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On Central Group Extensions and Homology

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0. Introduction

Given an extension of groups

$$N \rightarrowtail G \twoheadrightarrow Q \tag{0.1}$$

it is well-known (see [7, 8]) that there is an exact sequence in homology

$$H_2G \to H_2Q \to N/[G, N] \to H_1G \to H_1Q \to 0, \tag{0.2}$$

where $H_1G = G_{ab}$ is the abelianized group $G/[G, G]$. Ganea pointed out in [5] that, if N is *central* in G, we may extend the sequence (0.2) one place to the left, obtaining

$$N \otimes G_{ab} \to H_2G \to H_2Q \to N \to G_{ab} \to Q_{ab} \to 0. \tag{0.3}$$

Ganea's proof is topological, but uses no explicit spectral sequence technique. In this paper we exploit Ganea's topological approach, but by using spectral sequence techniques for fibre spaces, we extend (0.3) a further four places to the left. The feature which then enters into the sequence, beyond the ordinary homology groups of groups, is the group $H_4(N, 2)$, that is, the fourth homology group of the Eilenberg-Mac Lane complex $K(N, 2)$. Indeed, we associate naturally with the central extension (0.1) a homomorphism

$$\sigma: H_4(N, 2) \to N \otimes G_{ab} \tag{0.4}$$

which, in fact, factors as

$$H_4(N, 2) \xrightarrow{\bar{\sigma}} N \otimes N \to N \otimes G_{ab}, \tag{0.5}$$

where $\bar{\sigma}$ is intrinsic to N and natural, and the second homomorphism is induced by the evident map $N \to G_{ab}$. Then, in particular, we obtain the 8-term sequence

$$H_3G \to H_3Q \to \operatorname{coker} \sigma \to H_2G \to H_2Q \to N \to G_{ab} \to Q_{ab} \to 0. \tag{0.6}$$

Our full 10-term sequence (1.5) then involves a certain quotient, $\overline{H_3G}$, of H_3G, which we explicitly describe, and commences

$$H_4Q \to \ker \sigma \to \overline{H_3G} \to H_3Q \to \cdots. \tag{0.7}$$

We remark that the Ganea extension (0.3) of (0.2) and certain parts of our further extension can be established by an elementary method using free presentations of the groups concerned. This is done in a separate paper [2] which also discusses some elementary applications.[1])

[1] **Added in proof:** Y. NOMURA, *The Whitney Join and its Dual*, Osaka J. Math. *7* (1970), 353–373, uses topological methods to obtain an extension of (0.2), different from ours, back to H_3Q, even if N is not central in G.

In Section 3 of the present paper we show that the sequence (0.6) is relevant to the study by Bass [1] and Kervaire [6] of *perfect* (or *connected*) groups, that is, groups G such that the abelianized group G_{ab} is trivial. For we obtain immediately from it the results of Section 1 of [6] and the principal lemma of Section 2 of [6], namely that, if (0.1) is a central extension with G perfect, then $H_3G \to H_3Q$ is surjective. Kervaire also obtains an exact sequence in [6], relating to an extension

$$K \rightarrowtail G \twoheadrightarrow Q$$

of perfect groups, which suggests a definition of $K_3(\Lambda)$ in algebraic K-theory. In Section 3 of this paper we offer a commentary on this sequence in the form of a natural generalization which exploits again the fact that $H_3G \to H_3Q$ is surjective. With regard to algebraic K-theory, we note that an immediate proof of the exact sequence

$$K_2(\Lambda) \to K_2(\Lambda/\mathfrak{a}) \to K_1(\Lambda, \mathfrak{a}) \to K_1(\Lambda) \to K_1(\Lambda/\mathfrak{a})$$

of algebraic K-theory can be obtained from (0.2) and from well-known properties (Theorem 15.1 of [9] of $GL(\Lambda)$) and $GL(\Lambda/a)$; this proof is given in [2].

