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Homotopy Equivalences of Almost Smooth Manifolds

G. BRUMFIEL

§ 1. *Introduction.* Let M^k , $k \geq 6$, be a simply connected, oriented, closed combinatorial manifold with a differentiable structure in the complement of a point. Let $M_0^k = M^k - \text{interior}(D^k)$, where $D^k \subset M^k$ is a combinatorially embedded disc. M_0^k inherits a differentiable structure from $M^k - (p)$, hence ∂M_0^k belongs to Γ_{k-1} , the group of oriented differentiable structures on S^{k-1} . In general, $\partial M_0^k \in \Gamma_{k-1}$ is not a homotopy invariant of M^k . In this paper we study this non-invariance.

Specifically, let $B_h(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of homotopy smoothings of M_0 [18]. That is, $\Sigma^{k-1} \in B_h(M_0)$ if and only if there is a smooth manifold M'_0 , with $\partial M'_0 = \Sigma^{k-1}$, and a homotopy equivalence of pairs $h: M'_0, \partial M'_0 \rightarrow M_0, \partial M_0$. Then $B_h(M'_0) = B_h(M_0)$, and M^k is homotopy equivalent to a smooth manifold if and only if $0 \in B_h(M_0)$. We will give a homotopy theoretic description of the set of differences $\Delta_h(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_h(M_0)\} \subset \Gamma_{k-1}$, for certain classes of manifolds. If $\partial M_0 \in \Gamma_{k-1}$ is known, for example if $\partial M_0 = 0$, this determines $B_h(M_0)$. In any case, $B_h(M_0)$ and $\Delta_h(M_0)$ have the same number of elements.

Following Sullivan, two homotopy smoothings, $h: M'_0, \partial M'_0 \rightarrow M_0, \partial M_0$ and $g: M''_0, \partial M''_0 \rightarrow M_0, \partial M_0$, are called equivalent if there is a diffeomorphism $f: M'_0 \xrightarrow{\simeq} M''_0$ such that h is homotopic to gf . The set of equivalence classes is denoted $hS(M_0)$. In [18], Sullivan constructs a bijection $\theta: hS(M_0) \xrightarrow{\simeq} [M_0, F/0]$, where $F/0$ is the fibre of the map $BSO \rightarrow BSF$. Thus, if $h: M'_0 \rightarrow M_0$ represents an element of $hS(M_0)$, the formula $d\theta(M'_0, h) = \partial M'_0 - \partial M_0 \in \Gamma_{k-1}$ defines a map $d: [M_0, F/0] \rightarrow \Gamma_{k-1}$, and $\Delta_h(M_0) = \text{image}(d) \subset \Gamma_{k-1}$.

The group Γ_{k-1} can be described as follows. If $k \neq 2^j - 1$ or $2^j - 2$ then $\Gamma_{k-1} \simeq \simeq bP_k \oplus (\pi_{k-1}^s / \text{im}(J))$, where $bP_k \subset \Gamma_{k-1}$ is the cyclic subgroup of homotopy spheres that bound π -manifolds [9], [11], [15].

$\Gamma_{2^j-2} \simeq \text{kernel}(\pi_{2^j-2}^s \xrightarrow{\psi} Z_2)$, where ψ is the Arf invariant. $\psi \neq 0$ if and only if the element $h_{j-1}^2 \in \text{Ext}_A(Z_2, Z_2)$ is an infinite cycle in the Adams spectral sequence [6]. Mahowald has shown that h_{j-1}^2 is an infinite cycle if $j \leq 6$. Also, if $\psi \neq 1$, $\Gamma_{2^j-3} = \pi_{2^j-3}^s / \text{im}(J)$ ($= \pi_{2^j-3}^s$ if $j > 2$).

If k is odd then $bP_k = 0$. If k is even, the direct sum decomposition of Γ_{k-1} follows from properties of two homomorphisms, namely, the Kervaire-Milnor map $\varrho: \Gamma_{k-1} \rightarrow \pi_{k-1}^s / \text{im}(J)$, with kernel $(\varrho) = bP_k$ [15], and an invariant $f_R: \Gamma_{k-1} \rightarrow Z_2$ if $k = 4n + 2 \neq 2^j - 2$ [11], or $f_R: \Gamma_{k-1} \rightarrow Z_{\theta_n}$ if $k = 4n$, where $\theta_n = a_n \cdot 2^{2n-2} \cdot (2^{2n-1} - 1) \text{num}(B_n/4n)$, $a_n = 2$ if n is odd, $a_n = 1$ if n is even, and B_n is the Bernoulli number [9]. The restriction of f_R to $bP_k \subset \Gamma_{k-1}$ is an isomorphism. Thus a homotopy sphere $\Sigma^{k-1} \in \Gamma_{k-1}$ is determined by $\varrho(\Sigma^{k-1}) \in \pi_{k-1}^s / \text{im}(J)$ and $f_R(\Sigma^{k-1}) \in bP_k$.

The invariants $f_R: \Gamma_{4n-1} \rightarrow Z_{\theta_n}$ and $f_R: \text{bspin}_{8n+2} \rightarrow Z_2$ are natural, and can be computed where $\text{bspin}_{8n+2} \subset \Gamma_{8n+1}$ is the subgroup (of index 2) of homotopy spheres that bound spin manifolds. However, $f_R: \Gamma_{8n+5} \rightarrow Z_2$ and the extension $f_R: \Gamma_{8n+1} \rightarrow Z_2$ depend on choices, and can not be effectively computed. Thus our results on $\Delta_h(M_0^k)$ are complete only if $k \not\equiv 6 \pmod{8}$ and if, when $k \equiv 2 \pmod{8}$, M_0^k is a spin manifold.

The paper is arranged as follows. In §§ 2 and 3, we discuss Sullivan's work on homotopy smoothings and describe the composition $\varrho d: [M_0^k, F/0] \rightarrow \Gamma_{k-1} \rightarrow \pi_{k-1}^s / \text{im}(J)$. In § 4, we give some homotopy theoretic results on $F/0$. Many of the results in these three sections are well-known. In § 5, we compute the composition $f_R d: [M_0^{4n}, F/0] \rightarrow \Gamma_{4n-1} \rightarrow Z_{\theta_n}$. In § 6, we compute the composition $f_R d: [M_0^{8n+2}, F/0] \rightarrow \Gamma_{8n+1} \rightarrow Z_2$ for spin manifolds, M_0^{8n+2} . The main results of the paper are Propositions 4.4, 4.5, 5.1, 5.2 and 6.5.

In two appendixes, we give applications of the results of § 2 through § 6. In Appendix I, we set $M^{2k} = CP(k)$ and characterize those homotopy $(2k-1)$ -spheres which admit differentiable, fixed point free, S^1 actions. In Appendix II, we set $M^{k+1} = S^1 \times N^k$ and compute certain canonical subgroups of the inertia group, $I(N^k) \subset \Gamma_k$, of a smooth manifold N^k .

Many of the ideas in this paper are due to D. Sullivan. I am very grateful to him for many conversations.

§2. *Homotopy Smoothings.* We first sketch a definition of the bijection $\theta: hS(M_0) \xrightarrow{\cong} [M_0, F/0]$. Let $h: M'_0 \rightarrow M_0$ be a homotopy smoothing of M_0^k , and let \bar{h} be a homotopy inverse of h . Homotope the map h to a smooth embedding of M'_0 in the total space, $E(\xi_0)$, of the (stable) vector bundle $\xi_0 = \xi_0(h) = \bar{h}^*(\tau_{M'_0}) - \tau_{M_0}$ over M_0 where τ_{M_0} is the tangent bundle. Then the normal bundle of M'_0 in $E(\xi_0)$ is trivial and choosing a framing of M'_0 in $E(\xi_0)$ determines a fibre homotopy trivialization of ξ_0 . (In fact, it follows from the h -cobordism theorem that there is a diffeomorphism $H: M'_0 \times \mathbb{R}^q \xrightarrow{\cong} E(\xi_0^q)$, q large, homotopic to h .) This defines an element $\theta(h) \in [M_0, F/0]$, which depends only on the class of (M'_0, h) in $hS(M_0)$. By construction, the composition $M_0 \rightarrow F/O \rightarrow BS0$ represents $\xi_0(h) \in KO^0(M_0)$.

Now, h induces a bijection $h_*: hS(M'_0) \xrightarrow{\cong} hS(M_0)$, defined by $h_*(M''_0, g) = (M''_0, hg)$ where $g: M''_0 \rightarrow M'_0$. Also, there is the bijection $h^*: [M_0, F/0] \xrightarrow{\cong} [M'_0, F/0]$ induced by the homotopy equivalence $h: M'_0 \rightarrow M_0$. Since $F/0$ is an H -space, h^* is an isomorphism of groups. Consider the diagram

$$\begin{array}{ccc}
 hS(M_0) \xrightarrow{\theta} [M_0, F/0] & \xrightarrow{d} & \Gamma_{k-1} \\
 h_* \uparrow & \Downarrow h^* & \uparrow d \\
 hS(M'_0) \xrightarrow{\theta} [M'_0, F/0] & &
 \end{array} \tag{2.1}$$

This diagram is very non-commutative. In fact, if $g: M''_0 \rightarrow M'_0$ is a homotopy smoothing

of M'_0 then $d\theta(h_*(g)) = \partial M''_0 - \partial M_0 = (\partial M''_0 - \partial M'_0) + (\partial M'_0 - \partial M_0) = d\theta(g) + d\theta(h)$. We also have

PROPOSITION 2.2. *If $g \in hS(M'_0)$ then*

$$h^*\theta h_*(g) - \theta(g) = h^*\theta(h) \in [M'_0, F/0].$$

This can be equivalently stated as follows. Suppose

$$\begin{array}{ccc} M''_0 & \xrightarrow{f} & M_0 \\ g \searrow & & \nearrow h \\ & M'_0 & \end{array}$$

is a homotopy commutative diagram and f, g, h are all homotopy equivalences. Then $f = h_*(g)$ and applying the isomorphism \bar{h}^* to the equation in 2.2 gives

$$\theta(f) = \theta(h) + \bar{h}^*(\theta(g)) \in [M_0, F/0] \tag{2.3}$$

We will prove 2.3. In §§ 5 and § 6 we give formulas for the difference $d - dh^*$ and for the deviation of d from linearity (that is, in general d is not a homomorphism of groups).

Proof of 2.3. Choose a diffeomorphism $H: M'_0 \times \mathbf{R}^q \simeq E(\xi^q(\theta(h)))$ homotopic to h , and, in the diagram below, let $E(\bar{H})$ be the obvious bundle map covering $\bar{H} = H^{-1}$.

$$\begin{array}{ccc} E(\bar{H}^* \pi_1^*(\xi^q(\theta(g)))) & \xrightarrow{E(\bar{H})} & E(\pi_1^*(\xi^q(\theta(g)))) \\ \downarrow & & \downarrow \\ E(\xi^q(\theta(h))) & \xrightarrow{H} & M'_0 \times \mathbf{R}^q \\ \downarrow \pi & & \downarrow \pi_1 \\ M_0 & \xrightarrow{\bar{h}} & M'_0 \end{array}$$

Since $\pi_1 \bar{H} \simeq \bar{h} \pi$, it follows from the bundle covering homotopy theorem that there is a bundle isomorphism, B , covering the identity on $E(\xi^q(\theta(h)))$, and a bundle homotopy commutative diagram

$$\begin{array}{ccc} E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) & = & E(\pi^* \bar{h}^*(\xi^q(\theta(g)))) \xrightarrow{E(\bar{h}\pi)} E(\xi^q(\theta(g))) \\ & \downarrow B & \uparrow E(\pi_1) \\ & & E(\bar{H}^* \pi_1^*(\xi^q(\theta(g)))) \xrightarrow{E(\bar{H})} E(\pi_1^*(\xi^q(\theta(g)))) \\ & & = E(\xi^q(\theta(g))) \times \mathbf{R}^q. \end{array}$$

Let $G: M''_0 \times \mathbf{R}^q \simeq E(\xi^q(\theta(g)))$ be a diffeomorphism homotopic to g . Then $\bar{F} = (\bar{G} \times 1) E(\bar{H}) B: E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \simeq M''_0 \times \mathbf{R}^q \times \mathbf{R}^q$ is a diffeomorphism homotopic to $\bar{f} = \bar{g} \bar{h}$ where $\bar{G} = G^{-1}$. Thus the fibre homotopy trivialization

$$(\pi_2 \times \pi_3) \bar{F}: E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \rightarrow \mathbf{R}^q \times \mathbf{R}^q$$

represents $\theta(f)$. On the other hand, bundle homotopy commutativity of the diagram above implies that $(\pi_2 \times \pi_3) F$ is properly homotopic to $(\pi_2 \bar{G}E(\bar{h}) \times \pi_2 \bar{H}) \Delta$ where

$$\Delta: E(\bar{h}^*(\xi^q(\theta(g))) + \xi^q(\theta(h))) \rightarrow E(\bar{h}^*(\xi^q(\theta(g)))) \times E(\xi^q(\theta(h)))$$

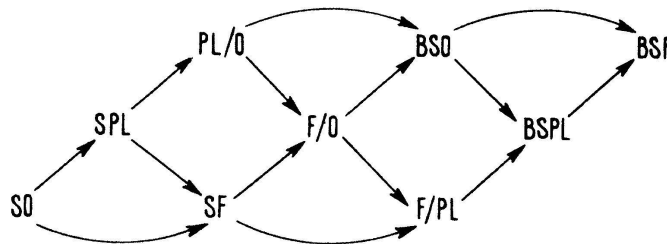
is the diagonal. Since $(\pi_2 \bar{G}E(\bar{h}) \times \pi_2 \bar{H}) \Delta$ represents $\bar{h}^*(\theta(g)) + \theta(h)$, we have shown that $\theta(f) = \bar{h}^*(\theta(g)) + \theta(h)$, as desired.

The tangential homotopy equivalence, that is, $h: M'_0 \rightarrow M_0$ with $h^*(\tau_{M_0}) = \tau_{M'_0}$ are particularly important. Let $B_{th}(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of manifolds M'_0 tangentially homotopy equivalent to M_0 , and let $\Delta_{th}(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_{th}(M_0)\} \subset \Gamma_{k-1}$.

There is a fibration $SF \xrightarrow{j} F/0 \xrightarrow{i} BS0$, where $SF = \lim_{\rightarrow} SF_q$ and SF_q is the space of base point preserving maps of degree one of S^{q-1} to itself. Thus, given $h: M'_0 \rightarrow M_0$, we have $h^*(\tau_{M_0}) = \tau_{M'_0}$ if and only if $\xi_0(h) = \bar{h}^*(\tau_{M_0}) - \tau_{M'_0} = 0 \in KO^0(M_0)$ or, equivalently, if and only if $\theta(h) \in \text{image}([M_0, SF] \xrightarrow{j^*} [M_0, F/0])$. Thus $\Delta_{th}(M_0) = d(\text{image}([M_0, SF] \rightarrow [M_0, F/0]))$.

