $b^{\beta}_{\alpha}(t)$ denotes $b_{\alpha\gamma}(t)\,g^{\beta\gamma}(t)$.

Then the exterior differential of the differential form (1.11) becomes as follows

$$
\begin{aligned}
d\left((n', \delta_1\tau, \delta n, ..., \delta n)\right) &= \left((\delta n', \delta_1\tau, \delta n, ..., \delta n)\right) \\
&\quad + \left((n', \delta(\delta_1)\tau, \delta n, ..., \delta n)\right) + \left((n', \delta_1\, d\tau, \delta n, ..., \delta n)\right),
\end{aligned}
\tag{1.12}
$$

because since S^{m+1} is a space of constant curvature, we have

$$
\left((n', \delta_1\tau, \delta\delta n, \delta n, ..., \delta n)\right) = 0.
$$

Because G is isometric, the quantity $n_i(t)\,\delta^i_1\sqrt{g^*(t)}$ is independent of t, where $g^*(t)$ means the determinant of $g_{\alpha\beta}(t)$, we have

$$
\left(((\delta n)', \delta_1\tau, \delta n, ..., \delta n)\right) = (-1)^m(m-1)!\,K'n_i(t)\,\delta^i_1\tau\, dA(t)
\tag{1.13}
$$

where $dA(t)$ is the area element of $W^m(t)$, and using (1.8), we obtain

$$
\begin{aligned}
\left((n', \delta_1\, d\tau, \delta n, ..., \delta n)\right) &= (-1)^m(m-1)! \\
&\quad \times \frac{1}{\sqrt{g^*(t)}}\,B^{\alpha\beta}(t)\,\tau_\alpha\tau_\beta\left(n_i(t)\,\delta^i_1\right)^2\sqrt{g^*(t)}\, dA(t)
\end{aligned}
\tag{1.14}
$$

where $B^{\alpha\beta}(t)$ means the cofactor of an element $b_{\beta\alpha}(t)$ in the determinant $|b_{\alpha\beta}(t)|$ divided by $g^*(t)$.

By making use of (1.10), (1.12), (1.13), (1.14) and the relation

$$
\delta(\delta^i_1) = \Gamma^i_{j1}x^j_\gamma\, du^\gamma,
$$

we have

$$
\begin{aligned}
d\left((n', \delta_1\tau, \delta n, ..., \delta n)\right) &= (-1)^m(m-1)!\,\Bigg\{K'n_i(t)\,\delta^i_1\tau\, dA(t) \\
&\quad + \frac{1}{\sqrt{g^*(t)}}\,B^{\alpha\beta}(t)\,\tau_\alpha\tau_\beta\left(n_i(t)\,\delta^i_1\right)^2\sqrt{g^*(t)}\, dA(t)\Bigg\} \\
&\quad + \left((n', \tau\Gamma_{j1}x^j_\gamma\, du^\gamma, \delta n, ..., \delta n)\right) \\
&\quad - \left((\Gamma_{j1}n^j(t)\,\tau_\gamma\, du^\gamma, \delta_1\tau, \delta n, ..., \delta n)\right).
\end{aligned}
$$

Next we shall prove that

$$((n', \tau\Gamma_{j1}x_\gamma^j \, du^\gamma, \delta n, ..., \delta n)) - ((\Gamma_{j1}n^j(t) \, \tau_\gamma \, du^\gamma, \delta_1\tau, \delta n, ..., \delta n)) = 0. \tag{1.15}$$

For the first term of the left-hand member of (1.15), making use of (1.8), we can see the following

$$
\begin{aligned}
((n', \tau\Gamma_{j1}x_\gamma^j \, du^\gamma, \delta n, ..., \delta n)) &= (-1)^{m-1} \tau n_l(t) \, \delta_1^l \\
&\times ((\Gamma_{j1}x_\gamma^j, g^{\alpha\beta}(t) \, \tau_\beta x_\alpha, x_{\alpha_1}, ..., x_{\alpha_{m-1}})) \\
&\times b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{m-1}}^{\alpha_{m-1}}(t) \, du^\gamma \wedge du^{\beta_1} \wedge ... \wedge du^{\beta_{m-1}}.
\end{aligned}
\tag{1.16}
$$

