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**Autor:** Eckmann, B. / Hilton, P.J.  
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# On Central Group Extensions and Homology

by B. ECKMANN and P. J. HILTON

## 0. Introduction

Given an extension of groups

$$N \twoheadrightarrow G \twoheadrightarrow Q \quad (0.1)$$

it is well-known (see [7, 8]) that there is an exact sequence in homology

$$H_2G \rightarrow H_2Q \rightarrow N/[G, N] \rightarrow H_1G \rightarrow H_1Q \rightarrow 0, \quad (0.2)$$

where  $H_1G = G_{ab}$  is the abelianized group  $G/[G, G]$ . Ganea pointed out in [5] that, if  $N$  is *central* in  $G$ , we may extend the sequence (0.2) one place to the left, obtaining

$$N \otimes G_{ab} \rightarrow H_2G \rightarrow H_2Q \rightarrow N \rightarrow G_{ab} \rightarrow Q_{ab} \rightarrow 0. \quad (0.3)$$

Ganea's proof is topological, but uses no explicit spectral sequence technique. In this paper we exploit Ganea's topological approach, but by using spectral sequence techniques for fibre spaces, we extend (0.3) a further four places to the left. The feature which then enters into the sequence, beyond the ordinary homology groups of groups, is the group  $H_4(N, 2)$ , that is, the fourth homology group of the Eilenberg-Mac Lane complex  $K(N, 2)$ . Indeed, we associate naturally with the central extension (0.1) a homomorphism

$$\sigma: H_4(N, 2) \rightarrow N \otimes G_{ab} \quad (0.4)$$

which, in fact, factors as

$$H_4(N, 2) \xrightarrow{\bar{\sigma}} N \otimes N \rightarrow N \otimes G_{ab}, \quad (0.5)$$

where  $\bar{\sigma}$  is intrinsic to  $N$  and natural, and the second homomorphism is induced by the evident map  $N \rightarrow G_{ab}$ . Then, in particular, we obtain the 8-term sequence

$$H_3G \rightarrow H_3Q \rightarrow \operatorname{coker} \sigma \rightarrow H_2G \rightarrow H_2Q \rightarrow N \rightarrow G_{ab} \rightarrow Q_{ab} \rightarrow 0. \quad (0.6)$$

Our full 10-term sequence (1.5) then involves a certain quotient,  $\overline{H_3G}$ , of  $H_3G$ , which we explicitly describe, and commences

$$H_4Q \rightarrow \ker \sigma \rightarrow \overline{H_3G} \rightarrow H_3Q \rightarrow \cdots. \quad (0.7)$$

We remark that the Ganea extension (0.3) of (0.2) and certain parts of our further extension can be established by an elementary method using free presentations of the groups concerned. This is done in a separate paper [2] which also discusses some elementary applications.<sup>1)</sup>

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<sup>1</sup> **Added in proof:** Y. NOMURA, *The Whitney Join and its Dual*, Osaka J. Math. 7 (1970), 353–373, uses topological methods to obtain an extension of (0.2), different from ours, back to  $H_3Q$ , even if  $N$  is not central in  $G$ .

In Section 3 of the present paper we show that the sequence (0.6) is relevant to the study by Bass [1] and Kervaire [6] of *perfect* (or *connected*) groups, that is, groups  $G$  such that the abelianized group  $G_{ab}$  is trivial. For we obtain immediately from it the results of Section 1 of [6] and the principal lemma of Section 2 of [6], namely that, if (0.1) is a central extension with  $G$  perfect, then  $H_3G \rightarrow H_3Q$  is surjective. Kervaire also obtains an exact sequence in [6], relating to an extension

$$K \twoheadrightarrow G \twoheadrightarrow Q$$

of perfect groups, which suggests a definition of  $K_3(A)$  in algebraic  $K$ -theory. In Section 3 of this paper we offer a commentary on this sequence in the form of a natural generalization which exploits again the fact that  $H_3G \rightarrow H_3Q$  is surjective. With regard to algebraic  $K$ -theory, we note that an immediate proof of the exact sequence

$$K_2(A) \rightarrow K_2(A/\mathfrak{a}) \rightarrow K_1(A, \mathfrak{a}) \rightarrow K_1(A) \rightarrow K_1(A/\mathfrak{a})$$

of algebraic  $K$ -theory can be obtained from (0.2) and from well-known properties (Theorem 15.1 of [9] of  $GL(A)$  and  $GL(A/\mathfrak{a})$ ); this proof is given in [2].

