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# Homotopy Equivalences of Almost Smooth Manifolds

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§ 1. *Introduction.* Let $M^k$, $k \geqslant 6$, be a simply connected, oriented, closed combinatorial manifold with a differentiable structure in the complement of a point. Let $M_0^k = M^k - \text{interior } (D^k)$, where $D^k \subset M^k$ is a combinatorially embedded disc. $M_0^k$ inherits a differentiable structure from $M^k - (p)$, hence $\partial M_0^k$ belongs to $\Gamma_{k-1}$, the group of oriented differentiable structures on $S^{k-1}$. In general, $\partial M_0^k \in \Gamma_{k-1}$ is not a homotopy invariant of $M^k$. In this paper we study this non-invariance.

Specifically, let $B_h(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of homotopy smoothings of $M_0$ [18]. That is, $\Sigma^{k-1} \in B_h(M_0)$ if and only if there is a smooth manifold $M_0'$, with $\partial M_0' = \Sigma^{k-1}$, and a homotopy equivalence of pairs $h: M_0', \partial M_0' \to M_0, \partial M_0$. Then $B_h(M_0') = B_h(M_0)$, and $M^k$ is homotopy equivalent to a smooth manifold if and only if $0 \in B_h(M_0)$. We will give a homotopy theoretic description of the set of differences $\Delta_h(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_h(M_0)\} \subset \Gamma_{k-1}$, for certain classes of manifolds. If $\partial M_0 \in \Gamma_{k-1}$ is known, for example if $\partial M_0 = 0$, this determines $B_h(M_0)$. In any case, $B_h(M_0)$ and $\Delta_h(M_0)$ have the same number of elements.

Following Sullivan, two homotopy smoothings, $h: M_0', \partial M_0' \to M_0, \partial M_0$ and $g: M_0'', \partial M_0'' \to M_0, \partial M_0$, are called equivalent if there is a diffeomorphism $f: M_0' \simeq M_0''$ such that $h$ is homotopic to $gf$. The set of equivalence classes is denoted $hS(M_0)$. In [18], Sullivan constructs a bijection $\theta: hS(M_0) \simeq [M_0, F/0]$, where $F/0$ is the fibre of the map $BS0 \to BSF$. Thus, if $h: M_0' \to M_0$ represents an element of $hS(M_0)$, the formula $d\theta(M_0', h) = \partial M_0' - \partial M_0 \in \Gamma_{k-1}$ defines a map $d: [M_0, F/0] \to \Gamma_{k-1}$, and $\Delta_h(M_0) = \text{image } (\mathrm{d}) \subset \Gamma_{k-1}$.

The group $\Gamma_{k-1}$ can be described as follows. If $k \neq 2^j - 1$ or $2^j - 2$ then $\Gamma_{k-1} \simeq \simeq bP_k \oplus (\pi_{k-1}^s / \mathrm{im}(J))$, where $bP_k \subset \Gamma_{k-1}$ is the cyclic subgroup of homotopy spheres that bound $\pi$-manifolds [9], [11], [15].

$\Gamma_{2^j-2} \simeq \text{kernel} (\pi_{2^j-2}^s \xrightarrow{\psi} Z_2)$, where $\psi$ is the Arf invariant. $\psi \neq 0$ if and only if the element $h_{j-1}^2 \in Ext_A(Z_2, Z_2)$ is an infinite cycle in the Adams spectral sequence [6]. Mahowald has shown that $h_{j-1}^2$ is an infinite cycle if $j \leqslant 6$. Also, if $\psi \neq 1$, $\Gamma_{2^j-3} = = \pi_{2^j-3}^s / \mathrm{im}(J)$ $(= \pi_{2^j-3}^s$ if $j > 2)$.

If $k$ is odd then $bP_k = 0$. If $k$ is even, the direct sum decomposition of $\Gamma_{k-1}$ follows from properties of two homomorphisms, namely, the Kervaire-Milnor map $\varrho: \Gamma_{k-1} \to \to \pi_{k-1}^s / \mathrm{im}(J)$, with kernel $(\varrho) = bP_k$ [15], and an invariant $f_R: \Gamma_{k-1} \to Z_2$ if $k = 4n + 2 \neq \neq 2^j - 2$ [11], or $f_R: \Gamma_{k-1} \to Z_{\theta_n}$ if $k = 4n$, where $\theta_n = a_n \cdot 2^{2n-2} \cdot (2^{2n-1} - 1) \text{ num} (B_n / 4n)$, $a_n = 2$ if $n$ is odd, $a_n = 1$ if $n$ is even, and $B_n$ is the Bernoulli number [9]. The restriction of $f_R$ to $bP_k \subset \Gamma_{k-1}$ is an isomorphism. Thus a homotopy sphere $\Sigma^{k-1} \in \Gamma_{k-1}$ is determined by $\varrho(\Sigma^{k-1}) \in \pi_{k-1}^s / \mathrm{im}(J)$ and $f_R(\Sigma^{k-1}) \in bP_k$.

The invariants $f_R:\Gamma_{4n-1}\to Z_{\theta_n}$ and $f_R:b\mathrm{spin}_{8n+2}\to Z_2$ are natural, and can be computed where $\mathrm{bspin}_{8n+2}\subset\Gamma_{8n+1}$ is the subgroup (of index 2) of homotopy spheres that bound spin manifolds. However, $f_R:\Gamma_{8n+5}\to Z_2$ and the extension $f_R:\Gamma_{8n+1}\to Z_2$ depend on choices, and can not be effectively computed. Thus our results on $\Delta_h(M_0^k)$ are complete only if $k\not\equiv 6 \pmod 8$ and if, when $k\equiv 2 \pmod 8$, $M_0^k$ is a spin manifold.

The paper is arranged as follows. In §§ 2 and 3, we discuss Sullivan's work on homotopy smoothings and describe the composition $\varrho d:[M_0^k, F/0]\to\Gamma_{k-1}\to\to\pi_{k-1}^s/\mathrm{im}(J)$. In § 4, we give some homotopy theoretic results on $F/0$. Many of the results in these three sections are well-known. In § 5, we compute the composition $f_R d:[M_0^{4n}, F/0]\to\Gamma_{4n-1}\to Z_{\theta_n}$. In § 6, we compute the composition $f_R d:[M_0^{8n+2}, F/0]\to\to\Gamma_{8n+1}\to Z_2$ for spin manifolds, $M_0^{8n+2}$. The main results of the paper are Propositions 4.4, 4.5, 5.1, 5.2 and 6.5.

In two appendixes, we give applications of the results of § 2 through § 6. In Appendix I, we set $M^{2k}=CP(k)$ and characterize those homotopy $(2k-1)$-spheres which admit differentiable, fixed point free, $S^1$ actions. In Appendix II, we set $M^{k+1}==S^1\times N^k$ and compute certain canonical subgroups of the inertia group, $I(N^k)\subset\Gamma_k$, of a smooth manifold $N^k$.

Many of the ideas in this paper are due to D. Sullivan. I am very grateful to him for many conversations.

§ 2. *Homotopy Smoothings.* We first sketch a definition of the bijection $\theta:hS(M_0)\cong\cong[M_0, F/0]$. Let $h:M_0'\to M_0$ be a homotopy smoothing of $M_0^k$, and let $\bar{h}$ be a homotopy inverse of $h$. Homotope the map $h$ to a smooth embedding of $M_0'$ in the total space, $E(\xi_0)$, of the (stable) vector bundle $\xi_0=\xi_0(h)=\bar{h}^*(\tau_{M_0'})-\tau_{M_0}$ over $M_0$ where $\tau_{M_0}$ is the tangent bundle. Then the normal bundle of $M_0'$ in $E(\xi_0)$ is trivial and choosing a framing of $M_0'$ in $E(\xi_0)$ determines a fibre homotopy trivialization of $\xi_0$. (In fact, it follows from the $h$-cobordism theorem that there is a diffeomorphism $H:M_0'\times\mathbf{R}^q\cong E(\xi_0^q)$, $q$ large, homotopic to $h$.) This defines an element $\theta(h)\in[M_0, F/0]$, which depends only on the class of $(M_0', h)$ in $hS(M_0)$. By construction, the composition $M_0\to F/O\to BS0$ represents $\xi_0(h)\in KO^0(M_0)$.

Now, $h$ induces a bijection $h_*:hS(M_0')\cong hS(M_0)$, defined by $h_*(M_0'', g)=(M_0'', hg)$ where $g:M_0''\to M_0'$. Also, there is the bijection $h^*:[M_0, F/0]\cong[M_0', F/0]$ induced by the homotopy equivalence $h:M_0'\to M_0$. Since $F/0$ is an $H$-space, $h^*$ is an isomorphism of groups. Consider the diagram

$$\begin{tikzcd}
hS(M_0) \arrow[r, "\theta", "\cong"'] & {[M_0, F/0]} \arrow[dr, "d"] \arrow[d, "h^*", "\wr"'] & \\
hS(M_0') \arrow[u, "h_*", "\wr"'] \arrow[r, "\theta", "\cong"'] & {[M_0', F/0]} \arrow[r, "d"'] & \Gamma_{k-1}
\end{tikzcd} \tag{2.1}$$

This diagram is very non-commutative. In fact, if $g:M_0''\to M_0'$ is a homotopy smoothing of $M_0'$ then $d\theta(h_*(g)) = \partial M_0'' - \partial M_0 = (\partial M_0'' - \partial M_0') + (\partial M_0' - \partial M_0) = d\theta(g) + d\theta(h)$. We also have

PROPOSITION 2.2. *If* $g \in hS(M_0')$ *then*

$$h^*\theta h_*(g) - \theta(g) = h^*\theta(h) \in [M_0', F/0].$$

This can be equivalently stated as follows. Suppose

$$\begin{tikzcd} M_0'' \arrow[rr, "f"] \arrow[dr, "g"'] & & M_0 \\ & M_0' \arrow[ur, "h"'] & \end{tikzcd}$$

is a homotopy commutative diagram and $f$, $g$, $h$ are all homotopy equivalences. Then $f = h_*(g)$ and applying the isomorphism $\bar{h}^*$ to the equation in 2.2 gives

$$\theta(f) = \theta(h) + \bar{h}^*(\theta(g)) \in [M_0, F/0] \tag{2.3}$$

We will prove 2.3. In §§ 5 and § 6 we give formulas for the difference $d - dh^*$ and for the deviation of $d$ from linearity (that is, in general $d$ is not a homomorphism of groups).

*Proof of* 2.3. Choose a diffeomorphism $H: M_0' \times \mathbf{R}^q \simeq E(\xi^q(\theta(h)))$ homotopic to $h$, and, in the diagram below, let $E(\bar{H})$ be the obvious bundle map covering $\bar{H} = H^{-1}$.

$$\begin{tikzcd} E(\bar{H}^*\pi_1^*(\xi^q(\theta(g)))) \arrow[r, "E(\bar{H})", "\simeq"'] \arrow[d] & E(\pi_1^*(\xi^q(\theta(g)))) \arrow[d] \\ E(\xi^q(\theta(h))) \arrow[r, "\bar{H}", "\simeq"'] \arrow[d, "\pi"] & M_0' \times \mathbf{R}^q \arrow[d, "\pi_1"] \\ M_0 \arrow[r, "\bar{h}"] & M_0' \end{tikzcd}$$

Since $\pi_1 \bar{H} \simeq \bar{h}\pi$, it follows from the bundle covering homotopy theorem that there is a bundle isomorphism, $B$, covering the identity on $E(\xi^q(\theta(h)))$, and a bundle homotopy commutative diagram

$$\begin{tikzcd} E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) = E(\pi^*\bar{h}^*(\xi^q(\theta(g)))) \arrow[r, "E(\bar{h}\pi)"] \arrow[d, "B", "\wr"'] & E(\xi^q(\theta(g))) \\ E(\bar{H}^*\pi_1^*(\xi^q(\theta(g)))) \arrow[r, "E(\bar{H})", "\simeq"'] & E(\pi_1^*(\xi^q(\theta(g)))) \arrow[u, "E(\pi_1)"'] \end{tikzcd}$$

$$= E(\xi^q(\theta(g))) \times \mathbf{R}^q.$$

Let $G: M_0'' \times \mathbf{R}^q \simeq E(\xi^q(\theta(g)))$ be a diffeomorphism homotopic to $g$. Then $F = (\bar{G} \times 1)\, E(\bar{H})\, B: E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \simeq M_0'' \times \mathbf{R}^q \times \mathbf{R}^q$ is a diffeomorphism homotopic to $f = \overline{gh}$ where $\bar{G} = G^{-1}$. Thus the fibre homotopy trivialization

$$(\pi_2 \times \pi_3)\, F: E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \to \mathbf{R}^q \times \mathbf{R}^q$$

represents $\theta(f)$. On the other hand, bundle homotopy commutativity of the diagram above implies that $(\pi_2 \times \pi_3)\, \bar{F}$ is properly homotopic to $(\pi_2 \bar{G}E(\bar{h}) \times \pi_2 \bar{H})\Delta$ where

$$\Delta: E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \to E(\bar{h}^*(\xi^q(\theta(g)))) \times E(\xi^q(\theta(h)))$$

is the diagonal. Since $(\pi_2 \bar{G}E(\bar{h}) \times \pi_2 \bar{H})\Delta$ represents $\bar{h}^*(\theta(g)) + \theta(h)$, we have shown that $\theta(f) = \bar{h}^*(\theta(g)) + \theta(h)$, as desired.

The tangential homotopy equivalence, that is, $h: M_0' \to M_0$ with $h^*(\tau_{M_0}) = \tau_{M_0'}$ are particularly important. Let $B_{th}(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of manifolds $M_0'$ tangentially homotopy equivalent to $M_0$, and let $\Delta_{th}(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_{th}(M_0)\} \subset \Gamma_{k-1}$.

There is a fibration $SF \xrightarrow{j} F/0 \xrightarrow{i} BS0$, where $SF = \lim_{\rightarrow} SF_q$ and $SF_q$ is the space of base point preserving maps of degree one of $S^{q-1}$ to itself. Thus, given $h: M_0' \to M_0$, we have $h^*(\tau_{M_0}) = \tau_{M_0'}$ if and only if $\xi_0(h) = \bar{h}^*(\tau_{M_0'}) - \tau_{M_0} = 0 \in K0^\circ(M_0)$ or, equivalently, if and only if $\theta(h) \in \operatorname{image}([M_0, SF] \xrightarrow{j^*} [M_0, F/0])$. Thus $\Delta_{th}(M_0) = d(\operatorname{image}([M_0, SF] \to [M_0, F/0]))$.