Section 2 is devoted to a study of $\bar{\sigma}: H_4(N, 2) \to N \otimes N$ and a companion homomorphism

$$\bar{\tau}: H_5(N, 2) \to \mathrm{Tor}(N, N).$$

We use the full 10-term exact sequence to compute $\bar{\sigma}$ and $\bar{\tau}$; we reobtain in the process the values of the groups $H_4(N, 2)$, $H_5(N, 2)$, first computed by Eilenberg-Mac Lane [3]. Their procedure in computing $H_4(N, 2)$ was to use the bar construction to identify $H_4(N, 2)$ with Whitehead's Γ-group, and then use Whitehead's calculations [10]. Our procedure in computing $H_4(N, 2)$ exploits the more general homomorphism σ and the factorization (0.5).

We remark in Section 1 that the topological situation giving rise to a 10-term exact sequence is much more general than that obtained from a central extension of groups, although the latter is in a sense universal. All we require is a fibration

$$F \to E \to B$$

in which F is connected and B is 1-connected with $H_3B = 0$. Thus, for example, such a sequence obtains whenever we have a fibration over S^2.

1. The Extended Exact Sequence

Let

$$N \rightarrowtail G \twoheadrightarrow Q \tag{1.1}$$

be a central group extension. There is then an operation of N on G, that is, a homomorphism

$$\varrho: N \times G \to G \tag{1.2}$$

which is simply given by the product operation in G, $\varrho(x, y) = xy$, $x \in N$, $y \in G$. Then ϱ induces a homomorphism, which we also denote ϱ, in homology,

$$\varrho: H_i(N \times G) \to H_i G, \quad i \geqslant 0. \tag{1.3}$$

Our main theorem is the following.

THEOREM 1.1: *Given the central group extension $N \rightarrowtail G \twoheadrightarrow Q$, there is a natural homomorphism*

$$\sigma: H_4(N, 2) \to N \otimes G_{ab} \tag{1.4}$$

and a natural exact sequence

$$\begin{aligned} &H_4 Q \to \ker \sigma \to H_3 G/\varrho((N \otimes H_2 G) \oplus \operatorname{Tor}(N, G_{ab})) \\ &\quad \to H_3 Q \to \operatorname{coker} \sigma \xrightarrow{\varrho} H_2 G \to H_2 Q \to N \to G_{ab} \to Q_{ab} \to 0. \end{aligned} \tag{1.5}$$

Before proving this theorem we make the following remark. In the third term of the sequence the denominator is to be understood as the image under ϱ of a subgroup of $H_3(N \times G)$. Now it is true that the Künneth formula does not split naturally; nevertheless, the quotient group of $H_3 G$ is described in natural, unambiguous fashion, since $\operatorname{Tor}(N, G_{ab})$ is embedded naturally in $H_3(N \times G)/N \otimes H_2 G$. Equivalently, one may observe that $H_2(G; H_1 N)$ is embedded naturally in $H_3(N \times G)$.

We now prove the theorem. Following Ganea [5], we base ourselves on the fibre sequence

$$K(G, 1) \to K(Q, 1) \to K(N, 2). \tag{1.6}$$

There is then an operation of $K(N, 1) = \Omega K(N, 2)$ on $K(G, 1)$ and this is precisely the operation derived from ϱ (1.2). This observation enables us to interpret certain differentials in the Serre spectral sequence associated with[2] (1.6). This spectral sequence relates, in its simplest form, to a fibration

$$F \to E \to B$$

in which B is 1-connected and F is 0-connected. Then, in the spectral sequence $\{E_r^{pq}\}$, we have

$$E_2^{pq} = H_p(B; H_q(F)),$$

[2]) Ganea [5] does not use the Serre spectral sequence directly, basing himself instead on his result [4] that a fibration $F \to E \to B$ yields a fibration $F * \Omega B \to E \cup CF \to B$. However, the action of ΩB on F is also implicit in this result.

and the sequence converges (finitely) to the graded group associated with H_*E, suitably filtered. Moreover the degree of d_r is $(-r, r-1)$.