Section 2 is devoted to a study of  $\bar{\sigma}: H_4(N, 2) \rightarrow N \otimes N$  and a companion homomorphism

$$\bar{\tau}: H_5(N, 2) \rightarrow \text{Tor}(N, N).$$

We use the full 10-term exact sequence to compute  $\bar{\sigma}$  and  $\bar{\tau}$ ; we reobtain in the process the values of the groups  $H_4(N, 2)$ ,  $H_5(N, 2)$ , first computed by Eilenberg-Mac Lane [3]. Their procedure in computing  $H_4(N, 2)$  was to use the bar construction to identify  $H_4(N, 2)$  with Whitehead's  $\Gamma$ -group, and then use Whitehead's calculations [10]. Our procedure in computing  $H_4(N, 2)$  exploits the more general homomorphism  $\sigma$  and the factorization (0.5).

We remark in Section 1 that the topological situation giving rise to a 10-term exact sequence is much more general than that obtained from a central extension of groups, although the latter is in a sense universal. All we require is a fibration

$$F \rightarrow E \rightarrow B$$

in which  $F$  is connected and  $B$  is 1-connected with  $H_3B = 0$ . Thus, for example, such a sequence obtains whenever we have a fibration over  $S^2$ .

## 1. The Extended Exact Sequence

Let

$$N \twoheadrightarrow G \twoheadrightarrow Q \tag{1.1}$$

be a central group extension. There is then an operation of  $N$  on  $G$ , that is, a homomorphism

$$\varrho: N \times G \rightarrow G \quad (1.2)$$

which is simply given by the product operation in  $G$ ,  $\varrho(x, y) = xy$ ,  $x \in N$ ,  $y \in G$ . Then  $\varrho$  induces a homomorphism, which we also denote  $\varrho$ , in homology,

$$\varrho: H_i(N \times G) \rightarrow H_i G, \quad i \geq 0. \quad (1.3)$$

Our main theorem is the following.

**THEOREM 1.1:** *Given the central group extension  $N \rightarrow G \rightarrow Q$ , there is a natural homomorphism*

$$\sigma: H_4(N, 2) \rightarrow N \otimes G_{ab} \quad (1.4)$$

and a natural exact sequence

$$\begin{aligned} H_4 Q &\rightarrow \ker \sigma \rightarrow H_3 G / \varrho((N \otimes H_2 G) \oplus \text{Tor}(N, G_{ab})) \\ &\rightarrow H_3 Q \rightarrow \text{coker } \sigma \xrightarrow{\varrho} H_2 G \rightarrow H_2 Q \rightarrow N \rightarrow G_{ab} \rightarrow Q_{ab} \rightarrow 0. \end{aligned} \quad (1.5)$$

Before proving this theorem we make the following remark. In the third term of the sequence the denominator is to be understood as the image under  $\varrho$  of a subgroup of  $H_3(N \times G)$ . Now it is true that the Künneth formula does not split naturally; nevertheless, the quotient group of  $H_3 G$  is described in natural, unambiguous fashion, since  $\text{Tor}(N, G_{ab})$  is embedded naturally in  $H_3(N \times G)/N \otimes H_2 G$ . Equivalently, one may observe that  $H_2(G; H_1 N)$  is embedded naturally in  $H_3(N \times G)$ .

We now prove the theorem. Following Ganea [5], we base ourselves on the fibre sequence

$$K(G, 1) \rightarrow K(Q, 1) \rightarrow K(N, 2). \quad (1.6)$$

There is then an operation of  $K(N, 1) = \Omega K(N, 2)$  on  $K(G, 1)$  and this is precisely the operation derived from  $\varrho$  (1.2). This observation enables us to interpret certain differentials in the Serre spectral sequence associated with<sup>2)</sup> (1.6). This spectral sequence relates, in its simplest form, to a fibration

$$F \rightarrow E \rightarrow B$$

in which  $B$  is 1-connected and  $F$  is 0-connected. Then, in the spectral sequence  $\{E_r^{pq}\}$ , we have

$$E_2^{pq} = H_p(B; H_q(F)),$$

---

<sup>2)</sup> Ganea [5] does not use the Serre spectral sequence directly, basing himself instead on his result [4] that a fibration  $F \rightarrow E \rightarrow B$  yields a fibration  $F * \Omega B \rightarrow E \cup CF \rightarrow B$ . However, the action of  $\Omega B$  on  $F$  is also implicit in this result.



and the sequence converges (finitely) to the graded group associated with  $H_*E$ , suitably filtered. Moreover the degree of  $d_r$  is  $(-r, r-1)$ .