Two other subsets of $B_h(M_0)$ are of geometric interest. Let $B_c(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of smooth manifolds $M_0''$ combinatorially equivalent to $M_0$, and let $B_{tc}(M_0) \subset B_c(M_0)$ be the subset of boundaries of those $M_0'$ such that some combinatorial equivalence $h: M_0' \to M_0$ preserves the (smooth) tangent bundles, that is, $h^*(\tau_{M_0}) = \tau_{M_0'}$ as vector bundles. Let $\Delta_c(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_c(M_0)\}$ and let $\Delta_{tc}(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_{tc}(M_0)\}$.

There are spaces $SPL$ and $PL/0$, and a braid of fibrations

$$\begin{tikzcd}
 & & PL/0 \arrow[dr] \arrow[rr, bend left] & & BS0 \arrow[dr] \arrow[rr, bend left] & & BSF \\
 & SPL \arrow[ur] \arrow[dr] & & F/0 \arrow[ur] \arrow[dr] & & BSPL \arrow[ur] & \\
S0 \arrow[ur] \arrow[rr, bend right] & & SF \arrow[ur] \arrow[rr, bend right] & & F/PL \arrow[ur] & &
\end{tikzcd}$$

From smoothing theory [14], it follows that $\Delta_c(M_0) = d(\operatorname{image}([M_0, PL/0] \to [M_0, F/0]))$ and that $\Delta_{tc}(M_0) = d(\operatorname{image}([M_0, SPL] \to [M_0, F/0]))$. Also, if $v \in [M_0^k, PL/0]$ then $dv = \partial^*(v) \in \pi_{k-1}(PL/0) = \Gamma_{k-1}$, where $\partial: S^{k-1} \to M_0^k$ represents the homotopy class of the inclusion of the boundary, $\partial M_0 \to M_0$.

In particular, $d: [M_0^k, PL/0] \to \Gamma_{k-1}$ and $d: [M_0^k, SPL] \to \Gamma_{k-1}$ are group homomorphisms. Also, $\Delta_c(M_0^k)$ and $\Delta_{tc}(M_0^k)$ are homotopy invariants of $M_0^k$.

Recall that for a simply connected, closed manifold, $M^k$, there is the surgery obstruction $s: [M^k, F/0] \to P_k$, where $P_k = \mathbf{Z}, 0, \mathbf{Z}_2, 0$ if $k \equiv 0, 1, 2, 3 \pmod 4$, respectively, defined as follows [18]. If $u \in [M^k, F/0]$, represent $u$ by a framing $f: M' \times \mathbf{R}^q \to E(\xi^q(u))$ of some manifold $M'$ in the total space of the bundle $\xi^q(u) = i_*(u)$ over $M$.

Then $s(u) \in P_k$ is the obstruction to constructing a homotopy equivalence $M'' \times \mathbf{R}^q \to \to E(\xi^q(u))$, framed cobordant to $M' \times \mathbf{R}^q$ in $E(\xi^q(u)) = E(\xi^q)$.

PROPOSITION 2.4 (Sullivan). *Suppose $u: M_0^k \to F/0$ extends to a map $\bar{u}: M^k \to F/0$. Then $du \in bP_k$. In fact, $du = bs(\bar{u})$ where $b: P_k \to bP_k$ is the natural projection.*

PROOF. Represent $\bar{u}$ by a framing of a connected sum $M' \# W$ in the vector bundle $E(\xi(\bar{u}))$ over $M$ where the projection $M_0' \to M_0$ is a homotopy equivalence and where $W$ is an almost parallelizable manifold. Then $s(\bar{u}) = -[W] \in P_n$ where $P_n$ is regarded as the group of cobordism classes of almost parallelizable $PL$ manifolds. By smoothing theory, in the complement of a point, $M' \# W$ inherits a smooth structure from $E(\xi(\bar{u}))$ and $\partial(M' \# W)_0 = \partial M_0$. Then $du = \partial M_0' - \partial M_0 = -\partial W_0 = bs(\bar{u}) \in bP_k$.

REMARK 2.5. If $k = 4n$ and $u \in [M^{4n}, F/0]$ is represented by $f: M' \times \mathbf{R}^q \to E(\xi^q)$, then

$$s(u) = (\tfrac{1}{8})(\operatorname{index}(M) - \operatorname{index}(M')) = (\tfrac{1}{8})\langle L(M)(1 - L(\xi)), [M^{4n}]\rangle \in \mathbf{Z}$$

since $\tau_{M'} = f^*(\tau_M + \xi)$.

If $k = 4n+2$ and $u \in [M^{4n+2}, F/0]$, there is also a cohomology formula for $s(u)$; namely,

$$s(u) = \langle v^2(M) \cdot u^*(K), [M]_2 \rangle \in \mathbf{Z}_2$$

where $v(M) = 1 + v_1(M) + v_2(M) + \ldots \in H^*(M, \mathbf{Z}_2)$ is the total *Wu* class, and $K = k_2 + k_6 + k_{10} + \ldots \in H^{4*+2}(F/0, \mathbf{Z}_2)$ is a suitable class [18].

## § 3. *The composition $\varrho d: [M_0^k, F/0] \to \Gamma_{k-1} \to \pi_{k-1}^s/\operatorname{im}(J)$*

Let $\partial: S^{k-1} \to M_0^k$ represent the homotopy class of the inclusion of the boundary, $\partial M_0^k \to M_0^k$. Then $\partial$ induces $\partial^*: [M_0^k, F/0] \to [S^{k-1}, F/0] = \pi_{k-1}(F/0)$. Further, image $(\partial^*)$ is contained in the torsion subgroup of $\pi_{k-1}(F/0)$, which is isomorphic to $\pi_{k-1}^s/\operatorname{im}(J)$.

PROPOSITION 3.1. *Let $u \in [M_0^k, F/0]$. Then*

$$\varrho(du) = \partial^*(u) \in \pi_{k-1}^s/\operatorname{im}(J) \subset \pi_{k-1}(F/0).$$

*Proof.* Let $u = \theta(h)$, where $h: M_0' \to M_0$. Then $u$ is represented by a fibre homotopy trivialization of $\xi_0(h) = \xi_0$, defined by a framing $H: M_0' \times \mathbf{R}^q \simeq E(\xi_0^q)$. The restriction of $\xi_0$ to $\partial M_0^k$ is trivial. For, if $k-1 \equiv 0$ or $4 \pmod 8$, the Pontrjagin class of $\xi_0 |_{\partial M_0^k}$ is zero, and if $k-1 \equiv 1$ or $2$ (mod. 8) $\xi_0 |_{\partial M_0^k}$ is fibre homotopically trivial. Thus, $H$ induces a framing $\partial H: \partial M_0' \times \mathbf{R}^q \simeq \partial M_0 \times \mathbf{R}^q$, which represents $\partial^*(u) \in \pi_{k-1}(F/0)$. It now follows from the definition of the Kervaire-Milnor map, $\varrho$, and a little smoothing theory, that $\partial^*(u)=\varrho(\partial M_0'-\partial M_0)=\varrho(\mathrm{du})$.

COROLLARY 3.2. *The composition* $\varrho d:[M_0^k, F/0]\to\pi_{k-1}^s/\mathrm{im}(J)$ *is a homomorphism of groups. Thus, if* $u, v\in[M_0^k, F/0]$ *then* $du+dv-d(u+v)\in bP_k\subset\Gamma_{k-1}$.

COROLLARY 3.3. *Let* $h:M_0'\to M_0$ *be any degree one map (not necessarily a homotopy equivalence). Then* $\varrho(dh^*(u))=\varrho(du)$, *where* $u\in[M_0, F/0]$ *and* $h^*:[M_0, F/0]\to[M_0', F/0]$. *Thus* $dh^*(u)-du\in bP_k\subset\Gamma_{k-1}$.

§ 4. *Discussion of* $F/0$. If we are to apply the results of § 2 and § 3 (and those in § 5 and § 6 below), we must be able to compute $[M_0^k, F/0]$. In general, this is difficult. The following discussion relates the group $[M_0^k, F/0]$ to more familiar homotopy invariants of $M_0^k$.

There are fibrations $S0\xrightarrow{\Omega J}SF\xrightarrow{j}F/0\xrightarrow{i}BS0\xrightarrow{J}BSF$. These induce an exact sequence of groups

$$K0^{-1}(X)\to[X, SF]\xrightarrow{j_*}[X, F/0]\xrightarrow{i_*}K0^0(X)\to J(X)\to 0$$

for any finite complex $X$. Further, since $SF_{q+1}$ is a component of $\Omega^q S^q$, $[X, SF]= =\lim_\to[S^q\wedge X, S^q]=\pi_0^s(X)$, as sets, where $\pi_0^s(X)$ is the $0^{\text{th}}$ stable cohomotopy group of $X$. Actually, $\pi_0^s(X)$ is a ring, and, as groups, $[X, SF]\simeq 1+\pi_0^s(X)$ where the addition on the right is given by $(1+\alpha)(1+\beta)=1+\alpha+\beta+\alpha\beta$ [13].

The Adams conjecture on $J:K0^0(X)\to J(X)$ can be stated as follows ([1]):

*4.1* Let $\xi\in K0^0(X)$. Then there is an integer, $e(k,\xi)$, such that $J(k^{e(k,\xi)}(\psi^k-1)(\xi))=0$ where $\psi^k$ is the Adams operation.

Since $K0^0(X)$ is finitely generated, we may choose $e(k,\xi)=e(k)$ independent of $\xi$. For any function $e(k)$, Adams has proved that kernel $(J)=i_*([X, F/0])$ is contained in the subgroup of $K0^0(X)$ generated by the elements $k^{e(k)}(\psi^k-1)(\xi)$, $\xi\in K0^0(X)$. The Adams conjecture 4.1 has recently been proved by Sullivan and Quillen.

PROPOSITION 4.2. *If* $K0^0(M^k)\to K0^0(M_0^k)$ *is surjective (e.g., if* $k-1\not\equiv 1$ *or* 2 (mod 8) *or if* $M^k$ *is a spin manifold), then each element* $w\in[M_0^k, F/0]$ *can be written as a sum,* $w=u+v$, *where* $u\in$image $([M^k, F/0])$ *and* $v\in$image $([M_0, SF)]$.

*Proof.* $J(\xi_0(w))=J(i_*(w))=0$. It follows that there is an element $\xi\in K0^0(M^k)$ such that $J(\xi)=0$ and $\xi|_{M_0}=\xi_0(w)=\xi_0$. Then $\xi=i_*(\bar{u})$ for some $\bar{u}\in[M^k, F/0]$. Let $u=\bar{u}|_{M_0}$. Then $w-u\in$kernel $(i_*)=$ image $(j_*)$, and 4.2 is proved.

*Remark* 4.3. It is a consequence of the Adams conjecture that for each prime $p$, there is a homotopy equivalence $(F/0)_{(p)}\sim BS0_{(p)}\times Cok(J)_{(p)}$ where $X_{(p)}$ denotes the localization of $X$ at $p$. Morevoer, $SJ_{(p)} \sim \mathrm{im}(J)_{(p)} \times Cok(J)_{(p)}$, and the map $j_{(p)}: SF_{(p)} \to (F/0)_{(p)}$ is a product map $j_{(p)} \times Id: \mathrm{im}(J)_{(p)} \times Cok(J)_{(p)} \to BS0_{(p)} \times Cok(J)_{(p)}$. This factoring of $(F/0)_{(p)}$ enables one to also establish the conclusion of 4.2 in the case $(k-1) \equiv 2 \pmod 8$.

PROPOSITION 4.4. *If $u, v \in [M_0^k, F/0]$, with $u \in$ image $([M_0^k, F/0])$ and $v \in$ image $([M_0, SF])$, then $d(u+v) = du + dv \in \Gamma_{k-1}$.*

*Proof.* Let $v = \theta(h)$, and let $h^*(u) = \theta(g)$ where $h: M_0' \to M_0$ and $g: M_0'' \to M_0'$ are homotopy equivalences. By 2.3, $\theta(f) = u + v$ where $f = hg: M_0'' \to M_0$. Thus, $d(u+v) = = \partial M_0'' - \partial M_0 = (\partial M_0'' - \partial M_0') + (\partial M_0' - \partial M_0) = dh^*(u) + dv$.

By the hypothesis, $h: M_0' \to \mathbf{M}_0$ is a tangential homotopy equivalence. Also, the maps $M_0' \xrightarrow{h} M_0 \xrightarrow{u} F/0$ extend to maps $M' \xrightarrow{h} M \xrightarrow{\bar{u}} F/0$. By Proposition 2.4, $du$ and $dh^*(u)$ belong to $bP_k \in \Gamma_{k-1}$. Since $h^*(L(M)) = L(M')$ and $h^*(v^2(M)) = v^2(M')$, it follows from the formulas in Remark 2.5 that $du = dh^*(u)$. Thus $d(u+v) = dh^*(u) + dv = = du + dv$.

The following is an immediate consequence of Propositions 2.4, 4.2, 4.4, and Remark 4.3, and is one of our main results.

PROPOSITION 4.5. *Assume that $k \not\equiv 2$ (mod.8) or that $M_0^k$ is a spin manifold. Then*

$$\Delta_h(M_0^k) = (\Delta_h(M_0^k) \cap bP_k) + \Delta_{th}(M_0^k) \subset \Gamma_{k-1}.$$

Here, by the sum of the two subsets, we mean all elements $\Sigma + \Sigma'$ where $\Sigma \in \Delta_h(M_0^k) \cap bP_k$ and $\Sigma' \in \Delta_{th}(M_0^k)$.

