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## The Spectra of Hyponormal Integral Operators<sup>1)</sup>

K. F. CLANCEY and C. R. PUTNAM

1. Recall that a bounded operator  $T = H + iJ$  on a Hilbert space  $\mathfrak{H}$  is said to be hyponormal if

$$T^*T - TT^* = D \geq 0, \text{ that is, } HJ - JH = -iC, C = \frac{1}{2}D \geq 0. \quad (1.1)$$

It is known that such operators behave to some extent like normal operators; in particular,  $\text{sp}(H)$  and  $\text{sp}(J)$  are just the (real) projections of  $\text{sp}(T)$  onto the real and imaginary axes; see Putnam [5b], p. 46.

Let  $H$  have the spectral resolution

$$H = \int \lambda dE_\lambda, \quad (1.2)$$

and let  $E(\Delta)$  be the projection operator associated with an open interval  $\Delta$ . For any bounded operator  $T$  (hyponormal or not), let  $T_\Delta = E(\Delta)TE(\Delta)$ , regarded as an operator on  $E(\Delta)\mathfrak{H}$  and with spectrum  $\text{sp}(T_\Delta)$ . Since  $H_\Delta J_\Delta - J_\Delta H_\Delta = -iC_\Delta$ , it is seen that  $T_\Delta$  is hyponormal on  $E(\Delta)\mathfrak{H}$  whenever  $T$  is hyponormal on  $\mathfrak{H}$ . It was shown in [5d] that if  $T$  is hyponormal, then

$$\text{sp}(T_\Delta) \subset \text{sp}(T). \quad (1.3)$$

In case the self-commutator  $D$  of  $T$  in (1.1) is compact, the relation (1.3) was proved by Clancey [2a].

A refinement of (1.3) was proved in [5f] to the following

$$\text{sp}(T_\Delta) \cap \{z: \text{Re}(z) \in \Delta\} = \text{sp}(T) \cap \{z: \text{Re}(z) \in \Delta\}, \quad (1.4)$$

$\Delta$  being any open interval. In view of the projection properties mentioned above, the real part of  $\text{sp}(T_\Delta)$  lies in the closure of  $\Delta$ . It was noted in [5f] that, as a consequence of (1.4),

$$\text{Im}[\text{sp}(T) \cap \{z: \text{Re}(z) = s\}] = \bigcap_{\Delta} \text{sp}(E(\Delta)JE(\Delta)), \quad s \in \Delta, \quad (1.5)$$

the intersection being over all open intervals  $\Delta$  containing  $s$ . This relation will be used below to determine the spectra of certain singular integral operators.

Suppose that

$$a(x), b(x) \in L^\infty(E), a(x) \text{ real, } b(x) \neq 0 \text{ a.e. on } E, \quad (1.6)$$

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where  $E$  is a bounded set of positive measure on the real line. Let  $T_0 = H_0 + iJ_0$  denote the bounded operator on  $L^2(E)$  defined by

$$(H_0f)(x) = xf(x) \quad \text{and} \quad (J_0f)(x) = - \left[ a(x)f(x) + \frac{b(x)}{i\pi} \int_E \frac{\bar{b}(t)}{t-x} f(t) dt \right], \quad (1.7)$$

where the integral is interpreted as a Cauchy principal value. It is easily verified that

$$H_0J_0 - J_0H_0 = -iC_0, \quad C_0f = \pi^{-1}(f, b)b, \quad (1.8)$$

so that  $C_0 \geq 0$  and hence  $T_0$  is hyponormal. It is seen that the range of  $C_0$  is spanned by the vector  $b \in L^2(E)$  and that  $H_0 = x$  has simple spectrum and that the vectors  $\{H_0^n b\}$ ,  $n=0, 1, 2, \dots$  span  $L^2(E)$ .