Since in the fibration (1.6) the base is 1-connected and the fibre 0-connected, we immediately obtain the right hand end of the sequence (1.5), beginning with  $H_2G \rightarrow H_2Q$ , i.e., the sequence (0.2) in the central case. Moreover, the homomorphism  $H_2G \rightarrow H_2Q$  is effectively just the passage from  $E_2^{02}$  to  $E_\infty^{02}$  in the spectral sequence. Now consider

$$E_2^{40} \xrightarrow{d_2} E_2^{21} \xrightarrow{d_2} E_2^{02}. \quad (1.7)$$

The first homomorphism provides the definition of  $\sigma: H_4(N, 2) \rightarrow N \otimes G_{ab}$ , while the second is the restriction of  $\varrho: H_2(N \times G) \rightarrow H_2G$  to  $N \otimes G_{ab}$ ; we also denote this restriction by  $\varrho$ . We thus have exactness

$$N \otimes G_{ab} \xrightarrow{\varrho} H_2G \rightarrow E_3^{02}.$$

However,  $E_2^{30} = H_3(N, 2) = 0$ , so  $E_3^{02} = E_4^{02} = E_\infty^{02}$  and we have the exact sequence, due to Ganea [5],

$$N \otimes G_{ab} \xrightarrow{\varrho} H_2G \rightarrow H_2Q. \quad (1.8)$$

Now  $E_3^{21} = \ker \varrho / \text{im } \sigma$ , hence

$$E_\infty^{21} = \ker \varrho / \text{im } \sigma. \quad (1.9)$$

Also

$$E_\infty^{30} = 0, \quad (1.10)$$

since  $E_2^{30} = 0$ , and

$$E_\infty^{12} = 0, \quad (1.11)$$

since  $E_2^{12} = 0$ . Thus we have an exact sequence

$$E_\infty^{03} \rightarrow H_3Q \rightarrow \ker \varrho / \text{im } \sigma, \quad (1.12)$$

which yields, with (1.8), the exact sequence

$$E_\infty^{03} \rightarrow H_3Q \rightarrow N \otimes G_{ab} / \text{im } \sigma \xrightarrow{\varrho} H_2G \rightarrow H_2Q, \quad (1.13)$$

where we again denote by  $\varrho$  the homomorphism induced by  $\varrho$ .

We now analyze the passage through the spectral sequence from  $E_2^{03}$  to  $E_\infty^{03}$ . We start with  $E_2^{03} = H_3G$ ; passing to  $E_3^{03}$ , we factor out  $\varrho(N \otimes H_2G)$ ; passing to  $E_4^{03}$ , we further factor out  $\varrho(\text{Tor}(N, G_{ab}))$ ; we thus have the exact sequence

$$E_\infty^{40} \rightarrow E_4^{40} \xrightarrow{d_4} H_3G / \varrho((N \otimes H_2G) \oplus \text{Tor}(N, G_{ab})) \rightarrow E_\infty^{03}. \quad (1.14)$$

Now, reverting to (1.7),  $E_3^{40} = \ker \sigma$ ; and  $E_3^{40} = E_4^{40}$  since  $E_2^{12} = 0$ . This completes the proof of the theorem since  $E_\infty^{40}$  is a quotient of  $H_4Q$ .

COROLLARY 1.2 (see also [2]): *Suppose that  $G$  is perfect (i.e.,  $G_{ab}=0$ ). Then*

a) *There is an exact sequence*

$$0 \rightarrow H_2G \rightarrow H_2Q \rightarrow N \rightarrow 0,$$

b)  *$H_3G \rightarrow H_3Q$  is surjective. Indeed, we have an exact sequence*

$$H_4Q \rightarrow H_4(N, 2) \rightarrow H_3G/\varrho(N \otimes H_2G) \rightarrow H_3Q \rightarrow 0.$$

We observe that we may pass immediately to a generalization of Theorem 1.1. We suppose given a fibration

$$F \rightarrow E \rightarrow B, \quad (1.15)$$

with  $F$  connected and  $B$  1-connected; and we suppose further that  $H_3B=0$ . We have an operation of  $\Omega B$  on  $F$ ,

$$\varrho: \Omega B \times F \rightarrow F,$$

and, by identifying  $H_1\Omega B$  with  $H_2B$ , we obtain induced homomorphisms

$$\varrho: H_2B \otimes H_iF \rightarrow H_{i+1}F, \quad \varrho: \text{Tor}(H_2B, H_{i-1}F) \rightarrow H_{i+1}F. \quad (1.16)$$