*Remark* 4.6. Note that the map $\partial^*: [M_0^k, SF] \to \pi_{k-1}(SF) = \pi_{k-1}^s$ is an invariant of the stable homotopy of $M_0^k$ and can be computed as

$$\partial^*: [S^q \wedge M_0^k, S^q] \to \pi_{q+k-1}(S^q) = \pi_{k-1}^s, \ q \text{ large}.$$

We will need the following familiar invariant. Consider the subgroup of elements $(\xi, \alpha) \in K0^0(X) \otimes \pi_{4k-1}(X)$ such that $ph_k(\xi) = 0 \in H^{4k}(X, Q)$ and $\alpha^* = 0: H^{4k-1}(X) \to \to H^{4k-1}(S^{4k-1})$. Let $\bar{X} = X \cup_\alpha e^{4k}$, and let $\bar{\xi} \in K0^0(\bar{X})$ restrict to $\xi \in K0^0(X)$. Then $ph_k(\bar{\xi}) \in p^*(H^{4k}(S^{4k}, Q)) = Q$, where $p: \bar{X} \to S^{4k}$ is the projection. Further, since $\bar{\xi}$ is well-defined modulo $p^*(K0^0(S^{4k}))$, $ph_k(\bar{\xi})$ is well-defined modulo $p^*(H^{4k}(S^{4k}, a_k Z))$. It follows that $e_R(\xi, \alpha) = (1/a_k)\, ph_k(\bar{\xi}) \in Q/Z$ is a well-defined homomorphism. Moreover, the diagram

$$\begin{array}{ccc} K0^0(X) \otimes \pi_{4k-1}(X) & \searrow^{e_R} & \\ \downarrow^{\wp \otimes s} & & Q/Z \\ K0^0(S^8 \wedge X) \otimes \pi_{4k+7}(S^8 \wedge X) & \nearrow_{e_R} & \end{array} \tag{4.7}$$

commutes (when $e_R$ is defined), where $\mathscr{P}$ is the periodicity isomorphism and $s$ is suspension. $e_R$ can be interpreted as a functional operation from $K0$-theory to cohomology. If $X = S^{8n}$ and $\xi \in K0^0(S^{8n})$ is a generator, we recover the Adams homomorphism $e_R: \pi_{8n+4k-1}(S^{8n}) \to Q/Z$ [2]. If $X = M_0^{4n}$ and $\alpha \in \pi_{4n-1}(M_0^{4n})$ represents the inclusion of the boundary, we get a homomorphism $e_R: K0^0(M_0^{4n}) \to Q/Z$.

The following K0-theory invariant of $F/0$ bundles will also be essential.

PROPOSITION 4.8. *There is an element* $\gamma \in 1 + K0^0(F/0)$ *such that* $ph(\gamma) = \hat{A} \in H^{**}(F/0, Q) \overset{i^*}{\simeq} H^{**}(BS0, Q)$. *Further, if* $u, v \in [X, F/0]$ *then* $\gamma(u+v) = \gamma(u) \cdot \gamma(v) \in 1 + K0^0(X)$, *where by* $\gamma(u)$ *we mean* $u^*(\gamma) \in 1 + K0^0(X)$.

*Proof.* The universal bundle over $F/0$ admits a unique spin structure. Thus, the Thom space $M(F/0)$ has two canoncial $K0$-theory orientations, namely, an orientation $U_1 \in K0^0(M(F/0))$ induced from $M$ Spin, with $ph(U_1) = \Phi(\hat{A}^{-1}) \in H^{**}(M(F/0), Q)$, and an orientation, $U_2$, with $ph(U_2) = \Phi(1)$, induced from the sphere spectrum via a fibre homotopy trivialization. Define $\gamma \in 1 + K0^0(F/0)$ by the equation $\gamma \cdot U_1 = U_2 \in K0^0(M(F/0))$. Then $\Phi(1) = ph(U_2) = ph(\gamma) ph(U_1) = \Phi(ph(\gamma) \cdot A^{-1})$, hence $ph(\gamma) = \hat{A}$.

The second statement follows from universal multiplicative properties of the orientations $U_1$ and $U_2$.

The final three results in this section are technical results about the invariants $e_R$ and $\gamma$ which we will need in §5.

Let $u \in [M_0^k, F/0]$ correspond to a homotopy equivalence $h: M_0' \to M_0$. Homotope $h$ to an embedding $h: M_0' \to M_0 \times \mathbf{R}^{8q}$. The normal bundle of $M_0'$ in $M_0 \times \mathbf{R}^{8q}$ is $h^*(-\xi_0(u))$, and we have the "collapsing map" $c: T(e_{M_0}^{8q}) \to T(h^*(-\xi_0)_{M_0'}^{8q})$. Since $\xi_0$ is a spin vector bundle there are Thom isomorphisms $\Phi_{K0}: K0(M_0') \overset{\sim}{\to} K0^0(T(h^*(-\xi_0)_{M_0'}^{8q}))$ and $\Phi_{K0} = \mathscr{P}: K0(M_0) \overset{\sim}{\to} K0^0(T(e_{M_0}^{8q}))$, and a Gysin homomorphism $h_*: K0(M_0') \to K0(M_0)$ defined by $h_*(x) = \mathscr{P}^{-1} c^* \Phi_{K0}(x)$.

PROPOSITION 4.9. *If* $u \in [M_0, F/0]$ *corresponds to* $h: M_0' \to M_0$ *then* $h_*(1) = \gamma(u) \in K0(M_0)$.

*Proof.* This follows from the definition of $\gamma(u)$ and the observation that the fibre homotopy trivialization

$$T(\xi_0^{8q} + e_{M_0}^{8q}) \xrightarrow{\bar{c}} T(h^*(\xi_0^{8q}) + h^*(-\xi_0^{8q})) = T(e_{M'_0}^{16q}) \xrightarrow{\pi} S^{16q}$$

represents $u \in [M_0, F/0]$, where $\bar{c}$ is defined by embedding $M_0 \times \mathbf{R}^{8q} \subset E(\xi_0^{8q}) \times \mathbf{R}^{8q}$ and extending $c$, and $\pi$ is the projection.

PROPOSITION 4.10(i) *Let* $u, v \in [M_0^{4n}, F/0]$. *If* $v \in [M_0, PL/0]$ *or* $v \in [M_0, SF]$, *then* $e_R(\gamma(u+v)) = e_R(\gamma(u)) + e_R(\gamma(v)) \in Q/\mathbf{Z}$.

(ii) *Suppose* $M_0^{4n}$ *is a spin manifold. If* $u \in [M_0^{4n}, SF]$ *or* $u \in [M_0^{4n}, PL/0]$, *then* $e_R(\gamma(u)) = e_R(\xi_0(u)) = 0$.

*Proof.* Let $\overline{\gamma(u)}, \overline{\gamma(v)} \in KO(M^{4n})$ extend $\gamma(u), \gamma(v) \in KO(M_0^{4n})$. By 4.8, $\gamma(u+v) = = \gamma(u) \cdot \gamma(v)$, so $\overline{\gamma(v)} \cdot \overline{\gamma(v)} \in KO(M^{4n})$ is an extension of $\gamma(u+v)$. Then

$$\begin{aligned} e_R(\gamma(u+v)) &= (1/a_n) \langle ph(\overline{\gamma(u)} \cdot \overline{\gamma(v)}), [M^{4n}] \rangle \\ &= (1/a_n) \langle ph(\overline{\gamma(u)})\, ph(\overline{\gamma(v)}), [M^{4n}] \rangle \in Q/\mathbf{Z}. \end{aligned}$$

From the assumption, it follows that $ph(\overline{\gamma(v)}) = 1 + ph_n(\overline{\gamma(v)})$; hence

$$\begin{aligned} &(1/a_n) \langle ph(\overline{\gamma(u)})\, ph(\overline{\gamma(v)}), [M^{4n}] \rangle \\ &\quad = (1/a_n) \langle ph_n(\overline{\gamma(u)}) + ph_n(\overline{\gamma(v)}), [M^{4n}] \rangle \in Q/\mathbf{Z}, \end{aligned}$$

and 4.10(i) follows immediately.

For 4.10(ii), note that the Thom space of the normal bundle of $M_0$, $T(\nu_{M_0}^{8q})$, has a canonical $K0$-orientation. This extends to some $K0$-orientation, $U$, of $T(\nu_M^{8q})$. Then, since there is a degree one map $S^{8q+4n} \to T(\nu_M^{8q})$, we have

$$(1/a_n) \langle ph(\overline{\gamma(u)} - 1)\, ph(U), [T(\nu_M)] \rangle \in \mathbf{Z}.$$

Since $ph(\overline{\gamma(u)}) - 1 = ph_n(\overline{\gamma(u)})$, it follows that

$$\begin{aligned} e_R(\gamma(u)) &= (1/a_n) \langle ph_n(\overline{\gamma(u)}), [M^{4n}] \rangle \\ &= (1/a_n) \langle ph_n(\overline{\gamma(u)})\, ph(U), [T(\nu_M)] \rangle = 0 \in Q/\mathbf{Z}. \end{aligned}$$

Similarly, $e_R(\xi_0(u)) = (1/a_n) \langle ph_n(\overline{\xi_0(u)})\, ph(U), [T(\nu_M)] \rangle = 0 \in Q/\mathbf{Z}$, and 4.10(ii) is proved.

PROPOSITION 4.11. *Let* $u \in [M_0^{4n}, SF]$. *Then* $e_R(\gamma(u)) = e_R(\partial^*(u))$ *where* $\partial^*(u) \in \pi_{4n-1}(SF) = \pi_{4n-1}^s$. *Moreover,* $e_R(\gamma(u))$ *has order a power of* 2.

*Proof.* Let $v: M_0 \times S^{8q} \to S^{8q}$ be the adjoint of $u: M_0 \to SF_{8q+1}$, and let $\alpha \in KO^0(S^{8q})$ be the generator. Then $\gamma(u) \cdot \pi^*(\alpha) = v^*(\alpha)$, where $\pi: M_0 \times S^{8q} \to S^{8q}$ is the projection. Thus $v^*(\alpha) - \pi^*(\alpha) = \mathscr{P}(\gamma(u) - 1) \in KO^0(S^{8q} \wedge M_0)$. It follows that there is a homotopy commutative diagram

$$\begin{tikzcd}
 & S^{8q} \wedge M_0^{4n} \arrow[rr] \arrow[dr, "p(\gamma(u)-1)"] \arrow[dd, "v-\pi"'] & & S^{8q} \wedge M^{4n} \arrow[dl] \arrow[dd] \arrow[dr] & \\
S^{8q+4n-1} \arrow[ur, "\partial"] \arrow[dr, "\partial^*(u)"'] & & BSO & & S^{8q+4n} \\
 & S^{8q} \arrow[ur, "\alpha"] \arrow[rr] & & S^{8q} \cup_{\partial^*(u)} e^{8q+4n} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

From the definitions and diagram 4.7, one sees that $e_R(\partial^*(u)) = e_R(\gamma(u))$.

For the second statement, it is only necessary to observe that there are spin manifolds, $N_0^{4n}$, with $\partial N_0^{4n}=S^{4n-1}$, and maps $g$: $N_0^{4n}$, $\partial N_0^{4n}\to M_0^{4n}$, $\partial M_0^{4n}$ of degree a power of 2, say $2^r$. Then $2^r e_R(\gamma(u))=2^r e_R(\partial^*(u))=e_R(2^r\partial^*(u))=e_R(\partial^*(g^*(u)))= =e_R(\gamma(g^*(u)))=0$, by 4.10(ii).

§5. *The composition* $f_R d$: $[M_0^{4n}, F/0]\to \mathbf{Z}_{\theta_n}$. The invariant $f_R:\Gamma_{4n-1}\to\mathbf{Z}_{\theta_n}$ is defined as follows. Given $\Sigma^{4n-1}\in\Gamma_{4n-1}$, let $\Sigma^{4n-1}=\partial W_0^{4n}$, where $W_0^{4n}$ is a smooth spin manifold such that the decomposable Pontryagin numbers of $W^{4n}$ vanish. Then

$$f_R(\Sigma^{4n-1})=(\tfrac{1}{8})\text{ index }(W^{4n})\in\mathbf{Z}/\theta_n\cdot\mathbf{Z}.$$

(It is proved in [9] that such manifolds $W_0^{4n}$ exist and that $f_R$ is well-defined.)

It will be convenient to regard $f_R$ as a homomorphism $f_R:\Gamma_{4n-1}\to Q/\mathbf{Z}$. Namely, define $f_R(\Sigma^{4n-1})=(\tfrac{1}{8}\theta_n)$ index $(W^{4n})\in Q/\mathbf{Z}$, where $W^{4n}$ is as above.

Recall that the $L$-genus is given by

$$L_n(p_1\dots p_n)=(8\theta_n p_n/a_n(2n-1)!\,j_n)+L_n(p_1\dots p_{n-1},0).$$

PROPOSITION 5.1. *Let* $u\in[M_0^{4n}, F/0]$. *Then*

$$f_R(du)=(\tfrac{1}{8}\theta_n)\langle L(M)\,(1-L(\xi)),[M^{4n}]\rangle\in Q/\mathbf{Z},$$

*where* $L(\xi)=L(p_1(\xi_0(u))\dots p_{n-1}(\xi_0(u)),p_n(\xi))$ *and* $p_n(\xi)/a_n(2n-1)!j_n\in Q/\mathbf{Z}$ *is d determined (formally) by the equations*

$$(1/a_n)\langle\hat{A}(\xi),[M^{4n}]\rangle=e_R(\gamma(u))\in Q/\mathbf{Z}$$

*and*

$$(1/a_n)\langle ph(\xi),[M^{4n}]\rangle=e_R(\xi_0(u))\in Q/\mathbf{Z}.$$

The proof of Proposition 5.1 will require some preliminary results.

First, note that since

$$(1/a_n)\hat{A}_n(p_1\dots p_n)=(-\text{num}(B_n/4n)\,p_n/a_n(2n-1)!\,j_n)+\hat{A}_n(p_1\dots p_{n-1},0)$$

and

$$(1/a_n)\,ph_n(p_1\dots p_n)=((-1)^{n-1}j_n p_n/a_n(2n-1)!\,j_n)+ph_n(p_1\dots p_{n-1},0),$$

and since num $(B_n/4n)$ and $j_n=$denom $(B_n/4n)$ are relatively prime, it follows that the equations in 5.1 for $p_n(\xi)/a_n(2n-1)!j_n\in Q/\mathbf{Z}$ have at most one solution.

Secondly, the computation of $p_n(\xi)/a_n(2n-1)!j_n$ in Proposition 5.1 is purely formal. That is, we do not assert the existence of a vector bundle $\xi$ with the properties indicated. However, Proposition 5.1 and Remark 2.5 are closely related. If $u\in[M_0^{4n}, F/0]$ extends to $\bar{u}\in[M^{4n}, F/0]$, then $\xi=\xi(\bar{u})$ is an extension of $\xi_0=\xi_0(u)$. Remark 2.5 asserts that

$f_R(du)=(\frac{1}{8}\theta_n)\langle L(M)(1-L(\xi)), [M^{4n}]\rangle\in Q/\mathbf{Z}$. Moreover, $\gamma(\bar{u})\in \tilde{K}O(M)$ extends $\gamma(u)\in \tilde{K}O(M_0)$, hence $e_R(\gamma(u))=(1/a_n)\ \langle ph(\gamma(\bar{u})), [M]\rangle=(1/a_n)\langle \hat{A}(\xi), [M]\rangle$ and also, of course, $e_R(\xi_0)=(1/a_n)\langle ph(\xi), [M]\rangle$.