Since in the fibration (1.6) the base is 1-connected and the fibre 0-connected, we immediately obtain the right hand end of the sequence (1.5), beginning with $H_2G \to H_2Q$, i.e., the sequence (0.2) in the central case. Moreover, the homomorphism $H_2G \to H_2Q$ is effectively just the passage from E_2^{02} to E_∞^{02} in the spectral sequence. Now consider

$$E_2^{40} \xrightarrow{d_2} E_2^{21} \xrightarrow{d_2} E_2^{02}. \tag{1.7}$$

The first homomorphism provides the definition of $\sigma : H_4(N, 2) \to N \otimes G_{ab}$, while the second is the restriction of $\varrho : H_2(N \times G) \to H_2G$ to $N \otimes G_{ab}$; we also denote this restriction by ϱ. We thus have exactness

$$N \otimes G_{ab} \xrightarrow{\varrho} H_2G \twoheadrightarrow E_3^{02}.$$

However, $E_2^{30} = H_3(N, 2) = 0$, so $E_3^{02} = E_4^{02} = E_\infty^{02}$ and we have the exact sequence, due to Ganea [5],

$$N \otimes G_{ab} \xrightarrow{\varrho} H_2G \to H_2Q. \tag{1.8}$$

Now $E_3^{21} = \ker \varrho / \operatorname{im} \sigma$, hence

$$E_\infty^{21} = \ker \varrho / \operatorname{im} \sigma. \tag{1.9}$$

Also

$$E_\infty^{30} = 0, \tag{1.10}$$

since $E_2^{30} = 0$, and

$$E_\infty^{12} = 0, \tag{1.11}$$

since $E_2^{12} = 0$. Thus we have an exact sequence

$$E_\infty^{03} \rightarrowtail H_3Q \twoheadrightarrow \ker \varrho / \operatorname{im} \sigma, \tag{1.12}$$

which yields, with (1.8), the exact sequence

$$E_\infty^{03} \rightarrowtail H_3Q \to N \otimes G_{ab} / \operatorname{im} \sigma \xrightarrow{\varrho} H_2G \to H_2Q, \tag{1.13}$$

where we again denote by ϱ the homomorphism induced by ϱ.

We now analyze the passage through the spectral sequence from E_2^{03} to E_∞^{03}. We start with $E_2^{03} = H_3G$; passing to E_3^{03}, we factor out $\varrho(N \otimes H_2G)$; passing to E_4^{03}, we further factor out $\varrho(\operatorname{Tor}(N, G_{ab}))$; we thus have the exact sequence

$$E_\infty^{40} \rightarrowtail E_4^{40} \xrightarrow{d_4} H_3G / \varrho((N \otimes H_2G) \oplus \operatorname{Tor}(N, G_{ab})) \twoheadrightarrow E_\infty^{03}. \tag{1.14}$$

Now, reverting to (1.7), $E_3^{40} = \ker \sigma$; and $E_3^{40} = E_4^{40}$ since $E_2^{12} = 0$. This completes the proof of the theorem since E_∞^{40} is a quotient of H_4Q.

COROLLARY 1.2 (see also [2]): *Suppose that G is perfect (i.e., $G_{ab}=0$). Then*
a) There is an exact sequence

$$0 \to H_2G \to H_2Q \to N \to 0,$$

b) $H_3G \to H_3Q$ is surjective. Indeed, we have an exact sequence

$$H_4Q \to H_4(N, 2) \to H_3G/\varrho(N \otimes H_2G) \to H_3Q \to 0.$$

We observe that we may pass immediately to a generalization of Theorem 1.1. We suppose given a fibration

$$F \to E \to B, \tag{1.15}$$

with F connected and B 1-connected; and we suppose further that $H_3B=0$. We have an operation of ΩB on F,

$$\varrho: \Omega B \times F \to F,$$

and, by identifying $H_1\Omega B$ with H_2B, we obtain induced homomorphisms

$$\varrho: H_2B \otimes H_iF \to H_{i+1}F, \quad \varrho: \mathrm{Tor}(H_2B, H_{i-1}F) \to H_{i+1}F. \tag{1.16}$$

Then Theorem 1.1 generalizes to assert a natural homomorphism

$$\sigma: H_4B \to H_2B \otimes H_1F \tag{1.17}$$

and a natural exact sequence

$$\begin{aligned} &H_4E \to \ker\sigma \to H_3F/\varrho(H_2B \otimes H_2F \oplus \mathrm{Tor}(H_2B, H_1F)) \\ &\quad \to H_3E \to \operatorname{coker}\sigma \xrightarrow{\varrho} H_2F \to H_2E \to H_2B \to H_1F \to H_1E \to 0. \end{aligned} \tag{1.18}$$

Indeed, the homomorphism σ (1.17) exists without the supplementary hypothesis that $H_3B=0$.

