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On a Marinescu Structure on $\mathscr{C}(X)$

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0.1. Introduction

For any completely regular topological space X the **R**-algebra $\mathscr{C}(X)$, the set of all continuous real-valued functions of X endowed with the pointwise defined operations, can be represented as the union of subalgebras, each of which is canonically identified with $\mathscr{C}(Y)$ for some locally compact space Y. On each of those subalgebras $\mathscr{C}(Y)$ there is a natural topology, namely the topology of compact convergence.

The collection of all filters on $\mathscr{C}(X)$ which have as a basis a convergent filter in one of those subalgebras, defines a certain type of convergence structure (Limitierung [1]) on $\mathscr{C}(X)$, a so called Marinescu structure. The algebra $\mathscr{C}(X)$ endowed with this structure is referred to as $\mathscr{C}_I(X)$.

A well-written study of Marinescu structures can be found in [7].

The purpose of this note is to give a description of $\mathscr{C}_I(X)$. Here we state some of the properties of $\mathscr{C}_I(X)$.

The evaluation map ω from the cartesian product $\mathscr{C}_I(X)\times X$ into the reals is continuous. Assigning to each set $A\subset X$ the set

$$\{f\in\mathscr{C}(X)\mid f(q)=0 \quad \text{for all } q\in A\},$$

we obtain a one-to-one correspondence between the collection of all closed (proper) ideals in $\mathscr{C}_I(X)$ and all non-empty closed subsets of X. In particular, every closed maximal ideal consists of all functions in $\mathscr{C}(X)$ vanishing on a fixed point p in X.

Of some interest to us is the initial topology on $\mathscr{C}(X)$ determined by all continuous seminorms of $\mathscr{C}_I(X)$. This topology turns out to be the topology of compact convergence.

As a consequence $\mathscr{C}_I(X)$ and $\mathscr{C}_{co}(X)$, the **R**-vector space $\mathscr{C}(X)$ together with above mentioned topology, have the same dual spaces. In addition we find that the properties of $\mathscr{C}_I(X)$ listed so far hold also for $\mathscr{C}_c(X)$, the algebra $\mathscr{C}(X)$ equipped with the continuous convergence structure [1]. We therefore investigate whether $\mathscr{C}_I(X)$ and $\mathscr{C}_c(X)$ are identical. On a space X having a countable neighbourhood basis for each point, the identity of $\mathscr{C}_I(X)$ and $\mathscr{C}_c(X)$ is equivalent to the local compactness of X. This is a corollary to the more general theorem 8.

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[1]) Parts of this paper are contained in the thesis of the second author.

1.1. Definition of the Convergence Structure

Let X be a completely regular topological space. We denote the Stone-Čech compactification of X by βX. It is well-known that every continuous map from X into a compact space C can be extended to a continuous map from βX into C. Since X is a dense subspace of βX, this extension is unique.

By $\mathscr{C}(X)$, we mean the $\mathbf{R}$-algebra of all continuous real-valued functions on X (under the pointwise defined operations). Every function f in $\mathscr{C}(X)$ can be regarded as a map from X into $\dot{\mathbf{R}}$, the one-point compactification of the reals. Hence we can extend f to a function from βX into $\dot{\mathbf{R}}$. Clearly if f is bounded, then the extension is still real-valued. For any $f \in \mathscr{C}(X)$, the extension of f to βX, as a function with values in $\dot{\mathbf{R}}$, is again denoted by f. Let $K_f \subset \beta X$ be the pre-image under f of the point $\infty \in \dot{\mathbf{R}}$. Since $f: \beta X \to \dot{\mathbf{R}}$ is continuous, K_f is a compact subset of βX. The function f restricted to X is of course real-valued, and thus K_f must be a subset of $\beta X \backslash X$, the complement of X in βX. For any space Y such that

$$X \subset Y \subset \beta X,$$

we identify each continuous real-valued function on Y with its restriction to X. Therefore given any compact set $K \subset \beta X \backslash X$, the algebra $\mathscr{C}(\beta X \backslash K)$ is contained in $\mathscr{C}(X)$. In particular, the subalgebra $\mathscr{C}(\beta X \backslash K_f)$ contains f. We now conclude that

$$\mathscr{C}(X) = \bigcup_{K \subset \beta X \backslash X} \mathscr{C}(\beta X \backslash K),$$

where K ranges through all compact subsets of $\beta X \backslash X$.

By $\mathscr{C}_{co}(\beta X \backslash K)$, we mean the algebra $\mathscr{C}(\beta X \backslash K)$ endowed with the topology of compact convergence. The convergence structure, being the subject of our investigation, is the finest of all convergence structures on $\mathscr{C}(X)$ making the inclusion maps from $\mathscr{C}_{co}(\beta X \backslash K)$ into $\mathscr{C}(X)$ continuous for every compact subset $K \subset \beta X \backslash X$. We denote the algebra $\mathscr{C}(X)$ together with this convergence structure by $\mathscr{C}_I(X)$, and notice that this is simply the inductive limit, in the category of convergence spaces, (see [7]) of the family

$$\{\mathscr{C}_{co}(\beta X \backslash K) : K \text{ a compact subset of } \beta X \backslash X\} \qquad (*)$$

with the ordering defined by inclusion. Of course the inclusion map from $\mathscr{C}_{co}(\beta X \backslash K)$ into $\mathscr{C}_{co}(\beta X \backslash K')$ is continuous whenever K is contained in K'. Since all the spaces considered in $(*)$ are locally convex topological $\mathbf{R}$-algebras, $\mathscr{C}_I(X)$ is indeed a Marinescu space as introduced by H. Jarchow in [7]. We leave it to the reader to verify that $\mathscr{C}_I(X)$ is a convergence $\mathbf{R}$-algebra [1], meaning that the operations are continuous.

