Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 46 (1971)

Artikel: Toward a Generalized Gauss-Bonnet Formula for Complete, Open Manifolds

Autor: Portnoy, Esther

DOI: https://doi.org/10.5169/seals-35525

Toward a Generalized Gauss-Bonnet Formula for Complete, Open Manifolds

By Esther Portnoy

Introduction

The classical Gauss-Bonnet theorem has been extended in at least two directions. In the 1940's an analogous theorem was proved for compact n-dimensional Riemannian polyhedra by C. B. Allendoerfer and A. Weil [1], and by S. S. Chern [2, 3]. For manifolds with smooth boundary this has the form

$$\int_M \Omega \, dV + \int_{\partial M} \Pi \, dA = \chi'(M)$$

(Ω and Π will be defined in Section 1; χ' is the inner characteristic).

In 1965, R. Finn [6] broadened the two-dimensional result by showing that for a large class of complete, open manifolds, it is possible to choose, in a natural way, a sequence of curves tending to the ideal boundary point at ∞ such that

$$\int_M K \, dA = 2\pi \left[\chi(M) - \lim \frac{\mathscr{L}^2}{4\pi \mathscr{A}} \right],$$

where $\mathscr{L}$ and $\mathscr{A}$ are the length and subtended area of the curves. Finn's results were extended in important ways by A. Huber [7].

The question then arises whether it is possible to prove a formula relating curvature integrals, topological invariants, and purely geometric quantities, for complete, open n-dimensional manifolds, subject only to regularity and integrability conditions. The object of this paper is to examine this question on tube regions, as defined in Section 2. The most promising geometric quantity appears to be the ratio $\mathscr{A}^{n/n-1}/n\mathscr{V}$, where $\mathscr{A}$ is the $(n-1)$-dimensional volume ("area") of one of a family of submanifolds diverging to an ideal boundary point, and $\mathscr{V}$ is the n-dimensional volume of a compact region bounded in part by that submanifold.

In Section 3 are given several examples of manifolds for which such a formula is derivable. In each case there is a fairly simple relationship among the curvature integrals, the Euler characteristics of the manifold and submanifolds, and the limiting value of the above ratio of volumes; however, the relationship varies from one example to the next. I have not found any example for which another geometric quantity enters into such a formula. It seems likely, then, that while such a formula may hold in general, its form must be more complicated than that for the two-dimensional case.

The boundary curvature Π can be decomposed into several terms, each depending to a different degree on the intrinsic curvatures of the submanifold and on the imbedding of the submanifold in the manifold. The term Ψ_0, which depends entirely on the imbedding, is examined in the remaining sections. It is shown in Section 4 that Ψ_0 generalizes the geodesic curvature of a curve in a two-dimensional manifold, in the sense that each is a relative Gauss-Kronecker curvature, analogous to the Gauss-Kronecker curvature of a hypersurface in euclidean space. In Section 6 it is shown that under certain convergence conditions the cross-sections of an orthogonal tube region are asymptotically submanifolds of constant mean curvature. If in addition the cross-sections are asymptotically umbilic submanifolds, then

$$\lim \int \Psi_0 \, dA = (-1)^{n-1} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

Section 7 contains examples and counterexamples for the theorem of Section 6.

Analysis of the terms $\Psi_1, \ldots, \Psi_{[(n-1)/2]}$ will hopefully yield a similar asymptotic formula for the complete boundary curvature Π; then by considering a manifold consisting of a compact manifold with smooth boundary, to each component of which is smoothly attached a tube region, one will obtain an extended Gauss-Bonnet formula.

1. The Gauss-Bonnet Formula in n Dimensions

Let M^n be an n-dimensional Riemannian manifold, and U a coordinate neighborhood in M^n with coordinates $(u^i)_{i=1}^n$. With respect to these coordinates define the metric tensor g_{ij} with determinant g, and the Riemannian curvature tensor

$$R_{ijkl} = g_{im}\Gamma^m_{jk,l} - g_{im}\Gamma^m_{jl,k} + \Gamma^m_{jk}\Gamma_{ml/i} - \Gamma^m_{jl}\Gamma_{mk/i}.$$

Then the Gauss curvature is defined by:

$$\Omega = \frac{(-1)^{n/2}}{2^n (2\pi)^{n/2} (n/2)!\, g} \sum \varepsilon(i)\, \varepsilon(j)\, R_{i(1)i(2)j(1)j(2)} \cdots R_{i(n-1)i(n)j(n-1)j(n)}$$
$$\text{for } n \text{ even},$$
$$\Omega = 0 \quad \text{for } n \text{ odd}.$$

$\varepsilon(i)$ is the Kronecker index, which is ± 1 if i is an even or odd permutation of $1, \ldots, n$, and otherwise 0. Ω is invariant under change of coordinates. For $n=2$, Ω is $1/2\pi$ times the usual two-dimensional Gauss curvature. On a hypersurface in euclidean space, Ω is a dimensional constant times the Gauss-Kronecker curvature for n even.