Two other subsets of $B_h(M_0)$ are of geometric interest. Let $B_c(M_0) \subset \Gamma_{k-1}$ be the set of boundaries of smooth manifolds M''_0 combinatorially equivalent to M_0 , and let $B_{tc}(M_0) \subset B_c(M_0)$ be the subset of boundaries of those M'_0 such that some combinatorial equivalence $h: M'_0 \rightarrow M_0$ preserves the (smooth) tangent bundles, that is, $h^*(\tau_{M_0}) = \tau_{M'_0}$ as vector bundles. Let $\Delta_c(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_c(M_0)\}$ and let $\Delta_{tc}(M_0) = \{\Sigma^{k-1} - \partial M_0 \mid \Sigma^{k-1} \in B_{tc}(M_0)\}$.

There are spaces SPL and $PL/0$, and a braid of fibrations



From smoothing theory [14], it follows that $\Delta_c(M_0) = d(\text{image}([M_0, PL/0] \rightarrow [M_0, F/0]))$ and that $\Delta_{tc}(M_0) = d(\text{image}([M_0, SPL] \rightarrow [M_0, F/0]))$. Also, if $v \in [M_0^k, PL/0]$ then $dv = \partial^*(v) \in \pi_{k-1}(PL/0) = \Gamma_{k-1}$, where $\partial: S^{k-1} \rightarrow M_0^k$ represents the homotopy class of the inclusion of the boundary, $\partial M_0 \rightarrow M_0$.

In particular, $d: [M_0^k, PL/0] \rightarrow \Gamma_{k-1}$ and $d: [M_0^k, SPL] \rightarrow \Gamma_{k-1}$ are group homomorphisms. Also, $\Delta_c(M_0^k)$ and $\Delta_{tc}(M_0^k)$ are homotopy invariants of M_0^k .

Recall that for a simply connected, closed manifold, M^k , there is the surgery obstruction $s: [M^k, F/0] \rightarrow P_k$, where $P_k = \mathbf{Z}, 0, \mathbf{Z}_2, 0$ if $k \equiv 0, 1, 2, 3 \pmod{4}$, respectively, defined as follows [18]. If $u \in [M^k, F/0]$, represent u by a framing $f: M' \times \mathbf{R}^q \rightarrow E(\xi^q(u))$ of some manifold M' in the total space of the bundle $\xi^q(u) = i_*(u)$ over M .

Then $s(u) \in P_k$ is the obstruction to constructing a homotopy equivalence $M' \times \mathbf{R}^q \rightarrow E(\xi^q(u))$, framed cobordant to $M' \times \mathbf{R}^q$ in $E(\xi^q(u)) = E(\xi^q)$.

PROPOSITION 2.4 (Sullivan). *Suppose $u: M_0^k \rightarrow F/0$ extends to a map $\bar{u}: M^k \rightarrow F/0$. Then $du \in bP_k$. In fact, $du = bs(\bar{u})$ where $b: P_k \rightarrow bP_k$ is the natural projection.*

PROOF. Represent \bar{u} by a framing of a connected sum $M' \# W$ in the vector bundle $E(\xi(\bar{u}))$ over M where the projection $M'_0 \rightarrow M_0$ is a homotopy equivalence and where W is an almost parallelizable manifold. Then $s(\bar{u}) = -[W] \in P_n$ where P_n is regarded as the group of cobordism classes of almost parallelizable PL manifolds. By smoothing theory, in the complement of a point, $M' \# W$ inherits a smooth structure from $E(\xi(\bar{u}))$ and $\partial(M' \# W)_0 = \partial M_0$. Then $du = \partial M'_0 - \partial M_0 = -\partial W_0 = bs(\bar{u}) \in bP_k$.

REMARK 2.5. If $k = 4n$ and $u \in [M^{4n}, F/0]$ is represented by $f: M' \times \mathbf{R}^q \rightarrow E(\xi^q)$, then

$$s(u) = \left(\frac{1}{8}\right) (\text{index}(M) - \text{index}(M')) = \left(\frac{1}{8}\right) \langle L(M) (1 - L(\xi)), [M^{4n}] \rangle \in \mathbf{Z}$$

since $\tau_{M'} = f^*(\tau_M + \xi)$.

If $k = 4n + 2$ and $u \in [M^{4n+2}, F/0]$, there is also a cohomology formula for $s(u)$; namely,

$$s(u) = \langle v^2(M) \cdot u^*(K), [M]_2 \rangle \in \mathbf{Z}_2$$

where $v(M) = 1 + v_1(M) + v_2(M) + \dots \in H^*(M, \mathbf{Z}_2)$ is the total Wu class, and $K = k_2 + k_6 + k_{10} + \dots \in H^{4*+2}(F/0, \mathbf{Z}_2)$ is a suitable class [18].

§ 3. *The composition $\varrho d: [M_0^k, F/0] \rightarrow \Gamma_{k-1} \rightarrow \pi_{k-1}^s / \text{im}(J)$*

Let $\partial: S^{k-1} \rightarrow M_0^k$ represent the homotopy class of the inclusion of the boundary, $\partial M_0^k \rightarrow M_0^k$. Then ∂ induces $\partial^*: [M_0^k, F/0] \rightarrow [S^{k-1}, F/0] = \pi_{k-1}(F/0)$. Further, image (∂^*) is contained in the torsion subgroup of $\pi_{k-1}(F/0)$, which is isomorphic to $\pi_{k-1}^s / \text{im}(J)$.

PROPOSITION 3.1. *Let $u \in [M_0^k, F/0]$. Then*

$$\varrho(du) = \partial^*(u) \in \pi_{k-1}^s / \text{im}(J) \subset \pi_{k-1}(F/0).$$

Proof. Let $u = \theta(h)$, where $h: M'_0 \rightarrow M_0$. Then u is represented by a fibre homotopy trivialization of $\xi_0(h) = \xi_0$, defined by a framing $H: M'_0 \times \mathbf{R}^q \simeq E(\xi_0^q)$. The restriction of ξ_0 to ∂M_0^k is trivial. For, if $k - 1 \equiv 0$ or $4 \pmod{8}$, the Pontrjagin class of $\xi_0|_{\partial M_0^k}$ is zero, and if $k - 1 \equiv 1$ or $2 \pmod{8}$ $\xi_0|_{\partial M_0^k}$ is fibre homotopically trivial. Thus, H induces a framing $\partial H: \partial M'_0 \times \mathbf{R}^q \simeq \partial M_0 \times \mathbf{R}^q$, which represents $\partial^*(u) \in \pi_{k-1}(F/0)$. It now

follows from the definition of the Kervaire-Milnor map, ϱ , and a little smoothing theory, that $\partial^*(u) = \varrho(\hat{c}M'_0 - \partial M_0) = \varrho(du)$.

COROLLARY 3.2. *The composition $\varrho d: [M_0^k, F/0] \rightarrow \pi_{k-1}^s / \text{im}(J)$ is a homomorphism of groups. Thus, if $u, v \in [M_0^k, F/0]$ then $du + dv - d(u+v) \in bP_k \subset \Gamma_{k-1}$.*

COROLLARY 3.3. *Let $h: M'_0 \rightarrow M_0$ be any degree one map (not necessarily a homotopy equivalence). Then $\varrho(dh^*(u)) = \varrho(du)$, where $u \in [M_0, F/0]$ and $h^*: [M_0, F/0] \rightarrow [M'_0, F/0]$. Thus $dh^*(u) - du \in bP_k \subset \Gamma_{k-1}$.*

§ 4. *Discussion of $F/0$.* If we are to apply the results of § 2 and § 3 (and those in § 5 and § 6 below), we must be able to compute $[M_0^k, F/0]$. In general, this is difficult. The following discussion relates the group $[M_0^k, F/0]$ to more familiar homotopy invariants of M_0^k .

There are fibrations $S0 \xrightarrow{\Omega J} SF \xrightarrow{j} F/0 \xrightarrow{i} BSO \xrightarrow{J} BSF$. These induce an exact sequence of groups

$$K0^{-1}(X) \rightarrow [X, SF] \xrightarrow{j_*} [X, F/0] \xrightarrow{i_*} K0^0(X) \rightarrow J(X) \rightarrow 0$$

for any finite complex X . Further, since SF_{q+1} is a component of $\Omega^q S^q$, $[X, SF] = \lim_{\leftarrow} [S^q \wedge X, S^q] = \pi_0^s(X)$, as sets, where $\pi_0^s(X)$ is the 0th stable cohomotopy group of X . Actually, $\pi_0^s(X)$ is a ring, and, as groups, $[X, SF] \simeq 1 + \pi_0^s(X)$ where the addition on the right is given by $(1 + \alpha)(1 + \beta) = 1 + \alpha + \beta + \alpha\beta$ [13].

The Adams conjecture on $J: K0^0(X) \rightarrow J(X)$ can be stated as follows ([1]):

4.1 Let $\xi \in K0^0(X)$. Then there is an integer, $e(k, \xi)$, such that $J(k^{e(k, \xi)}(\psi^k - 1)(\xi)) = 0$ where ψ^k is the Adams operation.

Since $K0^0(X)$ is finitely generated, we may choose $e(k, \xi) = e(k)$ independent of ξ . For any function $e(k)$, Adams has proved that $\text{kernel}(J) = i_*([X, F/0])$ is contained in the subgroup of $K0^0(X)$ generated by the elements $k^{e(k)}(\psi^k - 1)(\xi)$, $\xi \in K0^0(X)$. The Adams conjecture 4.1 has recently been proved by Sullivan and Quillen.

PROPOSITION 4.2. *If $K0^0(M^k) \rightarrow K0^0(M_0^k)$ is surjective (e.g., if $k-1 \not\equiv 1$ or $2 \pmod{8}$ or if M^k is a spin manifold), then each element $w \in [M_0^k, F/0]$ can be written as a sum, $w = u + v$, where $u \in \text{image}([M^k, F/0])$ and $v \in \text{image}([M_0, SF])$.*

Proof. $J(\xi_0(w)) = J(i_*(w)) = 0$. It follows that there is an element $\xi \in K0^0(M^k)$ such that $J(\xi) = 0$ and $\xi|_{M_0} = \xi_0(w) = \xi_0$. Then $\xi = i_*(\bar{u})$ for some $\bar{u} \in [M^k, F/0]$. Let $u = \bar{u}|_{M_0}$. Then $w - u \in \text{kernel}(i_*) = \text{image}(j_*)$, and 4.2 is proved.

Remark 4.3. It is a consequence of the Adams conjecture that for each prime p , there is a homotopy equivalence $(F/0)_{(p)} \sim BSO_{(p)} \times \text{Cok}(J)_{(p)}$ where $X_{(p)}$ denotes the

localization of X at p . Moreover, $SJ_{(p)} \sim \text{im}(J)_{(p)} \times \text{Cok}(J)_{(p)}$, and the map $j_{(p)}: SF_{(p)} \rightarrow (F/0)_{(p)}$ is a product map $j_{(p)} \times \text{Id}: \text{im}(J)_{(p)} \times \text{Cok}(J)_{(p)} \rightarrow BS0_{(p)} \times \text{Cok}(J)_{(p)}$. This factoring of $(F/0)_{(p)}$ enables one to also establish the conclusion of 4.2 in the case $(k-1) \equiv 2 \pmod{8}$.

PROPOSITION 4.4. *If $u, v \in [M_0^k, F/0]$, with $u \in \text{image}([M_0^k, F/0])$ and $v \in \text{image}([M_0, SF])$, then $d(u+v) = du + dv \in \Gamma_{k-1}$.*

Proof. Let $v = \theta(h)$, and let $h^*(u) = \theta(g)$ where $h: M'_0 \rightarrow M_0$ and $g: M''_0 \rightarrow M'_0$ are homotopy equivalences. By 2.3, $\theta(f) = u + v$ where $f = hg: M''_0 \rightarrow M_0$. Thus, $d(u+v) = \partial M''_0 - \partial M_0 = (\partial M''_0 - \partial M'_0) + (\partial M'_0 - \partial M_0) = dh^*(u) + dv$.

By the hypothesis, $h: M'_0 \rightarrow M_0$ is a tangential homotopy equivalence. Also, the maps $M'_0 \xrightarrow{h} M_0 \xrightarrow{u} F/0$ extend to maps $M' \xrightarrow{h} M \xrightarrow{u} F/0$. By Proposition 2.4, du and $dh^*(u)$ belong to $bP_k \in \Gamma_{k-1}$. Since $h^*(L(M)) = L(M')$ and $h^*(v^2(M)) = v^2(M')$, it follows from the formulas in Remark 2.5 that $du = dh^*(u)$. Thus $d(u+v) = dh^*(u) + dv = du + dv$.

The following is an immediate consequence of Propositions 2.4, 4.2, 4.4, and Remark 4.3, and is one of our main results.

PROPOSITION 4.5. *Assume that $k \not\equiv 2 \pmod{8}$ or that M_0^k is a spin manifold. Then*

$$\Delta_h(M_0^k) = (\Delta_h(M_0^k) \cap bP_k) + \Delta_{th}(M_0^k) \subset \Gamma_{k-1}.$$

Here, by the sum of the two subsets, we mean all elements $\Sigma + \Sigma'$ where $\Sigma \in \Delta_h(M_0^k) \cap bP_k$ and $\Sigma' \in \Delta_{th}(M_0^k)$.

