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Note on Direct Decompositions of Torsion-free Abelian Groups

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All the groups of this note are torsion-free abelian groups under addition.

Jónsson [5] was the first to point out that torsion-free groups of finite rank may have non-isomorphic direct decompositions into (directly) indecomposable groups. He discovered a few pathological phenomena, and using his techniques, Corner [1] furnished examples, both in the finite and in the countable rank cases, with a surprising flexibility even in the choice of the ranks of the indecomposable summands. For groups of countable rank, Corner [1] proved that the same group can have two, basically different direct decompositions: one with just two indecomposable summands and one with infinitely many components. It is not difficult to find more pathological decompositions (see e.g. Fuchs and Loonstra [4]). Unfortunately, no complete survey is known of the variety of direct decompositions a torsion-free group might have.

The aim of this note is to point out that a countable group can have continuously many, pairwise non-isomorphic, indecomposable summands. Moreover, we are going to prove the following two, more general theorems:

THEOREM 1. *For every infinite cardinal m less than the first strongly inaccessible aleph, there exists a torsion-free group A of rank m such that A has direct decompositions*

$$A = B_j \oplus C_j \quad \text{with} \quad B_j \cong C_j$$

and with j ranging over an index set J of cardinality $|J| = 2^m$ where the B_j are pairwise non-isomorphic and indecomposable.

THEOREM 2. *For every m as in Theorem 1 there is a torsion-free group A of rank m such that*

$$A = B_j \oplus C_j \quad (j \in J)$$

holds for an index set J of cardinality 2^m where all the B_j are indecomposable and isomorphic among themselves, while the C_j are indecomposable and pairwise non-isomorphic.

Recall that an infinite cardinal $m^* > \aleph_0$ is said to be strongly inaccessible if (a) $\sum_{i \in I} m_i < m^*$ whenever $m_i < m^*$ for each $i \in I$ and the index set I is of cardinality $< m^*$; (b) $2^n < m^*$ for cardinals $n < m^*$. It is known (this follows from the combination of the method of Fuchs [3] with a set-theoretical result by Corner [2]) that for every cardinal m , less than the first strongly inaccessible cardinal, there is a

(so-called rigid) system $\{X_i\}_{i \in I}$ of torsion-free groups X_i with the following properties:

- (i) $|X_i| = m$;
- (ii) $|I| = 2^m$;
- (iii) $\text{Hom}(X_i, X_k) \cong Z$ (= integers) or $= 0$ according as $i = k$ or $i \neq k$.

Notice that then all the X_i are indecomposable.

Proof of Theorem 1. For the sake of convenience we shall denote by $\{X_i, Y_i\}_{i \in I}$ a system of groups with properties (i), (iii) and $|I| = m$; we assume $0 \in I$. Let $\{X'_i, Y'_i\}_{i \in I}$ be another copy of the same system and $x_i \rightarrow x'_i, y_i \rightarrow y'_i$ fixed isomorphisms between X_i and X'_i, Y_i and Y'_i . Let p, q and r be different odd primes. In view of (iii), we can select $\bar{x}_i \in X_i, \bar{y}_i \in Y_i$ for all $i \in I$ such that \bar{x}_i is not divisible in X_i by p and r , and \bar{y}_i is not divisible in Y_i by q and r . Then the corresponding $\bar{x}'_i \in X'_i$ and $\bar{y}'_i \in Y'_i$ will have the same properties. Writing

$$X = \bigoplus_{i \in I} X_i \quad \text{and} \quad Y = \bigoplus_{i \in I} Y_i,$$

and similarly $X' = \bigoplus X'_i, Y' = \bigoplus Y'_i$, we define

$$B = \langle X \oplus Y, p^{-1}(\bar{x}_0 + \bar{x}_i), q^{-1}(\bar{y}_0 + \bar{y}_i) \text{ for all } i \neq 0; \\ r^{-1}(\bar{x}_i + \bar{y}_i) \text{ for all } i \in I \rangle$$

and

$$C = \langle X' \oplus Y', p^{-1}(\bar{x}'_0 + \bar{x}'_i), q^{-1}(\bar{y}'_0 + \bar{y}'_i) \text{ for all } i \neq 0; \\ r^{-1}(\bar{x}'_i + \bar{y}'_i) \text{ for all } i \in I \rangle.$$

It is then obvious that $B \cong C$.