Next let M^{n-1} be a regular submanifold of M^n. To simplify notation, choose coordinates near M^{n-1} so that $M^{n-1}=\{u^n=0\}$, while for $i<n$, $g_{in}\equiv 0$, $g_{nn,i}\equiv 0$. Let γ be the determinant of the metric tensor on M^{n-1}. For other quantities, such as R^*_{ijkl}, an asterisk will be used to denote a quantity defined in M^{n-1} as an $(n-1)$-dimensional Riemannian manifold with the metric induced from M^n; thus,

$$R^*_{ijkl} = R_{ijkl} - \Gamma^n_{jk}\Gamma_{nl/i} + \Gamma^n_{jl}\Gamma_{nk/i}$$

for $i, j, k, l<n$. Now define[1])

$$\Psi_l = \frac{(-1)^l(\sqrt{g_{nn}})^{n-1-2l}}{(n-1)!\gamma} \sum \varepsilon(i)\,\varepsilon(j)\, R^*_{i(1)i(2)j(1)j(2)} \cdots R^*_{i(2l-1)i(2l)j(2l-1)j(2l)} \Gamma^n_{i(2l+1)j(2l+1)} \cdots \Gamma^n_{i(n-1)j(n-1)}$$

and

$$\Pi = -\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \sum_{l=0}^{[(n-1)/2]} \frac{(-1)^l\,\Psi_l}{l!\,2^l} \sum_{k=l}^{[(n-1)/2]} \frac{(-1)^k(n-1)(n-3)\cdots(n-2k+1)}{2^k(k-l)!}.$$

Note in particular that if $n=2q+1$, the coefficient of Ψ_l in Π vanishes for $l<q$;

$$\Pi = -\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\,\frac{\Psi_q}{2^q} = -\tfrac{1}{2}\Omega^*.$$

For $n=2$,

$$\Pi = -\frac{1}{2\pi}\Psi_0 = -\frac{1}{2\pi}\,\frac{\sqrt{g_{22}}\Gamma^2_{11}}{g_{11}} = -\frac{1}{2\pi}\kappa_{g'},$$

$1/2\pi$ times the geodesic curvature of the curve M^1, considered as part of the boundary of a compact region in which $u^2\leqslant c$; that is, with the opposite orientation as the u^1-curve.

THEOREM. *Let K be a compact n-dimensional region in a Riemannian manifold M^n, such that ∂K consists of a finite number of smooth, regularly imbedded $(n-1)$-dimensional closed submanifolds of M^n. Near each component of ∂K choose coordinates so that ∂K is a submanifold $u^n=$*const., *and $\partial/\partial u^n$ is the outward-directed normal to*

---

[1]) The terms Ψ_l are related to the terms Φ_k of Allendoerfer-Weil by the formula

$$\Phi_k = \frac{(-1)^k(n-1)!}{2^k(n-1-2k)!}\sum_{l=0}^{k}\frac{(-1)^l\,\Psi_l}{2^l l!(k-l)!},$$

with a similar relationship holding for Chern's differential forms Φ_k.

∂K. Let $dV=\sqrt{g}\,du^1\dots du^n$ and $dA=\sqrt{\gamma}\,du^1\dots du^{n-1}$. Then

$$\int_K \Omega\, dV + \int_{\partial K} \Pi\, dA = \chi'(K),$$

where χ' is the inner Euler-Poincaré characteristic.

For the proof, see either Chern [2] or Allendoerfer-Weil [1]. The case in which ∂K is polyhedral rather than smooth can also be treated, but requires a more detailed analysis of Π.