Conversely, if  $T = H + iJ$  is any hyponormal operator on  $H$  satisfying

$$T^*T - TT^* = D \geq 0 \quad \text{and } D \text{ has rank one} \quad (1.9)$$

and

$$D = (\cdot, z)z \text{ and } \{H^n z\}, \quad n = 0, 1, 2, \dots, \text{ span } H, \quad (1.10)$$

then  $T$  is unitarily equivalent to a singular integral operator  $T_0 = H_0 + iJ_0$  defined by (1.7). This result was first proved by Xa Dao-xeng [7]; a simpler proof using a result in [5a] was given by Rosenblum [6], p. 326.

It may be noted that the operator  $T_0$  above is irreducible by virtue of the condition that  $b(x) \neq 0$  a.e. on  $E$ . To see this, note that if  $\Omega \neq 0$  reduces  $T_0$ , then  $\Omega$  reduces both  $H_0$  and  $J_0$ . If  $f \in \Omega$ ,  $f \neq 0$  (that is,  $f(x) \neq 0$  a.e.) and if  $(f, b) \neq 0$ , then  $(C_0f)(x) = \pi^{-1}(f, b)b(x) \neq 0$  a.e. on  $E$ , and hence  $\{(H_0^n C_0f)(x)\}$ ,  $n=0, 1, 2, \dots$ , span the space  $L^2(E)$ , that is,  $\Omega = L^2(E)$ . If  $(f, b) = 0$ , then, since  $f \neq 0$ ,  $C_0H_0^N f \neq 0$  for some positive integer  $N$ . Otherwise, by Weierstrass' theorem,  $f(x)b(x) \equiv 0$  a.e. and hence,  $f(x) = 0$  a.e., a contradiction. Thus, if  $g = H_0^N f \neq 0$ , one can proceed as above to show that  $\Omega = L^2(E)$ .

**THEOREM 1.** *Let  $T_0 = H_0 + iJ_0$  be the hyponormal operator on  $L^2(E)$  defined by (1.6) and (1.7). Then  $\text{sp}(T_0)$  is the set of numbers  $z = s + it$  ( $s, t$  real) for which*

$$\text{meas}_1 \{x \in E \cap \Delta : -a(x) - |b(x)|^2 - \varepsilon < t < -a(x) + |b(x)|^2 + \varepsilon\} > 0 \quad (1.11)$$

*holds for every  $\varepsilon > 0$  and for every open interval  $\Delta$  containing  $s$ .*

**THEOREM 2.** *Let  $T_0$  be defined as in Theorem 1. Then for almost all points  $x \in E$ , there exists some vertical segment  $\{x + iy : a_x \leq y \leq b_x\}$ , where  $a_x < b_x$ , belonging to the spectrum of  $T_0$ . In particular,  $\text{sp}(T_0)$  cannot be totally disconnected.*

Theorem 1 generalizes results of Clancey [2a], Theorem 1 and Putnam [5c]. Its proof will be given in section 2. In a formulation involving a “determining set” or “determining function”, Theorem 1 is contained in Clancey [2b] and Pincus [3c]. All of these proofs, including the one of the present paper, use results of either Pincus [3a] or Rosenblum [6] together with the relation (1.4) (or (1.5)) established in [5f]. It may also be noted that in [3c], the operator  $D$  of (1.9) is assumed only to be of trace class, rather than of rank one, and that  $\mathfrak{H}$  is the least subspace reducing  $T$  and containing the range of  $D$ .

A hyponormal operator  $T$  is said to be completely hyponormal on  $\mathfrak{H}$  if there is no non-trivial subspace of  $\mathfrak{H}$  which reduces  $T$  and on which  $T$  is normal. A set  $S$  of the complex plane is said to have positive density if for every open disk  $N$ ,

$$\text{meas}_2(S \cap N) > 0 \quad \text{whenever } S \cap N \neq \emptyset. \tag{1.12}$$

It was shown in [5d] that if  $T$  is completely hyponormal then its spectrum has positive density. The converse question of whether every compact set  $S$  is the spectrum of some completely hyponormal operator is unsettled, although some partial results have been obtained; see [5g], also Theorem 3 below and the remarks in section 4.

