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Autor: Fuchs, L.

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Note on Direct Decompositions of Torsion-free Abelian Groups

L. FUCHS (New Orleans)

All the groups of this note are torsion-free abelian groups under addition.

Jónsson [5] was the first to point out that torsion-free groups of finite rank may have non-isomorphic direct decompositions into (directly) indecomposable groups. He discovered a few pathological phenomena, and using his techniques, Corner [1] furnished examples, both in the finite and in the countable rank cases, with a surprising flexibility even in the choice of the ranks of the indecomposable summands. For groups of countable rank, Corner [1] proved that the same group can have two, basically different direct decompositions: one with just two indecomposable summands and one with infinitely many components. It is not difficult to find more pathological decompositions (see e.g. Fuchs and Loonstra [4]). Unfortunately, no complete survey is known of the variety of direct decompositions a torsion-free group might have.

The aim of this note is to point out that a countable group can have continuously many, pairwise non-isomorphic, indecomposable summands. Moreover, we are going to prove the following two, more general theorems:

THEOREM 1. *For every infinite cardinal* $\mathfrak{m}$ *less than the first strongly inaccessible aleph, there exists a torsion-free group* A *of rank* $\mathfrak{m}$ *such that* A *has direct decompositions*

$$A = B_j \oplus C_j \quad \textit{with} \quad B_j \cong C_j$$

and with j *ranging over an index set* J *of cardinality* $|J| = 2^{\mathfrak{m}}$ *where the* B_j *are pairwise non-isomorphic and indecomposable.*

THEOREM 2. *For every* $\mathfrak{m}$ *as in Theorem* 1 *there is a torsion-free group* A *of rank* $\mathfrak{m}$ *such that*

$$A = B_j \oplus C_j \quad (j \in J)$$

holds for an index set J *of cardinality* $2^{\mathfrak{m}}$ *where all the* B_j *are indecomposable and isomorphic among themselves, while the* C_j *are indecomposable and pairwise non-isomorphic.*

Recall that an infinite cardinal $\mathfrak{m}^* > \aleph_0$ is said to be strongly inaccessible if (a) $\sum_{i \in I} \mathfrak{m}_i < \mathfrak{m}^*$ whenever $\mathfrak{m}_i < \mathfrak{m}^*$ for each $i \in I$ and the index set I is of cardinality $< \mathfrak{m}^*$; (b) $2^{\mathfrak{n}} < \mathfrak{m}^*$ for cardinals $\mathfrak{n} < \mathfrak{m}^*$. It is known (this follows from the combination of the method of Fuchs [3] with a set-theoretical result by Corner [2]) that for every cardinal $\mathfrak{m}$, less than the first strongly inaccessible cardinal, there is a (so-called rigid) system $\{X_i\}_{i\in I}$ of torsion-free groups X_i with the following properties:

(i) $|X_i|=\mathfrak{m}$;

(ii) $|I|=2^{\mathfrak{m}}$;

(iii) $\operatorname{Hom}(X_i, X_k)\cong Z$ (=integers) or $=0$ according as $i=k$ or $i\neq k$.

Notice that then all the X_i are indecomposable.

Proof of Theorem 1. For the sake of convenience we shall denote by $\{X_i, Y_i\}_{i\in I}$ a system of groups with properties (i), (iii) and $|I|=\mathfrak{m}$; we assume $0\in I$. Let $\{X_i', Y_i'\}_{i\in I}$ be another copy of the same system and $x_i\to x_i'$, $y_i\to y_i'$ fixed isomorphisms between X_i and X_i', Y_i and Y_i'. Let p, q and r be different odd primes. In view of (iii), we can select $\bar{x}_i\in X_i$, $\bar{y}_i\in Y_i$ for all $i\in I$ such that $\bar{x}_i$ is not divisible in X_i by p and r, and $\bar{y}_i$ is not divisible in Y_i by q and r. Then the corresponding $\bar{x}_i'\in X_i'$ and $\bar{y}_i'\in Y_i'$ will have the same properties. Writing

$$X = \bigoplus_{i\in I} X_i \quad\text{and}\quad Y = \bigoplus_{i\in I} Y_i,$$

and similarly $X'=\oplus X_i'$, $Y'=\oplus Y_i'$, we define

$$\begin{aligned} B = \langle X\oplus Y,\ p^{-1}(\bar{x}_0+\bar{x}_i),\ q^{-1}(\bar{y}_0+\bar{y}_i) &\quad \text{for all} \quad i\neq 0;\\ r^{-1}(\bar{x}_i+\bar{y}_i) &\quad \text{for all} \quad i\in I\rangle \end{aligned}$$

and

$$\begin{aligned} C = \langle X'\oplus Y',\ p^{-1}(\bar{x}_0'+\bar{x}_i'),\ q^{-1}(\bar{y}_0'+\bar{y}_i') &\quad \text{for all} \quad i\neq 0;\\ r^{-1}(\bar{x}_i'+\bar{y}_i') &\quad \text{for all} \quad i\in I\rangle. \end{aligned}$$

It is then obvious that $B\cong C$.