Now B is indecomposable. For, if $B = G \oplus H$ then the full invariance of the subgroups X_i and Y_i in B implies that $X_i = (X_i \cap G) \oplus (X_i \cap H)$ and $Y_i = (Y_i \cap G) \oplus (Y_i \cap H)$, so by indecomposability, each of X_i, Y_i must be contained entirely either in G or in H . Arguing with the additional generators of B , standard techniques (see e.g. [3]) show that all of X_i and Y_i have to belong to the same component of B . This proves that B (and hence C) is indecomposable.

We define $A = B \oplus C$, and D as the divisible hull of A ; then $|A| = m$. We wish to change B and C in order to get other decompositions for A . For each $i \in I$, choose an integer k_i (to be specified later), and consider the following subgroups of A (recall that $x_i \rightarrow x'_i, y_i \rightarrow y'_i$ are fixed maps):

$$\left. \begin{aligned} U_i &= X_i; & V_i &= \{k_i y_i + (k_i^2 - 1) y'_i \mid y_i \in Y_i\}, \\ U'_i &= \{k_i x_i + x'_i \mid x_i \in X_i\}, & V'_i &= \{y_i + k_i y'_i \mid y_i \in Y_i\}. \end{aligned} \right\} \quad (1)$$

Then $x_i \rightarrow k_i x_i + x'_i, y_i \rightarrow k_i y_i + (k_i^2 - 1) y'_i, y_i \rightarrow y_i + k_i y'_i$ are isomorphisms. Let U, U', V, V' have the obvious meaning, and

$$\bar{u}_i = \bar{x}_i, \quad \bar{u}'_i = k_i \bar{x}_i + \bar{x}'_i, \quad \bar{v}_i = k_i \bar{y}_i + (k_i^2 - 1) \bar{y}'_i, \quad \bar{v}'_i = \bar{y}_i + k_i \bar{y}'_i.$$

We consider the following subgroups of D :

$$B^* = \langle U \oplus V, p^{-1}(\bar{u}_0 + \bar{u}_i), q^{-1}(\bar{v}_0 + \bar{v}_i) \text{ for all } i \neq 0; \\ r^{-1}(\bar{u}_i + k_i \bar{v}_i) \text{ for all } i \in I \rangle$$

and

$$C^* = \langle U' \oplus V', p^{-1}(\bar{u}'_0 + \bar{u}'_i), q^{-1}(\bar{v}'_0 + \bar{v}'_i) \text{ for all } i \neq 0; \\ r^{-1}(\bar{u}_i + k_i \bar{v}'_i) \text{ for all } i \in I \rangle.$$

From the definition it is readily seen that $B^* \cong C^*$. Notice that if the k_i are chosen so as to satisfy

$$\left. \begin{aligned} k_i &\equiv k_0 \pmod{pq} \\ k_i^2 &\equiv 1 \pmod{r} \end{aligned} \right\} \quad (2)$$

for all $i \in I$, then in A

$$\bar{u}'_0 + \bar{u}'_i = k_0(\bar{x}_0 + \bar{x}_i) + (k_i - k_0)\bar{x}_i + (\bar{x}'_0 + \bar{x}'_i)$$

is divisible by p ,

$$\bar{v}_0 + \bar{v}_i = k_0(\bar{y}_0 + \bar{y}_i) + (k_i - k_0)\bar{y}_i + (k_0^2 - 1)(\bar{y}'_0 + \bar{y}'_i) + (k_i^2 - k_0^2)\bar{y}'_i, \\ \bar{v}'_0 + \bar{v}'_i = (\bar{y}_0 + \bar{y}_i) + k_0(\bar{y}'_0 + \bar{y}'_i) + (k_i - k_0)\bar{y}'_i$$

are divisible by q , while

$$\bar{u}_i + k_i \bar{v}_i = (\bar{x}_i + \bar{y}_i) + (k_i^2 - 1)\bar{y}_i + k_i(k_i^2 - 1)\bar{y}'_i, \\ \bar{u}'_i + k_i \bar{v}'_i = k_i(\bar{x}_i + \bar{y}_i) + (\bar{x}'_i + \bar{y}'_i) + (k_i^2 - 1)\bar{y}'_i$$

are divisible by r . In other words, B^* and C^* are subgroups of A ; they are obviously disjoint. From (1) it is evident that all of X_i, X'_i, Y_i and Y'_i are contained in $B^* \oplus C^*$, and it is straightforward to check that all the other generators of B and C also belong to $B^* \oplus C^*$. Consequently, $A = B^* \oplus C^*$. Since no k_i can be divisible by r , the indecomposability of B^* can be established in the same way as was done above for B .