Remark 4.6. Note that the map $\partial^*: [M_0^k, SF] \rightarrow \pi_{k-1}(SF) = \pi_{k-1}^s$ is an invariant of the stable homotopy of M_0^k and can be computed as

$$\partial^*: [S^q \wedge M_0^k, S^q] \rightarrow \pi_{q+k-1}(S^q) = \pi_{k-1}^s, \quad q \text{ large.}$$

We will need the following familiar invariant. Consider the subgroup of elements $(\xi, \alpha) \in K0^0(X) \otimes \pi_{4k-1}(X)$ such that $ph_k(\xi) = 0 \in H^{4k}(X, Q)$ and $\alpha^* = 0: H^{4k-1}(X) \rightarrow H^{4k-1}(S^{4k-1})$. Let $\bar{X} = X \cup_{\alpha} e^{4k}$, and let $\bar{\xi} \in K0^0(\bar{X})$ restrict to $\xi \in K0^0(X)$. Then $ph_k(\bar{\xi}) \in p^*(H^{4k}(S^{4k}, Q)) = Q$, where $p: \bar{X} \rightarrow S^{4k}$ is the projection. Further, since $\bar{\xi}$ is well-defined modulo $p^*(K0^0(S^{4k}))$, $ph_k(\bar{\xi})$ is well-defined modulo $p^*(H^{4k}(S^{4k}, a_k Z))$. It follows that $e_R(\xi, \alpha) = (1/a_k) ph_k(\bar{\xi}) \in Q/Z$ is a well-defined homomorphism. Moreover, the diagram

$$\begin{array}{ccc} K0^0(X) \otimes \pi_{4k-1}(X) & & \\ \downarrow \varphi \otimes s & \searrow^{e_R} & Q/Z \\ K0^0(S^8 \wedge X) \otimes \pi_{4k+7}(S^8 \wedge X) & \nearrow_{e_R} & \end{array} \quad (4.7)$$

commutes (when e_R is defined), where \mathcal{P} is the periodicity isomorphism and s is suspension. e_R can be interpreted as a functional operation from $K0$ -theory to cohomology. If $X = S^{8n}$ and $\xi \in K0^0(S^{8n})$ is a generator, we recover the Adams homomorphism $e_R: \pi_{8n+4k-1}(S^{8n}) \rightarrow Q/Z$ [2]. If $X = M_0^{4n}$ and $\alpha \in \pi_{4n-1}(M_0^{4n})$ represents the inclusion of the boundary, we get a homomorphism $e_R: K0^0(M_0^{4n}) \rightarrow Q/Z$.

The following $K0$ -theory invariant of $F/0$ bundles will also be essential.

PROPOSITION 4.8. *There is an element $\gamma \in 1 + K0^0(F/0)$ such that $ph(\gamma) = \hat{A} \in H^{**}(F/0, Q) \simeq H^{**}(BSO, Q)$. Further, if $u, v \in [X, F/0]$ then $\gamma(u+v) = \gamma(u) \cdot \gamma(v) \in 1 + K0^0(X)$, where by $\gamma(u)$ we mean $u^*(\gamma) \in 1 + K0^0(X)$.*

Proof. The universal bundle over $F/0$ admits a unique spin structure. Thus, the Thom space $M(F/0)$ has two canonical $K0$ -theory orientations, namely, an orientation $U_1 \in K0^0(M(F/0))$ induced from M Spin, with $ph(U_1) = \Phi(\hat{A}^{-1}) \in H^{**}(M(F/0), Q)$, and an orientation, U_2 , with $ph(U_2) = \Phi(1)$, induced from the sphere spectrum via a fibre homotopy trivialization. Define $\gamma \in 1 + K0^0(F/0)$ by the equation $\gamma \cdot U_1 = U_2 \in K0^0(M(F/0))$. Then $\Phi(1) = ph(U_2) = ph(\gamma)ph(U_1) = \Phi(ph(\gamma) \cdot \hat{A}^{-1})$, hence $ph(\gamma) = \hat{A}$.

The second statement follows from universal multiplicative properties of the orientations U_1 and U_2 .

The final three results in this section are technical results about the invariants e_R and γ which we will need in §5.

Let $u \in [M_0^k, F/0]$ correspond to a homotopy equivalence $h: M'_0 \rightarrow M_0$. Homotope h to an embedding $h: M'_0 \rightarrow M_0 \times \mathbb{R}^{8q}$. The normal bundle of M'_0 in $M_0 \times \mathbb{R}^{8q}$ is $h^*(-\xi_0(u))$, and we have the “collapsing map” $c: T(e_{M_0}^{8q}) \rightarrow T(h^*(-\xi_0)_{M'_0}^{8q})$. Since ξ_0 is a spin vector bundle there are Thom isomorphisms $\Phi_{K0}: K0(M'_0) \simeq K0^0(T(h^*(-\xi_0)_{M'_0}^{8q}))$ and $\Phi_{K0} = \mathcal{P}: K0(M_0) \simeq K0^0(T(e_{M_0}^{8q}))$, and a Gysin homomorphism $h_*: K0(M'_0) \rightarrow K0(M_0)$ defined by $h_*(x) = \mathcal{P}^{-1}c^*\Phi_{K0}(x)$.

PROPOSITION 4.9. *If $u \in [M_0, F/0]$ corresponds to $h: M'_0 \rightarrow M_0$ then $h_*(1) = \gamma(u) \in K0(M_0)$.*

Proof. This follows from the definition of $\gamma(u)$ and the observation that the fibre homotopy trivialization

$$T(\xi_0^{8q} + e_{M_0}^{8q}) \xrightarrow{\bar{c}} T(h^*(\xi_0^{8q}) + h^*(-\xi_0^{8q})) = T(e_{M'_0}^{16q}) \xrightarrow{\pi} S^{16q}$$

represents $u \in [M_0, F/0]$, where \bar{c} is defined by embedding $M_0 \times \mathbb{R}^{8q} \subset E(\xi_0^{8q}) \times \mathbb{R}^{8q}$ and extending c , and π is the projection.

PROPOSITION 4.10(i) *Let $u, v \in [M_0^{4n}, F/0]$. If $v \in [M_0, PL/0]$ or $v \in [M_0, SF]$, then $e_R(\gamma(u+v)) = e_R(\gamma(u)) + e_R(\gamma(v)) \in Q/Z$.*

(ii) Suppose M_0^{4n} is a spin manifold. If $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$, then $e_R(\gamma(u)) = e_R(\xi_0(u)) = 0$.

Proof. Let $\overline{\gamma(u)}, \overline{\gamma(v)} \in K0(M^{4n})$ extend $\gamma(u), \gamma(v) \in K0(M_0^{4n})$. By 4.8, $\gamma(u+v) = \gamma(u) \cdot \gamma(v)$, so $\overline{\gamma(u)} \cdot \overline{\gamma(v)} \in K0(M^{4n})$ is an extension of $\gamma(u+v)$. Then

$$\begin{aligned} e_R(\gamma(u+v)) &= (1/a_n) \langle ph(\overline{\gamma(u)} \cdot \overline{\gamma(v)}), [M^{4n}] \rangle \\ &= (1/a_n) \langle ph(\overline{\gamma(u)}) ph(\overline{\gamma(v)}), [M^{4n}] \rangle \in Q/Z. \end{aligned}$$

From the assumption, it follows that $ph(\overline{\gamma(v)}) = 1 + ph_n(\overline{\gamma(v)})$; hence

$$\begin{aligned} (1/a_n) \langle ph(\overline{\gamma(u)}) ph(\overline{\gamma(v)}), [M^{4n}] \rangle \\ = (1/a_n) \langle ph_n(\overline{\gamma(u)}) + ph_n(\overline{\gamma(v)}), [M^{4n}] \rangle \in Q/Z, \end{aligned}$$

and 4.10(i) follows immediately.

For 4.10(ii), note that the Thom space of the normal bundle of M_0 , $T(v_{M_0}^{8q})$, has a canonical $K0$ -orientation. This extends to some $K0$ -orientation, U , of $T(v_M^{8q})$. Then, since there is a degree one map $S^{8q+4n} \rightarrow T(v_M^{8q})$, we have

$$(1/a_n) \langle ph(\overline{\gamma(u)} - 1) ph(U), [T(v_M)] \rangle \in \mathbf{Z}.$$

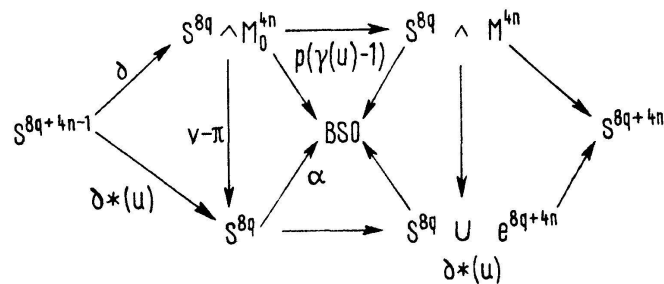
Since $ph(\overline{\gamma(u)}) - 1 = ph_n(\overline{\gamma(u)})$, it follows that

$$\begin{aligned} e_R(\gamma(u)) &= (1/a_n) \langle ph_n(\overline{\gamma(u)}), [M^{4n}] \rangle \\ &= (1/a_n) \langle ph_n(\overline{\gamma(u)}) ph(U), [T(v_M)] \rangle = 0 \in Q/Z. \end{aligned}$$

Similarly, $e_R(\xi_0(u)) = (1/a_n) \langle ph_n(\overline{\xi_0(u)}) ph(U), [T(v_M)] \rangle = 0 \in Q/Z$, and 4.10(ii) is proved.

PROPOSITION 4.11. *Let $u \in [M_0^{4n}, SF]$. Then $e_R(\gamma(u)) = e_R(\partial^*(u))$ where $\partial^*(u) \in \pi_{4n-1}(SF) = \pi_{4n-1}^s$. Moreover, $e_R(\gamma(u))$ has order a power of 2.*

Proof. Let $v: M_0 \times S^{8q} \rightarrow S^{8q}$ be the adjoint of $u: M_0 \rightarrow SF_{8q+1}$, and let $\alpha \in K0^0(S^{8q})$ be the generator. Then $\gamma(u) \cdot \pi^*(\alpha) = v^*(\alpha)$, where $\pi: M_0 \times S^{8q} \rightarrow S^{8q}$ is the projection. Thus $v^*(\alpha) - \pi^*(\alpha) = \mathcal{P}(\gamma(u) - 1) \in K0^0(S^{8q} \wedge M_0)$. It follows that there is a homotopy commutative diagram



From the definitions and diagram 4.7, one sees that $e_R(\partial^*(u)) = e_R(\gamma(u))$.

For the second statement, it is only necessary to observe that there are spin manifolds, N_0^{4n} , with $\partial N_0^{4n} = S^{4n-1}$, and maps $g: N_0^{4n}, \partial N_0^{4n} \rightarrow M_0^{4n}, \partial M_0^{4n}$ of degree a power of 2, say 2^r . Then $2^r e_R(\gamma(u)) = 2^r e_R(\partial^*(u)) = e_R(2^r \partial^*(u)) = e_R(\partial^*(g^*(u))) = e_R(\gamma(g^*(u))) = 0$, by 4.10(ii).

§5. The composition $f_R d: [M_0^{4n}, F/0] \rightarrow \mathbf{Z}_{\theta_n}$. The invariant $f_R: \Gamma_{4n-1} \rightarrow \mathbf{Z}_{\theta_n}$ is defined as follows. Given $\Sigma^{4n-1} \in \Gamma_{4n-1}$, let $\Sigma^{4n-1} = \partial W_0^{4n}$, where W_0^{4n} is a smooth spin manifold such that the decomposable Pontryagin numbers of W^{4n} vanish. Then

$$f_R(\Sigma^{4n-1}) = (\frac{1}{8}) \text{index}(W^{4n}) \in \mathbf{Z}/\theta_n \cdot \mathbf{Z}.$$

(It is proved in [9] that such manifolds W_0^{4n} exist and that f_R is well-defined.)

It will be convenient to regard f_R as a homomorphism $f_R: \Gamma_{4n-1} \rightarrow Q/Z$. Namely, define $f_R(\Sigma^{4n-1}) = (\frac{1}{8}\theta_n) \text{index}(W^{4n}) \in Q/Z$, where W^{4n} is as above.

Recall that the L -genus is given by

$$L_n(p_1 \dots p_n) = (8\theta_n p_n / a_n (2n - 1)! j_n) + L_n(p_1 \dots p_{n-1}, 0).$$

PROPOSITION 5.1. Let $u \in [M_0^{4n}, F/0]$. Then

$$f_R(du) = (\frac{1}{8}\theta_n) \langle L(M)(1 - L(\xi)), [M^{4n}] \rangle \in Q/Z,$$

where $L(\xi) = L(p_1(\xi_0(u)) \dots p_{n-1}(\xi_0(u)), p_n(\xi))$ and $p_n(\xi) / a_n (2n - 1)! j_n \in Q/Z$ is determined (formally) by the equations

$$(1/a_n) \langle \hat{A}(\xi), [M^{4n}] \rangle = e_R(\gamma(u)) \in Q/Z$$

and

$$(1/a_n) \langle ph(\xi), [M^{4n}] \rangle = e_R(\xi_0(u)) \in Q/Z.$$

The proof of Proposition 5.1 will require some preliminary results.

First, note that since

$$(1/a_n) \hat{A}_n(p_1 \dots p_n) = (-\text{num}(B_n/4n) p_n / a_n (2n - 1)! j_n) + \hat{A}_n(p_1 \dots p_{n-1}, 0)$$

and

$$(1/a_n) ph_n(p_1 \dots p_n) = ((-1)^{n-1} j_n p_n / a_n (2n - 1)! j_n) + ph_n(p_1 \dots p_{n-1}, 0),$$

and since $\text{num}(B_n/4n)$ and $j_n = \text{denom}(B_n/4n)$ are relatively prime, it follows that the equations in 5.1 for $p_n(\xi) / a_n (2n - 1)! j_n \in Q/Z$ have at most one solution.

Secondly, the computation of $p_n(\xi) / a_n (2n - 1)! j_n$ in Proposition 5.1 is purely formal. That is, we do not assert the existence of a vector bundle ξ with the properties indicated. However, Proposition 5.1 and Remark 2.5 are closely related. If $u \in [M_0^{4n}, F/0]$ extends to $\bar{u} \in [M^{4n}, F/0]$, then $\xi = \xi(\bar{u})$ is an extension of $\xi_0 = \xi_0(u)$. Remark 2.5 asserts that

$f_R(du) = (\frac{1}{8}\theta_n) \langle L(M)(1 - L(\xi)), [M^{4n}] \rangle \in Q/Z$. Moreover, $\gamma(\bar{u}) \in K0(M)$ extends $\gamma(u) \in K0(M_0)$, hence $e_R(\gamma(u)) = (1/a_n) \langle ph(\gamma(\bar{u})), [M] \rangle = (1/a_n) \langle \hat{A}(\xi), [M] \rangle$ and also, of course, $e_R(\xi_0) = (1/a_n) \langle ph(\xi), [M] \rangle$.

Recall that the image of the Adams homomorphism $e_R: \pi_{4n-1}^s \rightarrow Q/Z$ consists of integral multiples of $1/j_n = 1/\text{denom}(B_n/4n)$ [2]. Thus, there is a unique homomorphism $\tilde{e}_R: \pi_{4n-1}^s \rightarrow Q/Z$, defined by $\text{num}(B_n/4n) \tilde{e}_R(\alpha) = e_R(\alpha)$. If α is the image of the generator of $\pi_{4n-1}(S^0) = Z$, then $e_R(\alpha) = (B_n/4n) = \text{num}(B_n/4n)/\text{denom}(B_n/4n)$. Thus, \tilde{e}_R is a normalization of e_R , with $\tilde{e}_R(\alpha) = 1/j_n$.