Let $\varepsilon_{i_1...i_{m+1}}$ and $\varepsilon_{\alpha_1...\alpha_m}$ be the ε-symbol of S^{m+1} and of $W^m(t)$ respectively,

$$\varepsilon_{i_1...i_{m+1}} \overset{\text{def.}}{=} \sqrt{g} \, e_{i_1...i_{m+1}}, \qquad \varepsilon_{\alpha_1...\alpha_m} \overset{\text{def.}}{=} \sqrt{g^*(t)} \, e_{\alpha_1...\alpha_m},$$

the symbol $e_{i_1...i_{m+1}}$ meaning plus one or minus one, depending on whether the indices $i_1, ... \, i_{m+1}$ denote an even permutation of $1, 2, ..., m+1$ or odd permutation, and zero when at least any two indices have the same value, and also the symbol $e_{\alpha_1...\alpha_m}$ meaning similarly for the indices $\alpha_1, ..., \alpha_m$ running from 1 to m.

Making use of the relation

$$n_i(t) \, \varepsilon_{\alpha\alpha_1...\alpha_{m-1}} = \varepsilon_{ii_2...i_{m+1}} x_\alpha^{i_2} x_{\alpha_1}^{i_3} ... x_{\alpha_{m-1}}^{i_{m+1}}$$

we have

$$
\begin{aligned}
((n', \tau\Gamma_{j1}x_\gamma^j \, du^\gamma, \delta n, ..., \delta n)) &= (-1)^{m-1} \tau n_l(t) \, \delta_1^l \Gamma_{j1}^i n_i(t) \, x_\gamma^j \tau_\beta g^{\beta d}(t) \\
&\times \varepsilon_{\alpha\alpha_1...\alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{m-1}}^{\alpha_{m-1}}(t) \, du^\gamma \wedge du^{\beta_1} \wedge ... \wedge du^{\beta_{m-1}} \\
&= (-1)^{m-1} \tau n_l(t) \, \delta_1^l \Gamma_{j1}^i n_i(t) \, x_\gamma^j \tau_\beta \varepsilon_{\alpha_1...\alpha_{m-1}}^\beta \, \varepsilon^{\gamma\beta_1...\beta_{m-1}} b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{m-1}}^{\alpha_{m-1}}(t) \, dA(t)
\end{aligned}
$$

and we can see easily the following relation

$$
\begin{aligned}
\varepsilon_{\alpha_1...\alpha_{m-1}}^\beta \, \varepsilon^{\gamma\beta_1...\beta_{m-1}} b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{m-1}}^{\alpha_{m-1}}(t) &= \varepsilon^{\beta\gamma_1...\gamma_{m-1}} \, \varepsilon^{\gamma\beta_1...\beta_{m-1}} b_{\gamma_1\beta_1}(t) ... b_{\gamma_{m-1}\beta_{m-1}}(t) \\
&= (m-1)! \, B^{\beta\gamma}(t).
\end{aligned}
$$

Since $B^{\beta\gamma}(t)$ is the symmetric tensor, we have

$$
\begin{aligned}
((n', \tau\Gamma_{j1}x_\gamma^j du^\gamma, \delta n, ... \delta n)) &= (-1)^{m-1}(m-1)! \, \tau n_l(t) \\
&\times \delta_1^l \Gamma_{ji1} n^i(t) \, x_{(\gamma}^j \tau_{\beta)} B^{\beta\gamma}(t) \, dA(t)
\end{aligned}
\tag{1.17}
$$

where Γ_{ji1} means $g_{il}\Gamma_{j1}^l$ and the symbol $(\gamma\beta)$ denotes the symmetric part for the indices γ and β.