Recall that the image of the Adams homomorphism $e_R:\pi^s_{4n-1}\to Q/\mathbf{Z}$ consists of integral multiples of $1/j_n=1/\text{denom}\ (B_n/4n)$ [2]. Thus, there is a unique homomorphism $\tilde{e}_R:\pi^s_{4n-1}\to Q/\mathbf{Z}$, defined by num $(B_n/4n)\ \tilde{e}_R(\alpha)=e_R(\alpha)$. If $\alpha$ is the image of the generator of $\pi_{4n-1}(SO)=\mathbf{Z}$, then $e_R(\alpha)=(B_n/4n)=\text{num}\ (B_n/4n)/\text{denom}\ (B_n/4n)$. Thus, $\tilde{e}_R$ is a normalization of $e_R$, with $\tilde{e}_R(\alpha)=1/j_n$.

PROPOSITION 5.2. *If $u\in[M_0^{4n}, SF]$, then $f_R(du)=\tilde{e}_R(\partial^*(u))\in Q/\mathbf{Z}$. In particular, $f_R(du)$ has order a power of* 2.

*Proof.* Represent $u$ by a tangential homotopy equivalence $h_0:M_0'\to M_0$. Let $h$ denote the obvious extension $h:M'\to M$. Then $\tau_{M'}=h^*(\tau_M+p^*(\sigma))$ as $PL$ bundles, where $p:M^{4n}\to S^{4n}$ is a map of degree one and $\sigma\in\pi_{4n}(BSPL)$. Since $h_0$ is a tangential homotopy equivalence, and since index $(M')=$index $(M)$, it is easy to see that the Pontrjagin class $p_n(\sigma)=0$. That is, $\sigma$ is a torsion element of $\pi_{4n}(BSPL)$. Further, $J_{PL}(\sigma)=\partial^*(u)$, where $J_{PL}:\pi_{4n}(BSPL)\to\pi_{4n}(BSF)=\pi^s_{4n-1}$, and $\beta(\sigma)=du$, where $\beta:\pi_{4n}(BSPL)\to\pi_{4n-1}(PL/0)=\Gamma_{4n-1}$. It then follows from [9; Theorems 4.7, 4.8] that num $(B_n/4n)f_R(du)=e_R(\partial^*(u))$. This relation, together with 4.11, proves Proposition 5.2.

Note that if $u\in[M_0^{4n}, SF]$, then Proposition 5.1 asserts that $f_R(du)=-p_n(\xi)/a_n(2n-1)!j_n\in Q/\mathbf{Z}$, where

$$(1/a_n)\ \langle \hat{A}(\xi), [M^{4n}]\rangle = -\ \text{num}\,(B_n/4n)\ p_n(\xi)/a_n(2n-1)!\,j_n = e_R(\gamma(u))\in Q/\mathbf{Z}.$$

Thus 5.2 and 4.11 imply 5.1 in the case $u\in[M_0^{4n}, SF]$.

COROLLARY 5.3(i). *The map $d:[M_0^{4n}, SF]\to\Gamma_{4n-1}$ is a group homomorphism.*
(ii) *If $h:M_0'\to M_0$ is any degree one map, then the diagram*

$$\begin{tikzcd}[M_0, SF] \arrow[d, "h^*"] \arrow[dr, "d"] & \\ {[M_0', SF]} \arrow[r, "d"'] & \Gamma_{4n-1}\end{tikzcd}$$

*commutes.*

*Proof.* This follows from 5.2 and 3.1 since $f_R\oplus\varrho:\Gamma_{4n-1}\to\mathbf{Z}_{\theta_n}\oplus\ (\pi^s_{4n-1}/\text{im}(J))$ is an isomorphism.

COROLLARY 5.4. *If $u\in[M_0^{4n}, F/0]$ and $v\in[M_0^{4n}, SF]$, then $d(u+v)=du+dv$.*
*Proof.* This follows from 4.2, 4.4 and 5.3(i).
We can also prove Proposition 5.1. By 2.5 and 5.2, Proposition 5.1 is true if

$u \in$ image $([M^{4n}, F/0])$ or if $u \in$ image $([M_0^{4n}, SF])$. By 4.4, it suffices to prove that

$$(\tfrac{1}{8}\theta_n)\langle L(M)(1 - L(\xi(u+v))), [M^{4n}]\rangle$$

$$(\tfrac{1}{8}\theta_n)\langle L(M)(1 - L(\xi(u))), [M^{4n}]\rangle + (\tfrac{1}{8}\theta_n)\langle L(M)(1 - L(\xi(v))), [M^{4n}]\rangle$$

if $u \in$ image $([M^{4n}, F/0])$ and $v \in$ image $([M_0^{4n}, SF])$. Since $L(\xi(v)) = 8\theta_n p_n(\xi(v))/a_n(2n-1)!j_n$, this is equivalent to proving that $p_n(\xi(u+v))/a_n(2n-1)!j_n = p_n(\xi(u))/a_n(2n-1)!j_n + p_n(\xi(v))/a_n(2n-1)!j_n$. But, by 4.10(i), $e_R(\gamma(u+v)) = e_R(\gamma(u)) + e_R(\gamma(v))$, and, of course, $e_R(\xi_0(u+v)) = e_R(\xi_0(u+v)) = e_R(\xi_0(u)) + e_R(\xi_0(v))$. The equations given in 5.1 which determine $p_n(\xi)/a_n(2n-1)!j_n$ now yield the desired additivity result.

Remark 4.6 and Propositions 3.1 and 5.2 show that $\Delta_{th}(M_0^{4n})$ is computable in terms of the stable homotopy theory invariant $\partial^*: [S^q \wedge M_0^{4n}, S^q] \to \pi_{q+4n-1}(S^q) = \pi_{4n-1}^s$. Proposition 2.4 and Remark 2.5, together with the Adams conjecture, show that $\Delta_h(M_0^{4n}) \cap bP_{4n}$ is computable in terms of $L(M)$ and $ph(KO(M^{4n})) \subset H^{**}(M^{4n}, Q)$. Thus, $\Delta_h(M_0^{4n}) = (\Delta_h(M_0^{4n}) \cap bP_{4n}) + \Delta_{th}(M_0^{4n})$ is computable in terms of familiar invariants.

It is interesting that by using the Riemann-Roch theorem for spin maps, Proposition 5.1 can be proved without using Proposition 4.2 or the Adams conjecture. Then 3.1 and 5.1 provide, in a sense, a homotopy theoretic computation of the geometric map $d: [M_0^{4n}, F/0] \to \Gamma_{4n-1}$. However, use of the Adams conjecture gives the more practical description of $\Delta_h(M_0^{4n})$ above.

We now give some corollaries of the results above.

COROLLARY 5.5(i). *If $M_0^{4n}$ is a spin manifold and $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$, then $f_R(du) = 0$. Hence $du \in \pi_{4n-1}^s/im(J) \subset \Gamma_{4n-1}$.*
(ii) *If $M_0^{4n}$ is a weakly complex manifold and $u \in [M_0^{4n}, SF]$, then $a_n f_R(du) = 0$.*

*Proof.* In the notation of Proposition 5.1, it follows from 4.10(ii) that $p_n(\xi)/a_n(2n-1)!j_n = 0$. Hence, $L(\xi) = 1$ and $f_R(du) = 0$.

We will give an alternate proof of 5.5(i). Let $h: M_0' \to M_0$ represent $u$. Then $h^*(\tau_{M_0}) = \tau_{M_0'}$ as vector bundles if $u \in [M_0, SF]$, and as $PL$ bundles if $u \in [M_0, PL/0]$. In either case, $W_0 = M_0' \# (-M_0)$ is a spin manifold, $\partial W_0 = \partial M_0' - \partial M_0$, and all the Pontrjagin numbers of $W$, including $p_n(W)$, vanish. Then $f_R(du) = f_R(\partial M_0' - \partial M_0) = (\tfrac{1}{8}\theta_n)$ index $(W) = 0$.

5.5(ii) can be proved by an argument similar to the second proof of 5.5(i). Namely, if $M_0$ is weakly complex and $M_0'$, $W_0$ are as above, then $M_0'$ and $W_0$ are weakly complex, and all the Chern numbers of $W$ vanish. An invariant $f_c: \Gamma_{4n-1} \to Q/\mathbf{Z}$ is defined in [9], using weakly complex manifolds instead of spin manifolds, and $f_c = a_n f_R$. It follows that $0 = f_c(du) = a_n f_R(du)$.

COROLLARY 5.6. *If* $u \in [M_0^{4n}, PL/0]$, *then* num $(B_n/4n) f_R(du) = e_R(\gamma(u))$, *and* $f_R(du)$ *has order a power of* 2.

*Proof.* The first statement follows from Proposition 5.1, since $f_R(du) = -p_n(\xi)/a_n(2n-1)!j_n \in Q/\mathbf{Z}$ and $(1/a_n)\langle \hat{A}(\xi), [M^{4n}]\rangle = -\text{num } (B_n/4n) p_n(\xi)/a_n(2n-1)!j_n = e_R(\gamma(u)) \in Q/\mathbf{Z}$.

For the second statement, let $g: N_0^{4n}, \partial N_0^{4n} \to M_0^{4n}, \partial M_0^{4n}$ be a map of degree $2^r$ where $N_0^{4n}$ is a spin manifold. Then $2^r f_R(du) = f_R(dg^*(u)) = 0$ by 5.5(i).

COROLLARY 5.7. *If* $M_0^{4n}$ *is a spin manifold with* $f_R(\partial M_0^{4n}) \neq 0$ (*or if* $M_0^{4n}$ *is any manifold and* $f_R(\partial M_0^{4n})$ *has order not a power of* 2), *then* $0 \notin B_{th}(M_0^{4n})$ *and* $0 \notin B_c(M_0^{4n})$; *that is,* $M_0^{4n}$ *is not tangentially homotopy equivalent or combinatorially equivalent to a smooth manifold.*

*Proof.* This follows from 5.2 and 5.6.

Here is an example to show that $f_R d: [M_0^{4n}, SF] \to \mathbf{Z}_{\theta_n}$ is not zero in general. Adams has defined elements $\mu_k \in \pi_{8k+2}^s$ such that $2\mu_k = 0$, $\mu_k \eta \neq 0$ and $\mu_k \eta \in im(J) \subset \pi_{8k+3}^s$ [2]. If $M^{8k+4}$ is not a spin manifold (for example, $M^{8k+4} = \mathbf{C}P(4k+2)$), choose $x \in H^{8k+2}(M, Z_2)$ such that $S_q^2(x) \neq 0$ and let $g: M_0 \to S^{8k+2}$ be a map such that $g^*(\sigma) = x$, where $\sigma \in H^{8k+2}(S^{8k+2})$. Then the composition $S^{8k+3} \xrightarrow{\partial} M_0^{8k+4} \xrightarrow{g} S^{8k+2} \xrightarrow{\mu_k} SF$ represents $\partial^*(\mu_k g) = \mu_k \eta$, since $g\partial = \eta$. Since $\tilde{e}_R(\mu_k \eta) = \frac{1}{2} \in Q/\mathbf{Z}$, 5.2 implies $f_R(d(\mu_k g)) = \frac{1}{2} \in Q/Z$.

In [10] we showed that the element $\mu_k$ could, in fact, be defined in $\pi_{8k+2}(SPL)$. Thus, in the example above, we actually have $u = \mu_k g \in [M_0^{8k+4}, SPL]$ and $du \in \Delta_{tc}(M_0^{8k+6})$ is the element of order 2 in $bP_{8k+4}$. I do not know of an example of $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$ such that $a_n \cdot f_R(du) \neq 0$.

We next give a somewhat simpler formula for $f_R d: [M_0^{4n}, F/0] \to \mathbf{Z}_{\theta_n}$, when $M_0^{4n}$ is a spin manifold, generalizing 5.5(i).

COROLLARY 5.8. *Let* $u \in [M_0^{4n}, F/0]$, *where* $M_0^{4n}$ *is a spin manifold. Then* $f_R(du) = (\frac{1}{8}\theta_n) < L(M)(1 - L(\xi)), [M] > \in Q/\mathbf{Z}$, *where* $L(\xi)$ *is as in* 5.1 *and* $(p_n(\xi)/a_n(2n-1)!j_n) \in Q/\mathbf{Z}$ *is determined by the equations*

$$(1/a_n)\langle(\hat{A}(\xi) - 1)\hat{A}(M), [M]\rangle = 0 \in Q/\mathbf{Z}$$

*and*

$$(1/a_n)\langle ph(\xi)\hat{A}(M), [M]\rangle = 0 \in Q/\mathbf{Z}.$$

*Proof.* This follows from 4.4, 5.5(i), and 2.4, and the Riemann-Roch Theorem for manifolds with framed boundary.

The point of 5.8 is that for spin manifolds, $f_R(du)$ depends only on the Pontrjagin classes of $M_0^{4n}$ and $\xi_0(u)$, and not on the $K0$-theory invariants $\gamma(u)$ and $\xi_0(u)$. This is because if $W_0 = M_0' \# (-M_0)$ then $W_0$ is a spin manifold, $\partial W_0 = \partial M_0' - \partial M_0$, and the

Pontrjagin numbers of $W$, including $p_n(W)$, are functions of the Pontrjagin classes of $M_0$ and $\xi_0(u)$. Thus $f_R(du)=f_R(\partial W_0)$ can be computed in terms of Pontrjagin classes alone. 5.8 gives a specific formula.

In the next result, we study the deviation of $d: [M_0^{4n}, F/0] \to \Gamma_{4n-1}$ from linearity.

COROLLARY 5.9. *Let* $u, v \in [M_0^{4n}, F/0]$. *Then*

$$du + dv - d(u+v) = (\tfrac{1}{8})\langle L(M)(L(\xi_0(u)) - 1)(L(\xi_0(v)) - 1), [M]\rangle \in \mathbf{Z}/\theta_n\mathbf{Z} = bP_{4n}.$$

*Proof.* By 3.2, it suffices to prove that

$$f_R(du) + f_R(dv) - f_R(d(u+v)) = (\tfrac{1}{8}\theta_n)\langle L(M)(L(\xi_0(u)) - 1) \times (L(\xi_0(v)) - 1), [M]\rangle \in Q/\mathbf{Z}.$$

By 4.4 and 5.3(i), we may assume that $u$, $v \in$ image $([M^{4n}, F/0])$. The formula now follows from 2.4 since $L(\xi(u+v)) = L(\xi(u))\, L(\xi(v))$, hence

$$\begin{aligned} L(\xi(u+v)) - 1 &= (L(\xi(u)) - 1)(L(\xi(v)) - 1) + (L(\xi(u) - 1) + (L(\xi(v)) - 1) \\ &= (L(\xi_0(u)) - 1)(L(\xi_0(v)) - 1) + (L(\xi(u)) - 1) \\ &\quad + (L(\xi(v)) - 1). \end{aligned}$$

Finally, we investigate the non-commutativity of $d$ with maps.