PROPOSITION 5.2. *If $u \in [M_0^{4n}, SF]$, then $f_R(du) = \tilde{e}_R(\partial^*(u)) \in Q/Z$. In particular, $f_R(du)$ has order a power of 2.*

Proof. Represent u by a tangential homotopy equivalence $h_0: M'_0 \rightarrow M_0$. Let h denote the obvious extension $h: M' \rightarrow M$. Then $\tau_{M'} = h^*(\tau_M + p^*(\sigma))$ as PL bundles, where $p: M^{4n} \rightarrow S^{4n}$ is a map of degree one and $\sigma \in \pi_{4n}(BSPL)$. Since h_0 is a tangential homotopy equivalence, and since $\text{index}(M') = \text{index}(M)$, it is easy to see that the Pontrjagin class $p_n(\sigma) = 0$. That is, σ is a torsion element of $\pi_{4n}(BSPL)$. Further, $J_{PL}(\sigma) = \partial^*(u)$, where $J_{PL}: \pi_{4n}(BSPL) \rightarrow \pi_{4n}(BSF) = \pi_{4n-1}^s$, and $\beta(\sigma) = du$, where $\beta: \pi_{4n}(BSPL) \rightarrow \pi_{4n-1}(PL/0) = \Gamma_{4n-1}$. It then follows from [9; Theorems 4.7, 4.8] that $\text{num}(B_n/4n) f_R(du) = e_R(\partial^*(u))$. This relation, together with 4.11, proves Proposition 5.2.

Note that if $u \in [M_0^{4n}, SF]$, then Proposition 5.1 asserts that $f_R(du) = -p_n(\xi)/a_n(2n-1)!j_n \in Q/Z$, where

$$(1/a_n) \langle \hat{A}(\xi), [M^{4n}] \rangle = -\text{num}(B_n/4n) p_n(\xi)/a_n(2n-1)!j_n = e_R(\gamma(u)) \in Q/Z.$$

Thus 5.2 and 4.11 imply 5.1 in the case $u \in [M_0^{4n}, SF]$.

COROLLARY 5.3(i). *The map $d: [M_0^{4n}, SF] \rightarrow \Gamma_{4n-1}$ is a group homomorphism.*
 (ii) *If $h: M'_0 \rightarrow M_0$ is any degree one map, then the diagram*

$$\begin{array}{ccc} [M_0, SF] & \xrightarrow{d} & \Gamma_{4n-1} \\ \downarrow h^* & \searrow & \uparrow d \\ [M'_0, SF] & & \end{array}$$

commutes.

Proof. This follows from 5.2 and 3.1 since $f_R \oplus \varrho: \Gamma_{4n-1} \rightarrow Z_{\theta_n} \oplus (\pi_{4n-1}^s/\text{im}(J))$ is an isomorphism.

COROLLARY 5.4. *If $u \in [M_0^{4n}, F/0]$ and $v \in [M_0^{4n}, SF]$, then $d(u+v) = du + dv$.*

Proof. This follows from 4.2, 4.4 and 5.3(i).

We can also prove Proposition 5.1. By 2.5 and 5.2, Proposition 5.1 is true if

$u \in \text{image} ([M^{4n}, F/0])$ or if $u \in \text{image} ([M_0^{4n}, SF])$. By 4.4, it suffices to prove that

$$(\frac{1}{8}\theta_n) \langle L(M) (1 - L(\xi(u+v))), [M^{4n}] \rangle$$

$$(\frac{1}{8}\theta_n) \langle L(M) (1 - L(\xi(u))), [M^{4n}] \rangle + (\frac{1}{8}\theta_n) \langle L(M) (1 - L(\xi(v))), [M^{4n}] \rangle$$

if $u \in \text{image} ([M^{4n}, F/0])$ and $v \in \text{image} ([M_0^{4n}, SF])$. Since $L(\xi(v)) = 8\theta_n p_n(\xi(v)) / a_n(2n-1)!j_n$, this is equivalent to proving that $p_n(\xi(u+v)) / a_n(2n-1)!j_n = p_n(\xi(u)) / a_n(2n-1)!j_n + p_n(\xi(v)) / a_n(2n-1)!j_n$. But, by 4.10(i), $e_R(\gamma(u+v)) = e_R(\gamma(u)) + e_R(\gamma(v))$, and, of course, $e_R(\xi_0(u+v)) = e_R(\xi_0(u)) + e_R(\xi_0(v))$. The equations given in 5.1 which determine $p_n(\xi) / a_n(2n-1)!j_n$ now yield the desired additivity result.

Remark 4.6 and Propositions 3.1 and 5.2 show that $\Delta_{th}(M_0^{4n})$ is computable in terms of the stable homotopy theory invariant $\partial^*: [S^q \wedge M_0^{4n}, S^q] \rightarrow \pi_{q+4n-1}(S^q) = \pi_{4n-1}^s$. Proposition 2.4 and Remark 2.5, together with the Adams conjecture, show that $\Delta_h(M_0^{4n}) \cap bP_{4n}$ is computable in terms of $L(M)$ and $ph(KO(M^{4n})) \subset H^{**}(M^{4n}, Q)$. Thus, $\Delta_h(M_0^{4n}) = (\Delta_h(M_0^{4n}) \cap bP_{4n}) + \Delta_{th}(M_0^{4n})$ is computable in terms of familiar invariants.

It is interesting that by using the Riemann-Roch theorem for spin maps, Proposition 5.1 can be proved without using Proposition 4.2 or the Adams conjecture. Then 3.1 and 5.1 provide, in a sense, a homotopy theoretic computation of the geometric map $d: [M_0^{4n}, F/0] \rightarrow \Gamma_{4n-1}$. However, use of the Adams conjecture gives the more practical description of $\Delta_h(M_0^{4n})$ above.

We now give some corollaries of the results above.

COROLLARY 5.5(i). *If M_0^{4n} is a spin manifold and $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$, then $f_R(du) = 0$. Hence $du \in \pi_{4n-1}^s / \text{im}(J) \subset \Gamma_{4n-1}$.*

(ii) *If M_0^{4n} is a weakly complex manifold and $u \in [M_0^{4n}, SF]$, then $a_n f_R(du) = 0$.*

Proof. In the notation of Proposition 5.1, it follows from 4.10(ii) that $p_n(\xi) / a_n(2n-1)!j_n = 0$. Hence, $L(\xi) = 1$ and $f_R(du) = 0$.

We will give an alternate proof of 5.5(i). Let $h: M'_0 \rightarrow M_0$ represent u . Then $h^*(\tau_{M_0}) = \tau_{M'_0}$ as vector bundles if $u \in [M_0, SF]$, and as PL bundles if $u \in [M_0, PL/0]$. In either case, $W_0 = M'_0 \# (-M_0)$ is a spin manifold, $\partial W_0 = \partial M'_0 - \partial M_0$, and all the Pontrjagin numbers of W , including $p_n(W)$, vanish. Then $f_R(du) = f_R(\partial M'_0 - \partial M_0) = (\frac{1}{8}\theta_n) \text{index}(W) = 0$.

5.5(ii) can be proved by an argument similar to the second proof of 5.5(i). Namely, if M_0 is weakly complex and M'_0, W_0 are as above, then M'_0 and W_0 are weakly complex, and all the Chern numbers of W vanish. An invariant $f_c: \Gamma_{4n-1} \rightarrow Q/Z$ is defined in [9], using weakly complex manifolds instead of spin manifolds, and $f_c = a_n f_R$. It follows that $0 = f_c(du) = a_n f_R(du)$.

COROLLARY 5.6. *If $u \in [M_0^{4n}, PL/0]$, then $\text{num}(B_n/4n) f_R(du) = e_R(\gamma(u))$, and $f_R(du)$ has order a power of 2.*

Proof. The first statement follows from Proposition 5.1, since $f_R(du) = -p_n(\xi)/a_n(2n-1)!j_n \in Q/Z$ and $(1/a_n) \langle \hat{A}(\xi), [M^{4n}] \rangle = -\text{num}(B_n/4n) p_n(\xi)/a_n(2n-1)!j_n = e_R(\gamma(u)) \in Q/Z$.

For the second statement, let $g: N_0^{4n}, \partial N_0^{4n} \rightarrow M_0^{4n}, \partial M_0^{4n}$ be a map of degree 2ⁿ where N_0^{4n} is a spin manifold. Then $2f_R(du) = f_R(dg^*(u)) = 0$ by 5.5(i).

COROLLARY 5.7. *If M_0^{4n} is a spin manifold with $f_R(\partial M_0^{4n}) \neq 0$ (or if M_0^{4n} is any manifold and $f_R(\partial M_0^{4n})$ has order not a power of 2), then $0 \notin B_{th}(M_0^{4n})$ and $0 \notin B_c(M_0^{4n})$; that is, M_0^{4n} is not tangentially homotopy equivalent or combinatorially equivalent to a smooth manifold.*

Proof. This follows from 5.2 and 5.6.

Here is an example to show that $f_R d: [M_0^{4n}, SF] \rightarrow Z_{\theta_n}$ is not zero in general. Adams has defined elements $\mu_k \in \pi_{8k+2}^S$ such that $2\mu_k = 0, \mu_k \eta \neq 0$ and $\mu_k \eta \in \text{im}(J) \subset \pi_{8k+3}^S$ [2]. If M^{8k+4} is not a spin manifold (for example, $M^{8k+4} = CP(4k+2)$), choose $x \in H^{8k+2}(M, Z_2)$ such that $S_q^2(x) \neq 0$ and let $g: M_0 \rightarrow S^{8k+2}$ be a map such that $g^*(\sigma) = x$, where $\sigma \in H^{8k+2}(S^{8k+2})$. Then the composition $S^{8k+3} \xrightarrow{\partial} M_0^{8k+4} \xrightarrow{g} S^{8k+2} \xrightarrow{\mu_k} SF$ represents $\partial^*(\mu_k g) = \mu_k \eta$, since $g\partial = \eta$. Since $\tilde{e}_R(\mu_k \eta) = \frac{1}{2} \in Q/Z$, 5.2 implies $f_R(d(\mu_k g)) = \frac{1}{2} \in Q/Z$.

In [10] we showed that the element μ_k could, in fact, be defined in $\pi_{8k+2}(SPL)$. Thus, in the example above, we actually have $u = \mu_k g \in [M_0^{8k+4}, SPL]$ and $du \in \Delta_{ic}(M_0^{8k+6})$ is the element of order 2 in bP_{8k+4} . I do not know of an example of $u \in [M_0^{4n}, SF]$ or $u \in [M_0^{4n}, PL/0]$ such that $a_n \cdot f_R(du) \neq 0$.

We next give a somewhat simpler formula for $f_R d: [M_0^{4n}, F/0] \rightarrow Z_{\theta_n}$, when M_0^{4n} is a spin manifold, generalizing 5.5(i).

COROLLARY 5.8. *Let $u \in [M_0^{4n}, F/0]$, where M_0^{4n} is a spin manifold. Then $f_R(du) = (\frac{1}{8}\theta_n) \langle L(M)(1-L(\xi)), [M] \rangle \in Q/Z$, where $L(\xi)$ is as in 5.1 and $(p_n(\xi)/a_n(2n-1)!j_n) \in Q/Z$ is determined by the equations*

$$(1/a_n) \langle (\hat{A}(\xi) - 1) \hat{A}(M), [M] \rangle = 0 \in Q/Z$$

and

$$(1/a_n) \langle ph(\xi) \hat{A}(M), [M] \rangle = 0 \in Q/Z.$$

Proof. This follows from 4.4, 5.5(i), and 2.4, and the Riemann-Roch Theorem for manifolds with framed boundary.

The point of 5.8 is that for spin manifolds, $f_R(du)$ depends only on the Pontrjagin classes of M_0^{4n} and $\xi_0(u)$, and not on the KO -theory invariants $\gamma(u)$ and $\xi_0(u)$. This is because if $W_0 = M_0' \# (-M_0)$ then W_0 is a spin manifold, $\partial W_0 = \partial M_0' - \partial M_0$, and the

Pontrjagin numbers of W , including $p_n(W)$, are functions of the Pontrjagin classes of M_0 and $\xi_0(u)$. Thus $f_R(du) = f_R(\partial W_0)$ can be computed in terms of Pontrjagin classes alone. 5.8 gives a specific formula.

In the next result, we study the deviation of $d: [M_0^{4n}, F/0] \rightarrow \Gamma_{4n-1}$ from linearity.

COROLLARY 5.9. *Let $u, v \in [M_0^{4n}, F/0]$. Then*

$$du + dv - d(u + v) = (\frac{1}{8}) \langle L(M) (L(\xi_0(u)) - 1) (L(\xi_0(v)) - 1), [M] \rangle \in \mathbf{Z}/\theta_n \mathbf{Z} = bP_{4n}.$$

Proof. By 3.2, it suffices to prove that

$$f_R(du) + f_R(dv) - f_R(d(u + v)) = (\frac{1}{8}\theta_n) \langle L(M) (L(\xi_0(u)) - 1) \times (L(\xi_0(v)) - 1), [M] \rangle \in Q/\mathbf{Z}.$$

By 4.4 and 5.3(i), we may assume that $u, v \in \text{image}([M^{4n}, F/0])$. The formula now follows from 2.4 since $L(\xi(u + v)) = L(\xi(u)) L(\xi(v))$, hence

$$\begin{aligned} L(\xi(u + v)) - 1 &= (L(\xi(u)) - 1) (L(\xi(v)) - 1) + (L(\xi(u)) - 1) + (L(\xi(v)) - 1) \\ &= (L(\xi_0(u)) - 1) (L(\xi_0(v)) - 1) + (L(\xi(u)) - 1) \\ &\quad + (L(\xi(v)) - 1). \end{aligned}$$

Finally, we investigate the non-commutativity of d with maps.

COROLLARY 5.10. *Let $u \in [M_0^{4n}, F/0]$ and let $h: M'_0 \rightarrow M_0$ be a map of degree one. Then*

$$dh^*(u) - du = (\frac{1}{8}) \langle (h^*(L(M)) - L(M')) (h^*L(\xi_0(u)) - 1), [M'] \rangle \in \mathbf{Z}/\theta_n \mathbf{Z} = bP_{4n}.$$

Proof. By 3.3 it suffices to compute $f_R(dh^*(u)) - f_R(du)$. By 4.4 and 5.3(ii) we may assume that u extends to $\bar{u} \in [M^{4n}, F/0]$. Then, by 2.4

$$\begin{aligned} f_R(dh^*(u)) - f_R(du) &= (\frac{1}{8}\theta_n) \langle (h^*L(M) - L(M')) \cdot (L(\xi(h^*(u))) - 1), [M'] \rangle \\ &= (\frac{1}{8}\theta_n) \langle (h^*L(M) - L(M')) \cdot (L(\xi_0(h^*(u))) - 1), [M'] \rangle \in Q/\mathbf{Z}. \end{aligned}$$

COROLLARY 5.11. *If $h: M'_0 \rightarrow M_0$ is a degree one map of $4n$ -manifolds which corresponds rational Pontrajagin classes, then the diagram*

$$\begin{array}{ccc} [M_0, F/0] & \searrow_d & \\ h^* \downarrow & & \Gamma_{4n-1} \\ [M'_0, F/0] & \nearrow_d & \end{array}$$

commutes. Thus, if h is a homotopy equivalence which corresponds rational Pontrjagin classes then $\Delta_h(M_0) = \Delta_n(M'_0)$.