On the other hand, we calculate the second term of the left-hand member of (1.15). Since G is isometric, that is, $\partial g_{ij}/\partial x^1 = 0$, we have

$$
\begin{aligned}
\Gamma_{j1}^l n^j(t) \, n_l(t) &= \frac{1}{2} g^{lk} \left(\frac{\partial g_{kj}}{\partial x^1} + \frac{\partial g_{1k}}{\partial x^j} - \frac{\partial g_{j1}}{\partial x^k} \right) n^j(t) \, n_l(t) \\
&= \frac{1}{2} \frac{\partial g_{kj}}{\partial x^1} n^j(t) \, n^k(t) = 0,
\end{aligned}
\tag{1.18}
$$

and we can give the vector δ_1^i by the expression

$$\delta_1^i = n_l(t)\,\delta_1^l n^i(t) + \varphi^\beta x_\beta^i. \tag{1.19}$$

Substituting (1.19) in the second term of the left-hand member of (1.15) and making use of (1.18), we have

$$
\left.\begin{aligned}
&-\left(\left(\Gamma_{j1}n^j(t)\,\tau_\gamma\,du^\gamma,\,\delta_1\tau,\,\delta n,\,...,\,\delta n\right)\right)\\
&\quad = -(-1)^{m-1}\,\tau n_l(t)\,\delta_1^l\left(\left(\Gamma_{j1}n^j(t)\,\tau_\gamma,\,n,\,x_{\alpha_1},\,...,\,x_{\alpha_{m-1}}\right)\right)\\
&\quad\times b_{\beta_1}^{\alpha_1}(t)\dots b_{\beta_{m-1}}^{\alpha_{m-1}}(t)\,du^\gamma\wedge du^{\beta_1}\wedge\dots\wedge du^{\beta_{m-1}}.
\end{aligned}\right\} \tag{1.20}
$$

Let us take the relation

$$\varepsilon_{\alpha\alpha_1...\alpha_{m-1}}g^{\alpha\beta}(t)\,x_\beta^j g_{ij} = (-1)^m\,\varepsilon_{ii_2...i_{m+1}}x_{\alpha_1}^{i_2}\dots x_{\alpha_{m-1}}^{i_m}n^{i_{m+1}}.$$

Then we have

$$
\left.\begin{aligned}
&-\left(\left(\Gamma_{j1}n^j(t)\,\tau_\gamma\,du^\gamma,\,\delta_1\tau,\,\delta n,\,...,\,\delta n\right)\right)\\
&\quad = (-1)^{m-1}(m-1)!\,\tau n_l(t)\,\delta_1^l\Gamma_{ij1}n^i(t)\,x_{(\beta}^j\tau_{\gamma)}B^{\beta\gamma}(t)\,dA(t).
\end{aligned}\right\} \tag{1.21}
$$

Thus from (1.17), (1.21) and $\Gamma_{ij1}+\Gamma_{ji1}=\partial g_{ij}/\partial x^1=0$, we can arrive at (1.15) as follows

$$
\begin{aligned}
&\left(\left(n',\,\tau\Gamma_{j1}x_\gamma^j\,du^\gamma,\,\delta n,\,\cdots,\,\delta n\right)\right) - \left(\left(\Gamma_{j1}n^j(t)\,\tau_\gamma\,du^\gamma,\,\delta_1\tau,\,\delta n,\,...,\,\delta n\right)\right)\\
&\quad = (-1)^{m-1}(m-1)!\,\tau n_l(t)\,\delta_1^l\left(\Gamma_{ij1}+\Gamma_{ji1}\right)n^i(t)\,x_{(\gamma}^j\tau_{\beta)}B^{\beta\gamma}(t)\,dA(t) = 0.
\end{aligned}
$$