COROLLARY 5.10. *Let* $u \in [M_0^{4n}, F/0]$ *and let* $h: M_0' \to M_0$ *be a map of degree one. Then*

$$dh^*(u) - du = (\tfrac{1}{8})\langle (h^*(L(M)) - L(M'))(h^*L(\xi_0(u)) - 1), [M']\rangle \in \mathbf{Z}/\theta_n\mathbf{Z} = bP_{4n}.$$

*Proof.* By 3.3 it suffices to compute $f_R(dh^*(u)) - f_R(du)$. By 4.4 and 5.3(ii) we may assume that $u$ extends to $\bar{u} \in [M^{4n}, F/0]$. Then, by 2.4

$$\begin{aligned} f_R(dh^*(u)) - f_R(du) &= (\tfrac{1}{8}\theta_n)\langle (h^*L(M) - L(M'))\cdot(L(\xi(h^*(u)) - 1), [M']\rangle \\ &= (\tfrac{1}{8}\theta_n)\langle (h^*L(M) - L(M'))\cdot(L(\xi_0(h^*(u))) - 1), [M']\rangle \in Q/\mathbf{Z}. \end{aligned}$$

COROLLARY 5.11. *If* $h: M_0' \to M_0$ *is a degree one map of* $4n$*-manifolds which corresponds rational Pontrajagin classes, then the diagram*

$$\begin{tikzcd} {[M_0, F/0]} \arrow[dd, "h^*"'] \arrow[dr, "d"] & \\ & \Gamma_{4n-1} \\ {[M_0', F/0]} \arrow[ur, "d"'] & \end{tikzcd}$$

*commutes. Thus, if $h$ is a homotopy equivalence which corresponds rational Pontrjagin classes then* $\Delta_h(M_0) = \Delta_n(M_0')$.

§ 6. *The composition* $f_R d$: $[M_0^{8n+2}, F/0] \to \mathbf{Z}_2$. In this section we consider spin manifolds of dimension $8n+2$. The main result is Proposition 6.5.

In [4], $KO$-characteristic numbers $\pi^J(M^{8n+2}) \in \mathbf{Z}_2$, where $J=(j_1 \ldots j_r)$ and $\pi^J = \pi^{j_1} \ldots \pi^{j_r} \in KO^0(BSO)$ are defined for smooth spin manifolds. In [10], the definition is extend to almost smooth manifolds, provided that $J \neq (0)$. Roughly, this is done as follows.

Let $M_0^{8n+2}$ be a spin manifold with $\partial M_0^{8n+2} \in \Gamma_{8n+1}$. Since $v_{M_0}^{8q}$ is a spin vector bundle, the Thom space $T(v_{M_0}^{8q})$ has a canonical $KO$-orientation. This extends to a $KO$-orientation $U_M \in KO^0(T(v_M^{8q}))$. Also, $v_{M_0}$ extends to a vector bundle $v_M^*$ over $M$ and we have $v_M = v_M^* + p^*(\sigma)$ as $PL$ bundles, where $p: M^{8n+2} \to S^{8n+2}$ is a map of degree one and $\sigma \in \pi_{8n+2}(BSPL)$. Moreover, $v_M^*$ is well-defined by the additional assumption that $e_R J_{PL}(\sigma)=0$, where $J_{PL}: \pi_{8n+2}(BSPL) \to \pi_{8n+2}(BSF) = \pi_{8n+1}^s$ is the $PL$ $J$-homomorphism and $e_R: \pi_{8n+1}^s \to \mathbf{Z}_2$ is the homomorphism defined by Adams, which splits off image$(J)$ as a direct summand [2]. Set

$$\pi^J(M^{8n+2}) = c^* \Phi_{KO}(\pi^J(v_M^*)) \in KO^0(S^{8q+8n+2}) = \mathbf{Z}_2,$$

where $\Phi_{KO}: KO(M) \xrightarrow{\simeq} KO^0(T(v_M^{8q}))$ is the Thom isomorphism defined by multiplication by $U_M$, and $c: S^{8q+8n+2} \to T(v_M^{8q})$ is the map of degree one defined by an embedding $M^{8n+2} \to S^{8q+8n+2}$. If $J \neq (0)$, the $KO$-operation $\pi^J$ has filtration greater than zero, hence the product $\pi^J(v_M^*) \cdot U_M \in KO^0(T(v_M^{8q}))$ is independent of the choice of the extension $U_M$.

We will also use the notation

$$\pi^J(M^{8n+2}) = \langle \pi^J(v_M^*), [M]_{KO} \rangle \in \mathbf{Z}_2$$

where $[M]_{KO}$ is the fundamental $KO$-homology class dual to $U_M$.

E. Brown has defined a homomorphism $\psi: \Omega_{\mathrm{spin}}^{8n+2} - \mathbf{Z}_2$, extending the Kervaire-Arf invariant $\Omega_{\mathrm{framed}}^{8n+2} \to \mathbf{Z}_2$ [7]. In fact, Brown's definition of $\psi$ applies to $PL$ manifolds $M^{8n+2}$, with $w_1(M) = w_2(M) = 0$. From the main results of [4], it follows that for smooth $M^{8n+2}$,

$$\psi(M^{8n+2}) = \sum \alpha_J \cdot \pi^J(M^{8n+2}) + \sum \beta_I \cdot w^I(M^{8n+2}) \in \mathbf{Z}_2$$

where $\alpha_J, \beta_I \in \mathbf{Z}_2$, $J = (j_1 \ldots j_r)$, $1 < j_1 \leqslant \ldots \leqslant j_r$, and the $w^I$ are Stiefel-Whitney numbers.

LEMMA 6.1. *The coefficients* $\beta_I$, $\alpha_J$ *can be chosen such that* $\alpha_J = 0$ *if* $n(J) = j_1 + \ldots \ldots + j_r \neq 2n$ *and* $\Sigma_{n(J)=2n} \alpha_J \pi^J \equiv (L^{-1})_{2n}(0, \pi^2 \ldots \pi^{2n}) (\mathrm{mod.}\ 2)$ *where* $L = 1 + L_1 + L_2 + \ldots$ *is the Hirzebruch L-polynomial.*

*Proof.* We only outline the proof of this lemma, and refer to [4] and [8] for details. The homotopy elements in $\pi_{8n+2}$ ($M$ spin) which have Adams spectral sequence filtration greater than 2 are precisely the classes $\{M^{8n+2}\}$ with $w^I(M^{8n+2})= =\pi^J(M^{8n+2})=0$ for $n(J)\geqslant 2n$. It can be shown that $\psi(\{M^{8n+2}\})=0$ if $\{M^{8n+2}\}\in \in\Omega_{\text{spin}}^{8n+2}=\pi_{8n+2}(M\text{ spin})$ represents such a homotopy element. Thus $\alpha_J=0$ if $n(J)<2n$. If $n(J)= =2n+1$, then the $K0$-characteristic number $\pi^J$ coincides with a Stiefel-Whitney number for all $(8n+2)$-spin manifolds. Thus we may choose the coefficients $\beta^I$ such that $\alpha_J=0$. Finally, if $T^2$ is the torus with the exotic spin structure and $N^{8n}$ is a spin manifold, then $\psi(N^{8n}\times T^2)=\text{index }(N^{8n})$ (mod 2). Since the Stiefel-Whitney numbers of $N^{8n}\times T^2$ vanish, it follows that $\Sigma_{n(J)=2n}\,\alpha_J\pi^J=(L^{-1})_{2n}(0,\pi^2\ldots\pi^{2n})$.

Let $b\,\text{spin}_{8n+2}\subset\Gamma_{8n+1}$ be the subgroup consisting of homotopy spheres that bound spin manifolds. In [10], we showed that $\Gamma_{8n+1}=b\,\text{spin}_{8n+2}\oplus\mathbf{Z}_2$. An invariant $f_R: b\,\text{spin}_{8n+2}\to\mathbf{Z}_2$, splitting off $\mathbf{Z}_2=bP_{8n+2}\subset b\,\text{spin}_{8n+2}$ as a direct summand, can be defined as follows. Given $\Sigma^{8n+1}\in b\,\text{spin}_{8n+2}$, let $\Sigma^{8n+1}=\partial M_0^{8n+2}$, where $M_0^{8n+2}$ is a spin manifold such that all the Stiefel-Whitney numbers of $M^{8n+2}$ vanish. Then

$$f_R(\Sigma^{8n+1})=\psi(M^{8n+2})-(L^{-1})_{2n}(0,\pi^2\cdots\pi^{2n})\,(M^{8n+2})\in\mathbf{Z}_2\,.$$

Let $h: M_0'\to M_0$ be a homotopy equivalence with $\theta(h)=u\in[M_0^{8n+2},F/0]$. The spin structure on $M_0$ induces a spin structure on $M_0'$ and, since $h: M_0'\to M_0$ is a homotopy equivalence, $\psi(M')=\psi(M)$. Further $h^*(w^I(M))=w^I(M')$, hence

$$f_R(du)=f_R(\partial M_0'-\partial M_0)=(L^{-1})_{2n}(M)-(L^{-1})_{2n}(M')\in\mathbf{Z}_2\,.$$

We now seek a formula expressing the $K0$-characteristic numbers of $M'$ in terms of invariants of $M$ and of the map $u: M_0^{8n+2}\to F/0$.

PROPOSITION 6.2. *Let $u\in[M_0^{8n+2},F/0]$ correspond to the homotopy equivalence $h: M_0'\to M_0$, where $M_0$ is a spin manifold. Then*

$$\pi^J(M')=\langle\pi^J(v_M^*-\xi_0^*(u))\,\gamma^*(u),[M]_{K0}\rangle\in\mathbf{Z}_2$$

*where $h^*(v_M^*-\xi_0^*(u))=v_{M'}^*\in K0^0(M')$ and $\gamma^*(u)\in K0(M)$ extends $\gamma(u)\in K0(M_0)$.*

*Proof.* Homotope $h: M'\to M$ to an embedding $h: M'\to M\times\mathbf{R}^{8q}$. The $PL$ normal bundle of $M'$ in $M\times\mathbf{R}^{8q}$ is $h^*((-\xi)^{8q})$, where $h^*(v_M-\xi)=v_{M'}$. By the $h$-cobordism theorem, the embedding $h$ extends to a $PL$ isomorphism $H: E(h^*(-\xi)^{8q})\simeq M\times\mathbf{R}^{8q}$. Let $c_1=H^{-1}: T(e_M^{8q})\to T(h^*(-\xi)_{M'}^{8q})$ be the induced collapsing map.

Now, $\xi|_{M_0}=\xi_0(u)=\xi_0$ and the canonical $K0$-orientation of the Thom space $T(h^*(-\xi_0)_{M_0'}^{8q})$ extends to a $K0$-orientation $U\in K0^0(T(h^*(-\xi)_{M'}^{8q}))$. For, $h^*(-\xi)= =v_{M'}-h^*(v_M)=(v_{M'}^*-h^*(v_M^*))+(p')^*(\sigma'-\sigma)$, where $p': M'\to S^{8n+2}$, and the Thom space of the $PL$ bundle $\sigma'-\sigma$ over $S^{8n+2}$ is $K0$-orientable. Further, by 4.9, $c_1^*(U)\in \in K0^0(T(e_M^{8q}))$ restricts to $\Phi_{K0}(\gamma(u))\in K0^0(T(e_{M_0}^{8q}))$.

There is a homotopy commutative diagram

$$\begin{array}{ccccc} & & S^{16q+8n+2} & & \\ & {}^{c}\swarrow & & \searrow^{c'} & \\ T(\nu_M^{16q}) & & \Longrightarrow & & T(\nu_{M'}^{16q}) \\ \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle (h\times Id)\Delta} \\ T(\nu_M^{8q})\wedge T(e_M^{8q}) & & \xRightarrow{Id\wedge c_1} & & T(\nu_M^{8q})\wedge T(h^*(-\xi)_{M'}^{8q}) \end{array}$$

where the diagonal $\Delta: M\to M\times M$ and the composition $(h\times Id)\Delta: M'\to M'\times M'\to$ $\to M\times M'$ are covered by bundle maps $\Delta: \nu_M^{16q}\to \nu_M^{8q}\times e_M^{8q}$ and $(h\times Id)\Delta: \nu_{M'}^{16q}\to \nu_M^{8q}\times$ $\times h^*(-\xi)_{M'}^{8q}$.

The proof of homotopy commutativity is similar to the proof of 2.3 and will be ommitted.

We thus have

$$\begin{aligned} \pi^J(M') &= (c')^*(\pi^J(\nu_{M'}^*)\cdot U_{M'}) = (c')^*(h^*(\pi^J(\nu_M^*-\xi_0^*))\cdot U_{M'}) \\ &= (c')^*(\Delta^*(h\times Id)^*((\pi^J(\nu_M^*-\xi_0^*)\cdot U_M)\cdot U) \\ &= c^*(\Delta^*(\pi^J(\nu_M^*-\xi_0^*)\cdot U_M\cdot c_1^*(U))) \\ &= c^*(\pi^J(\nu_M^*-\xi_0^*)\cdot\gamma^*(u)\cdot\Delta^*(U_M\cdot\Phi_{KO}(1))) = c^*\Phi_{KO}(\pi^J(\nu_M^*-\xi_0^*)\cdot\gamma^*(u)) \end{aligned}$$

and Theorem 6.2 is proved.

LEMMA 6.3. *If* $n(J)=2n$ *then*

$$\langle\pi^J(\nu_M^*-\xi_0^*(u))\cdot\gamma^*(u),[M]_{KO}\rangle = \langle\pi^J(\nu_M^*)\cdot\gamma^*(u),[M]_{KO}\rangle\in\mathbf{Z}_2.$$

*Proof.* It suffices to prove that $\pi^J(\nu_M^*-\xi_0^*)\equiv\pi^J(\nu_M^*)\pmod 2$ in $KO^0(M)$.

First, $\pi^J(\nu_M^*-\xi_0^*)$ is independent of the choice of $\xi_0^*$, extending $\xi_0\in KO^0(M_0)$. For, if $\alpha=p^*(\sigma)$, where $\sigma\in KO^0(S^{8n+2})$, and $\eta\in KO^0(M)$ then $\pi^J(\eta+\alpha)=\Sigma\pi^{J'}(\eta)\pi(\alpha)$. But if $J''\neq(0)$, $\pi^{J'}(\eta)\pi^{J''}(\alpha)=0$ unless $J''=J$, and $\pi^J(\alpha)=0$ unless $J=(2n)$, since products of elements of high filtration vanish. But also $\pi^{(2n)}(\sigma)=0$ because $\sigma=\mu\eta^2$, where $\mu\in KO^0(S^{8n})$ and $\eta^2: S^{8n+2}\to S^{8n}$, and $\pi^{(2n)}(\mu)=(4n-1)!\mu$. Thus $\pi^J(\eta+\alpha)=\pi^J(\eta)$.