1.2. Completeness of $\mathscr{C}_I(X)$

A filter Θ on a commutative convergence group G is called Cauchy if $\Theta - \Theta$ converges to zero, where "$-$" denotes the difference operation in G. If every Cauchy filter in G converges to some element in G, then the group is said to be complete.

THEOREM 1. *For any completely regular topological space X, the convergence algebra $\mathscr{C}_I(X)$ is complete.*

Proof. Let Θ be a Cauchy filter on $\mathscr{C}_I(X)$. We must find a function $t \in \mathscr{C}_I(X)$ such that Θ converges to t. Here, we remark that a filter Ψ on $\mathscr{C}_I(X)$ converges to a function g in $\mathscr{C}_I(X)$ if and only if there is a compact $K \subset \beta X \backslash X$ such that $\mathscr{C}(\beta X \backslash K)$ contains g and Ψ has a base in $\mathscr{C}_{co}(\beta X \backslash K)$ which is a filter convergent to g in this space. Now the filter $\Theta - \Theta$ has a base Φ in $\mathscr{C}_{co}(\beta X \backslash K)$ with Φ convergent to zero for some compact $K \subset \beta X \backslash X$. Hence any element A of Φ contains $M - M$ where $M \in \Theta$. We will show that M itself is in $\mathscr{C}(\beta X \backslash K')$ for some compact $K' \subset \beta X \backslash X$. Let g be a fixed element in M. For each $f \in M$, the function $f - g$ is in $M - M$, and thus in $\mathscr{C}(\beta X \backslash K)$. This means that

$$f^{-1}(\infty) \subset g^{-1}(\infty) \cup K.$$

Therefore M is contained in $\mathscr{C}(\beta X \backslash K')$ where K' stands for $g^{-1}(\infty) \cup K$. It follows that Θ has a base in $\mathscr{C}(\beta X \backslash K')$, call it Θ'. Since

$$\mathscr{C}(\beta X \backslash K) \subset \mathscr{C}(\beta X \backslash K'),$$

the filter $\Theta' - \Theta'$ on $\mathscr{C}_{co}(\beta X \backslash K')$ has Φ as a base, and thus Θ' is a Cauchy filter in $\mathscr{C}_{co}(\beta X \backslash K')$. The completeness of $\mathscr{C}_{co}(\beta X \backslash K')$ implies that Θ' itself converges to some function $t \in \mathscr{C}(\beta X \backslash K')$. Hence Θ converges to t in $\mathscr{C}_I(X)$ as desired.

1.3. Closed Ideals in $\mathscr{C}_I(X)$

By an ideal, we mean of course a *proper* ideal. It is evident that for every non-empty subset S of X the ideal

$$I(S) = \{f \in \mathscr{C}(X) : f(S) = \{0\}\}$$

is closed in $\mathscr{C}_I(X)$. We conjecture that all closed ideals in $\mathscr{C}_I(X)$ are precisely of this form.

To prove this, let $J \subset \mathscr{C}_I(X)$ be a closed ideal. We call the set of all points $p \in X$ with the property that every function $f \in J$ vanishes on p the null-set of J, and denote this set by $N_X(J)$. It is exactly the intersection of all zero-sets $Z_X(f)$ where f runs through J. By $Z_X(f)$, we mean $\{x \in X : f(x) = 0\}$. Since for any function $f \in J$, there is a bounded function $g \in J$ such that $Z_X(f) = Z_X(g)$, we can represent $N_X(J)$ as

$$\bigcap_{g \in J^\circ} Z_X(g),$$

where J° denotes the collection af all bounded functions in J. Furthermore, the set J° is a closed ideal in $\mathscr{C}_{co}(\beta X)$, and is therefore of the form $I(N_{\beta X}(J^\circ))$ where $N_{\beta X}(J^\circ)$ is a non-empty subset of βX. Evidently the ideal $J \subset I(N_X(J))$. We will show that J is all of $I(N_X(J))$. First, we verify that J° contains all the bounded functions in $I(N_X(J))$. Since J° consists of all functions in $\mathscr{C}(\beta X)$ vanishing on $N_{\beta X}(J^\circ)$, it is enough to prove that any bounded element of $I(N_X(J))$ vanishes on $N_{\beta X}(J^\circ)$. Clearly we are done as soon as we know that $N_{\beta X}(J^\circ)$ is the closure of $N_X(J)$ in βX. Assume, to the contrary, that $N_{\beta X}(J^\circ)$ contains $\overline{N_X(J)}$, the closure in βX of $N_X(J)$, as a proper subset. For a point $q \in N_{\beta X}(J^\circ)$ outside of $\overline{N_X(J)}$, we choose in βX a closed neighborhood U of q disjoint from $\overline{N_X(J)}$. There exists a function $g \in \mathscr{C}(\beta X)$ such that $g(q) = 1$ and g vanishes on the complement of U. We assert that $g \in J \cap \mathscr{C}(\beta X \backslash K)$, where K denotes the compact set $U \cap N_{\beta X}(J^\circ)$ contained in $\beta X \backslash X$. Clearly $J \cap \mathscr{C}(\beta X \backslash K)$ is a closed ideal in $\mathscr{C}_{co}(\beta X \backslash K)$, and therefore consists of all functions vanishing on its null-set. Since the bounded functions in $J \cap \mathscr{C}(\beta X \backslash K)$ are precisely the elements of J°, we conclude that $N_{\beta X}(J^\circ) \cap \beta X \backslash K$ is the null-set of $J \cap \mathscr{C}(\beta X \backslash K)$. The function g vanishes on $N_{\beta X}(J^\circ) \cap \beta X \backslash K$, and therefore g is an element of $J \cap \mathscr{C}(\beta X \backslash K)$ as claimed. Thus we know $g \in J^\circ$. On the other hand, g is not an element of $I(N_{\beta X}(J^\circ))$, which is of course J°. Because of this contradiction, we conclude that $N_{\beta X}(J^\circ) = \overline{N_X(J)}$, and thus J° consists of all bounded functions in $I(N_X(J))$ where $N_X(J)$ is not empty. To complete the proof, let f be an arbitrary element of $I(N_X(J))$. There is a unit u in $\mathscr{C}(X)$ such that $f \cdot u$ is bounded. Hence $f \cdot u \in J^\circ$, and therefore $(f \cdot u) \cdot 1/u \in J$. This implies that $J = I(A_X(J))$.