Now let l be an integer such that

$$l \equiv 1 \pmod{pq} \quad \text{and} \quad l \equiv -1 \pmod{r}.$$

We fix $k_0 = 1$ and, for each $i \neq 0$, we let either $k_i = 1$ or $k_i = l$. Such a choice will satisfy conditions (2), thus for each choice of the $k_i (i \in I, i \neq 0)$ we get a decomposition $A = B^* \oplus C^*$ with indecomposable components $B^* \cong C^*$. Because of $|I| = m$, there are 2^m different ways of selecting $\{k_i\}$. Therefore the proof will be completed if we can verify that for a different choice, say $\{k_i^*\}$, the corresponding group B^{**} can not be isomorphic to B^* .

Suppose $\phi: B^* \rightarrow B^{**}$ is an isomorphism. Owing to (iii), ϕ must induce on each U_i and V_i an isomorphism with U_i^* and V_i^* ($\subseteq B^{**}$), respectively. In particular, ϕ acts on the selected elements $\bar{u}_i, \bar{v}_i \in B^*$ and $\bar{u}_i^*, \bar{v}_i^* \in B^{**}$ as follows:

$$\bar{u}_i \rightarrow \pm \bar{u}_i^* \quad \text{and} \quad \bar{v}_i \rightarrow \pm \bar{v}_i^* \quad \text{for all } i. \quad (3)$$

As divisibility by integers is preserved by ϕ , $p \mid \bar{u}_0 + \bar{u}_i \rightarrow \pm (\bar{u}_0^* \pm \bar{u}_i^*)$, $q \mid \bar{v}_0 + \bar{v}_i \rightarrow \pm (\bar{v}_0^* \pm \bar{v}_i^*)$ and $r \mid \bar{u}_0 + \bar{v}_0 \rightarrow \pm (\bar{u}_0^* \pm \bar{v}_0^*)$ imply that in (3) we must have the same sign throughout, say $\phi: \bar{u}_i \rightarrow \bar{u}_i^*$, $\bar{v}_i \rightarrow \bar{v}_i^*$. Hence we infer $\phi: \bar{u}_i + k_i \bar{v}_i \rightarrow \bar{u}_i^* + k_i \bar{v}_i^*$, thus $r \mid \bar{u}_i^* + k_i \bar{v}_i^*$ for every $i \in I$. Since $r \mid \bar{u}_i^* + k_i \bar{v}_i^*$ and \bar{v}_i^* is not divisible by r in A , we must have $k_i \equiv k_i^* \pmod{r}$ for every i . This is impossible if one of k_i, k_i^* is equal to 1 and the other is l . Hence different choices of the k_i yield non-isomorphic groups B^* , in fact. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\{W, Y_i\}_{i \in I}$ be a system of groups satisfying (i), (iii) and $|I| = m$; we may again assume $0 \in I$. Let p and q be two odd primes such that $p \neq q > 3$, and let $\bar{w} \in W, \bar{y}_i \in Y_i$ be chosen such that neither $q \mid \bar{w}$ in W , nor $p, q \mid \bar{y}_i$ in Y_i . Let $\{X_i\}_{i \in I}$ be another system with $Y_i \cong X_i$ under fixed isomorphisms $y_i \rightarrow x_i$ under which $\bar{y}_i \rightarrow \bar{x}_i$. We define $A = B \oplus C$ where

$$B = \langle \bigoplus_{i \in I} X_i, p^{-1}(\bar{x}_0 + \bar{x}_i) \text{ for all } i \neq 0 \rangle,$$

$$C = \langle W \oplus \bigoplus_{i \in I} Y_i, p^{-1}(\bar{y}_0 + \bar{y}_i) \text{ for all } i \neq 0, q^{-1}(\bar{w} + \bar{y}_i) \text{ for all } i \in I \rangle.$$