Secondly, since $J(\xi_0)=0$, $\xi_0=\Sigma_k k^e(\psi^k-1)(\xi_k)$ for some (arbitrarily) large integer $e$ and $\xi_k\in KO^0(M_0)$. Since $2\xi_2$ and $(\psi^k-1)\xi_k$, $k$ odd, extend to $KO^0(M)$ and since $\psi^{2k}-1=(\psi^2\psi^k-\psi^k)+(\psi^k-1)$, it suffices to prove $\pi^J(\eta_1+2^e(\psi^2-1)\eta_2)\equiv\pi^J(\eta_1)$ (mod 2) and $\pi^J(\eta_1+(\psi^k-1)\eta_k)\equiv\pi^J(\eta_1)$ (mod 2), $k$ odd, where $\eta_1,\eta_2\in KO^0(M)$ and $\eta_k\in KO^0(M_0)$.

If we set $\pi_t=\Sigma_{j\geq 0}\pi^j t^j$ then

$$\pi_t(\eta_1+2^e(\psi^2-1)\eta_2)=\pi_t(\eta_1)\cdot\pi_t((\psi^2-1)\eta_2)^{2^e}\equiv\pi_t(\eta_1)\pmod 2,$$

because $e$ is large, hence $2^e$-fold powers vanish in $KO^0(M)$. It follows that $\pi^J(\eta_1+2^e(\psi^2-1)\eta_2)\equiv\pi^J(\eta_1)\pmod 2$.

If $k$ is odd it suffices to prove that all products $x\cdot\pi^j(((\psi^k-1)\eta_k)\equiv 0 \pmod 2$, where $j\geqslant 1$, filtration $(x)=8n-4j$ if $j$ is even, and filtration $(x)=8n-4j-2$ if $j$ is odd. Now,

$$\pi_t((\psi^k-1)\eta)=1+[\pi^1(\psi^k(\eta))-\pi^1(\eta)]\,t$$
$$+[(\pi^2(\psi^k(\eta))-\pi^2(\eta))-\pi^1(\eta)(\pi^1(\psi^k(\eta))-\pi^1(\eta))]\,t^2+\cdots.$$

An easy induction shows that it suffices to prove $x\cdot(\pi^j(\psi^k(\eta))-\pi^j(\eta))\equiv 0\pmod 2$. But a computation in $KO^0(BSO)$ shows that

$$\pi^j\psi^k-k^{2j}\pi^j-(2k^{2j}(k^2-1)/4!)\,(\pi^{(j,1)}-j\pi^{j+1})$$

has filtration greater than $4j+4$. Since $k$ is odd, $2k^{2j}(k^2-1)/4!$ and $k^{2j}-1$ are even integers, hence

$$x\cdot(\pi^j(\psi^k(\eta))-\pi^j(\eta))=x\cdot((k^{2j}-1)\,\pi^j(\eta)-(2k^{2j}(k^2-1)/4!)$$
$$\times\,(\pi^{(j,1)}-j\pi^{j+1})\,(\eta))\equiv 0\ (\text{mod } 2.).$$

LEMMA 6.4. $\langle (L^{-1})_{2n}(v_M^*)(\gamma^*(u)-1).\,[M]_{KO}\rangle=\langle v_{4n}^2(M)\,w_2(\gamma(u)),[M]\rangle\in\mathbf{Z}_2$.

*Proof.* Let $\gamma^*(u)=1+\tilde{\gamma}$. Then $L_{2n}^{-1}(v_M^*)\tilde{\gamma}$ has filtration $8n+2$, and we have a homo topy commutative diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\Delta} & M\wedge M & \xrightarrow{L_{2n}^{-1}\wedge\gamma} & BSO\wedge BSO & \xrightarrow{\otimes} & BSO\\
\downarrow & & \downarrow & & \uparrow & & \uparrow\\
S^{8n+2} & \to & (M/M^{(8n-1)})\wedge M & \to & BSO\langle 8n\rangle\wedge BSO & \to & BSO\langle 8n+2\rangle.
\end{array}$$

The product $L_{2n}^{-1}(v_M^*)\cdot\tilde{\gamma}$ can thus be computed by evaluating the cohomology map $\mathbf{Z}_2=H^{8n+2}(BSO\langle 8n+2\rangle,\mathbf{Z}_2)\to H^{8n+2}(M,\mathbf{Z}_2)$ in the diagram. The results of [4] on the operations $\pi^J\colon BSO\to BSO\langle 8n\rangle$, $n(J)=2n$, can be used to show that this coincides with $\langle v_{4n}^2(M)\cdot w_2(\gamma(u)),\,[M]\rangle\in\mathbf{Z}_2$.

Note that since $(\gamma-1)\colon F/O\to BSO$ is a homotopy equivalence on the 5-skeltons, $w_2(\gamma(u))=u^*(k_2)$, where $u\in[M_0^{8n+2},F/O]$ and $k_2\in H^2(F/O,\mathbf{Z}_2)=\mathbf{Z}_2$ is the generator.

PROPOSITION 6.5. *Let $u\in[M_0^{8n+2},F/O]$, where $M_0^{8n+2}$ is a spin manifold. Then*

$$f_R(du)=\langle v_{4n}^2(M)\cdot u^*(k_2),\,[M]\rangle\in\mathbf{Z}_2.$$

*Proof.* This follows immediately from 6.2, 6.3, 6.4 and the formula

$$f_R(du)=(L^{-1})_{2n}(M)-(L^{-1})_{2n}(M').$$

COROLLARY 6.6. *$d:[M_0^{8n+2}, F/0] \to \Gamma_{8n+1}$ is a group homomorphism.*

*Proof.* This follows from 3.2 and 6.5 and the fact that $k_2 \in H^2(F/0, \mathbf{Z}_2)$ is primitive.

COROLLARY 6.7. *Let $h:M_0' \to M_0$ be a map of degree one. Then $dh^*(u) - du = \langle (v_{4n}^2(M') - h^*(v_{4n}^2(M))) \cdot h^* u^*(k_2), [M'] \rangle \in bP_{8n+2} = \mathbf{Z}_2$, where $u \in [M_0^{8n+2}, F/0]$. In particular, if $h$ is a tangential map or a homotopy equivalence, then $dh^*(u) = du$. Thus $\Delta_h(M_0)$ is a homotopy invariant of $8n+2$ spin manifolds.*

*Proof.* This follows from 3.3 and 6.5.

COROLLARY 6.8 *Let $u \in [M_0^{8n+2}, PL/0]$. Then $f_R(du) = 0$.*

*Proof.* $PL/0$ is 6-connected, hence $u^*(k_2) = 0$ and 6.8 follows from 6.5.

*Remark* 6.9. In § 5, we showed that for $4n$-spin manifolds, $f_R(\Delta_c(M_0^{4n})) = f_R(\Delta_{th}(M_0^{4n})) = 0$. For $(8n+2)$-spin manifolds, $f_R(\Delta_{th}(M_0^{8n+2}))$ need not be zero. For example, if $M_0^{8n+2} = (N^{8n} \times S^2)_0$ and index $(N^{8n})$ is odd, and $u: (N^{8n} \times S^2)_0 \xrightarrow{\pi_2} S^2 \xrightarrow{h^2} SF$, then $f_R(du) = 1$.

*Remark* 6.10. Let $M^{8n+2}$ be a closed, smooth spin manifold. The above results, along with Proposition 2.4, determine the exact sequence of Sullivan [18],

$$0 \to hS(M^{8n+2}) \xrightarrow{\theta} [M^{8n+2}, F/0] \xrightarrow{s} \mathbf{Z}_2 .$$

Namely, if $u \in [M^{8n+2}, F/0]$, then

$$s(u) = \langle v_{4n}^2(M) \cdot u^*(k_2), [M] \rangle \in \mathbf{Z}_2 .$$

Thus, the cohomology formula of 2.5 simplifies for $8n+2$ spin manifolds.

The Adams conjecture, and the resulting factoring $(F/0)_{(2)} = BS0_{(2)} \times (Cok\,J)_{(2)}$, implies that $s = 0$ if and only if $v_{4n}^2(M)\, w_2(\gamma) = 0$ for all $\gamma \in KO^0(M)$.

## Appendix I. $S^1$ actions on homotopy spheres

It is known that equivariant diffeomorphism classes of differentiable, fixed point free $S^1$ actions on homotopy $(2n-1)$-spheres, $n \geq 4$, correspond bijectively with equivalence classes of homotopy smoothings of $CP(n-1)$ [12]. The correspondence is defined as follows. If $S^1$ acts on $\Sigma^{2n-1}$, there is a diagram

$$\begin{array}{ccc} \Sigma^{2n-1} & \xrightarrow{\tilde{h}} & S^{2n-1} \\ \downarrow & & \downarrow \\ P^{2n-2} = \Sigma^{2n-1}/S^1 & \xrightarrow{h} & CP(n-1) = S^{2n-1}/S^1 \end{array} \tag{I.1}$$

where $h$ classifies the principal $S^1$ bundle over $P^{2n-2}$ given by the action of $S^1$ on $\Sigma^{2n-1}$. An easy spectral sequence argument shows that $h$ is a homotopy equivalence.

There are homotopy equivalences $\mathbf{C}P(n-1)\xrightarrow{i}\mathbf{C}P(n)_0\xrightarrow{\pi}\mathbf{C}P(n-1)$, since $\mathbf{C}P(n)_0$ is the total space of a $\mathbf{D}^2$ bundle, $H$, over $\mathbf{C}P(n-1)$. (If $\mathbf{C}P(n-1)$ is regarded as the space of lines in $\mathbf{C}^n$ then $H$ is the dual of the "canonical" line bundle.) Consider the diagram

$$\begin{array}{ccccc} hS(\mathbf{C}P(n-1)) & \xrightarrow{\theta} & [\mathbf{C}P(n-1), F/0] & \xrightarrow{s} & P_{2n-2} \\ \downarrow i_* & & \uparrow\wr i_* & & \\ hS_\psi\mathbf{C}P(n)_0) & \overset{\theta}{\cong} & [\mathbf{C}P(n)_0, F/0] & \xrightarrow{d} & \Gamma_{2n-1} \end{array} \tag{I.2}$$

where, if $h:P^{2n-2}\to\mathbf{C}P(n-1)$ then $i_*(P^{2n-2},h)$ is the homotopy equivalence $\tilde{h}:P_0^{2n}=E(h^*H)\to E(H)=\mathbf{C}P(n)_0$.

LEMMA I.3(i). *Diagram* I.2 *commutes.*

(ii) $d\theta i_*(P^{2n-2},h)=\Sigma^{2n-1}\in\Gamma_{2n-1}$, *where* $\Sigma^{2n-1}\to P^{2n-2}$ *is as in diagram* I.1.

(iii) $si^*\theta:hS(\mathbf{C}P(n)_0)\to P_{2n-2}$ *is the geometric obstruction to finding a codimension 2, homotopy* $\mathbf{C}P(n-1)$ *in a homotopy* $\mathbf{C}P(n)_0$.

The proof of I.3 is relatively straightforward and will be omitted. It follows from I.3 that the set of homotopy $(2n-1)$-spheres which admit free $S^1$ actions coincides with $d(\theta i_*(hS(\mathbf{C}P(n-1))))=d((si^*)^{-1}(0))\subset\Delta_h(\mathbf{C}P(n)_0)=B_h(\mathbf{C}P(n)_0)\subset\Gamma_{2n-1}$. Denote this set by $\tilde{B}_h(\mathbf{C}P(n)_0)$.

We now want to apply the results of § 2 through § 6 to compute $\tilde{B}_h(\mathbf{C}P(n)_0)$. First, it follows from the exact sequence

$$\begin{aligned} KO^{-1}(\mathbf{C}P(n)_0) &\to [\mathbf{C}P(n)_0, SF]\to[\mathbf{C}P(n)_0, F/0]\to KO^0(\mathbf{C}P(n)_0)\\ &\to J(\mathbf{C}P(n)_0)\to 0 \end{aligned}$$

and results of [3] that $[\mathbf{C}P(n)_0, F/0]=\mathbf{Z}^{[(n-1)/2]}\oplus[\mathbf{C}P(n)_0, SF]$, where $\mathbf{Z}^{[(n-1)/2]}\subset$ $\subset\mathrm{image}([\mathbf{C}P(n), F/0]\to[\mathbf{C}P(n)_0, F/0])$ and $\mathrm{image}(\mathbf{Z}^{[(n-1)/2]}\to KO^0(\mathbf{C}P(n)_0))$ is generated by elements $k^e(\psi^k-1)(\xi)$, $\xi\in KO^0(\mathbf{C}P(n)_0)$. In theory it is thus possible to compute the fibre homotopically trivial bundles over $\mathbf{C}P(n)_0$. We have done this for $n\leqslant 8$ [12]. Let $\omega=r(H-1)\in KO^0(\mathbf{C}P(n))$, where $r$ forgets the complex structure.

LEMMA I.4. *Kernel* $(KO^0(\mathbf{C}P(8)_0)\to J(\mathbf{C}P(8)_0)=\mathbf{Z}^3$ *has generators* $\xi_1=24\omega+$ $+98\omega^2+111\omega^3$, $\xi_2=240\omega^2+380\omega^3$, *and* $\xi_3=504\omega^3$. *If* $n<8$, *kernel* $(KO^0(\mathbf{C}P(n)^0)\to$ $\to J(\mathbf{C}P(n)_0))$ *is generated by* $\xi_1,\xi_2,\xi_3$ *restricted to* $KO^0(\mathbf{C}P(n)_0)$.

Next, we need to compute $si^*:[\mathbf{C}P(n)_0, F/0]\to P_{2n-2}$.

LEMMA I.5. *If* $n\equiv 1$ *or* 3 $(mod\,4)$ *and* $u\in[\mathbf{C}P(n)_0, F/0]$ *then* $si^*(u)=(\frac{1}{8})$ $\langle L(\mathbf{C}P(n-1))(1-L(\xi_0(i^*(u)))), [\mathbf{C}P(n-1)]\rangle\in\mathbf{Z}$.

In particular,

(i) $si^*([\mathbf{C}P(n)_0, SF])=0$

(ii) If $n=5$ and $\xi_0(i^*(u))=m\xi_1+n\xi_2$ then

$$si^*(u) = -4m^2 + 10m + 28n \in \mathbf{Z}.$$

In particular, if $si^*(u)=0$ then $10m\equiv 0 \pmod 4$, or, $m\equiv 0 \pmod 2$.