§ 6. *The composition* $f_{Rd}: [M_0^{8n+2}, F/0] \rightarrow \mathbf{Z}_2$. In this section we consider spin manifolds of dimension $8n+2$. The main result is Proposition 6.5.

In [4], $K0$ -characteristic numbers $\pi^J(M^{8n+2}) \in \mathbf{Z}_2$, where $J=(j_1 \dots j_r)$ and $\pi^J = \pi^{j_1} \dots \pi^{j_r} \in K0^0(BS0)$ are defined for smooth spin manifolds. In [10], the definition is extended to almost smooth manifolds, provided that $J \neq (0)$. Roughly, this is done as follows.

Let M_0^{8n+2} be a spin manifold with $\partial M_0^{8n+2} \in \Gamma_{8n+1}$. Since $v_{M_0}^{8q}$ is a spin vector bundle, the Thom space $T(v_{M_0}^{8q})$ has a canonical $K0$ -orientation. This extends to a $K0$ -orientation $U_M \in K0^0(T(v_M^{8q}))$. Also, v_{M_0} extends to a vector bundle v_M^* over M and we have $v_M = v_M^* + p^*(\sigma)$ as PL bundles, where $p: M^{8n+2} \rightarrow S^{8n+2}$ is a map of degree one and $\sigma \in \pi_{8n+2}(BSPL)$. Moreover, v_M^* is well-defined by the additional assumption that $e_R J_{PL}(\sigma) = 0$, where $J_{PL}: \pi_{8n+2}(BSPL) \rightarrow \pi_{8n+2}(BSF) = \pi_{8n+1}^s$ is the PL J -homomorphism and $e_R: \pi_{8n+1}^s \rightarrow \mathbf{Z}_2$ is the homomorphism defined by Adams, which splits off $\text{image}(J)$ as a direct summand [2]. Set

$$\pi^J(M^{8n+2}) = c^* \Phi_{K0}(\pi^J(v_M^*)) \in K0^0(S^{8q+8n+2}) = \mathbf{Z}_2,$$

where $\Phi_{K0}: K0(M) \simeq K0^0(T(v_M^{8q}))$ is the Thom isomorphism defined by multiplication by U_M , and $c: S^{8q+8n+2} \rightarrow T(v_M^{8q})$ is the map of degree one defined by an embedding $M^{8n+2} \rightarrow S^{8q+8n+2}$. If $J \neq (0)$, the $K0$ -operation π^J has filtration greater than zero, hence the product $\pi^J(v_M^*) \cdot U_M \in K0^0(T(v_M^{8q}))$ is independent of the choice of the extension U_M .

We will also use the notation

$$\pi^J(M^{8n+2}) = \langle \pi^J(v_M^*), [M]_{K0} \rangle \in \mathbf{Z}_2$$

where $[M]_{K0}$ is the fundamental $K0$ -homology class dual to U_M .

E. Brown has defined a homomorphism $\psi: \Omega_{\text{spin}}^{8n+2} - \mathbf{Z}_2$, extending the Kervaire-Arf invariant $\Omega_{\text{framed}}^{8n+2} \rightarrow \mathbf{Z}_2$ [7]. In fact, Brown's definition of ψ applies to PL manifolds M^{8n+2} , with $w_1(M) = w_2(M) = 0$. From the main results of [4], it follows that for smooth M^{8n+2} ,

$$\psi(M^{8n+2}) = \sum \alpha_J \cdot \pi^J(M^{8n+2}) + \sum \beta_I \cdot w^I(M^{8n+2}) \in \mathbf{Z}_2$$

where $\alpha_J, \beta_I \in \mathbf{Z}_2, J = (j_1 \dots j_r), 1 < j_1 \leq \dots \leq j_r$, and the w^I are Stiefel-Whitney numbers.

LEMMA 6.1. *The coefficients β_I, α_J can be chosen such that $\alpha_J = 0$ if $n(J) = j_1 + \dots + j_r \neq 2n$ and $\sum_{n(J)=2n} \alpha_J \pi^J \equiv (L^{-1})_{2n}(0, \pi^2 \dots \pi^{2^n}) \pmod{2}$ where $L = 1 + L_1 + L_2 + \dots$ is the Hirzebruch L -polynomial.*

Proof. We only outline the proof of this lemma, and refer to [4] and [8] for details. The homotopy elements in $\pi_{8n+2}(M \text{ spin})$ which have Adams spectral sequence

filtration greater than 2 are precisely the classes $\{M^{8n+2}\}$ with $w^I(M^{8n+2}) = \pi^J(M^{8n+2}) = 0$ for $n(J) \geq 2n$. It can be shown that $\psi(\{M^{8n+2}\}) = 0$ if $\{M^{8n+2}\} \in \Omega_{\text{spin}}^{8n+2} = \pi_{8n+2}(M \text{ spin})$ represents such a homotopy element. Thus $\alpha_J = 0$ if $n(J) < 2n$. If $n(J) = 2n + 1$, then the KO -characteristic number π^J coincides with a Stiefel-Whitney number for all $(8n + 2)$ -spin manifolds. Thus we may choose the coefficients β^I such that $\alpha_J = 0$. Finally, if T^2 is the torus with the exotic spin structure and N^{8n} is a spin manifold, then $\psi(N^{8n} \times T^2) = \text{index}(N^{8n}) \pmod{2}$. Since the Stiefel-Whitney numbers of $N^{8n} \times T^2$ vanish, it follows that $\sum_{n(J)=2n} \alpha_J \pi^J = (L^{-1})_{2n}(0, \pi^2 \dots \pi^{2n})$.

Let $b \text{ spin}_{8n+2} \subset \Gamma_{8n+1}$ be the subgroup consisting of homotopy spheres that bound spin manifolds. In [10], we showed that $\Gamma_{8n+1} = b \text{ spin}_{8n+2} \oplus \mathbb{Z}_2$. An invariant $f_R: b \text{ spin}_{8n+2} \rightarrow \mathbb{Z}_2$, splitting off $\mathbb{Z}_2 = bP_{8n+2} \subset b \text{ spin}_{8n+2}$ as a direct summand, can be defined as follows. Given $\Sigma^{8n+1} \in b \text{ spin}_{8n+2}$, let $\Sigma^{8n+1} = \partial M_0^{8n+2}$, where M_0^{8n+2} is a spin manifold such that all the Stiefel-Whitney numbers of M^{8n+2} vanish. Then

$$f_R(\Sigma^{8n+1}) = \psi(M^{8n+2}) - (L^{-1})_{2n}(0, \pi^2 \dots \pi^{2n})(M^{8n+2}) \in \mathbb{Z}_2.$$

Let $h: M'_0 \rightarrow M_0$ be a homotopy equivalence with $\theta(h) = u \in [M_0^{8n+2}, F/0]$. The spin structure on M_0 induces a spin structure on M'_0 and, since $h: M'_0 \rightarrow M_0$ is a homotopy equivalence, $\psi(M') = \psi(M)$. Further $h^*(w^I(M)) = w^I(M')$, hence

$$f_R(du) = f_R(\partial M'_0 - \partial M_0) = (L^{-1})_{2n}(M) - (L^{-1})_{2n}(M') \in \mathbb{Z}_2.$$

We now seek a formula expressing the KO -characteristic numbers of M' in terms of invariants of M and of the map $u: M_0^{8n+2} \rightarrow F/0$.

PROPOSITION 6.2. *Let $u \in [M_0^{8n+2}, F/0]$ correspond to the homotopy equivalence $h: M'_0 \rightarrow M_0$, where M_0 is a spin manifold. Then*

$$\pi^J(M') = \langle \pi^J(v_M^* - \xi_0^*(u)) \gamma^*(u), [M]_{KO} \rangle \in \mathbb{Z}_2$$

where $h^*(v_M^* - \xi_0^*(u)) = v_{M'}^* \in KO^0(M')$ and $\gamma^*(u) \in KO(M)$ extends $\gamma(u) \in KO(M_0)$.

Proof. Homotope $h: M' \rightarrow M$ to an embedding $h: M' \rightarrow M \times \mathbb{R}^{8q}$. The PL normal bundle of M' in $M \times \mathbb{R}^{8q}$ is $h^*((-\xi)^{8q})$, where $h^*(v_M - \xi) = v_{M'}$. By the h -cobordism theorem, the embedding h extends to a PL isomorphism $H: E(h^*(-\xi)^{8q}) \simeq M \times \mathbb{R}^{8q}$. Let $c_1 = H^{-1}: T(e_M^{8q}) \rightarrow T(h^*(-\xi)_{M'}^{8q})$ be the induced collapsing map.

Now, $\xi|_{M_0} = \xi_0(u) = \xi_0$ and the canonical KO -orientation of the Thom space $T(h^*(-\xi_0)_{M_0}^{8q})$ extends to a KO -orientation $U \in KO^0(T(h^*(-\xi)_{M'}^{8q}))$. For, $h^*(-\xi) = v_{M'} - h^*(v_M) = (v_{M'}^* - h^*(v_M^*)) + (p')^*(\sigma' - \sigma)$, where $p': M' \rightarrow S^{8n+2}$, and the Thom space of the PL bundle $\sigma' - \sigma$ over S^{8n+2} is KO -orientable. Further, by 4.9, $c_1^*(U) \in KO^0(T(e_M^{8q}))$ restricts to $\Phi_{KO}(\gamma(u)) \in KO^0(T(e_{M_0}^{8q}))$.

There is a homotopy commutative diagram

$$\begin{array}{ccc}
 & S^{16q+8n+2} & \\
 & \swarrow c & \searrow c' \\
 T(v_M^{16q}) & \xrightarrow{\quad} & T(v_{M'}^{16q}) \\
 \Delta \downarrow & & \downarrow (h \times Id)\Delta \\
 T(v_M^{8q}) \wedge T(e_M^{8q}) & \xrightarrow{Id \wedge c_1} & T(v_M^{8q}) \wedge T(h^*(-\xi)_{M'}^{8q})
 \end{array}$$

where the diagonal $\Delta: M \rightarrow M \times M$ and the composition $(h \times Id)\Delta: M' \rightarrow M' \times M' \rightarrow M \times M'$ are covered by bundle maps $\Delta: v_M^{16q} \rightarrow v_M^{8q} \times e_M^{8q}$ and $(h \times Id)\Delta: v_{M'}^{16q} \rightarrow v_M^{8q} \times h^*(-\xi)_{M'}^{8q}$.

The proof of homotopy commutativity is similar to the proof of 2.3 and will be omitted.

We thus have

$$\begin{aligned}
 \pi^J(M') &= (c')^*(\pi^J(v_{M'}^*) \cdot U_{M'}) = (c')^*(h^*(\pi^J(v_M^* - \xi_0^*)) \cdot U_{M'}) \\
 &= (c')^*(\Delta^*(h \times Id)^*((\pi^J(v_M^* - \xi_0^*) \cdot U_M) \cdot U)) \\
 &= c^*(\Delta^*(\pi^J(v_M^* - \xi_0^*) \cdot U_M \cdot c_1^*(U))) \\
 &= c^*(\pi^J(v_M^* - \xi_0^*) \cdot \gamma^*(u) \cdot \Delta^*(U_M \cdot \Phi_{K_0}(1))) = c^*\Phi_{K_0}(\pi^J(v_M^* - \xi_0^*) \cdot \gamma^*(u))
 \end{aligned}$$

and Theorem 6.2 is proved.

LEMMA 6.3. *If $n(J) = 2n$ then*

$$\langle \pi^J(v_M^* - \xi_0^*(u)) \cdot \gamma^*(u), [M]_{K_0} \rangle = \langle \pi^J(v_M^*) \cdot \gamma^*(u), [M]_{K_0} \rangle \in \mathbf{Z}_2.$$

Proof. It suffices to prove that $\pi^J(v_M^* - \xi_0^*) \equiv \pi^J(v_M^*) \pmod{2}$ in $KO^0(M)$.

First, $\pi^J(v_M^* - \xi_0^*)$ is independent of the choice of ξ_0^* , extending $\xi_0 \in KO^0(M_0)$. For, if $\alpha = p^*(\sigma)$, where $\sigma \in KO^0(S^{8n+2})$, and $\eta \in KO^0(M)$ then $\pi^J(\eta + \alpha) = \Sigma \pi^{J'}(\eta) \pi(\alpha)$. But if $J'' \neq (0)$, $\pi^{J'}(\eta) \pi^{J''}(\alpha) = 0$ unless $J'' = J$, and $\pi^J(\alpha) = 0$ unless $J = (2n)$, since products of elements of high filtration vanish. But also $\pi^{(2n)}(\sigma) = 0$ because $\sigma = \mu \eta^2$, where $\mu \in KO^0(S^{8n})$ and $\eta^2: S^{8n+2} \rightarrow S^{8n}$, and $\pi^{(2n)}(\mu) = (4n-1)!\mu$. Thus $\pi^J(\eta + \alpha) = \pi^J(\eta)$.

Secondly, since $J(\xi_0) = 0$, $\xi_0 = \Sigma_k k^e (\psi^k - 1) (\xi_k)$ for some (arbitrarily) large integer e and $\xi_k \in KO^0(M_0)$. Since $2\xi_2$ and $(\psi^k - 1)\xi_k$, k odd, extend to $KO^0(M)$ and since $\psi^{2k} - 1 = (\psi^2 \psi^k - \psi^k) + (\psi^k - 1)$, it suffices to prove $\pi^J(\eta_1 + 2^e(\psi^2 - 1)\eta_2) \equiv \pi^J(\eta_1) \pmod{2}$ and $\pi^J(\eta_1 + (\psi^k - 1)\eta_k) \equiv \pi^J(\eta_1) \pmod{2}$, k odd, where $\eta_1, \eta_2 \in KO^0(M)$ and $\eta_k \in KO^0(M_0)$.

If we set $\pi_t = \Sigma_{j \geq 0} \pi^j t^j$ then

$$\pi_t(\eta_1 + 2^e(\psi^2 - 1)\eta_2) = \pi_t(\eta_1) \cdot \pi_t((\psi^2 - 1)\eta_2)^{2^e} \equiv \pi_t(\eta_1) \pmod{2},$$

because e is large, hence 2^e -fold powers vanish in $KO^0(M)$. It follows that $\pi^j(\eta_1 + 2^e(\psi^2 - 1)\eta_2) \equiv \pi^j(\eta_1) \pmod{2}$.