We now have established

THEOREM 2. *An ideal J in $\mathscr{C}_I(X)$ is closed if and only if $J = I(N_X(J))$.*

COROLLARY 1. *A maximal ideal in $\mathscr{C}_I(X)$ is closed if and only if it consists of all functions in $\mathscr{C}(X)$ vanishing at a fixed point in X.*

For every point $p \in X$ there is a continuous $\mathbf{R}$-algebra homomorphism

$$i_X(p) \colon \mathscr{C}_I(X) \to \mathbf{R},$$

defined by $i_X(p)(f) = f(p)$ for every $f \in \mathscr{C}(X)$. Assigning to each point $p \in X$ the homomorphism $i_X(p)$, we obtain a map

$$i_X \colon X \to \mathscr{H}om\,\mathscr{C}_I(X),$$

where $\mathscr{H}om\,\mathscr{C}_I(X)$ denotes the set of all continuous $\mathbf{R}$-algebra homomorphisms from $\mathscr{C}_I(X)$ onto $\mathbf{R}$. Since an element of $\mathscr{H}om\,\mathscr{C}_I(X)$ is determined by its kernel, a closed maximal ideal in $\mathscr{C}_I(X)$, we deduce from corollary 1

COROLLARY 2. *The map i_X is surjective.*

1.4. The associated locally convex topology of $\mathscr{C}_I(X)$

First, let us demonstrate that, in general, $\mathscr{C}_I(X)$ is not topological; more precisely

THEOREM 3. *$\mathscr{C}_I(X)$ is topological if and only if X is locally compact. If X is locally compact, then $\mathscr{C}_I(X)=\mathscr{C}_{co}(X)$.*

Proof. If X is locally compact, then $\mathscr{C}(X)$ is of the form $\mathscr{C}(\beta X\backslash K)$, where $K=\beta X\backslash X$ is a compact subset of βX. The inclusion map from $\mathscr{C}_{co}(\beta X\backslash K')$ into $\mathscr{C}_{co}(X)$ is continuous for any compact set $K'\subset\beta X\backslash X$. Thus $\mathscr{C}_{co}(X)$ is the finest of all convergence structures making the inclusion maps continuous, i.e., $\mathscr{C}_I(X)$ coincides with $\mathscr{C}_{co}(X)$ and hence is topological.

Conversely, assume that $\mathscr{C}_I(X)$ is topological. Since the neighborhood filter of zero has a base in $\mathscr{C}(\beta X\backslash K)$ for some compact $K\subset\beta X\backslash X$ and every neighborhood of zero is absorbent, we have

$$\mathscr{C}(X)=\mathscr{C}(\beta X\backslash K).$$

If there were a compact $K'\subset\beta X\backslash X$ strictly containing K, then the neighborhood filter of zero in $\mathscr{C}_{co}(\beta X\backslash K')$ would be strictly coarser than the neighborhood filter of zero in $\mathscr{C}_{co}(\beta X\backslash K)$. This is apparent since two locally compact spaces Z and Z' are homeomorphic if and only if $\mathscr{C}_{co}(Z)$ and $\mathscr{C}_{co}(Z')$ are bicontinuously isomorphic (see [3]). Therefore K must be equal to $\beta X\backslash X$ which means X is locally compact.

In view of the fact that $\mathscr{C}_I(X)$ is not, in general, topological, we wish to determine the associated locally convex space $\mathscr{C}_{\tau I}(X)$ of $\mathscr{C}_I(X)$. The topology of $\mathscr{C}_{\tau I}(X)$ is generated by all the continuous seminorms on $\mathscr{C}_I(X)$.