(iii) If $n=7$ and $\xi_0(i^*(u))=m\xi_1+n\xi_2+q\xi_3$ then

$$si^*(u) = (-m(32m^2+301)/3) + 84m^2 + 224mn - 384n - 496q \in Z.$$

*Proof.* The formula for $s$ was given in Remark 2.5.
Statements (ii) and (iii) follow from I.4 and explicit computation of theL-polynomials in the formula.

LEMMA I.6. *If* $n\equiv 2 \pmod 4$ *and* $u\in[\mathbf{C}P(n)_0, F/0]$ *then* $si^*(u)=$ $\langle v_{n-2}^2(\mathbf{C}P(n-1))\, i^*u^*(k_2), [\mathbf{C}P(n-1)]\rangle \in \mathbf{Z}_2$.
Thus $si^*(u)=0$ if and only if $w_2(\gamma(i^*(u)))=i^*u^*(k_2)=0$, or equivalently, if and only if $p_1(\xi_0(i^*(u)))\equiv 0 \pmod{48}$. In particular,

(i) $si^*([\mathbf{C}P(n)_0, SF])=0$,

(ii) If $n=6$ and $\xi_0(i^*(u))=m\xi_1+n\xi_2$ then $si^*(u)=m \pmod 2$.

*Proof.* The formula follows from 6.5 and 6.10. If $n\equiv 2 \pmod 4$ then $v_{n-2}^2(\mathbf{C}P(n-1))\neq 0$ and the second statement follows. Statements (i) and (ii) also follow easily.

We do not have general results with which to compute $si^*$ if $n\equiv 0 \pmod 4$. The following conjecture is probably true.

*Conjecture* I.7(i). If $n\equiv 0 \pmod 4$, $n\neq 2^j$, then $si^*([\mathbf{C}P(n)_0, F/0])=0$.

(ii) There are elements $h_j^2\in\pi_{2^{j+1}-1}(SF)$ such that if $u:\mathbf{C}P(2^j)_0\xrightarrow{p\pi}S^{2^{j+1}-2}\xrightarrow{h^2_j}SF$ then $si^*(u)=1\in\mathbf{Z}_2$. The summand $\mathbf{Z}^{(2^{j-1}-1)}\subset[\mathbf{C}P(2^j)_0, F/0]$ can be chosen so that $si^*(\mathbf{Z}^{(2^{j-1}-1)})=0$.

I.7(ii) is true if $j\leqslant 6$. For example $h_1^2=\eta^2\in\pi_2^s$, $h_2^2=\nu^2\in\pi_6^s$, and $h_3^2=\sigma^2\in\pi_{14}^s$.

We can use the results 2.5, 3.1, 4.4, 5.2, and 6.10 to compute $d:[\mathbf{C}P(n)_0, F/0]=$ $=\mathbf{Z}^{[(n-1)/2]}\oplus[\mathbf{C}P(n)_0, SF]\to\Gamma_{2n-1}=bP_{2n}\oplus(\pi_{2n-1}^s/\mathrm{im}(J))$.

LEMMA I.8. *We have* $d(\mathbf{Z}^{[(n-1)/2]})\subset bP_{2n}$. *Specifically,*

(i) *If* $u\in\mathbf{Z}\subset[\mathbf{C}P(4)_0, F/0]$ *and* $\xi_0(u)=m\xi_1$ *then* $du=10m-4m^2\in\mathbf{Z}/28Z=bP_8$.

(ii) *If* $u\in\mathbf{Z}^2\subset[\mathbf{C}P(5)_0, F/0]$ *and* $\xi_0(u)=m\xi_1+n\xi_2$, *then* $du=m\in\mathbf{Z}/2\mathbf{Z}=bP_{10}$.

(iii) *If* $u\in\mathbf{Z}^2\subset[\mathbf{C}P(6)_0, F/0]$ *and* $\xi_0(u)=m\xi_1+n\xi_2$, *then* $du=(-m(32m^2+301)/3)$ $+84m^2+224mn-384n\in\mathbf{Z}/992Z=bP_{12}$.

(iv) *If* $u\in\mathbf{Z}^3\subset[\mathbf{C}P(7)_0, F/0]$ *then* $du=0$, *since* $bP_{14}=0$.

*Proof.* $\mathbf{Z}^{[(n-1)/2]} \subset \text{image}([CP(n), F/0] \to [CP(n)_0, F/0])$, hence the first statement follows from 2.4 and 6.10. Statements (i) and (iii) follow from I.5 and 2.4 and (ii) follows from I.6 and 6.10.

Specific formulas for $d(\mathbf{Z}^{[(n-1)/2]})$, $n \geq 8$, would only require extending the computations of I.4 and I.5.

Recall that as a set $[CP(n)_0, SF] = \pi_s^0(CP(n)_0)$. In [12] we computed the $p$-primary summand ${}_p\pi_s^0(CP(n)_0)$ and the map ${}_p\pi_s^0(CP(n)_0) \xrightarrow{\partial^*} {}_p\pi_s^0(S^{2n-1}) = {}_p\pi_{2n-1}^s$ for $n \leq (p^2+2p)(p-1)-2$, $p$ odd, and we computed ${}_2\pi_s^0(CP(n)_0) \xrightarrow{\partial^*} {}_2\pi_{2n-1}^s$ for $n \leq 11$. Thus, using 5.2 and 6.9, we also computed $d: [CP(n)_0, SF] \to \Gamma_{2n-1}$ if $n \equiv 0$, 1, or 2 (mod 4) or if $n = 2^j - 1$. (note that by 5.5(ii), $a_n f_R(d[CP(2n)_0, SF]) = 0$ and by 6.9, $f_R(d[CP(4n+1)_0, SF]) = 0$.) These results involve computations in stable homotopy theory and are too complicated to reproduce here. We will state the conclusions for $n \leq 7$.

LEMMA I.9(i). $[CP(4)_0, SF] = Z_2$ *and* $d([CP(4)_0, SF]) = 0$.
(ii) $[CP(5)_0, SF] = \mathbf{Z}_2^2$ *and* $d([CP(5)_0, SF]) = \mathbf{Z}_2 = \{\nu^3\} \subset (\pi_9^s/\text{im}(J)) \subset \Gamma_9$.
(iii) $[CP(6)_0, SF] = \mathbf{Z}_2^2 + \mathbf{Z}_3$ *and* $d([CP(6)_0, SF]) = \mathbf{Z}_2 \subset bP_{12} = \Gamma_{11}$.
(iv) $[CP(7)_0, SF] = \mathbf{Z}_2 + \mathbf{Z}_3$ *and* $d([CP(7)_0, SF]) = \mathbf{Z}_3 = \{\alpha_1\beta_1\} = \pi_{13}^s = \Gamma_{13}$.

The construction of the non-zero element of $d([CP(6)_0, SF])$ is described in § 5, following the proof of 5.7.

Finally, we combine the results I.5 through I.9 to describe the set of homotopy spheres of dimensions 7, 9, 11, and 13 which admit free $S^1$ actions. That is, we compute $\tilde{B}_h(CP(n)_0) = d((si^*)^{-1}(0)) \subset d([CP(n)_0, F/0]) = B_h(CP(n)_0) \subset \Gamma_{2n-1}$, for $n = 4$, 5, 6, and 7.

THEOREM I.10(i). $\Gamma_7 = bP_8 = Z/28Z$ *and* $\tilde{B}_h(CP(4)_0) = \{10m - 4m^2 / m \in \mathbf{Z}\} = \{0, 4, \pm 6, \pm 8, -10, 14\} \subset \mathbf{Z}/28\mathbf{Z}$.
(ii) $\Gamma_9 = bP_{10} \oplus (\pi_9^s/\text{im}(J)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2^2$ *and* $\tilde{B}_h(CP(5)_0) = \mathbf{Z}_2 = \{\nu^3\} \subset (\pi_9^s/\text{im}(J)) \subset \subset \Gamma_9$.
(iii) $\Gamma_{11} = bP_{12} = \mathbf{Z}/992\mathbf{Z}$ *and* $\tilde{B}_h(CP(6)_0)$
$= \{(-m(32m^2 + 301/3) + 84m^2 + 224mn - 384n \mid m, n \in \mathbf{Z}, m \text{ even}\} \subset Z/992Z$.
(iv) $\Gamma_{13} = \pi_{13}^s = \mathbf{Z}_3$ *and* $\tilde{B}_h(CP(7)_0) = \mathbf{Z}_3 = \{\alpha_1\beta_1\} = \Gamma_{13}$.

## Appendix II. Applications to inertia groups

Given a smooth manifold $N^k$, the inertia group of $N^k$, $I(N^k) \subset \Gamma_k$, is defined to be the group of homotopy spheres $\Sigma^k \in \Gamma_k$ such that there is a diffeomorphism $N^k \simeq N^k \# \Sigma^k$. Define $I_h(N^k) \subset I(N^k)$ to be the subgroup of homotopy spheres $\Sigma^k \in I(N^k)$ such that some diffeomorphism $N^k \simeq N^k \# \Sigma^k$ is homotopic to the identity. (By the "identity"

$N^k = N^k \# \Sigma^k$ we mean the obvious $PL$ identification.) Similarly, define $I_c(N^k) \subset I_h(N^k)$ to be the subgroup of homotopy spheres $\Sigma^k$ such that some diffeomorphism $N^k \cong N^k \# \Sigma^k$ is $PL$ isotopic to the identity. Equivalently, $\Sigma^k \in I_c(N^k)$ if the smoothings $N^k$ and $N^k \# \Sigma^k$ are concordant.

The group $\Gamma_k$ is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms of $S^{k-1}$. If $\Sigma^k \in \Gamma_k$ corresponds to the diffeomorphism $\sigma: S^{k-1} \cong S^{k-1}$ then $\Sigma^k \in I(N^k)$ if and only if there is a diffeomorphism $h: N_0^k \cong N_0^k$ such that $h|_{\partial N_0 = S^{k-1}} = \sigma$. Let $h: N^k \to N^k$ also denote the $PL$ extension of $h$ defined by coning $h|_{\partial N_0}$ over $D^k \subset N^k$. It is easy to see that the mapping torus of $h$, $T_h = N^k \times I/(x, 0) \equiv \equiv (h(x), 1)$, is an almost smooth manifold, with $\partial(T_h)_0 = \Sigma^k$. Further, $\Sigma^k \in I_h(N^k)$ (resp. $\Sigma^k \in I_c(N^k)$) if and only if $h$ can be chosen such that there is a homotopy equivalence (resp. a $PL$ isomorphism) $H: T_h \to N^k \times S^1$, with $H|_{N^k \times 0} = Id$. Then $H: (T_h)_0 \to \to (N^k \in S^1)_0$ is a homotopy smoothing of $(N^k \times S^1)_0$.

Now $N^k \times S^1$ is not simply connected. However, if $N^k$ is simply connected, the map $\theta: hS((N^k \times S^1)_0) \to [(N^k \times S^1)_0, F/0]$ is still useful. There is a natural decomposition $[(N^k \times S^1)_0, F/0] \simeq [N^k, F/0] \oplus [N_0^k \wedge S^1, F/0]$. The first summand contains the image under $\theta$ of the homotopy smoothings $g \times Id: (N' \times S^1)_0 \to (N \times S^1)_0$, where $g: N' \to N$ is a homotopy equivalence. The second summand corresponds bijectively with the homotopy smoothings described above, $H: (T_h)_0 \to (N^k \times S^1)_0$, $H|_{N^k \times 0} = Id$, where $h: N_0^k \cong N_0^k$ is a diffeomorphism homotopic to the identity. Denote this second set of homotopy smoothings of $(N^k \times S^1)_0$ by $\widetilde{hS}((N^k \times S^1)_0)$.

PROPOSITION II.1. *$I_h(N^k) = d(\theta(\widetilde{hS}(N^k \times S^1)_0)) = d([N_0^k \wedge S^1, F/0]) \subset \Gamma_k$. Also, $I_c(N^k) = d([N_0^k \wedge S^1, PL/0])$.*

*Proof.* This follows from the discussion in the three paragraphs above.

We can thus use the results of § 2 through § 6 to compute $I_h(N^k)$. If $u \in [N_0^k \wedge S^1, F/0]$, $k$ odd, the formulas in 5.1 and 6.5 for $f_R(du)$ simplify.

PROPOSITION II.2. *If $N^{8n+1}$ is a simply connected spin manifold and $u \in [N_0^{8n+1} \wedge S^1, F/0]$ then $f_R(du) = 0$. Thus $I_h(N^{8n+1})$ is contained in the summand $(\pi_{8n+1}^s / im(J)) \subset \Gamma_{8n+1}$ and $I_h(N^{8n+1}) \cong \varrho(I_h(N^{8n+1}))$ is a homotopy invariant of $N^{8n+1}$.*

*Proof.* Since $u^*(k_2) = 0$, the result follows from 6.5.

PROPOSITION II.3. *If $u \in [N_0^{4n-1} \wedge S^1, F/0]$ then*

$$f_R(du) = (-\tfrac{1}{8}) \left\langle L(N^{4n-1} \times S^1) \left( \sum_{k=1}^{n} (8\theta_k / a_k (2k-1)! j_k) p_k(\xi) \right), [N^{4n-1} \times S^1] \right\rangle$$
$$\in \mathbf{Z}/\theta_n \mathbf{Z},$$

*where $p_n(\xi)$ is as in 5.1 and $p_k(\xi) = p_k(\xi_0(u))$ if $k < n$.*

*Proof.* Since cohomology products vanish in $N^{4n-1}\wedge S^1$, we have $(1-L(\xi))=-(\sum_{k=1}^{n}(8\theta_k/a_k(2k-1)!j_k)p_k(\xi))$ and the result follows from 5.1. We point out that $p_n(\xi)$ is determined by the equations $(-\operatorname{num}(B_n/4n)/a_n(2n-1)!j_n)p_n(\xi)=e_R(\gamma(u))\in Q/Z$ and $((-1)^{n-1}j_n/a_n(2n-1)!j_n)\,p_n(\xi)=e_R(\xi_0(u))\in Q/\mathbf{Z}$.