If k is odd it suffices to prove that all products $x \cdot \pi^j((\psi^k - 1)\eta_k) \equiv 0 \pmod{2}$, where $j \geq 1$, filtration $(x) = 8n - 4j$ if j is even, and filtration $(x) = 8n - 4j - 2$ if j is odd. Now,

$$\begin{aligned} \pi_t((\psi^k - 1)\eta) &= 1 + [\pi^1(\psi^k(\eta)) - \pi^1(\eta)]t \\ &\quad + [(\pi^2(\psi^k(\eta)) - \pi^2(\eta)) - \pi^1(\eta)(\pi^1(\psi^k(\eta)) - \pi^1(\eta))]t^2 + \dots \end{aligned}$$

An easy induction shows that it suffices to prove $x \cdot (\pi^j(\psi^k(\eta)) - \pi^j(\eta)) \equiv 0 \pmod{2}$. But a computation in $KO^0(BSO)$ shows that

$$\pi^j \psi^k - k^{2j} \pi^j - (2k^{2j}(k^2 - 1)/4!) (\pi^{(j, 1)} - j\pi^{j+1})$$

has filtration greater than $4j + 4$. Since k is odd, $2k^{2j}(k^2 - 1)/4!$ and $k^{2j} - 1$ are even integers, hence

$$\begin{aligned} x \cdot (\pi^j(\psi^k(\eta)) - \pi^j(\eta)) &= x \cdot ((k^{2j} - 1)\pi^j(\eta) - (2k^{2j}(k^2 - 1)/4!) \\ &\quad \times (\pi^{(j, 1)} - j\pi^{j+1})(\eta)) \equiv 0 \pmod{2}. \end{aligned}$$

LEMMA 6.4. $\langle (L^{-1})_{2n}(v_M^*)(\gamma^*(u) - 1) \cdot [M]_{KO} \rangle = \langle v_{4n}^2(M)w_2(\gamma(u)), [M] \rangle \in \mathbf{Z}_2$.

Proof. Let $\gamma^*(u) = 1 + \tilde{\gamma}$. Then $L_{2n}^{-1}(v_M^*)\tilde{\gamma}$ has filtration $8n + 2$, and we have a homotopy commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\Delta} & M \wedge M & \xrightarrow{L_{2n}^{-1} \wedge \gamma} & BSO \wedge BSO & \xrightarrow{\otimes} & BSO \\ \downarrow & & \downarrow & & \uparrow & & \uparrow \\ S^{8n+2} & \rightarrow & (M/M^{(8n-1)}) \wedge M & \rightarrow & BSO \langle 8n \rangle \wedge BSO & \rightarrow & BSO \langle 8n + 2 \rangle. \end{array}$$

The product $L_{2n}^{-1}(v_M^*) \cdot \tilde{\gamma}$ can thus be computed by evaluating the cohomology map $\mathbf{Z}_2 = H^{8n+2}(BSO \langle 8n + 2 \rangle, \mathbf{Z}_2) \rightarrow H^{8n+2}(M, \mathbf{Z}_2)$ in the diagram. The results of [4] on the operations $\pi^J: BSO \rightarrow BSO \langle 8n \rangle$, $n(J) = 2n$, can be used to show that this coincides with $\langle v_{4n}^2(M) \cdot w_2(\gamma(u)), [M] \rangle \in \mathbf{Z}_2$.

Note that since $(\gamma - 1): F/0 \rightarrow BSO$ is a homotopy equivalence on the 5-skeltons, $w_2(\gamma(u)) = u^*(k_2)$, where $u \in [M_0^{8n+2}, F/0]$ and $k_2 \in H^2(F/0, \mathbf{Z}_2) = \mathbf{Z}_2$ is the generator.

PROPOSITION 6.5. Let $u \in [M_0^{8n+2}, F/0]$, where M_0^{8n+2} is a spin manifold. Then

$$f_R(du) = \langle v_{4n}^2(M) \cdot u^*(k_2), [M] \rangle \in \mathbf{Z}_2.$$

Proof. This follows immediately from 6.2, 6.3, 6.4 and the formula

$$f_R(du) = (L^{-1})_{2n}(M) - (L^{-1})_{2n}(M').$$

COROLLARY 6.6. $d: [M_0^{8n+2}, F/0] \rightarrow \Gamma_{8n+1}$ is a group homomorphism.

Proof. This follows from 3.2 and 6.5 and the fact that $k_2 \in H^2(F/0, \mathbf{Z}_2)$ is primitive.

COROLLARY 6.7. Let $h: M'_0 \rightarrow M_0$ be a map of degree one. Then $dh^*(u) - du = \langle (v_{4n}^2(M') - h^*(v_{4n}^2(M))) \cdot h^*u^*(k_2), [M'] \rangle \in bP_{8n+2} = \mathbf{Z}_2$, where $u \in [M_0^{8n+2}, F/0]$. In particular, if h is a tangential map or a homotopy equivalence, then $dh^*(u) = du$. Thus $\Delta_h(M_0)$ is a homotopy invariant of $8n+2$ spin manifolds.

Proof. This follows from 3.3 and 6.5.

COROLLARY 6.8 Let $u \in [M_0^{8n+2}, PL/0]$. Then $f_R(du) = 0$.

Proof. $PL/0$ is 6-connected, hence $u^*(k_2) = 0$ and 6.8 follows from 6.5.

Remark 6.9. In § 5, we showed that for $4n$ -spin manifolds, $f_R(\Delta_c(M_0^{4n})) = f_R(\Delta_{th}(M_0^{4n})) = 0$. For $(8n+2)$ -spin manifolds, $f_R(\Delta_{th}(M_0^{8n+2}))$ need not be zero. For example, if $M_0^{8n+2} = (N^{8n} \times S^2)_0$ and $\text{index}(N^{8n})$ is odd, and $u: (N^{8n} \times S^2)_0 \xrightarrow{\pi_2} S^2 \xrightarrow{h^2} SF$, then $f_R(du) = 1$.

Remark 6.10. Let M^{8n+2} be a closed, smooth spin manifold. The above results, along with Proposition 2.4, determine the exact sequence of Sullivan [18],

$$0 \rightarrow hS(M^{8n+2}) \xrightarrow{\theta} [M^{8n+2}, F/0] \xrightarrow{s} \mathbf{Z}_2.$$

Namely, if $u \in [M^{8n+2}, F/0]$, then

$$s(u) = \langle v_{4n}^2(M) \cdot u^*(k_2), [M] \rangle \in \mathbf{Z}_2.$$

Thus, the cohomology formula of 2.5 simplifies for $8n+2$ spin manifolds.

The Adams conjecture, and the resulting factoring $(F/0)_{(2)} = BSO_{(2)} \times (Cok J)_{(2)}$, implies that $s=0$ if and only if $v_{4n}^2(M) w_2(\gamma) = 0$ for all $\gamma \in KO^0(M)$.

Appendix I. S^1 actions on homotopy spheres

It is known that equivariant diffeomorphism classes of differentiable, fixed point free S^1 actions on homotopy $(2n-1)$ -spheres, $n \geq 4$, correspond bijectively with equivalence classes of homotopy smoothings of $CP(n-1)$ [12]. The correspondence is defined as follows. If S^1 acts on Σ^{2n-1} , there is a diagram

$$\begin{array}{ccc} \Sigma^{2n-1} & \xrightarrow{\tilde{h}} & S^{2n-1} \\ \downarrow & & \downarrow \\ P^{2n-2} = \Sigma^{2n-1}/S^1 & \xrightarrow{h} & CP(n-1) = S^{2n-1}/S^1 \end{array} \tag{I.1}$$

where h classifies the principal S^1 bundle over P^{2n-2} given by the action of S^1 on Σ^{2n-1} . An easy spectral sequence argument shows that h is a homotopy equivalence.

There are homotopy equivalences $CP(n-1) \xrightarrow{i} CP(n)_0 \xrightarrow{\pi} CP(n-1)$, since $CP(n)_0$ is the total space of a D^2 bundle, H , over $CP(n-1)$. (If $CP(n-1)$ is regarded as the space of lines in C^n then H is the dual of the “canonical” line bundle.) Consider the diagram

$$\begin{array}{ccc} hS(CP(n-1)) \xrightarrow{\theta} [CP(n-1), F/0] \xrightarrow{s} P_{2n-2} & & \\ \downarrow i_* & \uparrow \iota_* & \\ hS_{\psi} CP(n)_0 \xrightarrow{\theta} [CP(n)_0, F/0] \xrightarrow{d} \Gamma_{2n-1} & & \end{array} \tag{I.2}$$

where, if $h: P^{2n-2} \rightarrow CP(n-1)$ then $i_*(P^{2n-2}, h)$ is the homotopy equivalence $\tilde{h}: P_0^{2n} = E(h^*H) \rightarrow E(H) = CP(n)_0$.

LEMMA I.3(i). *Diagram I.2 commutes.*

(ii) $d\theta i_*(P^{2n-2}, h) = \Sigma^{2n-1} \in \Gamma_{2n-1}$, where $\Sigma^{2n-1} \rightarrow P^{2n-2}$ is as in diagram I.1.

(iii) $si^*\theta: hS(CP(n)_0) \rightarrow P_{2n-2}$ is the geometric obstruction to finding a codimension 2, homotopy $CP(n-1)$ in a homotopy $CP(n)_0$.

The proof of I.3 is relatively straightforward and will be omitted. It follows from I.3 that the set of homotopy $(2n-1)$ -spheres which admit free S^1 actions coincides with $d(\theta i_*(hS(CP(n-1)))) = d((si^*)^{-1}(0)) \subset \Delta_h(CP(n)_0) = B_h(CP(n)_0) \subset \Gamma_{2n-1}$.

Denote this set by $\tilde{B}_h(CP(n)_0)$.

We now want to apply the results of § 2 through § 6 to compute $\tilde{B}_h(CP(n)_0)$. First, it follows from the exact sequence

$$\begin{aligned} K0^{-1}(CP(n)_0) &\rightarrow [CP(n)_0, SF] \rightarrow [CP(n)_0, F/0] \rightarrow K0^0(CP(n)_0) \\ &\rightarrow J(CP(n)_0) \rightarrow 0 \end{aligned}$$

and results of [3] that $[CP(n)_0, F/0] = \mathbf{Z}^{[(n-1)/2]} \oplus [CP(n)_0, SF]$, where $\mathbf{Z}^{[(n-1)/2]} \subset \text{image}([CP(n), F/0] \rightarrow [CP(n)_0, F/0])$ and $\text{image}(\mathbf{Z}^{[(n-1)/2]} \rightarrow K0^0(CP(n)_0))$ is generated by elements $k^e(\psi^k - 1)(\xi)$, $\xi \in K0^0(CP(n)_0)$. In theory it is thus possible to compute the fibre homotopically trivial bundles over $CP(n)_0$. We have done this for $n \leq 8$ [12]. Let $\omega = r(H-1) \in K0^0(CP(n))$, where r forgets the complex structure.

LEMMA I.4. *Kernel $(K0^0(CP(8)_0) \rightarrow J(CP(8)_0)) = \mathbf{Z}^3$ has generators $\xi_1 = 24\omega + 98\omega^2 + 111\omega^3$, $\xi_2 = 240\omega^2 + 380\omega^3$, and $\xi_3 = 504\omega^3$. If $n < 8$, kernel $(K0^0(CP(n)_0) \rightarrow J(CP(n)_0))$ is generated by ξ_1, ξ_2, ξ_3 restricted to $K0^0(CP(n)_0)$.*

Next, we need to compute $si^*: [CP(n)_0, F/0] \rightarrow P_{2n-2}$.

LEMMA I.5. *If $n \equiv 1$ or $3 \pmod{4}$ and $u \in [CP(n)_0, F/0]$ then $si^*(u) = (\frac{1}{8}) \langle L(CP(n-1))(1 - L(\xi_0(i^*(u)))) \rangle \in \mathbf{Z}$.*

In particular,

- (i) $si^*([\mathbf{CP}(n)_0, SF]) = 0$
- (ii) If $n=5$ and $\xi_0(i^*(u)) = m\xi_1 + n\xi_2$ then

$$si^*(u) = -4m^2 + 10m + 28n \in \mathbf{Z}.$$

In particular, if $si^*(u) = 0$ then $10m \equiv 0 \pmod{4}$, or, $m \equiv 0 \pmod{2}$.

- (iii) If $n=7$ and $\xi_0(i^*(u)) = m\xi_1 + n\xi_2 + q\xi_3$ then

$$si^*(u) = (-m(32m^2 + 301)/3) + 84m^2 + 224mn - 384n - 496q \in \mathbf{Z}.$$

Proof. The formula for s was given in Remark 2.5.

Statements (ii) and (iii) follow from I.4 and explicit computation of the L-polynomials in the formula.

LEMMA I.6. *If $n \equiv 2 \pmod{4}$ and $u \in [\mathbf{CP}(n)_0, F/0]$ then $si^*(u) = \langle v_{n-2}^2(\mathbf{CP}(n-1)) i^*u^*(k_2), [\mathbf{CP}(n-1)] \rangle \in \mathbf{Z}_2$.*

Thus $si^*(u) = 0$ if and only if $w_2(\gamma(i^*(u))) = i^*u^*(k_2) = 0$, or equivalently, if and only if $p_1(\xi_0(i^*(u))) \equiv 0 \pmod{48}$. In particular,

- (i) $si^*([\mathbf{CP}(n)_0, SF]) = 0$,
- (ii) If $n=6$ and $\xi_0(i^*(u)) = m\xi_1 + n\xi_2$ then $si^*(u) = m \pmod{2}$.

Proof. The formula follows from 6.5 and 6.10. If $n \equiv 2 \pmod{4}$ then $v_{n-2}^2(\mathbf{CP}(n-1)) \neq 0$ and the second statement follows. Statements (i) and (ii) also follow easily.

We do not have general results with which to compute si^* if $n \equiv 0 \pmod{4}$. The following conjecture is probably true.

Conjecture I.7(i). If $n \equiv 0 \pmod{4}$, $n \neq 2^j$, then $si^*([\mathbf{CP}(n)_0, F/0]) = 0$.

(ii) There are elements $h_j^2 \in \pi_{2^{j+1}-1}(SF)$ such that if $u: \mathbf{CP}(2^j)_0 \xrightarrow{P\pi} S^{2^{j+1}-2} \xrightarrow{h_j^2} SF$ then $si^*(u) = 1 \in \mathbf{Z}_2$. The summand $\mathbf{Z}^{(2^{j-1}-1)} \subset [\mathbf{CP}(2^j)_0, F/0]$ can be chosen so that $si^*(\mathbf{Z}^{(2^{j-1}-1)}) = 0$.