2. The Homomorphism σ

Let us consider the homomorphism $\sigma: H_4B \to H_2B \otimes H_1F$ of (1.17) in full generality. By considering the diagram

$$\begin{array}{ccccc} \Omega B & \to & EB & \to & B \\ \downarrow & & \downarrow & & \downarrow \\ F & \to & E & \to & B \end{array} \tag{2.1}$$

we obtain the commutative diagram

$$\begin{array}{ccc} H_4B & \xrightarrow{\sigma} & H_2B \otimes H_1\Omega B \\ \| & & \downarrow \\ H_4B & \xrightarrow{\sigma} & H_2B \otimes H_1F, \end{array} \tag{2.2}$$

so that it is sufficient to analyze the homomorphism $\sigma: H_4B \to H_2B \otimes H_1\Omega B$ or, equivalently,

$$\sigma: H_4B \to H_2B \otimes H_2B, \tag{2.3}$$

arising from the path-space fibration $\Omega B \to EB \to B$.

Now let $\pi = \pi_2 B = H_2 B$; then another application of naturality shows that if $\eta: B \to K(\pi, 2)$ is the fundamental class, then σ in (2.3) factors as

$$H_4B \xrightarrow{\eta} H_4(\pi, 2) \xrightarrow{\sigma} \pi \otimes \pi .$$

Thus we will be content to take $B = K(\pi, 2)$ and thus to analyze $\sigma = \bar{\sigma}: H_4(\pi, 2) \to \pi \otimes \pi$; this is, in any case, our real concern in this note. We will use (1.5) to carry out the calculation of $\bar{\sigma}$ and will at the same time compute $H_4(\pi, 2)$. This group has, of course, originally been computed by Eilenberg-Mac Lane [3], but we will base ourselves simply on (1.5).

We thus consider the homomorphism

$$\bar{\sigma}: H_4(\pi, 2) \to \pi \otimes \pi . \tag{2.4}$$

The sequence (1.5) reduces, for the central extension $\pi \rightarrowtail \pi \twoheadrightarrow 1$, to

$$0 \to \ker \bar{\sigma} \to H_3\pi/\varrho((\pi \otimes H_2\pi) \oplus \operatorname{Tor}(\pi, \pi)) \to 0 \to \operatorname{coker} \bar{\sigma} \to H_2\pi \to 0 . \tag{2.5}$$

Assume now that π is cyclic; then $H_2\pi = 0$ and coker $\bar{\sigma} = 0$, $\bar{\sigma}$ is surjective. If $\pi = \mathbf{Z}$, then ker $\bar{\sigma} = 0$, so $\bar{\sigma}$ is an isomorphism, and (2.4) is then an isomorphism $\bar{\sigma}: \mathbf{Z} \cong \mathbf{Z}$.

Now let $\pi = \mathbf{Z}_m$. Then $H_3\pi = \mathbf{Z}_m$ and we will prove below the key lemma.