Finally we have

$$
\left.\begin{aligned}
&\frac{(-1)^m}{(m-1)!}\,d\left((n',\,\delta_1\tau,\,\delta n,\,...,\,\delta n)\right) = K'n_i(t)\,\delta_1^i\tau\,dA(t)\\
&\quad + \frac{1}{\sqrt{g^*(t)}}\,B^{\alpha\beta}(t)\,\tau_\alpha\tau_\beta\left(n_i(t)\,\delta_1^i\right)^2\sqrt{g^*(t)}\,dA(t).
\end{aligned}\right\} \tag{1.22}
$$

Integrating both members of (1.22) over the interval $0\le t\le 1$, we get

$$
\left.\begin{aligned}
&\frac{(-1)^m}{(m-1)!}\,d\int_0^1\left((n',\,\delta_1\tau,\,\delta n,\,...,\,\delta n)\right)dt = (\bar K - K)\,n_i(0)\,\delta_1^i\tau\,dA(0)\\
&\quad + \sqrt{g^*(0)}\int_0^1 g^*(t)^{-1/2}B^{\alpha\beta}(t)\,dt\tau_\alpha\tau_\beta\left(n_i(0)\,\delta_1^i\right)^2dA(0).
\end{aligned}\right\} \tag{1.23}
$$

Furthermore integrating both members of (1.23) over W^m and applying Stokes' theorem, since W^m is closed, we have

$$
\iint_{W^m}(\bar K - K)\,n_i(0)\,\delta_1^i\tau\,dA(0)
$$

$$
+ \iint_{W^m}\sqrt{g^*(0)}\int_0^1 g^*(t)^{-1/2}\,B^{\alpha\beta}(t)\,dt\tau_\alpha\tau_\beta\left(n_i(0)\,\delta_1^i\right)^2dA(0) = 0,
$$

making use of the hypothesis $\bar{K} = K$, we obtain

$$\iint\limits_{W^m} \sqrt{g^*(0)} \int\limits_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) \, dt \tau_\alpha \tau_\beta \big(n_i(0)\, \delta_1^i\big)^2 \, dA(0) = 0.$$

On the other hand, from that the second fundamental form of $W^m(t)$ is positive definite everywhere in $W^m(t)$, $0 \leq t \leq 1$, the quantity

$$\sqrt{g^*(0)} \int\limits_0^1 g^*(t)^{-1/2} B^{\alpha\beta}(t) \, dt v_\alpha v_\beta$$

becomes positive definite. From that two hypersurfaces W^m and $\bar{W}^m$ do not contain a piece of a hypersurface covered by the orbits of transformations, a point on W^m such that $n_i(0)\delta_1^i = 0$ must be an isolate point. Moreover since τ is a continuous function of W^m, we have

$$\tau = \text{constant}$$

for all points of W^m. Consequently we can arrive at the following result

$$W^m \equiv \bar{W}^m \bmod G.$$

THEOREM 1.2. *Let H_r and $\bar{H}_r$ be the r-th mean curvature of W^m and $\bar{W}^m$ respectively. Assume that the second fundamental form of $W^m(t)$, $0 \leq t \leq 1$, is positive definite. If the relation*

$$H_r = \bar{H}_r$$

holds for each point $p \in W^m$, then W^m and $\bar{W}^m$ are congruent mod G.

Proof. We consider the following differential form of degree $m - 1$ attached to each point p on the hypersurface $W^m(t)$

$$\begin{aligned}
\big((n', \delta_1\tau, \underbrace{\delta n, \ldots, \delta n}_{r-1}, dx, \ldots, dx)\big) &\overset{\text{def.}}{\equiv} \sqrt{g}\big(n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx\big) \\
&= (-1)^{r-1} \sqrt{g}\big(n', \delta_1\tau, x_{\alpha_1}, \ldots, x_{\alpha_{r-1}}, x_{\beta_r} \cdots x_{\beta_{m-1}}\big) \\
&\quad \times b_{\beta_r}^{\alpha_r}(t) \ldots b_{\beta\partial_{-1}}^{\alpha_\nu - 1}(t) \, du^{\beta_1} \wedge \ldots \wedge du^{\beta_{r-1}} \wedge du^{\beta_r} \wedge \ldots \wedge du^{\beta_{m-1}}
\end{aligned} \qquad (1.24)$$