Let

$$p\colon \mathscr{C}_I(X)\to\mathbf{R}$$

be a continuous seminorm. We construct a seminorm $\tilde{p}$ which majorizes p and is more convenient to work with. For a compact set $K\subset\beta X\backslash X$, we denote by p_K the restriction of p to $\mathscr{C}(\beta X\backslash K)$. Clearly

$$p_K\colon \mathscr{C}_{co}(\beta X\backslash K)\to\mathbf{R}$$

is continuous. Therefore we can find a compact set $Q_K\subset\beta X\backslash K$ such that a constant multiple α of the seminorm

$$s_{Q_K}\colon \mathscr{C}_{co}(\beta X\backslash K)\to\mathbf{R},$$

defined by $s_{Q_K}(f)=\sup_{q\in Q_K}|f(q)|$, majorizes p_K. This implies that for any function $f\in\mathscr{C}(\beta X\backslash K)$,

$$\tilde{p}_K(f)=\sup\{p_K(g)\colon |g|\leqslant|f| \quad\text{and}\quad g\in\mathscr{C}(\beta X\backslash K)\}$$

is a real number less than or equal to $\alpha s_{Q_K}(f)$. Since for every function $g\in\mathscr{C}(X)$ the relation $|g|\leqslant|f|$ implies that $g\in\mathscr{C}(\beta X\backslash K)$, we know that

$$\tilde{p}(f)=\sup\{p(g)\colon |g|\leqslant|f| \quad\text{and}\quad g\in\mathscr{C}(X)\}$$

is identical to $\tilde{p}_K(f)$. Of course every function in $\mathscr{C}(X)$ is an element of $\mathscr{C}(\beta X\backslash K)$ for some compact $K\subset\beta X\backslash X$. It is not difficult to verify that the maps

$$\tilde{p}\colon \mathscr{C}_I(X)\to\mathbf{R}$$

and

$$\tilde{p}_K\colon \mathscr{C}_{co}(\beta X\backslash K)\to\mathbf{R} \quad\text{for any compact}\quad K\subset\beta X\backslash K,$$

sending each $f\in\mathscr{C}(X)$ to $\tilde{p}(f)$ and each $f\in\mathscr{C}(\beta X\backslash K)$ to $\tilde{p}_K(f)$ respectively, are seminorms. Since $\tilde{p}$ restricted to $\mathscr{C}(\beta X\backslash K)$ is $\tilde{p}_K$, we conclude that $\tilde{p}$ itself is a continuous seminorm. Furthermore, $\tilde{p}$ has the following properties

$$\tilde{p}(f)=\tilde{p}(|f|) \quad\text{for all}\quad f\in\mathscr{C}(X)$$

and

$$\tilde{p}(f)\leqslant\tilde{p}(g) \quad\text{for all}\quad f,g\in\mathscr{C}(X) \quad\text{with}\quad |f|\leqslant|g|.$$

LEMMA 1. *The kernel P of $\tilde{p}$, the set of all functions $f\in\mathscr{C}(X)$ with $\tilde{p}(f)=0$, is a closed ideal in $\mathscr{C}_I(X)$ consisting of all elements in $\mathscr{C}(X)$ vanishing on a compact subset of X.*

Proof. P is clearly a linear subspace of $\mathscr{C}(X)$. To show it is an ideal, let $g\in P$. For an arbitrary element $f\in\mathscr{C}(X)$, we consider

$$((-\mathbf{n}\vee f)\wedge\mathbf{n})$$

where $\mathbf{n}$ denotes the function of constant value $n\in\mathbf{N}$. Now

$$\tilde{p}(g\cdot((-\mathbf{n}\vee f)\wedge\mathbf{n}))\leqslant\tilde{p}(g\cdot\mathbf{n})=n\cdot\tilde{p}(g)$$

and hence $g\cdot((-\mathbf{n}\vee f)\wedge\mathbf{n})\in P$. The Fréchet filter generated by the sequence

$$(g\cdot((-\mathbf{n}\vee f)\wedge\mathbf{n}))_{n\in\mathbf{N}}$$

converges to $g\cdot f$ in $\mathscr{C}_I(X)$. Since P is obviously closed, $g\cdot f$ is an element of P. Thus P is a closed ideal in $\mathscr{C}_I(X)$, and therefore consists of all functions in $\mathscr{C}(X)$ vanishing on its non-empty null-set $Q\subset X$ (see theorem 2). It only remains to prove that Q is compact. We can express P as the union of the kernels of $\tilde{p}_K$ for all compact $K\subset\beta X\backslash X$.

On the other hand, the kernel P_K of $\tilde{p}_K$ contains the kernel H_K of s_{Q_K}. Hence we have

$$N_{\beta X\setminus K}(P_K) \subset N_{\beta X\setminus K}(H_K).$$

But $N_{\beta X\setminus K}(H_K)$ is nothing else but Q_K. Since Q is contained in the intersection of the null-sets of P_K,

$$Q \subset \bigcap_K Q_K,$$

where K runs through all compact subsets of $\beta X\setminus X$. The fact that $\bigcap_K Q_K$ is a compact subset of X implies that Q is compact.