LEMMA 2.1:

$$\varrho \operatorname{Tor}(\mathbf{Z}_m, \mathbf{Z}_m) = 2\mathbf{Z}_m .$$

Granted this lemma, we immediately deduce that, if $\pi = \mathbf{Z}_m$, then

$$\ker \bar{\sigma} = \begin{cases} 0, & m \text{ odd} \\ \mathbf{Z}_2, & m \text{ even} . \end{cases}$$

Thus, if $\pi = \mathbf{Z}_m$, m odd, (2.4) is then an isomorphism $\bar{\sigma}: \mathbf{Z}_m \cong \mathbf{Z}_m$. If m is even, we must determine the group extension to compute $H_4(\mathbf{Z}_m, 2)$. To do this we consider the central extension $\mathbf{Z}_m \overset{\iota}{\rightarrowtail} \mathbf{Z}_{m^2} \twoheadrightarrow \mathbf{Z}_m$. Then (1.5) yields

$$0 \to \ker \sigma \to \text{quotient of } \mathbf{Z}_{m^2} \to \cdots . \tag{2.6}$$

Moreover, by (2.2), $\sigma: H_4(Z_m, 2) \to \mathbf{Z}_m \otimes \mathbf{Z}_{m^2}$ factors as

$$H_4(\mathbf{Z}_m, 2) \xrightarrow{\bar{\sigma}} \mathbf{Z}_m \otimes \mathbf{Z}_m \xrightarrow{1 \otimes \iota} \mathbf{Z}_m \otimes \mathbf{Z}_{m^2} .$$

But plainly $1 \otimes \iota = 0$, so $\sigma = 0$, ker $\sigma = H_4(\mathbf{Z}_m, 2)$, so that, by (2.6), $H_4(\mathbf{Z}_m, 2)$ is cyclic, and we have proved that if $\pi = \mathbf{Z}_m$, m even, then (2.4) is an epimorphism $\bar{\sigma}: \mathbf{Z}_{2m} \twoheadrightarrow \mathbf{Z}_m$. Summing up, we have

THEOREM 2.2:
(i) *If* $\pi = \mathbf{Z}$, *then* $H_4(\pi, 2) = \mathbf{Z}$ *and* (2.4) *is an isomorphism;*
(ii) *if* $\pi = \mathbf{Z}_m$, *m odd, then* $H_4(\pi, 2) = \mathbf{Z}_m$ *and* (2.4) *is an isomorphism;*
(iii) *if* $\pi = \mathbf{Z}_m$, *m even, then* $H_4(\pi, 2) = \mathbf{Z}_{2m}$ *and* (2.4) *is an epimorphism.*

We now prove Lemma 2.1. Let x generate $\pi = \mathbf{Z}_m$ and let

$$\cdots \xrightarrow{f} \mathbf{Z}\pi \xrightarrow{g} \mathbf{Z}\pi \xrightarrow{f} \mathbf{Z}\pi \xrightarrow{\varepsilon} \mathbf{Z} \to 0, \quad f = x - 1, \quad g = x^{m-1} + x^{m-2} + \cdots + x + 1,$$

be the usual π-resolution of $\mathbf{Z}$. That is, we take the resolution $\mathbf{P} \twoheadrightarrow \mathbf{Z}$ in which $P_n = \mathbf{Z}\pi$, generated by a_n, and

$$\begin{aligned} \partial_n a_n &= f a_{n-1}, \quad n \text{ odd}, \\ &= g a_{n-1}, \quad n \text{ even}, \quad n > 0, \\ \varepsilon a_0 &= 1. \end{aligned}$$

Then $\mathbf{P} \otimes \mathbf{P}$ is a $\pi \times \pi$-resolution of $\mathbf{Z}$ and we seek a chain map $\varphi: \mathbf{P} \otimes \mathbf{P} \to \mathbf{P}$, compatible with the augmentations and with the multiplication map $\varrho: \pi \times \pi \to \pi$. Proceeding step-by-step, we find that we may define ϕ as follows in dimensions $\leqslant 3$:

$$\begin{aligned} &\phi_0(a_0 \otimes a_0) = a_0, \\ &\phi_1(a_1 \otimes a_0) = \phi_1(a_0 \otimes a_1) = a_1, \\ &\phi_2(a_2 \otimes a_0) = \phi_2(a_0 \otimes a_2) = a_2, \ \phi_2(a_1 \otimes a_1) = 0, \\ &\phi_3(a_3 \otimes a_0) = \phi_3(a_2 \otimes a_1) = \phi_3(a_1 \otimes a_2) = \phi_3(a_0 \otimes a_3) = a_3. \end{aligned}$$

Now $\operatorname{Tor}(\mathbf{Z}_m, \mathbf{Z}_m) \subseteq H_3(\mathbf{Z}_m \times \mathbf{Z}_m)$ is generated by $a_2 \otimes a_1 + a_1 \otimes a_2$. Thus, the image of $\operatorname{Tor}(\mathbf{Z}_m, \mathbf{Z}_m)$ in $H_3\mathbf{Z}_m$ under ϱ is the subgroup generated by $2a_3$, that is, $2\mathbf{Z}_m$.