The exterior differential of the differential form (1.24) becomes as follows

$$d\big((n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)\big) = \big((\delta n', \delta_1\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)\big)$$
$$+ \big((n', \delta(\delta_1)\,\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)\big) + \big((n', \delta_1 \, d\tau, \delta n, \ldots, \delta n, dx, \ldots, dx)\big)$$

because since S^{m+1} is a space of constant curvature, it follows that

$$\big((n', \delta_1\tau, \delta\delta n, \ldots, \delta n, dx, \ldots, dx)\big) = 0,$$

and also we have

$$((n', \delta_1 \tau, \delta n, ..., \delta n, \delta dx, ..., dx)) = 0.$$

Making use of (1.8), we have

$$
\begin{aligned}
&((n', \delta_1 \, d\tau, \delta n, ..., \delta n, dx, ..., dx)) \\
&= (-1)^{r-1} g^{\alpha\beta}(t) \tau_\alpha n_i(t) \delta_1^i \left((\delta_1 \tau_\gamma, x_\beta, x_{\alpha_1}, ..., x_{\alpha_{r-1}}, x_{\alpha_r}, ..., x_{\alpha_{m-1}})\right) \\
&\quad \times b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{r-1}}^{\alpha_{r-1}}(t) \, du^\gamma \wedge du^{\beta_1} \wedge ... \wedge du^{\beta_{r-1}} \wedge du^{\alpha_r} \\
&\quad \wedge ... \wedge du^{\alpha_{m-1}} \\
&= (-1)^{r-1} g^{\alpha\beta}(t) \, \varepsilon_{\beta\alpha_1...\alpha_{r-1}\alpha_r...\alpha_{m-1}} \varepsilon^{\gamma\beta_1...\beta_{r-1}\alpha_r...\alpha_{m-1}} \\
&\quad \times b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{r-1}}^{\alpha_{r-1}}(t) \left(n_i(t) \delta_1^i\right)^2 \tau_\alpha \tau_\gamma \, dA(t).
\end{aligned}
\tag{1.25}
$$

On the other hand, from (1.10) we get

$$
\begin{aligned}
&((\delta n', \delta_1 \tau, \delta n, ..., \delta n, dx, ..., dx)) \\
&= (((\delta n)', \delta_1 \tau, \delta n, ..., \delta n, dx, ..., dx)) \\
&\quad - ((\Gamma_{j1} n^j(t) \tau_\gamma \, du^\gamma, \delta_1 \tau, \delta n, ..., \delta n, dx, ..., dx)).
\end{aligned}
\tag{1.26}
$$

And after some calculations, we have

$$(-1)^r m! H'_r n_i(t) \delta_1^i \, dA(t) = r((\delta_1, (\delta n)', \delta n, ..., \delta n, dx, ..., dx)), \tag{1.27}$$

because $n_i(t) \delta_1^i \, dA(t)$ is independent of t and $dx'^i = \delta_1^i \, d\tau$, that is, the same direction to δ_1. Moreover we can prove similarly the following relation as the proof of (1.15)

$$
\begin{aligned}
&((n', \delta(\delta_1) \tau, \delta n, ..., \delta n, dx, ..., dx)) \\
&\quad - ((\Gamma_{j1} n^j(t) \tau_\gamma \, du^\gamma, \delta_1 \tau, \delta n, ..., \delta n, dx, ..., dx)) = 0.
\end{aligned}
\tag{1.28}
$$