Next, we will show that $\tilde{p}$ is majorized by a constant multiple of the supremum seminorm s over Q. Let $f \in \mathscr{C}(X)$, and consider

$$g = ((-\mathbf{s}(\mathbf{f}) \vee f) \wedge \mathbf{s}(\mathbf{f})).$$

By the previous lemma, we have

$$\tilde{p}(f-g) = 0.$$

Furthermore,

$$|\tilde{p}(f) - \tilde{p}(g)| \leqslant \tilde{p}(f-g),$$

and hence $\tilde{p}(f) = \tilde{p}(g)$. From the inequality $|g| \leqslant \mathbf{s}(\mathbf{f})$, we conclude that

$$\tilde{p}(f) \leqslant \tilde{p}(\mathbf{s}(\mathbf{f})) = s(f)\,\tilde{p}(\mathbf{1}).$$

Therefore we have proved

THEOREM 4. *The associated locally convex space of $\mathscr{C}_I(X)$ is $\mathscr{C}_{co}(X)$.*

The associated locally convex space of $\mathscr{C}_I(X)$ coincides with the locally convex inductive limit of the family

$$\{\mathscr{C}_{co}(\beta X\setminus K) \colon K \text{ is a compact subset of } \beta X\setminus X\}.$$

Thus we may state

COROLLARY 1. *The locally convex inductive limit of the family*

$$\{\mathscr{C}_{co}(\beta X\setminus K) \colon K \text{ is a compact subset of } \beta X\setminus X\}$$

is $\mathscr{C}_{co}(X)$.

For any convergence vector space E over $\mathbf{R}$, its dual $\mathscr{L}(E)$ is identical with the dual of the associated locally convex space of E. Therefore

COROLLARY 2. $\mathscr{L}(\mathscr{C}_I(X)) = \mathscr{L}(\mathscr{C}_{co}(X))$.

1.5. Functorial Properties of $\mathscr{C}_I(X)$

Let X and Y denote completely regular topological spaces. Every continuous map

$$t: X \to Y$$

induces a homomorphism

$$t^*: \mathscr{C}_I(Y) \to \mathscr{C}_I(X),$$

defined by $t^*(f) = f \circ t$ for every $f \in \mathscr{C}(Y)$. To see that t^* is continuous, we consider the restrictions

$$t_K^*: \mathscr{C}_{co}(\beta Y \backslash K) \to \mathscr{C}_I(X)$$

where t_K^* denotes $t^* \mid \mathscr{C}(\beta Y \backslash K)$, and verify that t_K^* is continuous for every compact set $K \subset \beta Y \backslash Y$. To this end, we extend t to a map

$$\bar{t}: \beta X \to \beta Y.$$

For each compact $K \subset \beta Y \backslash Y$, we know $\bar{t}^{-1}(K)$ is a compact subset of $\beta X \backslash X$. Furthermore, for a compact $K \subset \beta Y \backslash Y$ the map t_K^* is induced by

$$\bar{t} \mid (\beta X \backslash \bar{t}^{-1}(K)): \beta X \backslash \bar{t}^{-1}(K) \to \beta Y \backslash K,$$

which we denote by t_K. That is, $t_K^*(f) = f \circ t_K$ for all $f \in \mathscr{C}(\beta Y \backslash K)$. Clearly

$$t_K^*: \mathscr{C}_{co}(\beta Y \backslash K) \to \mathscr{C}_{co}(\beta X \backslash \bar{t}^{-1}(K))$$

is continuous for every compact $K \subset \beta Y \backslash Y$, and therefore t^* itself is continuous.

On the other hand, let

$$u: \mathscr{C}_I(Y) \to \mathscr{C}_I(X)$$

be a continuous **R**-algebra homomorphism sending unity to unity. We will now show that u is of the form t^* where t maps X into Y continuously. The homomorphism u induces a continuous map

$$u^*: \mathscr{H}om_s \mathscr{C}_I(X) \to \mathscr{H}om_s \mathscr{C}_I(Y)$$

defined by $u^*(h) = h \circ u$ for every $h \in \mathscr{H}om\, \mathscr{C}_I(X)$. The index s denotes the topology of pointwise convergence. Corollary 2 of theorem 2 implies that the map $i_Z: Z \to \mathscr{H}om_s \mathscr{C}_I(Z)$ is a homeomorphism for any completely regular topological space Z. Thus we have a continuous map t from X into Y defined by $t = i_Y^{-1} \circ u^* \circ i_X$. Now it is easy to verify that t^* is equal to u.

To summarize these facts, we state

THEOREM 5. *A homomorphism*

$$u: \mathscr{C}_I(Y) \to \mathscr{C}_I(X)$$

taking unity to unity is continuous if and only if there exists a continuous map $t: X \to Y$ *such that* $u = t^*$.

For maps $t: X \to Y$ and $s: Y \to Z$ between completely regular topological spaces, we have the obvious identities

$$(s \circ t)^* = t^* \circ s^*$$

and

$$\mathrm{id}_X^* = \mathrm{id}_{\mathscr{C}(X)}.$$

1.6. Realcompact Spaces

Let X be a completely regular topological space. As before, the zero-set $Z_{\beta X}(f)$ of a function $f \in \mathscr{C}(\beta X)$ means the set of all points $p \in \beta X$ where f vanishes.

Here, we consider the collection

$$(**) \qquad \{\mathscr{C}_{co}(\beta X \backslash Z_{\beta X}) : Z_{\beta X} \subset \beta X \backslash X \text{ is a zero-set}\}$$

This is a subfamily of the family of all topological algebras $\mathscr{C}_{co}(\beta X \backslash K)$ for K a compact subset of $\beta X \backslash X$. As in section 1.1, it is clear that the union of all $\mathscr{C}(\beta X \backslash Z_{\beta X})$ for $Z_{\beta X}$ a zero-set outside of X is again $\mathscr{C}(X)$. Under the natural ordering (as in section 1.1), the collection $(**)$ is an inductive system, and we denote the inductive limit of this system by $\mathscr{C}_{I'}(X)$.