Then Theorem 1.1 generalizes to assert a natural homomorphism

$$\sigma: H_4B \rightarrow H_2B \otimes H_1F \quad (1.17)$$

and a natural exact sequence

$$\begin{aligned} H_4E &\rightarrow \ker \sigma \rightarrow H_3F/\varrho(H_2B \otimes H_2F \oplus \text{Tor}(H_2B, H_1F)) \\ &\rightarrow H_3E \rightarrow \text{coker } \sigma \xrightarrow{\varrho} H_2F \rightarrow H_2E \rightarrow H_2B \rightarrow H_1F \rightarrow H_1E \rightarrow 0. \end{aligned} \quad (1.18)$$

Indeed, the homomorphism  $\sigma$  (1.17) exists without the supplementary hypothesis that  $H_3B=0$ .

## 2. The Homomorphism $\sigma$

Let us consider the homomorphism  $\sigma: H_4B \rightarrow H_2B \otimes H_1F$  of (1.17) in full generality. By considering the diagram

$$\begin{array}{ccccc} \Omega B & \rightarrow & EB & \rightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ F & \rightarrow & E & \rightarrow & B \end{array} \quad (2.1)$$

we obtain the commutative diagram

$$\begin{array}{ccc} H_4B & \xrightarrow{\sigma} & H_2B \otimes H_1\Omega B \\ \parallel & & \downarrow \\ H_4B & \xrightarrow{\sigma} & H_2B \otimes H_1F, \end{array} \quad (2.2)$$

so that it is sufficient to analyze the homomorphism  $\sigma: H_4 B \rightarrow H_2 B \otimes H_1 \Omega B$  or, equivalently,

$$\sigma: H_4 B \rightarrow H_2 B \otimes H_2 B, \quad (2.3)$$

arising from the path-space fibration  $\Omega B \rightarrow EB \rightarrow B$ .

Now let  $\pi = \pi_2 B = H_2 B$ ; then another application of naturality shows that if  $\eta: B \rightarrow K(\pi, 2)$  is the fundamental class, then  $\sigma$  in (2.3) factors as

$$H_4 B \xrightarrow{\eta} H_4(\pi, 2) \xrightarrow{\sigma} \pi \otimes \pi.$$

Thus we will be content to take  $B = K(\pi, 2)$  and thus to analyze  $\sigma = \bar{\sigma}: H_4(\pi, 2) \rightarrow \pi \otimes \pi$ ; this is, in any case, our real concern in this note. We will use (1.5) to carry out the calculation of  $\bar{\sigma}$  and will at the same time compute  $H_4(\pi, 2)$ . This group has, of course, originally been computed by Eilenberg-Mac Lane [3], but we will base ourselves simply on (1.5).

We thus consider the homomorphism

$$\bar{\sigma}: H_4(\pi, 2) \rightarrow \pi \otimes \pi. \quad (2.4)$$

The sequence (1.5) reduces, for the central extension  $\pi \rightarrow \pi \rightarrow 1$ , to

$$0 \rightarrow \ker \bar{\sigma} \rightarrow H_3 \pi / \varrho((\pi \otimes H_2 \pi) \oplus \text{Tor}(\pi, \pi)) \rightarrow 0 \rightarrow \text{coker } \bar{\sigma} \rightarrow H_2 \pi \rightarrow 0. \quad (2.5)$$

Assume now that  $\pi$  is cyclic; then  $H_2 \pi = 0$  and  $\text{coker } \bar{\sigma} = 0$ ,  $\bar{\sigma}$  is surjective. If  $\pi = \mathbb{Z}$ , then  $\ker \bar{\sigma} = 0$ , so  $\bar{\sigma}$  is an isomorphism, and (2.4) is then an isomorphism  $\bar{\sigma}: \mathbb{Z} \cong \mathbb{Z}$ .

Now let  $\pi = \mathbb{Z}_m$ . Then  $H_3 \pi = \mathbb{Z}_m$  and we will prove below the key lemma.