Note that by 5.9, $d\colon[N_0^k\wedge S^1,\ F/0]\to\Gamma_k$ is a group homomorphism if $k=4n-1$. Actually, if $u,\ v\in[N_0^k\wedge S^1,\ F/0]$ correspond to $H\colon(T_h)_0\to(N^k\times S^1)_0$ and $G\colon(T_g)_0\to(N^k\times S^1)_0$, respectively, where $h,\ g\colon N_0^k\xrightarrow{\simeq}N_0^k$ are diffeomorphisms, then $d(u+v)\in\Gamma_k$ corresponds to the diffeomorphism $(h\,|_{\partial N_0})\cdot(g\,|_{\partial N_0})\colon S^{k-1}\xrightarrow{\simeq}S^{k-1}$. Since this composite diffeomorphism also corresponds to $du+dv$, we have that $d\colon[N_0^k\wedge S^1]\to\Gamma_k$ is a group homomorphism for all $N^k$.

There is a braid of four interlocking exact sequences

$$\begin{tikzcd}[column sep=small]
& & \pi_{k+1}(F/0) \arrow[rr, bend left] \arrow[dr] & & P_{k+1} \arrow[rr, bend left] \arrow[dr, "b"] & & hS(N^k) \arrow[rr, bend left] \arrow[dr, "\theta"] & & hS(N_0^k) \arrow[rr, bend left] \arrow[dr, "d\theta"] & & \pi_{k-1}(F/0) \\
& \Gamma_{k+1} \arrow[ur, "\rho"] \arrow[dr, "\alpha"] & & {[N^k\wedge S^1, F/0]} \arrow[ur, "s"] \arrow[dr] & & \Gamma_k \arrow[ur, "\alpha"] \arrow[dr, "\rho"] & & {[N^k, F/0]} \arrow[ur] \arrow[dr, "s"] & & \Gamma_{k-1} \arrow[ur, "\rho"] & \\
P_{k+2} \arrow[ur, "b"] \arrow[rr, bend right] & & \widetilde{hS}(N^k\times S^1) \arrow[ur, "\theta"] \arrow[rr, bend right] & & \widetilde{hS}((N^k\times S^1)_0) \arrow[ur, "d\theta"] \arrow[rr, bend right] & & \pi_k(F/0) \arrow[ur] \arrow[rr, bend right] & & P_k \arrow[ur, "b"] & &
\end{tikzcd}$$

Here, $\alpha\colon\Gamma_k\to hS(N^k)$ is defined by $\alpha(\Sigma^k)=(N^k\#\Sigma^k,\ \mathrm{Id}\#(\text{point}))\in hS(N^k)$, $\Sigma^k\in\Gamma_k$. Since kernel $(\alpha)\cap bP_{k+1}=bs([N^k\wedge S^1,\ F/0])=d\theta(\widetilde{hS}((N^k\times S^1)_0))\cap bP_{k+1}=I_h(N^k)\cap\cap bP_{k+1}$, we see that $I_h(N^k)$ is very useful for computing $hS(N^k)$.

If we replace $F/0$ by $PL/0$, the cofibrations $S^{k-1}\to N_0^k\to N^k\to S^k\to N_0\wedge S^1$ yield an exact sequence $[N_0^k\wedge S^1,\ PL/0]\xrightarrow{d}\Gamma_k\to[N^k,\ PL/0]\to[N_0^k,\ PL/0]\xrightarrow{d}\Gamma_{k-1}$. Since $[N^k, PL/0]$ and $[N_0^k,\ PL/0]$ correspond to concordance classes of smoothings of $N^k$ and $N_0^k$, respectively, it is clear that $I_c(N^k)=d([N_0^k\wedge S^1,\ PL/0])=\{\Sigma^k\in\Gamma_k\mid$ the smoothings $N^k$ and $N^k\#\Sigma^k$ are concordant$\}$. The following is also clear.

PROPOSITION II.4. *$I_c(N^k)$ is a homotopy invariant of $N^k$.*

There are natural subgroups $I_{th}(N^k)\subset I_h(N^k)$ and $I_{tc}(N^k)\subset I_c(N^k)$ defined by $I_{th}(N^k)=d([N_0^k\wedge S^1,\ SF])$ and $I_{tc}(N^k)=d([N_0^k\wedge S^1,\ SPL])$. Geometrically, $I_{th}(N^k)\subset\Gamma_k$ (resp. $I_{tc}(N^k)\subset\Gamma_k$) corresponds to those diffeomorphisms $\sigma\colon S^{k-1}\xrightarrow{\simeq}S^{k-1}$ such that there is a diffeomorphism $h\colon N_0^k\xrightarrow{\simeq}N_0^k$, with $h\,|_{\partial N_0}=\sigma$, and a tangential homotopy equivalence (resp. $PL$ equivalence preserving the smooth tangent bundles) $H\colon(T_h)_0\to(N^k\times S^1)_0$ with $H\,|_{N^k\times 0}=Id$.

PROPOSITION II.5(i). *$f_R(I_c(N^{4n-1}))$ and $f_R(I_{th}(N^{4n-1}))\subset Z_{\theta_n}$ are 2-primary groups.*

(ii) *If $N^{4n-1}$ is a spin manifold then $f_R(I_c(N^{4n-1}))=f_R(I_{th}(N^{4n-1}))=0$*

(iii) *$I_{th}(N^{4n-1})$ and $I_{tc}(N^{4n-1})$ are homotopy invariants.*

*Proof.* These results follow from 5.2, 5.5, and 5.6. It follows from the construction given after the proof of 5.7 that if $w_2(N^{8k+3})\neq 0$ then the element of order 2 in $bP_{8k+4}$ belongs to $I_{tc}(N^{8k+3})$.

PROPOSITION II.6. *$I_{th}(N^{8n+1})\simeq \varrho I_{th}(N^{8n+1})$ and $I_{tc}(N^{8n+1})\simeq \varrho I_{tc}(N^{8n+1})$ are homotopy invariants of $(8n+1)$-spin manifolds.*

*Proof.* This follows from II.2.

Next we consider manifolds with a trivial stable normal bundle ($\pi$-manifolds) or a fibre homotopically trivial stable normal bundle (*fht*-manifolds).

LEMMA II.7. *$M^k$ is an fht-manifold if and only if there is a $\pi$-manifold $M'$ and a degree one map $M'\to M$.*

*Proof.* By transverse regularity, such a manifold $M'$, with $M'\times R^q\subset E(v_M^q)$, exists if and only if there is a fibre homotopy trivialization $T(v_M^q)\to S^q$.

Boardman and Vogt have shown that $PL/0$ and $F/0$ are infinite loop spaces [5]. It follows easily that the suspension maps $\pi_*(F/0)\to\pi_*^s(F/0)=\Omega_*^{\text{framed}}(F/0)$ and $\pi_*(PL/0)\to\pi_*^s(PL/0)=\Omega_*^{\text{framed}}(PL/0)$ are monomorphisms onto direct summands.

LEMMA II.8. *If $M^k$ is an almost smooth,* fht-manifold *then $\Delta_c(M^k)=0$ and $\Delta_h(M^k)\subset bP_k$. If $k=8n+2$ then $\Delta_h(M^k)=0$.*

*Proof.* Let $u\in[M_0^k, PL/0]$ and let $h:M_0'\to M_0$ be a degree one map where $M'$ is a $\pi$-manifold. Then by the above remark $du=\partial^*(u)=\partial^*h^*(u)=0\in\pi_{k-1}(PL/0)=\Gamma_{k-1}$. Similarly, if $u\in[M_0^k, F/0]$ then by 3.1 $\varrho(du)=\partial^*(u)=\partial^*h^*(u)=0\in\pi_{k-1}(F/0)$. The second statement follows from the first and the fact that the surgery obstruction $s:[M^{8n+2}, F/0]\to Z_2$ is given by $s(u)=\langle v_{4n}^2(M)u^*(k_2), [M]\rangle=0$, since the Wu class $v_{4n}(M)=0$.

PROPOSITION II.9. *If $N^k$ is a smooth, fht-manifold then $I_c(N^k)=0$ and $I_h(N^k)\subset bP_{k+1}$. If $k=8n+1$ then $I_h(N^k)=0$. If $N^k$ is a $\pi$-manifold and $k\not\equiv 5$* (mod 8) *then $I_h(N^k)=0$.*

*Proof.* The first two statements follow from II.8 since $N^k\times S^1$ is an *fht*-manifold. If $N^{4n-1}$ is a $\pi$-manifold and $u\in[N_0^{4n-1}\wedge S^1, F/0]$ then $f_R(du)=0$ by 5.8. Thus $I_h(N^k)=I_h(N^k)\cap bP_{k+1}=0$ if $k\equiv 1$, 3, or 7 (mod 8) and the third statement follows. (I am grateful to D. Sullivan for pointing out the first statement of II.9.)

Finally, as an example, we compute, $I_h(\mathbf{C}P(3)\times S^1)\subset\Gamma_7=bP_8=\mathbf{Z}_{28}$. ($\mathbf{C}P(3)\times S^1$ is not simply connected, but our methods remain valid for special cases with simple fundamental groups.) Now $(\mathbf{C}P(3)\times S^1)\wedge S^1$ is homotopy equivalent to $(\mathbf{C}P(3)\wedge S^2)\vee(\mathbf{C}P(3)\wedge S^1)\vee S^2$. Thus, since $KO^0(\mathbf{C}P(3)\wedge S^1)=0$, image$([(\mathbf{C}P(3)\times S^1)\wedge S^1, F/0]\to KO^0((\mathbf{C}P(3)\times S^1)\wedge S^1))=$image$([\mathbf{C}P(3)\wedge S^2,$ $F/0]\to KO^0(\mathbf{C}P(3)\wedge S^2))=\mathbf{Z}^2$, with generators $\xi_1$ and $\xi_2$ which satisfy $P(\xi_1)=1+$

$+p_1(\xi_1)+p_2(\xi_1)=1+48(z\cdot\sigma)+32\cdot 15(z^3\cdot\sigma)$ and $P(\xi_2)=1+32\cdot 45(z^3\cdot\sigma)$, where $z\in H^2(\mathbf{C}P(3), \mathbf{Z})$ and $\sigma\in H^2(S^2, \mathbf{Z})$ are generators. Thus if $u\in[(\mathbf{C}P(3)\times S^1)_0\wedge S^1, F/0]$ extends to $\bar{u}\in[(\mathbf{C}P(3)\times S^1)\wedge S^1, F/0]$ and $\xi=\xi(\bar{u})=m\xi_1+n\xi_2$ then

$$\begin{aligned} du = s(\bar{u}) &= (\tfrac{1}{8})\langle L(\mathbf{C}P(3)\times S^1\times S^1)(1-L(\xi)), [\mathbf{C}P(3)\times S^1\times S^1]\rangle \\ &= (-\tfrac{1}{8})\langle(1+(\tfrac{4}{3})z^2)((48m/3)(z\sigma)+(7(32\cdot 15m+32\cdot 45n)/45)(z^3\sigma), \\ &\quad [\mathbf{C}P(3)\times S^1\times S^1]\rangle = -12m-28n\in\mathbf{Z}/28\mathbf{Z}. \end{aligned}$$

It follows that $I_h(\mathbf{C}P(3)\times S^1)=\mathbf{Z}_7\subset\mathbf{Z}_{28}$.

*Remark* II.10. R. Lee [16] has shown that every self-homotopy equivalence of $\mathbf{C}P(n)\times S^1$ is homotopic to a diffeomorphism. If a manifold $M^k$ has this property it is easy to see that $I_h(M^k)=I(M^k)$. Thus $I(\mathbf{C}P(3)\times S^1)=\mathbf{Z}_7\subset\mathbf{Z}_{28}$.

*Remark* II.11. Let $\pi_0^+(\mathrm{Diff}(\mathbf{C}P(\mathrm{n})))$ denote the group of pseudo-isotopy classes of diffeomorphisms of $\mathbf{C}P(n)$ which leave fixed a generator of $H^2(\mathbf{C}P(n), \mathbf{Z})$. Lee has shown that $\pi_0^+(\mathrm{Diff}\,\mathbf{C}P(n))$ is isomorphic to the equivariant diffeomorphism classes of differentiable, semi-free $S^1$ actions on homotopy $(2n+2)$-spheres, with fixed point set $S^0$. (A group action is semi-free if it is free outside the fixed point set.) It follows from results of Sullivan that the natural map $\Gamma_7=\pi_0(\mathrm{Diff}(S^6))\xrightarrow{\gamma}\pi_0^+(\mathrm{Diff}(\mathbf{C}P(3)))$ is a surjection, where, if $\Sigma^7\in\Gamma_7$ corresponds to a diffeomorphism $\sigma: D^6\xrightarrow{\simeq}D^6$, with $\sigma|_{S^5}=\mathrm{Id}$, then $\gamma(\Sigma^7)|_{D^6}=\sigma$ and $\gamma(\Sigma^7)|_{\mathbf{C}P(3)-D^6}=\mathrm{Id}$, where $D^6\subset\mathbf{C}P(3)$. It is not difficult to see that the mapping torus of $\gamma(\Sigma^7)$ is $(\mathbf{C}P(3)\times S^1)\#\Sigma^7$. Hence, $\gamma(\Sigma^7)=0\in\pi_0^+(\mathrm{Diff}(\mathbf{C}P(3)))$ if and only if $\gamma(\Sigma^7)$ is pseudo-isotopic to the identity, or equivalently, if and only if there is a diffeomorphism $(\mathbf{C}P(3)\times S^1)\#\Sigma^7=T_{\gamma(\Sigma^7)}\xrightarrow{\simeq}\mathbf{C}P(3)\times S^1$ which is the identity on $\mathbf{C}P(3)\times 0$. Since any diffeomorphism $(\mathbf{C}P(3)\times S^1)\#\Sigma^7\xrightarrow{\simeq}\mathbf{C}P(3)\times S^1$ is pseudo-isotopic to one which fixes $\mathbf{C}P(3)\times 0$ [19; Lemma 4], this proves that $\mathrm{kernel}(\gamma)=I(\mathbf{C}P(3)\times S^1)=\mathbf{Z}_7\subset\mathbf{Z}_{28}$ and that $\pi_0^+(\mathrm{Diff}(\mathbf{C}P(3)))=\mathbf{Z}_4$.

*Remark* II.12. For each integer $j$ there is a manifold $P_j^6$ homotopy equivalent to $\mathbf{C}P(3)$ with $p_1(P_j^6)=(4+24j)z^2$. Thus if $u\in[(P_j^6\times S^1)_0\wedge S^1, F/0]$ with $\xi(\bar{u})=m\xi_1+n\xi_2$ then $du=s(\bar{u})=-(12+16j)m-28n\in\mathbf{Z}/28\mathbf{Z}$. It follows that $I_h(P_j^6\times S^1)=0$ if $j\equiv 1\pmod 7$ and $I_h(P_j^6\times S^1)=\mathbf{Z}_7$ if $j\not\equiv 1\pmod 7$. In particular, $I_h(N^k)$ is not a homotopy invariant of $N^k$.

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