2. Tube Regions in n Dimensions

A *tube region* is an n-dimensional manifold M^n diffeomorphic to $M^{n-1}\times[0,\infty)$, where M^{n-1} is a closed, compact $(n-1)$-dimensional manifold. Coordinates on a tube region will always be chosen such that $(u^i)_{i=1}^{n-1}$ are local coordinates on M^{n-1}, and $u^n=t\in[0,\infty)$. Note that for a tube region $\chi(M^n)=\chi(M^{n-1})$, and that for a compact subset of the form $K=M^{n-1}\times[T_1,T_2]$,

$$\chi'(K)=\chi(K)-\chi(\partial K)=\chi(M^{n-1})-2\chi(M^{n-1})=-\chi(M^{n-1}).$$

An *orthogonal tube region* is a tube region which is also a Riemannian manifold such that, with the above convention on coordinates, for each $i<n$, $g_{in}\equiv 0$ and $g_{nn,i}\equiv 0$; that is, the u^n-curves are geodesics orthogonal to the submanifolds $u^n=$const. For orthogonal tube regions it will often be convenient to introduce the parameter $s=s_n$, arc length along the u^n-curves; this is possible because $\partial s/\partial t=\sqrt{g_{nn}}$ is independent of $u^1,\dots,u^{n-1}$. We will generally be dealing with complete orthogonal tube regions, which means that $s\to\infty$ as $t\to\infty$.

We adopt the following notation:

$$\Gamma(t)=M^{n-1}(t)=\{u^n=t\},\qquad \sum(t)=M^{n-1}\times[0,t]$$

$$\gamma = \text{metric determinant on } M^{n-1}$$

$$= g/g_{nn}\quad \text{for orthogonal tube regions}$$

$$\mathscr{A}(t)=\int_{\Gamma(t)}\sqrt{\gamma}\,du^1\cdots du^{n-1}$$

$$\mathscr{V}(t)=\int_{\Sigma(t)}\sqrt{g}\,du^1\cdots du^n$$

$$=\int_{u^n=0}^{t}\mathscr{A}(u^n)\sqrt{g_{nn}}\,du^n\quad\text{for orthogonal tube regions}$$

$$\alpha=\lim_{t\to\infty}\frac{\mathscr{A}^{n/n-1}(t)}{n\mathscr{V}(t)}\quad\text{if that limit exists.}$$

In an orthogonal tube region,

$$\frac{d}{ds}\mathscr{V}=\frac{d\mathscr{V}}{dt}\frac{dt}{ds}=\mathscr{A}, \quad \text{while}$$

$$\frac{d\mathscr{A}}{ds}=\int\limits_{\Gamma(t)}\frac{1}{\sqrt{g_{nn}}}(\operatorname{tr}\Gamma^i_{jn})\,dA. \quad \text{Therefore}$$

$$\frac{d}{ds}\mathscr{A}^{1/n-1}=\frac{1}{n-1}\mathscr{A}^{2-n/n-1}\int\limits_{\Gamma(t)}\frac{1}{\sqrt{g_{nn}}}(\operatorname{tr}\Gamma^i_{jn})\,dA.$$

LEMMA 1. *If M^n is a complete, orthogonal tube region, and* $\lim d/ds\,\mathscr{A}^{1/n-1}$ *exists, then*

$$\alpha^{n-1}=\lim\frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}=\left(\lim\frac{d}{ds}\mathscr{A}^{1/n-1}\right)^{n-1}.$$

Proof. Since $\mathscr{V}$ is a continuously differentiable, strictly increasing function of s, with $d\mathscr{V}/ds=\mathscr{A}\neq 0$, we may take $\mathscr{V}$ as a new parameter along the u^n-curves. The limit of $\mathscr{V}$ as $s\to\infty$ is either finite or infinite. In the former case, certainly $\mathscr{A}\to 0$, and $d/ds\,\mathscr{A}^{1/n-1}\to 0$, thus the lemma holds. In the case that $\mathscr{V}\to\infty$, we recall that if $f'(x)\to l$ as $x\to\infty$ then $f(x)/x\to l$ also; here,

$$\frac{d}{d\mathscr{V}}\mathscr{A}^{n/n-1}=\frac{d}{ds}\mathscr{A}^{n/n-1}\frac{ds}{d\mathscr{V}}=n\mathscr{A}\frac{d}{ds}\mathscr{A}^{1/n-1}\frac{1}{\mathscr{A}}=n\frac{d}{ds}\mathscr{A}^{1/n-1}$$

implies that

$$\left(\lim\frac{d}{ds}\mathscr{A}^{1/n-1}\right)^{n-1}=\left(\frac{1}{n}\lim\frac{d}{d\mathscr{V}}\mathscr{A}^{n/n-1}\right)^{n-1}=\left(\frac{1}{n}\lim\frac{\mathscr{A}^{n/n-1}}{\mathscr{V}}\right)^{n-1}=\lim\frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

3. Generalized Gauss-Bonnet Formulas for Special Tube Regions

This section contains generalized Gauss-Bonnet formulas for several particular types of tube regions. All the manifolds are assumed to be complete, and coordinates are chosen according to the conventions in Section 2. Note that the constant $C_{(n)}$ of Case III is the leading coefficient of the polynomial $P_{(n)}$ of Case II. Case III includes a counterexample showing that Cohn-Vossen's inequality does not have an n-dimensional analogue for these curvatures.