I.7(ii) is true if $j \leq 6$. For example $h_1^2 = \eta^2 \in \pi_2^s$, $h_2^2 = v^2 \in \pi_6^s$, and $h_3^2 = \sigma^2 \in \pi_{14}^s$.

We can use the results 2.5, 3.1, 4.4, 5.2, and 6.10 to compute $d: [\mathbf{CP}(n)_0, F/0] = \mathbf{Z}^{[(n-1)/2]} \oplus [\mathbf{CP}(n)_0, SF] \rightarrow \Gamma_{2n-1} = bP_{2n} \oplus (\pi_{2n-1}^s / \text{im}(J))$.

LEMMA I.8. *We have $d(\mathbf{Z}^{[(n-1)/2]}) \subset bP_{2n}$. Specifically,*

- (i) If $u \in \mathbf{Z} \subset [\mathbf{CP}(4)_0, F/0]$ and $\xi_0(u) = m\xi_1$ then $du = 10m - 4m^2 \in \mathbf{Z}/28\mathbf{Z} = bP_8$.
- (ii) If $u \in \mathbf{Z}^2 \subset [\mathbf{CP}(5)_0, F/0]$ and $\xi_0(u) = m\xi_1 + n\xi_2$, then $du = m \in \mathbf{Z}/2\mathbf{Z} = bP_{10}$.
- (iii) If $u \in \mathbf{Z}^2 \subset [\mathbf{CP}(6)_0, F/0]$ and $\xi_0(u) = m\xi_1 + n\xi_2$, then $du = (-m(32m^2 + 301)/3) + 84m^2 + 224mn - 384n \in \mathbf{Z}/992\mathbf{Z} = bP_{12}$.
- (iv) If $u \in \mathbf{Z}^3 \subset [\mathbf{CP}(7)_0, F/0]$ then $du = 0$, since $bP_{14} = 0$.

Proof. $\mathbf{Z}^{[(n-1)/2]} \subset \text{image}([\mathbf{CP}(n), F/0] \rightarrow [\mathbf{CP}(n)_0, F/0])$, hence the first statement follows from 2.4 and 6.10. Statements (i) and (iii) follow from I.5 and 2.4 and (ii) follows from I.6 and 6.10.

Specific formulas for $d(\mathbf{Z}^{[(n-1)/2]})$, $n \geq 8$, would only require extending the computations of I.4 and I.5.

Recall that as a set $[\mathbf{CP}(n)_0, SF] = \pi_s^0(\mathbf{CP}(n)_0)$. In [12] we computed the p -primary summand ${}_p\pi_s^0(\mathbf{CP}(n)_0)$ and the map ${}_p\pi_s^0(\mathbf{CP}(n)_0) \xrightarrow{\partial^*} {}_p\pi_s^0(S^{2n-1}) = {}_p\pi_{2n-1}^s$ for $n \leq (p^2 + 2p)(p-1) - 2$, p odd, and we computed ${}_2\pi_s^0(\mathbf{CP}(n)_0) \xrightarrow{\partial^*} {}_2\pi_{2n-1}^s$ for $n \leq 11$. Thus, using 5.2 and 6.9, we also computed $d: [\mathbf{CP}(n)_0, SF] \rightarrow \Gamma_{2n-1}$ if $n \equiv 0, 1$, or $2 \pmod{4}$ or if $n = 2^j - 1$. (note that by 5.5(ii), $a_n f_R(d[\mathbf{CP}(2n)_0, SF]) = 0$ and by 6.9, $f_R(d[\mathbf{CP}(4n+1)_0, SF]) = 0$.) These results involve computations in stable homotopy theory and are too complicated to reproduce here. We will state the conclusions for $n \leq 7$.

LEMMA I.9(i). $[\mathbf{CP}(4)_0, SF] = \mathbf{Z}_2$ and $d([\mathbf{CP}(4)_0, SF]) = 0$.

(ii) $[\mathbf{CP}(5)_0, SF] = \mathbf{Z}_2^2$ and $d([\mathbf{CP}(5)_0, SF]) = \mathbf{Z}_2 = \{v^3\} \subset (\pi_9^s/\text{im}(J)) \subset \Gamma_9$.

(iii) $[\mathbf{CP}(6)_0, SF] = \mathbf{Z}_2^2 + \mathbf{Z}_3$ and $d([\mathbf{CP}(6)_0, SF]) = \mathbf{Z}_2 \subset bP_{12} = \Gamma_{11}$.

(iv) $[\mathbf{CP}(7)_0, SF] = \mathbf{Z}_2 + \mathbf{Z}_3$ and $d([\mathbf{CP}(7)_0, SF]) = \mathbf{Z}_3 = \{\alpha_1\beta_1\} = \pi_{13}^s = \Gamma_{13}$.

The construction of the non-zero element of $d([\mathbf{CP}(6)_0, SF])$ is described in § 5, following the proof of 5.7.

Finally, we combine the results I.5 through I.9 to describe the set of homotopy spheres of dimensions 7, 9, 11, and 13 which admit free S^1 actions. That is, we compute $\tilde{B}_h(\mathbf{CP}(n)_0) = d((si^*)^{-1}(0)) \subset d([\mathbf{CP}(n)_0, F/0]) = B_h(\mathbf{CP}(n)_0) \subset \Gamma_{2n-1}$, for $n = 4, 5, 6$, and 7.

THEOREM I.10(i). $\Gamma_7 = bP_8 = \mathbf{Z}/28\mathbf{Z}$ and $\tilde{B}_h(\mathbf{CP}(4)_0) = \{10m - 4m^2/m \in \mathbf{Z}\} = \{0, 4, \pm 6, \pm 8, -10, 14\} \subset \mathbf{Z}/28\mathbf{Z}$.

(ii) $\Gamma_9 = bP_{10} \oplus (\pi_9^s/\text{im}(J)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2^2$ and $\tilde{B}_h(\mathbf{CP}(5)_0) = \mathbf{Z}_2 = \{v^3\} \subset (\pi_9^s/\text{im}(J)) \subset \Gamma_9$.

(iii) $\Gamma_{11} = bP_{12} = \mathbf{Z}/992\mathbf{Z}$ and $\tilde{B}_h(\mathbf{CP}(6)_0) = \{(-m(32m^2 + 301/3) + 84m^2 + 224mn - 384n \mid m, n \in \mathbf{Z}, m \text{ even})\} \subset \mathbf{Z}/992\mathbf{Z}$.

(iv) $\Gamma_{13} = \pi_{13}^s = \mathbf{Z}_3$ and $\tilde{B}_h(\mathbf{CP}(7)_0) = \mathbf{Z}_3 = \{\alpha_1\beta_1\} = \Gamma_{13}$.

Appendix II. Applications to inertia groups

Given a smooth manifold N^k , the inertia group of N^k , $I(N^k) \subset \Gamma_k$, is defined to be the group of homotopy spheres $\Sigma^k \in \Gamma_k$ such that there is a diffeomorphism $N^k \simeq N^k \# \Sigma^k$. Define $I_h(N^k) \subset I(N^k)$ to be the subgroup of homotopy spheres $\Sigma^k \in I(N^k)$ such that some diffeomorphism $N^k \simeq N^k \# \Sigma^k$ is homotopic to the identity. (By the ‘‘identity’’

$N^k = N^k \# \Sigma^k$ we mean the obvious PL identification.) Similarly, define $I_c(N^k) \subset I_h(N^k)$ to be the subgroup of homotopy spheres Σ^k such that some diffeomorphism $N^k \simeq N^k \# \Sigma^k$ is PL isotopic to the identity. Equivalently, $\Sigma^k \in I_c(N^k)$ if the smoothings N^k and $N^k \# \Sigma^k$ are concordant.

The group Γ_k is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms of S^{k-1} . If $\Sigma^k \in \Gamma_k$ corresponds to the diffeomorphism $\sigma: S^{k-1} \simeq S^{k-1}$ then $\Sigma^k \in I(N^k)$ if and only if there is a diffeomorphism $h: N_0^k \simeq N_0^k$ such that $h|_{\partial N_0 = S^{k-1}} = \sigma$. Let $h: N^k \rightarrow N^k$ also denote the PL extension of h defined by coning $h|_{\partial N_0}$ over $D^k \subset N^k$. It is easy to see that the mapping torus of h , $T_h = N^k \times I/(x, 0) \equiv (h(x), 1)$, is an almost smooth manifold, with $\partial(T_h)_0 = \Sigma^k$. Further, $\Sigma^k \in I_h(N^k)$ (resp. $\Sigma^k \in I_c(N^k)$) if and only if h can be chosen such that there is a homotopy equivalence (resp. a PL isomorphism) $H: T_h \rightarrow N^k \times S^1$, with $H|_{N^k \times 0} = Id$. Then $H: (T_h)_0 \rightarrow (N^k \times S^1)_0$ is a homotopy smoothing of $(N^k \times S^1)_0$.

Now $N^k \times S^1$ is not simply connected. However, if N^k is simply connected, the map $\theta: hS((N^k \times S^1)_0) \rightarrow [(N^k \times S^1)_0, F/0]$ is still useful. There is a natural decomposition $[(N^k \times S^1)_0, F/0] \simeq [N^k, F/0] \oplus [N_0^k \wedge S^1, F/0]$. The first summand contains the image under θ of the homotopy smoothings $g \times Id: (N' \times S^1)_0 \rightarrow (N \times S^1)_0$, where $g: N' \rightarrow N$ is a homotopy equivalence. The second summand corresponds bijectively with the homotopy smoothings described above, $H: (T_h)_0 \rightarrow (N^k \times S^1)_0$, $H|_{N^k \times 0} = Id$, where $h: N_0^k \simeq N_0^k$ is a diffeomorphism homotopic to the identity. Denote this second set of homotopy smoothings of $(N^k \times S^1)_0$ by $\tilde{h}S((N^k \times S^1)_0)$.

PROPOSITION II.1. $I_h(N^k) = d(\theta(\tilde{h}S(N^k \times S^1)_0)) = d([N_0^k \wedge S^1, F/0]) \subset \Gamma_k$. Also, $I_c(N^k) = d([N_0^k \wedge S^1, PL/0])$.

Proof. This follows from the discussion in the three paragraphs above.

We can thus use the results of § 2 through § 6 to compute $I_h(N^k)$. If $u \in [N_0^k \wedge S^1, F/0]$, k odd, the formulas in 5.1 and 6.5 for $f_R(du)$ simplify.

PROPOSITION II.2. *If N^{8n+1} is a simply connected spin manifold and $u \in [N_0^{8n+1} \wedge S^1, F/0]$ then $f_R(du) = 0$. Thus $I_h(N^{8n+1})$ is contained in the summand $(\pi_{8n+1}^s/im(J)) \subset \Gamma_{8n+1}$ and $I_h(N^{8n+1}) \simeq_{\mathcal{Q}} (I_h(N^{8n+1}))$ is a homotopy invariant of N^{8n+1} .*

Proof. Since $u^*(k_2) = 0$, the result follows from 6.5.

PROPOSITION II.3. *If $u \in [N_0^{4n-1} \wedge S^1, F/0]$ then*

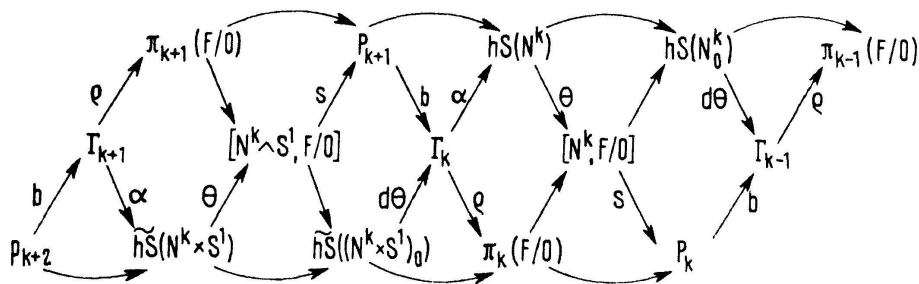
$$f_R(du) = \left(-\frac{1}{8}\right) \left\langle L(N^{4n-1} \times S^1) \left(\sum_{k=1}^n (8\theta_k/a_k(2k-1)!j_k) p_k(\xi) \right), [N^{4n-1} \times S^1] \right\rangle \\ \in \mathbf{Z}/\theta_n \mathbf{Z},$$

where $p_n(\xi)$ is as in 5.1 and $p_k(\xi) = p_k(\xi_0(u))$ if $k < n$.

Proof. Since cohomology products vanish in $N^{4n-1} \wedge S^1$, we have $(1 - L(\xi)) = -(\sum_{k=1}^n (8\theta_k/a_k(2k-1)!j_k)p_k(\xi))$ and the result follows from 5.1. We point out that $p_n(\xi)$ is determined by the equations $(-\text{num}(B_n/4n)/a_n(2n-1)!j_n)p_n(\xi) = e_R(\gamma(u)) \in \mathbb{Q}/\mathbb{Z}$ and $((-1)^{n-1}j_n/a_n(2n-1)!j_n)p_n(\xi) = e_R(\xi_0(u)) \in \mathbb{Q}/\mathbb{Z}$.

Note that by 5.9, $d: [N_0^k \wedge S^1, F/0] \rightarrow \Gamma_k$ is a group homomorphism if $k=4n-1$. Actually, if $u, v \in [N_0^k \wedge S^1, F/0]$ correspond to $H: (T_h)_0 \rightarrow (N^k \times S^1)_0$ and $G: (T_g)_0 \rightarrow (N^k \times S^1)_0$, respectively, where $h, g: N_0^k \xrightarrow{\cong} N_0^k$ are diffeomorphisms, then $d(u+v) \in \Gamma_k$ corresponds to the diffeomorphism $(h|_{\partial N_0}) \cdot (g|_{\partial N_0}): S^{k-1} \xrightarrow{\cong} S^{k-1}$. Since this composite diffeomorphism also corresponds to $du + dv$, we have that $d: [N_0^k \wedge S^1] \rightarrow \Gamma_k$ is a group homomorphism for all N^k .

There is a braid of four interlocking exact sequences



Here, $\alpha: \Gamma_k \rightarrow hS(N^k)$ is defined by $\alpha(\Sigma^k) = (N^k \# \Sigma^k, \text{Id} \# (\text{point})) \in hS(N^k)$, $\Sigma^k \in \Gamma_k$. Since $\text{kernel}(\alpha) \cap bP_{k+1} = bs([N^k \wedge S^1, F/0]) = d\theta(\tilde{hS}((N^k \times S^1)_0)) \cap bP_{k+1} = I_h(N^k) \cap bP_{k+1}$, we see that $I_h(N^k)$ is very useful for computing $hS(N^k)$.