Remark. We may identify $\bar{\sigma}: H_4(\pi, 2) \to \pi \otimes \pi$ with an element of $H^4(\pi, 2; \pi \otimes \pi)$ since $H_3(\pi, 2) = 0$. Since $\bar{\sigma}$ is just d_2 in the Serre spectral sequence, the standard identification of the differential shows that $\bar{\sigma} = \eta_1 \eta_2$, where $\eta \in H^2(\pi, 2; \pi)$ is the fundamental class and η_1, η_2 copy η into the first and second factors of $\pi \otimes \pi$. This remark may be used to establish the next theorem.

Let $\pi = N \otimes N'$. Then plainly $\bar{\sigma}: H_4(\pi, 2) \to \pi \otimes \pi$ maps $N \otimes N'$ to $N \otimes N' \oplus N' \otimes N$.

THEOREM 2.3:

$$\bar{\sigma}(x \otimes x') = x \otimes x' + x' \otimes x, \ x \in N, \ x' \in N'.$$

Proof. We note from (2.5) that $\varrho: \pi \otimes \pi \to H_2\pi$ induces an isomorphism coker $\bar{\sigma} \cong H_2\pi$. It is plain that $\varrho(x\otimes x') = x\otimes x'$ and $\varrho(x'\otimes x) = -x\otimes x'$. This establishes the theorem, in view of the naturality of $\bar{\sigma}$ with respect to π.

We thus have a complete determination of $H_4(\pi, 2)$ and $\bar{\sigma}$ for any finitely generated abelian group π. Of course we may, if we wish, use a direct limit argument to extend the determination of $\bar{\sigma}$ to any abelian group π.

We close this section by mentioning a companion homomorphism to σ, namely, $\tau: H_5(N, 2) \to \mathrm{Tor}(N, G_{ab})$ in the situation of (1.1). This is just $d_2: E_2^{50} \to E_2^{31}$ in the Serre spectral sequence of (1.6). Again it follows by naturality that τ is just

$$H_5(N, 2) \xrightarrow{\bar{\tau}} \mathrm{Tor}(N, N) \to \mathrm{Tor}(N, G_{ab}),$$

where the second homomorphism is induced by $N \to G_{ab}$. Thus, reverting to our previous notation, we consider the central extension $\pi \rightarrowtail \pi \twoheadrightarrow 1$ and the resulting

$$\bar{\tau}: H_5(\pi, 2) \to \mathrm{Tor}(\pi, \pi). \tag{2.7}$$

We will be content to study (2.7) when π is cyclic.

THEOREM 2.4:
(i) *If* $\pi = \mathbf{Z}$, *then* $H_5(\pi, 2) = 0$;
(ii) *if* $\pi = \mathbf{Z}_m$, *m odd, then* $H_5(\pi, 2) = 0$;
(iii) *if* $\pi = \mathbf{Z}_m$, *m even, then* $H_5(\pi, 2) = \mathbf{Z}_2$ *and* $\bar{\tau}$ *is a monomorphism.*

Proof. (i) is well-known. We will prove (ii) and (iii) simultaneously. In the Serre spectral sequence we have $\bar{\tau} = d_2: E_2^{50} \to E_2^{31}$. Then ker $\bar{\tau} = E_3^{50}$. Now $E_2^{22} = 0$ since π is cyclic; $E_2^{13} = 0$; and $E_2^{04} = 0$ since π is cyclic. Thus $E_3^{50} = E_\infty^{50} = 0$, and $\bar{\tau}$ is a monomorphism.