Then putting

$$(m-1)! \, c_{(r)}^{\alpha\beta} = \varepsilon_{\alpha_1...\alpha_{r-1}\alpha_r...\alpha_{m-1}}^\alpha \varepsilon^{\beta\beta_1...\beta_{r-1}\alpha_r...\alpha_{m-1}} b_{\beta_1}^{\alpha_1}(t) ... b_{\beta_{r-1}}^{\alpha_{r-1}}(t)$$

and using (1.25), (1.26), (1.27) and (1.28), we have

$$
\begin{aligned}
&d((n', \delta_1 \tau, \delta n, ..., \delta n, dx, ..., dx)) \\
&= \frac{(-1)^{r-1}}{r} m! H'_r n_i(t) \delta_1^i \tau \, dA(t) \\
&\quad + (-1)^{r-1} (m-1)! \, c_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta \left(n_i(t) \delta_1^i\right)^2 dA(t).
\end{aligned}
\tag{1.29}
$$

Integrating both members of (1.29) over the interval $0 \le t \le 1$, and putting

$$C_{(r)}^{\alpha\beta} = g^*(0)^{1/2} \int_0^1 g^*(t)^{-1/2} \, c_{(r)}^{\alpha\beta} \, dt,$$

we have

$$
\begin{aligned}
&m(\bar{H}_r - H_r) n_i(0) \delta_1^i \tau dA(0) + rC_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta \left(n_i(0) \delta_1^i\right)^2 dA(0) \\
&= \frac{r(-1)^{r-1}}{(m-1)!} d \int_0^1 ((n', \delta_1 \tau, \delta n, ..., \delta n, dx, ..., dx)) \, dt.
\end{aligned}
\tag{1.30}
$$

Furthermore integrating both members of (1.30) over W^m and applying Stokes' theorem

$$\frac{m}{r} \iint_{W^m} (\bar{H}_r - H_r) \, n_i(0) \, \delta_1^i \tau \, dA(0) + \iint_{W^m} (n_i(0) \, \delta_1^i)^2 \, C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta \, dA(0)$$

$$= \frac{(-1)^{r-1}}{(m-1)!} \int_{\partial W^m} \int_0^1 ((n', \delta_1\tau, \delta n, ..., \delta n, dx, ..., dx)) \, dt \,.$$

Since W^m is closed, we have

$$\frac{m}{r} \iint_{W^m} (\bar{H}_r - H_r) \, n_i(0) \, \delta_1^i \tau \, dA(0) + \iint_{W^m} (n_i(0) \, \delta_1^i)^2 \, C_{(r)}^{\alpha\beta} \tau_\alpha \tau_\beta \, dA(0) = 0 \,,$$

using the hypotheses $H_r = \bar{H}_r$ and that the second fundamental form of $W^m(t)$, $0 \le t \le 1$, is positive definite, and from that two hypersurfaces W^m and $\bar{W}^m$ do not contain a piece of a hypersurface covered by the orbits of transformations, we can arrive at

$$\tau_\alpha = 0$$

for all points of W^m, consequently we have

$$\tau = \text{constant}$$

for all points of W^m. Accordingly we can see the following result

$$W^m = \bar{W}^m \bmod G \,.$$

This proof follows to the method of that due to K. Voss [2].

Remark. In an euclidean space, if G is translation group, that is, a special isometric transformation group, Theorem 1.1 and Theorem 1.2 just coincide with theorems of K. Voss given in the introduction.

REFERENCES

[1] HOPF, H., and VOSS, K., *Ein Satz aus der Flächentheorie im Grossen*, Archiv. der Math. *3* (1952), 187–192.

[2] VOSS, K., *Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen*, Math. Ann. *131* (1956), 180–218.

[3] AEPPLI, A., *Einige Ähnlichkeits- und Symmetriesätze für differenzierbare Flächen im Raum*, Comment. Math. Helv. *33* (1959), 174–195.

[4] KATSURADA, Y. *Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part I: Method based on Stokes' Theorem)*, Comment. Math. Helv. *43* (1968), 176–194.

[5] HOPF H., and KATSURADA, Y., *Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part II: Method based on a Maximum Principle)*, Comment. Math. Helv. *43* (1968), 217–223.

[6] EISENHART, L. P. *Continuous Groups of Transformations*, (Princeton, London, 1934).