CASE I: *n odd.* If M^n is an odd-dimensional orthogonal tube region,

$$\int\limits_{M^n}\Omega\,dV+\int\limits_{\Gamma(0)}\Pi\,dA=-\tfrac{1}{2}\chi(M^{n-1}).$$

Proof. This is obtained directly from the definition $\Omega=0$ and the remark of Section 1 that $\Pi=-\frac{1}{2}\Omega^*$ for n odd.

CASE II: *Spherical symmetry.* If M is a complete, spherically symmetric manifold (as defined below), then

$$\int_M \Omega\, dV + \int_{\Gamma(0)} \Pi\, dA = P\left(\lim \frac{\mathscr{A}^{n/n-1}}{n\mathscr{V}}\right),$$

where P is a polynomial with coefficients depending on n.

Proof. A spherically symmetric tube region M^n is given by assigning a spherically symmetric metric in a region $\varepsilon \leqslant |x| < \infty$ of $\mathbf{R}^n$, that is, such that orthogonal transformations of $\mathbf{R}^n$ induce isometries of M^n. Then M^n is an orthogonal tube region. The rays from the origin correspond to geodesics, and the orthogonal trajectories of these rays, the spheres about the origin, are isometric to $(n-1)$-spheres of appropriate radius. We identify the rays as the u^n-curves, and define $r(u^n)>0$, the *radius* of $\Gamma(u^n)$, by the relation

$$\mathscr{A}(u^n) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} [r(u^n)]^{n-1}.$$

Because of the symmetry, it is only necessary to compute Π at one point of each submanifold $u^n=$const., and this computation can be greatly simplified by choice of coordinates. In a neighborhood of the ray $(0,\ldots,0,x^n)$ choose coordinates $u^i=x^i/|x|$, $i=1,\ldots,n-1$; $u^n=|x|$; and define the auxiliary variable $y=x^n/|x|=[1-\sum_{i=1}^{n-1}(u^i)^2]^{1/2}$.

Define the map $f: M^n \to \mathbf{R}^{n+1}$ by

$$f(u^1,\ldots,u^n) = (r(u^n)\,u^1,\ldots,r(u^n)\,u^{n-1}, r(u^n)\,y, u^n).$$

$f|_{\Gamma(u^n)}$ is an isometry. Thus for $i, j, k, l<n$,

$$g_{ij} = r^2\left(\delta_{ij} + \frac{u^i u^j}{y^2}\right)$$

$$R^*_{ijkl}(0,\ldots,0,u^n) = r^2(u^n)\,(\delta_{il}\,\delta_{jk} - \delta_{jl}\,\delta_{ik})$$

$$\Psi_l(0,\ldots,0,u^n) = (-1)^{n-1}2^l\left(\frac{r'}{\sqrt{g_{nn}}}\right)^{n-1-2l}\frac{1}{r^{n-1}}.$$

Thus

$$\int \Pi\, dA = \begin{cases} \dfrac{n!}{p!}\displaystyle\sum_{l=0}^{p-1}\frac{(-1)^l}{l!}\left(\frac{r'}{\sqrt{g_{nn}}}\right)^{n-1-2l}\sum_{k=l}^{p-1}\frac{(-1)^k(p-k)!}{4^k(k-l)!\,(n-2k)!} & \text{if } n=2p,\\ -1 \quad \text{if} \quad n \text{ is odd}. \end{cases}$$

Now

$$\mathscr{A}^{1/n-1}=\left[\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\right]^{1/n-1}r,\qquad \frac{d}{ds}\mathscr{A}^{1/n-1}=\left[\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\right]^{1/n-1}\frac{r'}{\sqrt{g_{nn}}}.$$