LEMMA 2.1:

$$\varrho \text{ Tor}(\mathbb{Z}_m, \mathbb{Z}_m) = 2\mathbb{Z}_m.$$

Granted this lemma, we immediately deduce that, if  $\pi = \mathbb{Z}_m$ , then

$$\ker \bar{\sigma} = \begin{cases} 0, & m \text{ odd} \\ \mathbb{Z}_2, & m \text{ even} \end{cases}.$$

Thus, if  $\pi = \mathbb{Z}_m$ ,  $m$  odd, (2.4) is then an isomorphism  $\bar{\sigma}: \mathbb{Z}_m \cong \mathbb{Z}_m$ . If  $m$  is even, we must determine the group extension to compute  $H_4(\mathbb{Z}_m, 2)$ . To do this we consider the central extension  $\mathbb{Z}_m \xrightarrow{1} \mathbb{Z}_{m^2} \rightarrow \mathbb{Z}_m$ . Then (1.5) yields

$$0 \rightarrow \ker \sigma \rightarrow \text{quotient of } \mathbb{Z}_{m^2} \rightarrow \cdots. \quad (2.6)$$

Moreover, by (2.2),  $\sigma: H_4(\mathbb{Z}_m, 2) \rightarrow \mathbb{Z}_m \otimes \mathbb{Z}_{m^2}$  factors as

$$H_4(\mathbb{Z}_m, 2) \xrightarrow{\bar{\sigma}} \mathbb{Z}_m \otimes \mathbb{Z}_m \xrightarrow{1 \otimes \iota} \mathbb{Z}_m \otimes \mathbb{Z}_{m^2}.$$

But plainly  $1 \otimes \iota = 0$ , so  $\sigma = 0$ ,  $\ker \sigma = H_4(\mathbb{Z}_m, 2)$ , so that, by (2.6),  $H_4(\mathbb{Z}_m, 2)$  is cyclic,

and we have proved that if  $\pi = \mathbf{Z}_m$ ,  $m$  even, then (2.4) is an epimorphism  $\bar{\sigma}: \mathbf{Z}_{2m} \twoheadrightarrow \mathbf{Z}_m$ . Summing up, we have

**THEOREM 2.2:**

- (i) If  $\pi = \mathbf{Z}$ , then  $H_4(\pi, 2) = \mathbf{Z}$  and (2.4) is an isomorphism;
- (ii) if  $\pi = \mathbf{Z}_m$ ,  $m$  odd, then  $H_4(\pi, 2) = \mathbf{Z}_m$  and (2.4) is an isomorphism;
- (iii) if  $\pi = \mathbf{Z}_m$ ,  $m$  even, then  $H_4(\pi, 2) = \mathbf{Z}_{2m}$  and (2.4) is an epimorphism.

We now prove Lemma 2.1. Let  $x$  generate  $\pi = \mathbf{Z}_m$  and let

$$\cdots \xrightarrow{f} \mathbf{Z}\pi \xrightarrow{g} \mathbf{Z}\pi \xrightarrow{f} \mathbf{Z}\pi \xrightarrow{\varepsilon} \mathbf{Z} \rightarrow 0, \quad f = x - 1, \quad g = x^{m-1} + x^{m-2} + \cdots + x + 1,$$

be the usual  $\pi$ -resolution of  $\mathbf{Z}$ . That is, we take the resolution  $\mathbf{P} \rightarrow \mathbf{Z}$  in which  $P_n = \mathbf{Z}\pi$ , generated by  $a_n$ , and

$$\begin{aligned} \partial_n a_n &= f a_{n-1}, \quad n \text{ odd}, \\ &= g a_{n-1}, \quad n \text{ even}, \quad n > 0, \\ \varepsilon a_0 &= 1. \end{aligned}$$

Then  $\mathbf{P} \otimes \mathbf{P}$  is a  $\pi \times \pi$ -resolution of  $\mathbf{Z}$  and we seek a chain map  $\varphi: \mathbf{P} \otimes \mathbf{P} \rightarrow \mathbf{P}$ , compatible with the augmentations and with the multiplication map  $\varrho: \pi \times \pi \rightarrow \pi$ . Proceeding step-by-step, we find that we may define  $\phi$  as follows in dimensions  $\leq 3$ :

$$\begin{aligned} \phi_0(a_0 \otimes a_0) &= a_0, \\ \phi_1(a_1 \otimes a_0) &= \phi_1(a_0 \otimes a_1) = a_1, \\ \phi_2(a_2 \otimes a_0) &= \phi_2(a_0 \otimes a_2) = a_2, \quad \phi_2(a_1 \otimes a_1) = 0, \\ \phi_3(a_3 \otimes a_0) &= \phi_3(a_2 \otimes a_1) = \phi_3(a_1 \otimes a_2) = \phi_3(a_0 \otimes a_3) = a_3. \end{aligned}$$

Now  $\text{Tor}(\mathbf{Z}_m, \mathbf{Z}_m) \subseteq H_3(\mathbf{Z}_m \times \mathbf{Z}_m)$  is generated by  $a_2 \otimes a_1 + a_1 \otimes a_2$ . Thus, the image of  $\text{Tor}(\mathbf{Z}_m, \mathbf{Z}_m)$  in  $H_3 \mathbf{Z}_m$  under  $\varrho$  is the subgroup generated by  $2a_3$ , that is,  $2\mathbf{Z}_m$ .