It is easy to see that $\mathscr{C}_{I'}(X)$ is actually the finest convergence structure on $\mathscr{C}(X)$ obtainable as an inductive limit of a subfamily of the family of all $\mathscr{C}_{co}(\beta X \backslash K)$ for K a compact subset of $\beta X \backslash X$. Of course the identity,

$$(***) \qquad \mathrm{id}: \mathscr{C}_{I'}(X) \to \mathscr{C}_I(X),$$

is continuous. Our main concern in this section is to determine under what conditions this identity is a homeomorphism.

If every compact subset of $\beta X \backslash X$ is contained in a zero-set in $\beta X \backslash X$, then clearly the identity $(***)$ is a homeomorphism. Conversely, assume that

$$\mathrm{id}: \mathscr{C}_I(X) \to \mathscr{C}_{I'}(X)$$

is continuous. Therefore we have a continuous injection

$$\mathrm{id}^*: \mathscr{H}om_s \mathscr{C}_{I'}(X) \to \mathscr{H}om_s \mathscr{C}_I(X),$$

where $\mathscr{H}om_s \mathscr{C}_{I'}(X)$ denotes the set of all continuous $\mathbf{R}$-algebra homomorphism from $\mathscr{C}_{I'}(X)$ onto $\mathbf{R}$ together with the topology of pointwise convergence. For both X and its Hewitt realcompactification vX the convergence algebras $\mathscr{C}_{I'}(X)$ and $\mathscr{C}_{I'}(vX)$ are identical, since any zero-set contained in $\beta X\backslash X$ is already contained in $\beta X\backslash vX$ (see [6], p. 118). Thus

$$\mathscr{H}om_s\mathscr{C}_{I'}(X) = \mathscr{H}om_s\mathscr{C}_{I'}(vX).$$

In view of (I), we conclude that the map

$$i_{vX}\colon vX \to \mathscr{H}om_s\mathscr{C}_{I'}(X)$$

is continuous. This tells us that $\mathrm{id}^* \circ i_{vX}$ maps vX injectively into $\mathscr{H}om_s\mathscr{C}_I(X)$, which is homeomorphic to X. Hence X must be realcompact.

To continue our investigation, without loss of generality we can regard X as a realcompact space. Since by assumption

$$\mathrm{id}\colon \mathscr{C}_I(X) \to \mathscr{C}_{I'}(X)$$

is continuous, we know that the inclusion map from $\mathscr{C}_{co}(\beta X\backslash K)$ into $\mathscr{C}_{I'}(X)$ is continuous for any compact $K \subset \beta X\backslash X$. Thus the neighborhood filter of zero in $\mathscr{C}_{co}(\beta X\backslash K)$ has a basis in $\mathscr{C}_{co}(\beta X\backslash Z_{\beta X})$ for some zero-set contained in $\beta X\backslash X$. Because every neighborhood of zero in $\mathscr{C}_{co}(\beta X\backslash K)$ is absorbent, $\mathscr{C}(\beta X\backslash Z_{\beta X}) \supset \mathscr{C}(\beta X\backslash K)$ meaning that $Z_{\beta X} \supset K$. To summarize, we have extablished the following

THEOREM 6. *Let X be a realcompact space. $\mathscr{C}_I(X)$ is identical to $\mathscr{C}_{I'}(X)$ if and only if every compact set in $\beta X\backslash X$ is contained in some zero-set in $\beta X\backslash X$.*

We note that in the case of a realcompact locally compact space X, the convergence algebra $\mathscr{C}_I(X)$ coincides with $\mathscr{C}_{I'}(X)$ if and only if $\beta X\backslash X$ is a zero-set, i.e., X is σ-compact.

More generally, assume that $\mathscr{C}_{I'}(X)$ is topological for a realcompact space X. By arguing as in section 1.4, we conclude that X is of the form $\beta X\backslash Z_{\beta X}$ for some zero-set $Z_{\beta X}$. This means that X is σ-compact and locally compact.

Therefore, we can state

THEOREM 7. *Let X be a realcompact space. The convergence algebra $\mathscr{C}_{I'}(X)$ is topological if and only if X is locally compact and σ-compact.*

As an example of a realcompact space X for which $\mathscr{C}_I(X)$ and $\mathscr{C}_{I'}(X)$ do not coincide, consider the reals together with the discrete topology.

1.7. Universal Representation of $\mathscr{C}_I(X)$

For a completely regular topological space X, the homomorphism

$$d\colon \mathscr{C}_I(X) \to \mathscr{C}_c(\mathscr{H}om_c\mathscr{C}_I(X)),$$

defined by $d(f)(h)=h(f)$ for all $f\in\mathscr{C}(X)$ and all $h\in\mathscr{H}om\,\mathscr{C}_I(X)$, is called the universal representation [2] of $\mathscr{C}_I(X)$. The subscript c indicates the continuous convergence structure (*Limitierung der stetigen Konvergenz* [1]) on the sets $\mathscr{H}om\,\mathscr{C}_I(X)$ and $\mathscr{C}(\mathscr{H}om_c\mathscr{C}_I(X))$.

We first investigate the continuous convergence structure on $\mathscr{H}om\,\mathscr{C}_I(X)$.