Therefore convergence (in even dimensions) of $\int \Pi\, dA$ is equivalent to convergence of $d/ds\,\mathscr{A}^{1/n-1}$, since both are equivalent to convergence of $r'/\sqrt{g_{nn}}$. If $\lim r'/\sqrt{g_{nn}}$ exists and is finite, then define the polynomial

$$P_{(n)}(x)=\begin{cases}\dfrac{(n-1)!}{\pi^p}\displaystyle\sum_{l=0}^{p-1}\frac{(-1)^l}{l!}\left[\frac{2\pi^p}{(p-1)!}\right]^{2l/n-1}x^{n-1-2l}\sum_{k=l}^{p-1}\frac{(-1)^k(p-k)!}{4^k(k-l)!(n-2k)!}\\ \qquad\text{if}\quad n=2p,\\ -1\quad\text{if}\quad n\text{ is odd}.\end{cases}$$

We have then shown

$$\int_M \Omega\, dV+\int_{\Gamma(0)}\Pi\, dA=-\chi(S^{n-1})-P_{(n)}(\alpha).$$

CASE III: *Flat cross-sections.* Suppose that M^n is an orthogonal tube region such that each of the submanifolds $u^n=$const. is flat. Then under the hypotheses of the theorem in Section 6,

$$\int_M \Omega\, dV+\int_{\Gamma(0)}\Pi\, dA=C\lim\frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}},$$

where C is a constant defined below.

Proof. For $l>0$, $\Psi_l=0$; thus

$$\begin{aligned}\Pi&=-\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\Psi_0\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(n-1)(n-3)\cdots(n-2k+1)}{2^k k!}\\ &=-\frac{\Gamma\left(\frac{n}{2}\right)(\sqrt{g_{nn}})^{n-1}}{2\pi^{n/2}\gamma}\det\Gamma^n_{ij}\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(n-1)(n-3)\cdots(n-2k+1)}{2^k k!}\\ &=\frac{(-1)^n\Gamma\left(\frac{n}{2}\right)\det\Gamma^j_{in}}{2\pi^{n/2}(\sqrt{g_{nn}})^{n-1}}\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(n-1)(n-3)\cdots(n-2k+1)}{2^k k!}.\end{aligned}$$

This vanishes for n odd (as is implied by the discussion above, Case I); for $n=2p$,

$$\Pi = \frac{(n-1)!\det\Gamma^j_{in}}{\pi^p(\sqrt{g_{nn}})^{n-1}} \sum_{k=0}^{p-1} \frac{(-1)^k(p-k)!}{4^k k!(n-2k)!}.$$

Now Lemma 1 asserts that if $\lim \mathscr{A}^n/(n\mathscr{V})^{n-1}$ exists, it must equal

$$\left(\frac{1}{n-1}\right)^{n-1} \lim \frac{\mathscr{A}^{2-n}}{(\sqrt{g_{nn}})^{n-1}} \left[\int \operatorname{tr}\Gamma^j_{in}\,dA\right]^{n-1}.$$

Therefore, if the eigen values of the $(n-1)\times(n-1)$ matrix $(\mathscr{A}^{1/n-1}\,\Gamma^j_{in}/\sqrt{g_{nn}}\;1)$ all converge to the same finite value, we will have

$$\lim \int \Pi\,dA = \frac{(n-1)!}{\pi^p} \sum_{k=0}^{p-1} \frac{(-1)^k(p-k)!}{4^k k!(n-2k)!} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}, \quad \overset{\text{def}}{=} C_{(n)} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

The details of the proof are omitted here, since the situation is identical to that treated in the theorem of Section 6. However, a special case may be of interest.

Consider the following manifold, whose cross-section is a torus:

$$M^n = \{(r_1(t)\cos\theta_1, r_1(t)\sin\theta_1, \ldots, r_{n-1}(t)\cos\theta_{n-1}, r_{n-1}(t)\sin\theta_{n-1}, t)\}.$$

Setting $u^i=\theta_i$ for $i=1,\ldots,n-1$ and $u^n=t$, we have, for $i,j<n$:

$$g_{ij} = \delta_{ij}r_i^2, \quad g_{in}=0, \quad g_{nn} = 1 + \sum_{i=1}^{n-1}(r_i')^2$$

$$\Gamma^j_{in} = \frac{r_i'}{r_i}\delta_{ij}, \quad \mathscr{A}(t) = (2\pi)^{n-1}\,r_1\cdots r_{n-1}$$

$$\int \Pi\,dA = \frac{(n-1)!(2\pi)^{n-1}}{\pi^p(\sqrt{g_{nn}})^{n-1}} r_1'\cdots r_{n-1}' \quad = C_{(n)}(2\pi)^{n-1}\beta_1\cdots\beta_{n-1},$$

where

$$\beta_i = \frac{r_i'}{\sqrt{g_{nn}}} = \frac{d}{ds}r_i.$$

$$\left(\frac{d}{ds}\mathscr{A}^{1/n-1}\right)^{n-1} = \left(\frac{2\pi}{n-1}\right)^{n-1} r_1\cdots r_{n-1}\left(\sum_{j=1}^{n-1}\frac{\beta_j}{r_j}\right)^{n-1}.$$