Let s and t be integers satisfying $ps - qt = 1$, and set

$$U_i = \{\alpha_i x_i + \beta_i y_i \mid y_i \in Y_i\}, \quad V_i = \{\gamma_i x_i + \delta_i y_i \mid y_i \in Y_i\}$$

for all $i \in I$ such that

$$\text{either } \alpha_i = s, \quad \beta_i = t, \quad \gamma_i = q, \quad \delta_i = p, \quad (4)$$

$$\text{or } \alpha_i = s + l_1 p, \quad \beta_i = t + l_2 p, \quad \gamma_i = q, \quad \delta_i = 2p, \quad (5)$$

where the integers l_1 and l_2 are chosen so as to have $l_2 q - 2l_1 p = s$. Thus $\alpha_i \delta_i - \beta_i \gamma_i = 1$ for both cases.

Using the obvious notations $\bar{u}_i = \alpha_i \bar{x}_i + \beta_i \bar{y}_i$, $\bar{v}_i = \gamma_i \bar{x}_i + \delta_i \bar{y}_i$, let us define

$$B^* = \langle \bigoplus_{i \in I} U_i, p^{-1}(\bar{u}_0 + \bar{u}_i) \text{ for all } i \neq 0 \rangle,$$

$$C^* = \langle W \oplus \bigoplus_{i \in I} V_i, p^{-1}(\bar{v}_0 + \bar{v}_i) \text{ for all } i \neq 0, q^{-1}(\delta_i \bar{w} + \bar{v}_i) \text{ for all } i \in I \rangle.$$

Since $\bar{u}_0 + \bar{u}_i$ and $\bar{v}_0 + \bar{v}_i$ are divisible by p in A , and since $\delta_i \bar{w} + \bar{v}_i = q x_i + \delta_i (\bar{w} + \bar{y}_i)$ are divisible by q in A , it is clear that B^* and C^* are subgroups of A . They generate their direct sum $B^* \oplus C^*$ in A . Owing to $\alpha_i \delta_i - \beta_i \gamma_i = 1$, all of X_i and Y_i are contained in

$B^* \oplus C^*$, and so are all the additional generators of B and C , as readily checked. We thus have $A = B^* \oplus C^*$ where, obviously, $B^* \cong B$. The indecomposability of the groups B, C, C^*, \dots can easily be established.

Since $|I| = m$, there are 2^m different ways of choosing the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ as described in (4) and (5). In order to complete the proof of the theorem, it will therefore suffice to prove that different choices yield non-isomorphic groups C^* .

Let C^{**} be defined in terms of $\alpha_i^*, \beta_i^*, \gamma_i^*, \delta_i^*$ as generated by $W \oplus \bigoplus_i V_i^*$, $p^{-1}(\bar{v}_0^* + \bar{v}_i^*)$ for all $i \neq 0$ and $q^{-1}(\delta_i^* \bar{w} + \bar{v}_i^*)$ for all $i \in I$. Any isomorphism $\phi: C^* \rightarrow C^{**}$ must induce an automorphism on W and isomorphisms $V_i^* \rightarrow V_i^{**}$ for every $i \in I$ which must act on the selected elements as $w \rightarrow \pm w, \bar{v}_i \rightarrow \pm \bar{v}_i^*$. Investigating the divisibility of $\bar{v}_0 + \bar{v}_i \rightarrow \pm (\bar{v}_0^* \pm \bar{v}_i^*)$ by p , we conclude that the signs of \bar{v}_i^* must be the same, say $+1$, for all i . From $q \mid \delta_i \bar{w} + \bar{v}_i \rightarrow \pm \delta_i \bar{w} + \bar{v}_i^*, q \mid \delta_i^* \bar{w} + \bar{v}_i^*$ and $q \nmid \bar{w}$ we obtain that $\delta_i^* \equiv \pm \delta_i \pmod{q}$. In view of (4) and (5) this is impossible unless $\delta_i^* = \delta_i$ for all i . Q.E.D.

It is easy to see that for $m = \aleph_0$, all the groups X_i, Y_i, W in the construction can be chosen to be of rank 1, and for an arbitrary m , to be of rank n where $n \leq m \leq 2^n$.

Using Pontrjagin's duality theory, we conclude that, *to every cardinal m less than the first strongly inaccessible aleph, there exists a connected compact group of cardinality 2^m which has 2^m non-isomorphic closed summands*. Moreover, as a closer examination of the invariants reveals, we may add that *these summands are algebraically all isomorphic*.

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