The space $\mathscr{H}om_c\mathscr{C}_c(X)$ is homeomorphic to X [3], and thus the continuous convergence structure on $\mathscr{H}om\,\mathscr{C}_c(X)$ is the topology of pointwise convergence. Since the evaluation map

$$\omega\colon \mathscr{C}_I(X)\times X\to\mathbf{R}$$

(defined by $\omega(f,p)=f(p)$ for all $f\in\mathscr{C}(X)$ and all $p\in X$) is continuous, the identity

$$\mathrm{id}\colon \mathscr{C}_I(X)\to\mathscr{C}_c(X)$$

is continuous. Furthermore, the sets $\mathscr{H}om\,\mathscr{C}_I(X)$ and $\mathscr{H}om\,\mathscr{C}_c(X)$ are identical (corollary 2 of theorem 2) which means that

$$\mathrm{id}\colon \mathscr{H}om_c\mathscr{C}_c(X)\to\mathscr{H}om_c\mathscr{C}_I(X)$$

is continuous. On the other hand the identity map from $\mathscr{H}om_c\mathscr{C}_I(X)$ into $\mathscr{H}om_s\mathscr{C}_I(X)$ is clearly continuous (the subscript s indicates the topology of pointwise convergence). It follows that

$$\mathscr{H}om_c\mathscr{C}_I(X)=\mathscr{H}om_s\mathscr{C}_I(X),$$

which is homeomorphic to X via the map i_X defined earlier. Therefore

$$i_X^*\colon \mathscr{C}_c(\mathscr{H}om_c\mathscr{C}_I(X))\to\mathscr{C}_c(X)$$

is a bicontinuous isomorphism, and of course $i_X^*\circ d$ is the identity map on $\mathscr{C}(X)$.

Our main problem is thus to determine whether $\mathscr{C}_I(X)$ and $\mathscr{C}_c(X)$ coincide. So far, we can say the following

THEOREM 8. *Let X be a completely regular topological space. If there is a point q in X having a countable base of neighborhoods and no compact neighborhood, then $\mathscr{C}_c(X)$ cannot be an inductive limit of topological vector spaces over* $\mathbf{R}$.

Proof. Any inductive limit of topological vector spaces over $\mathbf{R}$ has the property that for each filter Φ converging to zero, there exists a coarser filter Φ' convergent to zero with

$$\lambda\cdot\Phi'=\Phi'$$

for every real number λ unequal to zero.

Our aim is to show that under the assumption of the theorem, $\mathscr{C}_c(X)$ fails to satisfy this condition.

Let $\{Q_m\}_{m\in\mathbf{N}}$ be a countable collection of open sets in X that form a base for the neighborhood filter at q. We define inductively a certain system of nested neighborhoods of q. Let $N_1 = X$ and let $\{O_{1,\alpha}\}$ be an open covering of X with no finite subcovering. Set

$$U_1 = O_1^q \cap Q_1,$$

where O_1^q is a member of $\{O_{1,\alpha}\}$ containing q. Assume that the closed respectively open neighborhoods N_i and U_i are defined. Choose N_{i+1} to be a closed neighborhood of q contained in U_i, and let $\{O_{i+1,\alpha}\}$ be a covering of N_{i+1} by open sets in X having no finite subcovering. We pick U_{i+1} to be an open neighborhood of q contained in

$$O_{i+1}^q \cap Q_{i+1} \cap N_{i+1},$$

where O_{i+1}^q is a member of $\{O_{i+1,\alpha}\}$ with $q \in O_{i+1}^q$. With this system of respectively closed and open neighborhoods of q,

$$N_1 \subset U_1 \subset N_2 \subset U_2 \ldots,$$

we construct a filter Θ that does not satisfy the condition mentioned above. Let

$$T_n = \left\{f \in \mathscr{C}(X): f(N_n) \subset \left[\frac{-1}{n}, \frac{1}{n}\right]\right\}$$

and let

$$T_x = \{f \in \mathscr{C}(X): f(W_x) = \{0\}\}$$

for $x \neq q$, where we choose W_x as follows: Since $x \neq q$, the point x lies in N_r but not in N_{r+1} for some natural number r. Let W_x be a closed neighborhood of x contained in

$$\bigcap_{j=1}^{r} O_j^x \cap N_{r+1}$$

where O_j^x is a member of the covering system $\{O_{j,\alpha}\}$ containing x. It is clear that the sets $\{T_n : n \in \mathbf{N}\}$ and $\{T_x : x \in X \text{ and } x \neq q\}$ generate a filter Θ convergent to zero in $\mathscr{C}_c(X)$. Assume that there exists a coarser filter Θ' in $\mathscr{C}_c(X)$ convergent to zero with

$$\lambda \cdot \theta' = \theta'$$

for every real number $\lambda \neq 0$. To the interval $[-1, 1]$, there is a set $F' \in \Theta'$ and a neighborhood N_k of q such that

$$F'(N_k) = \{f(p): f \in F' \text{ and } p \in N_k\}$$

is a subset of $[-1, 1]$. For λ equal to $1/2k$, we have

$$\frac{1}{2k} F'(N_k) \subset \left[\frac{-1}{2k}, \frac{1}{2k}\right],$$

and $(1/2k)\, F' \in \Theta'$. Thus $(1/2k)\, F'$ contains a finite intersection of elements of the form T_n and T_x, say

$$\bigcap_{n \in \tilde{N}} T_n \cap \bigcap_{x \in \tilde{X}} T_x,$$

where $\tilde{N}$ is a finite subset of $\mathbf{N}$ and $\tilde{X}$ is a finite subset of $X \backslash \{q\}$. Now we claim that