Since $E_2^{12} = 0$, coker $\bar{\tau} = E_3^{31}$. Thus we have the exact sequence

$$H_5(\pi, 2) \rightarrowtail \mathbf{Z}_m \twoheadrightarrow E_3^{31}.$$

Also, we have $\bar{\sigma}: E_2^{40} \to E_2^{21}$ and we know that

$$E_3^{40} = \ker \bar{\sigma} = \begin{cases} 0, & m \text{ odd} \\ \mathbf{Z}_2, & m \text{ even}, \end{cases}$$

by Theorem 2.2. Then $E_3^{40} = E_4^{40}$ since $E_2^{12} = 0$, so we have an exact sequence

$$E_3^{31} \overset{d_3}{\rightarrowtail} E_3^{03} \twoheadrightarrow E_4^{03},$$

and

$$E_4^{40} \overset{d_4}{\rightarrowtail} E_4^{03}.$$

Finally, $E_2^{22} = 0$ since π is cyclic and $E_2^{03} = \mathbf{Z}_m$, so $E_3^{03} = \mathbf{Z}_m$. Putting all these facts together yields the theorem.

We remark that this theorem does have some relevance to (1.5); for in the third term, when we factor out the image of Tor (N, G_{ab}), we actually pass through coker τ, where $\tau: H_5(N, 2) \to \mathrm{Tor}(N, G_{ab})$.

3. The Kervaire Exact Sequence

In [6], Kervaire associates with a short exact sequence of perfect groups

$$1 \to K \to G \to Q \to 1 \tag{3.1}$$

an exact sequence

$$H_3K_0 \to H_3G_0 \to H_3Q_0 \to H_2K \to H_2G \to H_2Q \to 0, \tag{3.2}$$

provided that Q operates trivially on H_2K; here, G_0 (for example) denotes the *universal cover* of G, that is, there is a central extension

$$N \rightarrowtail G_0 \twoheadrightarrow G,$$

G_0 is perfect and $H_2G_0 = 0$.

We wish to remark in this section that we may, just as easily, obtain an exact sequence like (3.2) for any covering

$$\tilde{K} \to \tilde{G} \to \tilde{Q} \to 1$$

of (3.1). We first explain what we mean by a *covering* of (3.1). From (3.1) we obtain the exact sequence

$$H_2K \to H_2G \to H_2Q \to 0; \tag{3.3}$$

this follows (see [6]) from the Hochschild-Serre spectral sequence and the fact that K is perfect. Now let

$$U \to V \to W \to 0 \tag{3.4}$$

be an exact subsequence of (3.3); that is, (3.4) is exact and

$$\begin{array}{ccccccc} U & \to & V & \to & W & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_2K & \to & H_2G & \to & H_2Q & \to & 0 \end{array}$$

commutes. Let $\tilde{K}$, $\tilde{G}$, $\tilde{Q}$ be the covers of K, G, Q corresponding to U, V, W. Then there is a sequence

$$\tilde{K} \to \tilde{G} \to \tilde{Q}$$

covering $K \to G \to Q$ and an elementary and familiar argument establishes the exactness of

$$\tilde{K} \to \tilde{G} \to \tilde{Q} \to 1. \tag{3.5}$$

Thus, we call (3.5) the covering of (3.1) corresponding to (3.4). As special cases we have the universal cover of Kervaire, when (3.4) is the zero sequence; and (3.1) itself when (3.4) coincides with (3.3). We may set the covering of (3.1) in evidence by means of the commutative diagram

$$\begin{array}{ccccc} \tilde{K} & \to & \tilde{G} & \twoheadrightarrow & \tilde{Q} \\ \downarrow & & \downarrow & & \downarrow \\ K & \rightarrowtail & G & \twoheadrightarrow & Q \end{array} \tag{3.6}$$

THEOREM 3.1. *Let* (3.5) *be a covering of the sequence* (3.1) *of perfect groups. If* Q *operates trivially on* H_2K, *the sequence*

$$H_3(\tilde{K}) \to H_3(\tilde{G}) \to H_3(\tilde{Q}) \to H_2K \to H_2G \to H_2Q \to 0$$

is exact.