If we replace $F/0$ by $PL/0$, the cofibrations $S^{k-1} \rightarrow N_0^k \rightarrow N^k \rightarrow S^k \rightarrow N_0 \wedge S^1$ yield an exact sequence $[N_0^k \wedge S^1, PL/0] \xrightarrow{d} \Gamma_k \rightarrow [N^k, PL/0] \rightarrow [N_0^k, PL/0] \xrightarrow{d} \Gamma_{k-1}$. Since $[N^k, PL/0]$ and $[N_0^k, PL/0]$ correspond to concordance classes of smoothings of N^k and N_0^k , respectively, it is clear that $I_c(N^k) = d([N_0^k \wedge S^1, PL/0]) = \{\Sigma^k \in \Gamma_k \mid \text{the smoothings } N^k \text{ and } N^k \# \Sigma^k \text{ are concordant}\}$. The following is also clear.

PROPOSITION II.4. $I_c(N^k)$ is a homotopy invariant of N^k .

There are natural subgroups $I_{th}(N^k) \subset I_h(N^k)$ and $I_{tc}(N^k) \subset I_c(N^k)$ defined by $I_{th}(N^k) = d([N_0^k \wedge S^1, SF])$ and $I_{tc}(N^k) = d([N_0^k \wedge S^1, SPL])$. Geometrically, $I_{th}(N^k) \subset \Gamma_k$ (resp. $I_{tc}(N^k) \subset \Gamma_k$) corresponds to those diffeomorphisms $\sigma: S^{k-1} \xrightarrow{\cong} S^{k-1}$ such that there is a diffeomorphism $h: N_0^k \xrightarrow{\cong} N_0^k$, with $h|_{\partial N_0} = \sigma$, and a tangential homotopy equivalence (resp. PL equivalence preserving the smooth tangent bundles) $H: (T_h)_0 \rightarrow (N^k \times S^1)_0$ with $H|_{N^k \times 0} = \text{Id}$.

PROPOSITION II.5(i). $f_R(I_c(N^{4n-1}))$ and $f_R(I_{th}(N^{4n-1})) \subset Z_{\theta_n}$ are 2-primary groups.

(ii) If N^{4n-1} is a spin manifold then $f_R(I_c(N^{4n-1})) = f_R(I_{th}(N^{4n-1})) = 0$

(iii) $I_{th}(N^{4n-1})$ and $I_{tc}(N^{4n-1})$ are homotopy invariants.

Proof. These results follow from 5.2, 5.5, and 5.6. It follows from the construction given after the proof of 5.7 that if $w_2(N^{8k+3}) \neq 0$ then the element of order 2 in bP_{8k+4} belongs to $I_{tc}(N^{8k+3})$.

PROPOSITION II.6. $I_{th}(N^{8n+1}) \simeq_{\varrho} I_{th}(N^{8n+1})$ and $I_{tc}(N^{8n+1}) \simeq_{\varrho} I_{tc}(N^{8n+1})$ are homotopy invariants of $(8n+1)$ -spin manifolds.

Proof. This follows from II.2.

Next we consider manifolds with a trivial stable normal bundle (π -manifolds) or a fibre homotopically trivial stable normal bundle (*fht*-manifolds).

LEMMA II.7. M^k is an *fht*-manifold if and only if there is a π -manifold M' and a degree one map $M' \rightarrow M$.

Proof. By transverse regularity, such a manifold M' , with $M' \times R^q \subset E(v_M^q)$, exists if and only if there is a fibre homotopy trivialization $T(v_M^q) \rightarrow S^q$.

Boardman and Vogt have shown that $PL/0$ and $F/0$ are infinite loop spaces [5]. It follows easily that the suspension maps $\pi_*(F/0) \rightarrow \pi_*^s(F/0) = \Omega_*^{\text{framed}}(F/0)$ and $\pi_*(PL/0) \rightarrow \pi_*^s(PL/0) = \Omega_*^{\text{framed}}(PL/0)$ are monomorphisms onto direct summands.

LEMMA II.8. If M^k is an almost smooth, *fht*-manifold then $\Delta_c(M^k) = 0$ and $\Delta_h(M^k) \subset bP_k$. If $k = 8n + 2$ then $\Delta_h(M^k) = 0$.

Proof. Let $u \in [M_0^k, PL/0]$ and let $h: M'_0 \rightarrow M_0$ be a degree one map where M' is a π -manifold. Then by the above remark $du = \partial^*(u) = \partial^*h^*(u) = 0 \in \pi_{k-1}(PL/0) = \Gamma_{k-1}$. Similarly, if $u \in [M_0^k, F/0]$ then by 3.1 $\varrho(du) = \partial^*(u) = \partial^*h^*(u) = 0 \in \pi_{k-1}(F/0)$. The second statement follows from the first and the fact that the surgery obstruction $s: [M^{8n+2}, F/0] \rightarrow \mathbb{Z}_2$ is given by $s(u) = \langle v_{4n}^2(M)u^*(k_2), [M] \rangle = 0$, since the Wu class $v_{4n}(M) = 0$.

PROPOSITION II.9. If N^k is a smooth, *fht*-manifold then $I_c(N^k) = 0$ and $I_h(N^k) \subset bP_{k+1}$. If $k = 8n + 1$ then $I_h(N^k) = 0$. If N^k is a π -manifold and $k \not\equiv 5 \pmod{8}$ then $I_h(N^k) = 0$.

Proof. The first two statements follow from II.8 since $N^k \times S^1$ is an *fht*-manifold. If N^{4n-1} is a π -manifold and $u \in [N_0^{4n-1} \wedge S^1, F/0]$ then $f_R(du) = 0$ by 5.8. Thus $I_h(N^k) = I_h(N^k) \cap bP_{k+1} = 0$ if $k \equiv 1, 3, \text{ or } 7 \pmod{8}$ and the third statement follows. (I am grateful to D. Sullivan for pointing out the first statement of II.9.)

Finally, as an example, we compute, $I_h(\mathbb{C}P(3) \times S^1) \subset \Gamma_7 = bP_8 = \mathbb{Z}_{28}$. ($\mathbb{C}P(3) \times S^1$ is not simply connected, but our methods remain valid for special cases with simple fundamental groups.) Now $(\mathbb{C}P(3) \times S^1) \wedge S^1$ is homotopy equivalent to $(\mathbb{C}P(3) \wedge S^2) \vee (\mathbb{C}P(3) \wedge S^1) \vee S^2$. Thus, since $K^0(\mathbb{C}P(3) \wedge S^1) = 0$, $\text{image}([(\mathbb{C}P(3) \times S^1) \wedge S^1, F/0] \rightarrow K^0((\mathbb{C}P(3) \times S^1) \wedge S^1)) = \text{image}([\mathbb{C}P(3) \wedge S^2, F/0] \rightarrow K^0(\mathbb{C}P(3) \wedge S^2)) = \mathbb{Z}^2$, with generators ξ_1 and ξ_2 which satisfy $P(\xi_1) = 1 +$

$+p_1(\xi_1)+p_2(\xi_1)=1+48(z\cdot\sigma)+32\cdot 15(z^3\cdot\sigma)$ and $P(\xi_2)=1+32\cdot 45(z^3\cdot\sigma)$, where $z\in H^2(\mathbf{CP}(3), \mathbf{Z})$ and $\sigma\in H^2(S^2, \mathbf{Z})$ are generators. Thus if $u\in[(\mathbf{CP}(3)\times S^1)_0\wedge S^1, F/0]$ extends to $\bar{u}\in[(\mathbf{CP}(3)\times S^1)\wedge S^1, F/0]$ and $\xi=\xi(\bar{u})=m\xi_1+n\xi_2$ then

$$\begin{aligned} du &= s(\bar{u}) = \left(\frac{1}{8}\right) \langle L(\mathbf{CP}(3)\times S^1\times S^1)(1-L(\xi)), [\mathbf{CP}(3)\times S^1\times S^1] \rangle \\ &= \left(-\frac{1}{8}\right) \langle (1+\left(\frac{4}{3}\right)z^2)\left((48m/3)(z\sigma)+\left(7(32\cdot 15m+32\cdot 45n)/45\right)(z^3\sigma),\right. \\ &\quad \left.[\mathbf{CP}(3)\times S^1\times S^1] \right\rangle = -12m-28n\in\mathbf{Z}/28\mathbf{Z}. \end{aligned}$$

It follows that $I_h(\mathbf{CP}(3)\times S^1)=\mathbf{Z}_7\subset\mathbf{Z}_{28}$.

Remark II.10. R. Lee [16] has shown that every self-homotopy equivalence of $\mathbf{CP}(n)\times S^1$ is homotopic to a diffeomorphism. If a manifold M^k has this property it is easy to see that $I_h(M^k)=I(M^k)$. Thus $I(\mathbf{CP}(3)\times S^1)=\mathbf{Z}_7\subset\mathbf{Z}_{28}$.

Remark II.11. Let $\pi_0^+(\text{Diff}(\mathbf{CP}(n)))$ denote the group of pseudo-isotopy classes of diffeomorphisms of $\mathbf{CP}(n)$ which leave fixed a generator of $H^2(\mathbf{CP}(n), \mathbf{Z})$. Lee has shown that $\pi_0^+(\text{Diff } \mathbf{CP}(n))$ is isomorphic to the equivariant diffeomorphism classes of differentiable, semi-free S^1 actions on homotopy $(2n+2)$ -spheres, with fixed point set S^0 . (A group action is semi-free if it is free outside the fixed point set.) It follows from results of Sullivan that the natural map $\Gamma_7=\pi_0(\text{Diff}(S^6))\xrightarrow{\gamma}\pi_0^+(\text{Diff}(\mathbf{CP}(3)))$ is a surjection, where, if $\Sigma^7\in\Gamma_7$ corresponds to a diffeomorphism $\sigma:D^6\rightarrow D^6$, with $\sigma|_{S^5}=\text{Id}$, then $\gamma(\Sigma^7)|_{D^6}=\sigma$ and $\gamma(\Sigma^7)|_{\mathbf{CP}(3)-D^6}=\text{Id}$, where $D^6\subset\mathbf{CP}(3)$. It is not difficult to see that the mapping torus of $\gamma(\Sigma^7)$ is $(\mathbf{CP}(3)\times S^1)\# \Sigma^7$. Hence, $\gamma(\Sigma^7)=0\in\pi_0^+(\text{Diff}(\mathbf{CP}(3)))$ if and only if $\gamma(\Sigma^7)$ is pseudo-isotopic to the identity, or equivalently, if and only if there is a diffeomorphism $(\mathbf{CP}(3)\times S^1)\# \Sigma^7=T_{\gamma(\Sigma^7)}\rightarrow\cong\mathbf{CP}(3)\times S^1$ which is the identity on $\mathbf{CP}(3)\times 0$. Since any diffeomorphism $(\mathbf{CP}(3)\times S^1)\# \Sigma^7\rightarrow\cong\mathbf{CP}(3)\times S^1$ is pseudo-isotopic to one which fixes $\mathbf{CP}(3)\times 0$ [19; Lemma 4], this proves that $\text{kernel}(\gamma)=I(\mathbf{CP}(3)\times S^1)=\mathbf{Z}_7\subset\mathbf{Z}_{28}$ and that $\pi_0^+(\text{Diff}(\mathbf{CP}(3)))=\mathbf{Z}_4$.

Remark II.12. For each integer j there is a manifold P_j^6 homotopy equivalent to $\mathbf{CP}(3)$ with $p_1(P_j^6)=(4+24j)z^2$. Thus if $u\in[(P_j^6\times S^1)_0\wedge S^1, F/0]$ with $\xi(\bar{u})=m\xi_1+n\xi_2$ then $du=s(\bar{u})=-(12+16j)m-28n\in\mathbf{Z}/28\mathbf{Z}$. It follows that $I_h(P_j^6\times S^1)=0$ if $j\equiv 1\pmod{7}$ and $I_h(P_j^6\times S^1)=\mathbf{Z}_7$ if $j\not\equiv 1\pmod{7}$. In particular, $I_h(N^k)$ is not a homotopy invariant of N^k .

REFERENCES

[1] J. F. ADAMS, *On the groups J(X)-I*, Topology 2 (1963) 181-195.
 [2] J. F. ADAMS, *On the groups J(X)-IV*, Topology 5 (1966), 21-71.

- [3] J. F. ADAMS and G. WALKER, *On complex Stiefel manifolds*, Proc. Cambridge Phil. Soc. 6 (1965), 81–103.
- [4] D. W. ANDERSON, E. H. BROWN, JR., and F. P. PETERSON, *The structure of the spin cobodism ring*, Ann. of Math. 86 (1967), 271–298.
- [5] J. M. BOARDMAN and R. M. VOGT, *Homotopy everything H -spaces*, Bull. Amer. Math. Soc. 74 (1968), 1117–1122.
- [6] W. BROWDER, *The Arf invariant and its generalization*, Ann. of Math. 90 (1969), 157–186.
- [7] E. H. BROWN, JR., *Note on an invariant of Kervaire*, Michigan Math. J. 12 (1965), 23–24.
- [8] E. H. BROWN, JR. and F. P. PETERSON, *The Kervaire invariant of $(8k + 2)$ -manifolds*, Amer. J. Math. 88 (1966), 815–826.
- [9] G. BRUMFIEL, *On the homotopy groups of BPL and PL/O*, Ann. of Math. 88 (1968), 291–311.
- [10] G. BRUMFIEL, *On the homotopy groups of BPL and PL/O-II*, Topology 8 (1969), 305–311.
- [11] G. BRUMFIEL, *On the homotopy groups of BPL and PL/O-III*, Michigan Math. J. 17 (1970), 217–224.
- [12] G. BRUMFIEL, *Differentiable S^1 actions on homotopy spheres*, mimeographed (Univ. of Calif., Berkeley, 1968).
- [13] B. HARRIS, *J -homomorphisms and cobordism groups*, Inventiones Math. 7 (1969), 313–320.
- [14] M. HIRSCH and B. MAZUR, *Smoothings of piecewise linear manifolds*, mimeographed (Cambridge University, 1964).
- [15] M. A. KERVAIRE and J. W. MILNOR, *Groups of homotopy spheres*, Ann. of Math. 77 (1963), 504–537.
- [16] R. LEE, *Thesis*, Univ. of Michigan 1968.
- [17] M. MAHOWALD, *On the order of the image of J* , Topology, 6 (1967), 371–378.
- [18] D. SULLIVAN, *Smoothings homotopy equivalences*, Univ. of Warwick, 1966.
- [19] C. T. C. WALL, *Unknotting tori*, Proc. Cambridge Phil. Soc., 61 (1965), 659–664.

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