*Remark.* We may identify  $\bar{\sigma}: H_4(\pi, 2) \rightarrow \pi \otimes \pi$  with an element of  $H^4(\pi, 2; \pi \otimes \pi)$  since  $H_3(\pi, 2) = 0$ . Since  $\bar{\sigma}$  is just  $d_2$  in the Serre spectral sequence, the standard identification of the differential shows that  $\bar{\sigma} = \eta_1 \eta_2$ , where  $\eta \in H^2(\pi, 2; \pi)$  is the fundamental class and  $\eta_1, \eta_2$  copy  $\eta$  into the first and second factors of  $\pi \otimes \pi$ . This remark may be used to establish the next theorem.

Let  $\pi = N \otimes N'$ . Then plainly  $\bar{\sigma}: H_4(\pi, 2) \rightarrow \pi \otimes \pi$  maps  $N \otimes N'$  to  $N \otimes N' \oplus N' \otimes N$ .

**THEOREM 2.3:**

$$\bar{\sigma}(x \otimes x') = x \otimes x' + x' \otimes x, \quad x \in N, \quad x' \in N'.$$

*Proof.* We note from (2.5) that  $\varrho: \pi \otimes \pi \rightarrow H_2 \pi$  induces an isomorphism

$\text{coker } \bar{\sigma} \cong H_2\pi$ . It is plain that  $\varrho(x \otimes x') = x \otimes x'$  and  $\varrho(x' \otimes x) = -x \otimes x'$ . This establishes the theorem, in view of the naturality of  $\bar{\sigma}$  with respect to  $\pi$ .

We thus have a complete determination of  $H_4(\pi, 2)$  and  $\bar{\sigma}$  for any finitely generated abelian group  $\pi$ . Of course we may, if we wish, use a direct limit argument to extend the determination of  $\bar{\sigma}$  to any abelian group  $\pi$ .

We close this section by mentioning a companion homomorphism to  $\sigma$ , namely,  $\tau: H_5(N, 2) \rightarrow \text{Tor}(N, G_{ab})$  in the situation of (1.1). This is just  $d_2: E_2^{50} \rightarrow E_2^{31}$  in the Serre spectral sequence of (1.6). Again it follows by naturality that  $\tau$  is just

$$H_5(N, 2) \xrightarrow{\bar{\tau}} \text{Tor}(N, N) \rightarrow \text{Tor}(N, G_{ab}),$$

where the second homomorphism is induced by  $N \rightarrow G_{ab}$ . Thus, reverting to our previous notation, we consider the central extension  $\pi \rightarrow \pi \rightarrow 1$  and the resulting

$$\bar{\tau}: H_5(\pi, 2) \rightarrow \text{Tor}(\pi, \pi). \quad (2.7)$$

We will be content to study (2.7) when  $\pi$  is cyclic.

#### THEOREM 2.4:

- (i) If  $\pi = \mathbf{Z}$ , then  $H_5(\pi, 2) = 0$ ;
- (ii) if  $\pi = \mathbf{Z}_m$ ,  $m$  odd, then  $H_5(\pi, 2) = 0$ ;
- (iii) if  $\pi = \mathbf{Z}_m$ ,  $m$  even, then  $H_5(\pi, 2) = \mathbf{Z}_2$  and  $\bar{\tau}$  is a monomorphism.

*Proof.* (i) is well-known. We will prove (ii) and (iii) simultaneously. In the Serre spectral sequence we have  $\bar{\tau} = d_2: E_2^{50} \rightarrow E_2^{31}$ . Then  $\ker \bar{\tau} = E_3^{50}$ . Now  $E_2^{22} = 0$  since  $\pi$  is cyclic;  $E_2^{13} = 0$ ; and  $E_2^{04} = 0$  since  $\pi$  is cyclic. Thus  $E_3^{50} = E_\infty^{50} = 0$ , and  $\bar{\tau}$  is a monomorphism.