Proof. A straightforward application of the Hochschild-Serre spectral sequence for the extension $K \rightarrowtail G \twoheadrightarrow Q$ $(E_2^{pq} = H_p(Q; H_qK))$, using the facts that $E_2^{p1} = 0$ since K is perfect and $E_2^{12} = 0$ since Q is perfect and operates trivially on H_2K, yields the exact sequence

$$H_3K \to H_3G \to H_3Q \to H_2K \to H_2G \to H_2Q \to 0.$$

Now let C be the kernel of $\tilde{K} \to \tilde{G}$. Then it is clear from (3.6) that C lies in the kernel of $\tilde{K} \to K$ and hence is central in $\tilde{K}$. Moreover, $\tilde{K}/C$ is also perfect. We obtain from (3.6) the commutative diagram

$$\begin{array}{ccccc} \tilde{K}/C & \rightarrowtail & \tilde{G} & \twoheadrightarrow & \tilde{Q} \\ \downarrow & & \downarrow & & \downarrow \\ K & \rightarrowtail & G & \twoheadrightarrow & Q. \end{array}$$

We claim that $H_2(\tilde{K}/C) \to H_2K$ is monomorphic. This follows from Corollary 1.2 since the kernel of $\tilde{K}/C \twoheadrightarrow K$ is central. Thus, by the naturality of the operation, we infer that $\tilde{Q}$ operates trivially on $H_2(\tilde{K}/C)$. Thus, we have the commutative diagram, with exact rows,

$$\begin{array}{ccccccccccccc} H_3(\tilde{K}/C) & \to & H_3\tilde{G} & \to & H_3\tilde{Q} & \to & H_2(\tilde{K}/C) & \to & H_2\tilde{G} & \to & H_2\tilde{Q} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H_3K & \longrightarrow & H_3G & \to & H_3Q & \longrightarrow & H_2K & \longrightarrow & H_2G & \to & H_2Q & \to & 0. \end{array} \tag{3.7}$$

Now the first three vertical arrows in (3.7) are surjective by Corollary 1.2, and the last three are injective. Since, again by Corollary 1.2, $H_3\tilde{K} \to H_3(\tilde{K}/C)$ is surjective, the theorem is proved.

We close by offering an example to show that, if $\tilde{G}$ covers G, then $H_3\tilde{G} \to H_3G$ may fail to be injective, that is, fail to be an isomorphism. Let B be the binary icosahedral group.[3]) Then B is perfect and the center of B is $\mathbf{Z}_2$ and we have the central extension

$$\mathbf{Z}_2 \rightarrowtail B \twoheadrightarrow A_5$$

where A_5 is the alternating group of degree 5. Moreover, B is the fundamental group of a Poincaré space with universal covering space S^3, so that $H_2B=0$, $H_3B=\mathbf{Z}_{120}$. Thus B is the universal cover of the group A_5 and plainly $H_3B \to H_3A_5$ is not injective, since the order of A_5 is 60. Indeed, the exact sequence b) of Corollary 1.2 reads, in this case,

$$H_4A_5 \to \mathbf{Z}_4 \to \mathbf{Z}_{120} \to \mathbf{Z}_{30} \to 0$$

(from the exact sequence, H_3A_5 must be Z_{60} or Z_{30}; since the 2-Sylow subgroup of A_5 is $\mathbf{Z}_2 \times \mathbf{Z}_2$, it follows that the 2-component of H_3A_5 has exponent 2, so $H_3A_5=\mathbf{Z}_{30}$). Part a) of Corollary 1.2 simply asserts that $H_2A_5=\mathbf{Z}_2$.

The failure of $H_3\tilde{G} \to H_3G$ to be an isomorphism, on the one hand, vindicates Kervaire's definition of π_2G as H_3G_0. On the other hand, it does introduce a somewhat unsatisfactory feature into the analogy with covering spaces of connected spaces.

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[3]) We are indebted to R. Beyl for drawing our attention, in a slightly different context, to this example of a perfect group.