For any set  $S$ , let  $S^-$  denote its closure and  $\text{int}(S)$  its interior. There will be proved the following

**THEOREM 3.** *If  $S$  is any compact set for which*

$$S = (\text{int}(S))^- \tag{1.13}$$

*(so that, in particular,  $S$  has positive density), then there exists a singular integral operator  $T_0 = H_0 + iJ_0$  defined by (1.6) and (1.7) for which*

$$\text{sp}(T_0) = S. \tag{1.14}$$

**2. Proof of Theorem 1.** It follows from Pincus [3a], p. 375, that  $t \in \text{sp}(J_0)$ , where  $J_0$  is defined by (1.7), if and only if

$$\text{meas}_1 \{x \in E: -a(x) - |b(x)|^2 - \varepsilon < t < -a(x) + |b(x)|^2 + \varepsilon\} > 0$$

for every  $\varepsilon > 0$ . (In this connection, see also Rosenblum [6], p. 323; also the remarks in Pincus and Rovnyak [4], p. 620.) If the multiplication operator  $H = H_0 = x$  of (1.7) has the spectral resolution (1.2) then for any open interval  $\Delta$  (for which  $E \cap \Delta \neq \emptyset$ ),  $E(\Delta) J_0 E(\Delta)$  is simply the integral operator  $J_0$  restricted to  $E \cap \Delta$ . It follows that the condition  $t \in \text{sp}(E(\Delta) J_0 E(\Delta))$  reduces to (1.11), and Theorem 1 now follows from (1.5).

*Proof of Theorem 2.* Since  $b(x) \neq 0$  a.e. on  $E$ , then

$$E = \bigcup_{n=1}^{\infty} E_n, \text{ a.e., where } E_n = \{x \in E : |b(x)|^2 > 1/n\} \text{ for } n = 1, 2, \dots$$

Hence,  $E_1 \subset E_2 \subset \dots$  and  $\text{meas}_1(E - E_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Choose  $N$  so large that  $\text{meas}_1(E_n) > 0$  for  $n \geq N$ . Thus, at almost all  $x \in E_n$ , where  $n \geq N$ ,  $E_n$  has metric density 1. For such an  $x$ , let  $L = \text{ess lim sup } a(t)$ , where  $t \rightarrow x$  and  $t$  is restricted to  $E_n$ . Then, in every open interval containing  $x$  and for every  $\varepsilon > 0$ , there exists a subset of  $E$  of positive measure for which  $|a(x) - L| < \varepsilon$  and  $|b(x)|^2 > 1/N$ . It follows from the criterion of (1.11) that the segment  $x + iy$ , where  $L - 1/N \leq y \leq L + 1/N$ , belongs to the spectrum of  $T_0$ .

**3. Proof of Theorem 3.** For any Borel set  $\alpha$  of the line, let  $S(\alpha)$  denote the set  $S(\alpha) = S \cap \{z : \text{Re}(z) \in \alpha\}$ . For  $k = 1, 2, \dots$ , let  $\Pi_k$  denote a grid of squares in the complex plane with sides parallel to the axes and of length  $2^{-k}$ . We assume that the squares contain their lower and left sides and that  $z = 0$  is a lower left corner of some square in each grid. Since  $S$  is compact then the projection on the  $x$ -axis of  $S$  is contained in some interval  $[c, d]$ . Now choose a disjoint family  $\{K_p\}$ ,  $p = 1, 2, \dots$ , of Cantor sets of positive measure in  $[c, d]$  so that

$$\text{meas}_1\left(\bigcup_{p=1}^q K_p\right) \rightarrow d - c \text{ as } q \rightarrow \infty. \tag{3.1}$$