$$N_k \not\subset \bigcup_{x \in \tilde{X}} W_x \cup N_{k+1}.$$

Our construction guarantees that for a fixed W_x, either W_x is a subset of the complement of N_k or W_x is contained in an element of the open covering $\{O_{k,\alpha}\}$. Furthermore, N_{k+1} is contained in O_k^q. Since the open covering $\{O_{k,\alpha}\}$ has no finite subcovering, the claim is true. Therefore, we can find a function $g \in \mathscr{C}(X)$ vanishing on $\bigcup_{x \in X} W_x \cup N_{k+1}$ with g taking on the value $1/k$ for some point in N_k and $\|g\| \leqslant 1/k$. This function is certainly not in $(1/2k)\, F'$ but it is in $\bigcup_{n \in \tilde{N}} T_n \cap \bigcup_{x \in \tilde{X}} T_x$, and this contradiction establishes the theorem.

2.1. Consequences for $\mathscr{C}_c(X)$

In this section, we demonstrate consequences of the theory developed in 1.1 to 1.7 in investigating closed ideals in $\mathscr{C}_c(Y)$ for a convergence space Y, and in determining both the associated locally convex topological space of $\mathscr{C}_c(X)$ and the dual space of $\mathscr{C}_c(X)$, where X is a completely regular topological space. The results we obtain can be found in [4] and [5] respectively; however, the proofs given here are simpler than those provided in [4] and [5].

First, we look at closed ideals in $\mathscr{C}_c(Y)$.

Let Y be an arbitrary convergence space. To this space we associate a completely regular topological space as follows: Any two points $p, q \in Y$ are said to be equivalent if $f(p) = f(q)$ for all real-valued continuous functions f. As usual, the set of all these functions is denoted by $\mathscr{C}(Y)$. The quotient set defined by the above equivalence relation is called Y'. Any function $f \in \mathscr{C}(Y)$ defines a function

$$f' : Y' \to \mathbf{R}$$

by sending each $\bar{p} \in Y'$ to $f(p)$. The initial topology induced by the family

$$\{f' : f \in \mathscr{C}(Y)\}$$

is, of course, completely regular. The set Y' together with this topology is again denoted by Y'.

The obvious projection

$$\pi: Y \to Y'$$

induces an isomorphism (with respect to the usual **R**-algebra structure)

$$\pi^*: \mathscr{C}(Y') \to \mathscr{C}(Y)$$

defined by $\pi^*(g) = g \circ \pi$ for all $g \in \mathscr{C}(Y')$. This isomorphism is continuous if both algebras carry the continuous convergence structure. Hence for any closed ideal J in $\mathscr{C}_c(Y)$ (the algebra $\mathscr{C}(Y)$ together with the continuous convergence structure), the ideal $\pi^{*-1}(J) \subset \mathscr{C}_c(Y')$ is closed. Since the identity map,

$$\mathrm{id}: \mathscr{C}_I(Y') \to \mathscr{C}_c(Y')$$

is continuous, we conclude that $\pi^{*-1}(J)$ is closed in $\mathscr{C}_I(Y')$. Therefore, we know by theorem 2 that it is of the form $I(N)$ where $N \subset Y'$ is a closed non-empty subset. It is clear that $I(\pi^{-1}(N)) = J$. Since an ideal of the form $I(M)$ for any non-empty subset of Y is closed in $\mathscr{C}_c(Y)$, we have the following result

THEOREM 9. *For any convergence space Y, an ideal J in $\mathscr{C}_c(Y)$ is closed if and only if it is of the form $I(N_Y(J))$.*

Another application of the theory developed in chapter 1 is the following theorem

THEOREM 10. *Let X be a completely regular topological space. The associated locally convex space of $\mathscr{C}_c(X)$ is $\mathscr{C}_{co}(X)$.*

Proof. Clearly the identity from $\mathscr{C}_c(X)$ into the locally convex topological vector space $\mathscr{C}_{co}(X)$ is continuous. Since

$$\mathrm{id}: \mathscr{C}_I(X) \to \mathscr{C}_c(X)$$

is also continuous, in view of theorem 4 the proof is complete.

By reasoning as in the proof of the last theorem, we obtain

THEOREM 11. *For any completely regular space X the spaces $\mathscr{L}(\mathscr{C}_I(X))$, $\mathscr{L}(\mathscr{C}_c(X))$, and $\mathscr{L}(\mathscr{C}_{co}(X))$ are identical.*

REFERENCES

[1] Binz, E. and Keller, H. H., *Funktionenräume in der Kategorie der Limesräume*, Ann. Acad. Sci. Fenn. A, I, *383* (1966), 1–21.

[2] Binz, E., *Bemerkungen zu limitierten Funktionenalgebren*, Math. Ann. *175* (1968), 169–184.

[3] ——, *Zu den Beziehungen zwischen c-einbettbaren Limesräumen und ihren limitierten Funktionenalgebren*, Math. Ann. *181* (1969), 45–52.

[4] *On Closed Ideals in Convergence Function Algebras,* Math. Ann. *182* (1969), 145–153.
[5] BUTZMANN, H.-P., *Dualitäten in* $\mathscr{C}_c(X)$, Thesis, Universität Mannheim.
[6] GILLMAN, L. and JERISON, M., *Rings of Continuous Functions* (Van Nostrand, Princeton, 1960).
[7] JARCHOW, H., *Marinescu-Räume,* Comment. Math. Helv. *44* (1969), 138–163.

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