Since  $E_2^{12} = 0$ ,  $\text{coker } \bar{\tau} = E_3^{31}$ . Thus we have the exact sequence

$$H_5(\pi, 2) \rightarrow \mathbf{Z}_m \rightarrow E_3^{31}.$$

Also, we have  $\bar{\sigma}: E_2^{40} \rightarrow E_2^{21}$  and we know that

$$E_3^{40} = \ker \bar{\sigma} = \begin{cases} 0, & m \text{ odd} \\ \mathbf{Z}_2, & m \text{ even,} \end{cases}$$

by Theorem 2.2. Then  $E_3^{40} = E_4^{40}$  since  $E_2^{12} = 0$ , so we have an exact sequence

$$E_3^{31} \xrightarrow{d_3} E_3^{03} \rightarrow E_4^{03},$$

and

$$E_4^{40} \xrightarrow{d_4} E_4^{03}.$$

Finally,  $E_2^{22} = 0$  since  $\pi$  is cyclic and  $E_2^{03} = \mathbf{Z}_m$ , so  $E_3^{03} = \mathbf{Z}_m$ . Putting all these facts together yields the theorem.

We remark that this theorem does have some relevance to (1.5); for in the third term,

when we factor out the image of  $\text{Tor}(N, G_{ab})$ , we actually pass through coker  $\tau$ , where  $\tau: H_5(N, 2) \rightarrow \text{Tor}(N, G_{ab})$ .

### 3. The Kervaire Exact Sequence

In [6], Kervaire associates with a short exact sequence of perfect groups

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1 \quad (3.1)$$

an exact sequence

$$H_3K_0 \rightarrow H_3G_0 \rightarrow H_3Q_0 \rightarrow H_2K \rightarrow H_2G \rightarrow H_2Q \rightarrow 0, \quad (3.2)$$

provided that  $Q$  operates trivially on  $H_2K$ ; here,  $G_0$  (for example) denotes the *universal cover* of  $G$ , that is, there is a central extension

$$N \twoheadrightarrow G_0 \twoheadrightarrow G,$$

$G_0$  is perfect and  $H_2G_0 = 0$ .

We wish to remark in this section that we may, just as easily, obtain an exact sequence like (3.2) for any covering

$$\tilde{K} \rightarrow \tilde{G} \rightarrow \tilde{Q} \rightarrow 1$$

of (3.1). We first explain what we mean by a *covering* of (3.1). From (3.1) we obtain the exact sequence

$$H_2K \rightarrow H_2G \rightarrow H_2Q \rightarrow 0; \quad (3.3)$$

this follows (see [6]) from the Hochschild-Serre spectral sequence and the fact that  $K$  is perfect. Now let

$$U \rightarrow V \rightarrow W \rightarrow 0 \quad (3.4)$$

be an exact subsequence of (3.3); that is, (3.4) is exact and

$$\begin{array}{ccccccc} U & \rightarrow & V & \rightarrow & W & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_2K & \rightarrow & H_2G & \rightarrow & H_2Q & \rightarrow & 0 \end{array}$$

commutes. Let  $\tilde{K}, \tilde{G}, \tilde{Q}$  be the covers of  $K, G, Q$  corresponding to  $U, V, W$ . Then there is a sequence

$$\tilde{K} \rightarrow \tilde{G} \rightarrow \tilde{Q}$$

covering  $K \rightarrow G \rightarrow Q$  and an elementary and familiar argument establishes the exactness of

$$\tilde{K} \rightarrow \tilde{G} \rightarrow \tilde{Q} \rightarrow 1. \quad (3.5)$$

Thus, we call (3.5) the covering of (3.1) corresponding to (3.4). As special cases we have the universal cover of Kervaire, when (3.4) is the zero sequence; and (3.1) itself when (3.4) coincides with (3.3). We may set the covering of (3.1) in evidence by means of the commutative diagram

$$\begin{array}{ccccc} \tilde{K} & \rightarrow & \tilde{G} & \twoheadrightarrow & \tilde{Q} \\ \downarrow & & \downarrow & & \downarrow \\ K & \twoheadrightarrow & G & \twoheadrightarrow & Q \end{array} \quad (3.6)$$

**THEOREM 3.1.** *Let (3.5) be a covering of the sequence (3.1) of perfect groups. If  $Q$  operates trivially on  $H_2K$ , the sequence*

$$H_3(\tilde{K}) \rightarrow H_3(\tilde{G}) \rightarrow H_3(\tilde{Q}) \rightarrow H_2K \rightarrow H_2G \rightarrow H_2Q \rightarrow 0$$