The question then is, under what conditions the following limits exist and are equal:

$$\lim\beta_1\cdots\beta_{n-1}, \quad \lim r_1\cdots r_{n-1}\left(\frac{1}{n-1}\sum_{j=1}^{n-1}\frac{\beta_j}{r_j}\right)^{n-1}. \tag{3.1}$$

If each β_i has a positive limit, then $r_i \to \infty$ for each i, and it follows by l'Hospital's rule that

$$\lim \beta_i \frac{r_j}{r_i} = \lim \beta_i \lim \frac{r_j}{r_i} = \lim \beta_i \lim \frac{\beta_j}{\beta_i} = \lim \beta_j .$$

Therefore the limits in (3.1) exist and are equal.

This is clearly not a necessary condition; for example, suppose $r_i = t^{\alpha_i}$. Somewhat more generally, suppose

$$r_i(s) = s^{\alpha_i} \exp\left\{\int_1^s \frac{\varepsilon_i(y)}{y}\, dy\right\}$$

where $\varepsilon_i(s) \to 0$ as $s \to \infty$. If $\exp\{\int_1^s 1/y \sum \varepsilon_i(y)\, dy\}$ converges to some finite, nonzero value, then:

(1) both limits in (3.1) are zero if $\sum \alpha_i < n-1$;

(2) the limits exist and are equal if $\sum \alpha_i = n-1$;

(3) the limits do not exist if $\sum \alpha_i > n-1$, unless one or more of the α_i vanish.

A more interesting case arises when one considers the above space with the same coordinates, but assigns the metric

$$g_{ij} = \delta_{ij}\, t^{2\alpha_i}, \qquad g_{in} = 0, \qquad g_{nn} = 1 .$$

Then if $\sum \alpha_i = n-1$, we have

$$\beta_1 \cdots \beta_{n-1} = \alpha_1 \cdots \alpha_{n-1},$$

$$r_1 \cdots r_{n-1} \left(\frac{1}{n-1} \sum_{j=1}^{n-1} \frac{\beta_j}{r_j}\right)^{n-1} = \left(\frac{1}{n-1} \sum_{i=1}^{n-1} \alpha_i\right)^{n-1} = 1,$$

and these are not in general equal. Now the sign of $C_{(n)}$ depends on n: $C_{(2)} = 1/2\pi$, $C_{(4)} = -1/4\pi^2$, $C_{(6)} = 3/8\pi^3, \ldots$. However, for $n > 2$ it is possible to choose the α_i so that

$$\alpha_1 \cdots \alpha_{n-1} < \left(\frac{1}{n-1} \sum_{i=1}^{n-1} \alpha_i\right)^{n-1};$$

and for $n > 3$, the opposite inequality can be obtained by choosing some of the α_i negative. Therefore, for $n \geqslant 4$ and even, the difference

$$\left[\int_M \Omega\, dV + \int_{\Gamma(0)} \Pi\, dA\right] - \chi(M)$$

may have either sign; therefore an extension of Cohn-Vossen's inequality is impossible without further conditions on the manifold.

CASE IV: *Flat manifolds.* Let M be a tube region of dimension $n=2p$, which is flat outside some compact set. Then

$$\int_M \Omega\, dV + \int_{\Gamma(0)} \Pi\, dA = -\frac{(p-1)!}{2\pi^p} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

We begin with a geometric characterization of the boundary curvature for this case.

LEMMA 2. *Let M^n be a flat manifold of dimension $n=2p$, and N a smooth, $(n-1)$-dimensional submanifold. Choose coordinates near N as in Section 1, and so that $g_{nn}\equiv 1$ near N. Then*

$$\int_N \Pi\, dA = \frac{(p-1)!}{2\pi^p (n-1)!} \frac{d^{n-1}\mathscr{A}}{(du^n)^{n-1}}.$$

Proof of the lemma. In the specified coordinates, we have

$$R^*_{ijkl} = R_{ijkl} + g_{nn}(\Gamma^n_{jk}\Gamma^n_{il} - \Gamma^n_{jl}\Gamma^n_{ik}) = \Gamma^n_{jk}\Gamma^n_{il} - \Gamma^n_{jl}\Gamma_{ik},$$

$$\Psi_l = (-1)^{n-1} 2^l \det \Gamma^i_{jn},$$

and

$$\Pi = \frac{(p-1)!}{2\pi^p} \det \Gamma^i_{jn}.$$

For $P\in N$, let P_r be the point whose coordinates are $u^i(P_r)=u^i(P)$, $i=1,\dots,n-1$, $u^n(P_r)=u^n(P)+r$; and let $N_r=\{P_r : P\in N\}$.