Denote by  $R_1, \dots, R_{n_1}$  the elements of  $\Pi_1$  satisfying

$$R_j \subset \text{int}(S) \equiv \Omega_1, \quad j = 1, \dots, n_1, \tag{3.2}$$

and let  $R'_1, \dots, R'_{n_1}$  be respective smaller concentric closed squares of side  $2^{-2}$ . Then for  $j = 1, \dots, n_1$ , let  $K_{p_j}$  be the first  $K_p$  satisfying

$$\text{meas}_2(S(K_{p_j}) \cap R_j) > 0 \quad \text{and} \quad p_j > p_{j-1}. \tag{3.3}$$

Set  $A_j = S(K_{p_j}) \cap R'_j$  and let  $D_j$  be the projection on the  $x$ -axis of  $A_j$ . Clearly, the set  $\Omega_2 = \Omega_1 - \bigcup_{j=1}^{n_1} A_j$  is open. Denote by  $R_j$ , for  $j = n_1 + 1, \dots, n_1 + n_2$ , the squares in  $\Pi_2$  satisfying

$$R_j \subset \Omega_2, \quad j = n_1 + 1, \dots, n_1 + n_2. \tag{3.4}$$

Again, form concentric squares  $R'_{n_1+1}, \dots, R'_{n_1+n_2}$  of side  $2^{-3}$  and, for  $j = n_1 + 1, \dots, n_1 + n_2$ , let  $K_{p_j}$  be the first  $K_p$  satisfying (3.3). Repeat the process of forming  $A_j$  and  $D_j$  for  $j = n_1 + 1, \dots, n_1 + n_2$  and set  $\Omega_3 = \Omega_2 - \bigcup_{j=1}^{n_1+n_2} A_j$ . If this process is continued for each  $q$  and grid  $\Pi_q$  one obtains a family of closed sets  $\{A_j\}$ ,  $j = 1, 2, \dots$ , satisfying

$$\text{closure}\left(\bigcup_{j=1}^{\infty} A_j\right) = S. \tag{3.5}$$

Now define functions  $a(x)$  and  $b(x)$  on  $\bigcup D_j$  by setting

$$\left. \begin{aligned} - a(x) &= (\text{value of } y\text{-coordinate of the center of } R'_j) \text{ on } D_j, \\ b(x) &= (\text{one-half the length of the side of } R'_j)^{1/2} \text{ on } D_j. \end{aligned} \right\} \quad (3.6)$$

Then if  $T_0$  is the singular integral operator given by (1.7) and (3.6) acting on  $L^2(\bigcup D_j)$  it follows from Theorem 1 that relation (1.14) holds.

**4. Remarks.** It was shown in [5g] that there exist irreducible hyponormal operators satisfying (1.9) and having totally disconnected spectra. (An example was also given in [5e].) In view of the last part of Theorem 1, such an operator  $T=H+iJ$  cannot be of the type  $T_0=H_0+iJ_0$  defined by (1.6) and (1.7). That is, by the result of Xa Dao-xeng, since  $T$  satisfies (1.9), then relation (1.10) fails to hold.

It was shown in Theorem 3 that any compact set equal to the closure of its interior is the spectrum of some singular integral operator  $T_0=H_0+iJ_0$  defined by (1.6) and (1.7). Of course, the spectrum of a general such operator need not be of this type; indeed, if  $a(x)=0$  and if  $b(x)$  is the characteristic function of a Cantor set  $E$  of positive measure, then (cf. Theorem 1) the spectrum of  $T_0$  is the set  $E \times [-1, 1]$ .

It is interesting to note that although the spectrum of  $T_0$  cannot be totally disconnected, nevertheless, it may be a Mergelyan Swiss cheese. (Recall that this is a set  $X=D-\bigcup_{n=1}^{\infty} D_n$  where  $D$  is the closed unit disk and the  $D_n$  are open disjoint disks in  $D$  with radii  $r_n$  satisfying  $\sum r_n < \infty$ , and for which  $X$  is nowhere dense; see Zalcman [8], p. 69.) The proof of this assertion depends upon a result of W. K. Allard (see Brennan [1], p. 13) that almost every cross-section of a Swiss cheese is the union of a finite number of disjoint closed intervals; for details, see Clancey [2b].

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