*is exact.*

*Proof.* A straightforward application of the Hochschild-Serre spectral sequence for the extension  $K \twoheadrightarrow G \twoheadrightarrow Q$  ( $E_2^{pq} = H_p(Q; H_qK)$ ), using the facts that  $E_2^{p1} = 0$  since  $K$  is perfect and  $E_2^{12} = 0$  since  $Q$  is perfect and operates trivially on  $H_2K$ , yields the exact sequence

$$H_3K \rightarrow H_3G \rightarrow H_3Q \rightarrow H_2K \rightarrow H_2G \rightarrow H_2Q \rightarrow 0.$$

Now let  $C$  be the kernel of  $\tilde{K} \rightarrow \tilde{G}$ . Then it is clear from (3.6) that  $C$  lies in the kernel of  $\tilde{K} \rightarrow K$  and hence is central in  $\tilde{K}$ . Moreover,  $\tilde{K}/C$  is also perfect. We obtain from (3.6) the commutative diagram

$$\begin{array}{ccccc} \tilde{K}/C & \twoheadrightarrow & \tilde{G} & \twoheadrightarrow & \tilde{Q} \\ \downarrow & & \downarrow & & \downarrow \\ K & \twoheadrightarrow & G & \twoheadrightarrow & Q \end{array}$$

We claim that  $H_2(\tilde{K}/C) \rightarrow H_2K$  is monomorphic. This follows from Corollary 1.2 since the kernel of  $\tilde{K}/C \rightarrow K$  is central. Thus, by the naturality of the operation, we infer that  $\tilde{Q}$  operates trivially on  $H_2(\tilde{K}/C)$ . Thus, we have the commutative diagram, with exact rows,

$$\begin{array}{ccccccccc} H_3(\tilde{K}/C) & \rightarrow & H_3\tilde{G} & \rightarrow & H_3\tilde{Q} & \rightarrow & H_2(\tilde{K}/C) & \rightarrow & H_2\tilde{G} & \rightarrow & H_2\tilde{Q} & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H_3K & \longrightarrow & H_3G & \longrightarrow & H_3Q & \longrightarrow & H_2K & \longrightarrow & H_2G & \longrightarrow & H_2Q & \longrightarrow & 0. \end{array} \quad (3.7)$$

Now the first three vertical arrows in (3.7) are surjective by Corollary 1.2, and the last three are injective. Since, again by Corollary 1.2,  $H_3\tilde{K} \rightarrow H_3(\tilde{K}/C)$  is surjective, the theorem is proved.

We close by offering an example to show that, if  $\tilde{G}$  covers  $G$ , then  $H_3\tilde{G} \rightarrow H_3G$  may fail to be injective, that is, fail to be an isomorphism. Let  $B$  be the binary icosahedral

group.<sup>3)</sup> Then  $B$  is perfect and the center of  $B$  is  $\mathbf{Z}_2$  and we have the central extension

$$\mathbf{Z}_2 \twoheadrightarrow B \twoheadrightarrow A_5$$

where  $A_5$  is the alternating group of degree 5. Moreover,  $B$  is the fundamental group of a Poincaré space with universal covering space  $S^3$ , so that  $H_2B=0$ ,  $H_3B=\mathbf{Z}_{120}$ . Thus  $B$  is the universal cover of the group  $A_5$  and plainly  $H_3B \rightarrow H_3A_5$  is not injective, since the order of  $A_5$  is 60. Indeed, the exact sequence b) of Corollary 1.2 reads, in this case,

$$H_4A_5 \rightarrow \mathbf{Z}_4 \rightarrow \mathbf{Z}_{120} \rightarrow \mathbf{Z}_{30} \rightarrow 0$$

(from the exact sequence,  $H_3A_5$  must be  $\mathbf{Z}_{60}$  or  $\mathbf{Z}_{30}$ ; since the 2-Sylow subgroup of  $A_5$  is  $\mathbf{Z}_2 \times \mathbf{Z}_2$ , it follows that the 2-component of  $H_3A_5$  has exponent 2, so  $H_3A_5 = \mathbf{Z}_{30}$ ). Part a) of Corollary 1.2 simply asserts that  $H_2A_5 = \mathbf{Z}_2$ .

The failure of  $H_3\tilde{G} \rightarrow H_3G$  to be an isomorphism, on the one hand, vindicates Kervaire's definition of  $\pi_2G$  as  $H_3G_0$ . On the other hand, it does introduce a somewhat unsatisfactory feature into the analogy with covering spaces of connected spaces.

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<sup>3)</sup> We are indebted to R. Beyl for drawing our attention, in a slightly different context, to this example of a perfect group.