$$\mathscr{A}(N_r) = \int_N \sqrt{\gamma(P_r)}\, du^1 \cdots du^{n-1} = \int_N \frac{\sqrt{\gamma(P_r)}}{\sqrt{\gamma(P)}}\, dA.$$

Let $\xi^j_{(i)}(s)\,\partial/\partial u^j$ be the parallel translate of $\partial/\partial u^i(P)$ along the u^n-curve to P_s. Since parallel translation preserves inner products,

$$g_{ij}(P) = \frac{\partial}{\partial u^i}(P) \frac{\partial}{\partial u^j}(P) = \xi^k_{(i)}(r)\, \xi^l_{(j)}(r)\, g_{kl}(P_r),$$

whence

$$\frac{\sqrt{\gamma(P_r)}}{\sqrt{\gamma(P)}} = [\det \xi^k_{(i)}(r)]^{-1}, \qquad i, k = 1, \dots, n-1.$$

Now the equations of parallel transport are

$$\frac{d\xi^j_{(i)}}{du^n} = -\,\xi^k_{(i)}\Gamma^j_{kn},$$

and the hypothesis that M^n is flat gives the differential equation

$$\Gamma^i_{nk,\,n} = -\,\Gamma^j_{nk}\Gamma^i_{jn}.$$

Together these imply

$$\frac{d^m\xi^j_{(i)}}{(du^n)^m} = (-1)^m m!\,\xi^{k_1}_{(i)}\Gamma^{k_2}_{k_1n}\cdots\Gamma^j_{k_mn}$$

and, for r small enough to ensure convergence, the matrix equation

$$[\xi^j_{(i)}(r)] = \sum_{m=0}^{\infty}(-1)^m r^m[\Gamma^j_{in}(P)]^m = [I + r(\Gamma^j_{in}(P))]^{-1}.$$

Thus, for r sufficiently small,

$$\frac{\sqrt{\gamma(P_r)}}{\sqrt{\gamma(P)}} = \det[I + r\Gamma^i_{nj}(P)],$$

and

$$\mathscr{A}(N_r) = \int_N \det[I + r\Gamma^i_{nj}]\,dA;$$

finally,

$$\frac{d^{n-1}\mathscr{A}}{(du^n)^{n-1}} = (n-1)!\int_N \det\Gamma^i_{nj}\,dA \qquad = \frac{2\pi^p(n-1)!}{(p-1)!}\int_N \Pi\,dA.$$

This completes the proof of Lemma 2.

Now if M^n is an even-dimensional orthogonal tube region which is flat outside some compact set, Lemma 2 together with the compact Gauss-Bonnet formula implies that for u^n sufficiently large, $d^{n-1}\mathscr{A}/(du^n)^{n-1}$ is a constant; thus

$$\mathscr{A}(u^n) = a_1 + 2a_2u^n + \cdots + na\ \ (u^n)^{n-1}$$

and

$$\mathscr{V}(u^n) = a_0 + a_1u^n + \cdots + a_n(u^n)^n.$$

Therefore,

$$\lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}} = na_n = \frac{1}{(n-1)!}\frac{d^{n-1}\mathscr{A}}{(du^n)^{n-1}} = \frac{2\pi^p}{(p-1)!}\lim \int \Pi\, dA .$$

CASE V: *Similar cross-sections.* Let M be an orthogonal tube region with similar cross-sections. Then

$$\int_M \Omega\, dV + \int_{\Gamma(0)} \Pi\, dA = P\left(\lim \frac{\mathscr{A}^{n/n-1}}{n\mathscr{V}}\right),$$

where P is a polynomial whose coefficients depend on n and on the intrinsic geometry of the cross-section. (Note that spherical symmetry is a special case.)