Now B is indecomposable. For, if $B=G\oplus H$ then the full invariance of the subgroups X_i and Y_i in B implies that $X_i=(X_i\cap G)\oplus(X_i\cap H)$ and $Y_i=(Y_i\cap G)\oplus(Y_i\cap H)$, so by indecomposability, each of X_i, Y_i must be contained entirely either in G or in H. Arguing with the additional generators of B, standard techniques (see e.g. [3]) show that all of X_i and Y_i have to belong to the same component of B. This proves that B (and hence C) is indecomposable.

We define $A=B\oplus C$, and D as the divisible hull of A; then $|A|=\mathfrak{m}$. We wish to change B and C in order to get other decompositions for A. For each $i\in I$, choose an integer k_i (to be specified later), and consider the following subgroups of A (recall that $x_i\to x_i'$, $y_i\to y_i'$ are fixed maps):

$$\left.\begin{aligned} &U_i = X_i;\quad V_i=\{k_iy_i+(k_i^2-1)y_i' \mid y_i\in Y_i\},\\ &U_i'=\{k_ix_i+x_i' \mid x_i\in X_i\},\quad V_i'=\{y_i+k_iy_i' \mid y_i\in Y_i\}. \end{aligned}\right\} \tag{1}$$

Then $x_i\to k_ix_i+x_i'$, $y_i\to k_iy_i+(k_i^2-1)y_i'$, $y_i\to y_i+k_iy_i'$ are isomorphisms. Let U, U', V, V' have the obvious meaning, and

$$\bar{u}_i=\bar{x}_i,\quad \bar{u}_i'=k_i\bar{x}_i+\bar{x}_i',\quad \bar{v}_i=k_i\bar{y}_i+(k_i^2-1)\bar{y}_i',\quad \bar{v}_i'=\bar{y}_i+k_i\bar{y}_i'.$$

We consider the following subgroups of D:

$$B^* = \langle U \oplus V, p^{-1}(\bar{u}_0 + \bar{u}_i), q^{-1}(\bar{v}_0 + \bar{v}_i) \quad \text{for all} \quad i \neq 0;$$
$$r^{-1}(\bar{u}_i + k_i\bar{v}_i) \quad \text{for all} \quad i \in I\rangle$$

and

$$C^* = \langle U' \oplus V', p^{-1}(\bar{u}_0' + \bar{u}_i'), q^{-1}(\bar{v}_0' + \bar{v}_i') \quad \text{for all} \quad i \neq 0;$$
$$r^{-1}(\bar{u}_i + k_i\bar{v}_i') \quad \text{for all} \quad i \in I\rangle.$$

From the definition it is readily seen that $B^* \cong C^*$. Notice that if the k_i are chosen so as to satisfy

$$\left.\begin{aligned} k_i &\equiv k_0 \pmod{pq} \\ k_i^2 &\equiv 1 \pmod{r} \end{aligned}\right\} \tag{2}$$

for all $i \in I$, then in A

$$\bar{u}_0' + \bar{u}_i' = k_0(\bar{x}_0 + \bar{x}_i) + (k_i - k_0)\bar{x}_i + (\bar{x}_0' + \bar{x}_i')$$

is divisible by p,

$$\bar{v}_0 + \bar{v}_i = k_0(\bar{y}_0 + \bar{y}_i) + (k_i - k_0)\bar{y}_i + (k_0^2 - 1)(\bar{y}_0' + \bar{y}_i') + (k_i^2 - k_0^2)\bar{y}_i',$$
$$\bar{v}_0' + \bar{v}_i' = (\bar{y}_0 + \bar{y}_i) + k_0(\bar{y}_0' + \bar{y}_i') + (k_i - k_0)\bar{y}_i'$$

are divisible by q, while

$$\bar{u}_i + k_i\bar{v}_i = (\bar{x}_i + \bar{y}_i) + (k_i^2 - 1)\bar{y}_i + k_i(k_i^2 - 1)\bar{y}_i',$$
$$\bar{u}_i' + k_i\bar{v}_i' = k_i(\bar{x}_i + \bar{y}_i) + (\bar{x}_i' + \bar{y}_i') + (k_i^2 - 1)\bar{y}_i'$$

are divisible by r. In other words, B^* and C^* are subgroups of A; they are obviously disjoint. From (1) it is evident that all of X_i, X_i', Y_i and Y_i' are contained in $B^* \oplus C^*$, and it is straightforward to check that all the other generators of B and C also belong to $B^* \oplus C^*$. Consequently, $A = B^* \oplus C^*$. Since no k_i can be divisible by r, the indecomposability of B^* can be established in the same way as was done above for B.

Now let l be an integer such that

$$l \equiv 1 \pmod{pq} \quad \text{and} \quad l \equiv -1 \pmod{r}.$$

We fix $k_0 = 1$ and, for each $i \neq 0$, we let either $k_i = 1$ or $k_i = l$. Such a choice will satisfy conditions (2), thus for each choice of the $k_i (i \in I, i \neq 0)$ we get a decomposition $A = B^* \oplus C^*$ with indecomposable components $B^* \cong C^*$. Because of $|I| = \mathfrak{m}$, there are $2^{\mathfrak{m}}$ different ways of selecting $\{k_i\}$. Therefore the proof will be completed if we can verify that for a different choice, say $\{k_i^*\}$, the corresponding group B^{**} can not be isomorphic to B^*.

Suppose $\phi: B^* \to B^{**}$ is an isomorphism. Owing to (iii), ϕ must induce on each U_i and V_i an isomorphism with U_i^* and $V_i^* (\subseteq B^{**})$, respectively. In particular, ϕ acts on the selected elements $\bar{u}_i, \bar{v}_i \in B^*$ and $\bar{u}_i^*, \bar{v}_i^* \in B^{**}$ as follows:

$$\bar{u}_i \to \pm \bar{u}_i^* \quad \text{and} \quad \bar{v}_i \to \pm \bar{v}_i^* \quad \text{for all} \quad i. \tag{3}$$

As divisibility by integers is preserved by ϕ, $p \mid \bar{u}_0 + \bar{u}_i \to \pm(\bar{u}_0^* \pm \bar{u}_i^*)$, $q \mid \bar{v}_0 + \bar{v}_i \to \pm(\bar{v}_0^* \pm \bar{v}_i^*)$ and $r \mid \bar{u}_0 + \bar{v}_0 \to \pm(\bar{u}_0^* \pm \bar{v}_0^*)$ imply that in (3) we must have the same sign throughout, say $\phi: \bar{u}_i \to \bar{u}_i^*$, $\bar{v}_i \to \bar{v}_i^*$. Hence we infer $\phi: \bar{u}_i + k_i \bar{v}_i \to \bar{u}_i^* + k_i \bar{v}_i^*$, thus $r \mid \bar{u}_i^* + k_i \bar{v}_i^*$ for every $i \in I$. Since $r \mid \bar{u}_i^* + k_i^* \bar{v}_i^*$ and $\bar{v}_i^*$ is not divisible by r in A, we must have $k_i \equiv k_i^* \pmod{r}$ for every i. This is impossible if one of k_i, k_i^* is equal to 1 and the other is l. Hence different choices of the k_i yield non-isomorphic groups B^*, in fact. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\{W, Y_i\}_{i \in I}$ be a system of groups satisfying (i), (iii) and $|I| = \mathfrak{m}$; we may again assume $0 \in I$. Let p and q be two odd primes such that $p \neq q > 3$, and let $\bar{w} \in W$, $\bar{y}_i \in Y_i$ be chosen such that neither $q \mid \bar{w}$ in W, nor $p, q \mid \bar{y}_i$ in Y_i. Let $\{X_i\}_{i \in I}$ be another system with $Y_i \cong X_i$ under fixed isomorphisms $y_i \to x_i$ under which $\bar{y}_i \to \bar{x}_i$. We define $A = B \oplus C$ where

$$B = \langle \bigoplus_{i \in I} X_i, p^{-1}(\bar{x}_0 + \bar{x}_i) \quad \text{for all} \quad i \neq 0 \rangle,$$

$$C = \langle W \oplus \bigoplus_{i \in I} Y_i, p^{-1}(\bar{y}_0 + \bar{y}_i) \quad \text{for all} \quad i \neq 0, q^{-1}(\bar{w} + \bar{y}_i) \quad \text{for all} \quad i \in I \rangle.$$

Let s and t be integers satisfying $ps - qt = 1$, and set

$$U_i = \{\alpha_i x_i + \beta_i y_i \mid y_i \in Y_i\}, \quad V_i = \{\gamma_i x_i + \delta_i y_i \mid y_i \in Y_i\}$$

for all $i \in I$ such that

$$\text{either} \quad \alpha_i = s, \quad \beta_i = t, \quad \gamma_i = q, \quad \delta_i = p, \tag{4}$$

$$\text{or} \quad \alpha_i = s + l_1 p, \quad \beta_i = t + l_2 p, \quad \gamma_i = q, \quad \delta_i = 2p, \tag{5}$$

where the integers l_1 and l_2 are chosen so as to have $l_2 q - 2 l_1 p = s$. Thus $\alpha_i \delta_i - \beta_i \gamma_i = 1$ for both cases.

Using the obvious notations $\bar{u}_i = \alpha_i \bar{x}_i + \beta_i \bar{y}_i$, $\bar{v}_i = \gamma_i \bar{x}_i + \delta_i \bar{y}_i$, let us define

$$B^* = \langle \bigoplus_{i \in I} U_i, p^{-1}(\bar{u}_0 + \bar{u}_i) \quad \text{for all} \quad i \neq 0 \rangle,$$

$$C^* = \langle W \oplus \bigoplus_{i \in I} V_i, p^{-1}(\bar{v}_0 + \bar{v}_i) \quad \text{for all} \quad i \neq 0, q^{-1}(\delta_i \bar{w} + \bar{v}_i) \quad \text{for all} \quad i \in I \rangle.$$

Since $\bar{u}_0 + \bar{u}_i$ and $\bar{v}_0 + \bar{v}_i$ are divisible by p in A, and since $\delta_i \bar{w} + \bar{v}_i = q x_i + \delta_i(\bar{w} + \bar{y}_i)$ are divisible by q in A, it is clear that B^* and C^* are subgroups of A. They generate their direct sum $B^* \oplus C^*$ in A. Owing to $\alpha_i \delta_i - \beta_i \gamma_i = 1$, all of X_i and Y_i are contained in

$B^* \oplus C^*$, and so are all the additional generators of B and C, as readily checked. We thus have $A = B^* \oplus C^*$ where, obviously, $B^* \cong B$. The indecomposability of the groups $B, C, C^*, \ldots$ can easily be established.

Since $|I| = \mathfrak{m}$, there are $2^{\mathfrak{m}}$ different ways of choosing the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ as described in (4) and (5). In order to complete the proof of the theorem, it will therefore suffice to prove that different choices yield non-isomorphic groups C^*.

Let C^{**} be defined in terms of $\alpha_i^*, \beta_i^*, \gamma_i^*, \delta_i^*$ as generated by $W \oplus \oplus_i V_i^*$, $p^{-1}(\bar{v}_0^* + \bar{v}_i^*)$ for all $i \neq 0$ and $q^{-1}(\delta_i^* \bar{w} + \bar{v}_i^*)$ for all $i \in I$. Any isomorphism $\phi: C^* \to C^{**}$ must induce an automorphism on W and isomorphisms $V_i^* \to V_i^{**}$ for every $i \in I$ which must act on the selected elements as $w \to \pm w$, $\bar{v}_i \to \pm \bar{v}_i^*$. Investigating the divisibility of $\bar{v}_0 + \bar{v}_i \to \pm(\bar{v}_0^* \pm \bar{v}_i^*)$ by p, we conclude that the signs of $\bar{v}_i^*$ must be the same, say $+1$, for all i. From $q \mid \delta_i \bar{w} + \bar{v}_i \to \pm \delta_i \bar{w} + \bar{v}_i^*$, $q \mid \delta_i^* \bar{w} + \bar{v}_i^*$ and $q \nmid \bar{w}$ we obtain that $\delta_i^* \equiv \pm \delta_i \pmod{q}$. In view of (4) and (5) this is impossible unless $\delta_i^* = \delta_i$ for all i. Q.E.D.

It is easy to see that for $\mathfrak{m} = \aleph_0$, all the groups X_i, Y_i, W in the construction can be chosen to be of rank 1, and for an arbitrary $\mathfrak{m}$, to be of rank $\mathfrak{n}$ where $\mathfrak{n} \leqq \mathfrak{m} \leqq 2^{\mathfrak{n}}$.

Using Pontrjagin's duality theory, we conclude that, *to every cardinal* $\mathfrak{m}$ *less than the first strongly inaccessible aleph, there exists a connected compact group of cardinality* $2^{\mathfrak{m}}$ *which has* $2^{\mathfrak{m}}$ *non-isomorphic closed summands.* Moreover, as a closer examination of the invariants reveals, we may add that *these summands are algebraically all isomorphic.*

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Tulane University
New Orleans, Louisiana, USA

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