Proof. Similarity of the cross-sections means that, for $i, j<n$,

$$g_{ij}(u^1,\dots,u^{n-1},u^n) = r^2(u^n)\, g_{ij}(u^1,\dots,u^{n-1},0).$$

Here we will assume $r\in C^2$ and $r(0)=1$. If $r\not\equiv 1$, then r' is somewhere nonzero; we may assume $r'(0)\neq 0$. Then

$$R^*_{ijkl}(u^1,\dots,u^{n-1},u^n) = r^2(u^n)\, R^*_{ijkl}(u^1,\dots,u^{n-1},0),$$

and

$$\begin{aligned}\Gamma^n_{ij}(u^1,\dots,u^{n-1},u^n) &= -\frac{rr'}{g_{nn}}\, g_{ij}(u^1,\dots,u^{n-1},0),\\ &= \frac{rr'}{r'(0)}\frac{g_{nn}(0)}{g_{nn}}\,\Gamma^n_{ij}(u^1,\dots,u^{n-1},0).\end{aligned}$$

It follows that

$$\Psi_l(u^1,\dots,u^{n-1},u^n) = r^{-n+1}\left(\frac{r'}{\sqrt{g_{nn}}}\bigg/\frac{r'(0)}{\sqrt{g_{nn}(0)}}\right)^{n-1-2l}\Psi_l(u^1,\dots,u^{n-1},0),$$

$$\int_{\Gamma(u^n)} \Psi_l\, dA = \left(\frac{r'}{\sqrt{g_{nn}}}\bigg/\frac{r'(0)}{\sqrt{g_{nn}(0)}}\right)^{n-1-2l}\int_{\Gamma(0)} \Psi_l\, dA .$$

Thus $\int \Pi\, dA$ is a polynomial in $r'/\sqrt{g_{nn}}=dr/ds$. But $\mathscr{A}(u^n)=r^{n-1}\mathscr{A}(0)$, so if $\lim d/ds\, \mathscr{A}^{1/n-1}$ exists, it follows that

$$\lim \frac{r'}{\sqrt{g_{nn}}} = \mathscr{A}^{-1/n-1}(0)\lim \frac{d}{ds}\mathscr{A}^{1/n-1} \qquad = \mathscr{A}^{-1/n-1}(0)\lim \frac{\mathscr{A}^{n/n-1}}{n\mathscr{V}},$$

so that $\lim \int \Pi\, dA$ is a polynomial of degree $p-1$ in $\lim \mathscr{A}^{n/n-1}/n\mathscr{V}$.

The coefficients of this polynomial depend intrinsically on the cross-section, as follows:

$$\lim \int \Psi_l \, dA = \left(\mathscr{A}^{-1/n-1}(0) \lim \frac{\mathscr{A}^{n/n-1}}{n\mathscr{V}}\right)^{n-1-2l} \left(\frac{\sqrt{g_{nn}(0)}}{r'(0)}\right)^{n-1-2l} \int\limits_{\Gamma(0)} \Psi_l \, dA,$$

$$\begin{aligned}
\left(\frac{\sqrt{g_{nn}(0)}}{r'(0)}\right)^{n-1-2l} & \Psi_l(u^1, \ldots, u^{n-1}, 0) \\
&= \frac{(-1)^l}{(n-1)!\,\gamma} \left(\frac{g_{nn}(0)}{r'(0)}\right)^{n-1-2l} \sum \varepsilon(i)\,\varepsilon(j)\, R^*_{i(1)i(2)j(1)j(2)} \\
&\qquad \cdots R^*_{i(2l-1)i(2l)j(2l-1)j(2l)}\, \Gamma^n_{i(2l+1)j(2l+1)} \cdots \Gamma^n_{i(n-1)j(n-1)} \\
&= \frac{(-1)^{n-1+l}}{(n-1)!\,\gamma} \sum \varepsilon(i)\,\varepsilon(j)\, R^*_{i(1)i(2)j(1)j(2)} \cdots R^*_{i(2l-1)i(2l)j(2l-1)j(2l)} \\
&\qquad g_{i(2l+1)j(2l+1)} \cdots g_{i(n-1)j(n-1)}, \quad \overset{\text{def}}{=} \mathscr{A}^{(n-1-2l)/(n-1)}(0)\, \Theta_l.
\end{aligned}$$

Thus

$$\lim \int \Psi_l \, dA = \lim \left(\frac{\mathscr{A}^{n/n-1}}{n\mathscr{V}}\right)^{n-1-2l} \int\limits_{\Gamma(0)} \Theta_l \, dA.$$

4. Relative Gauss-Kronecker Curvature

If M is a smooth, oriented hypersurface in $\mathbf{R}^n$, the Gauss-Kronecker curvature of M at a point P is found by comparing the volume element on M at P to the volume element induced on the unit $(n-1)$-sphere by the Gauss map $\varphi : M \to S^{n-1}$, where $\varphi(P)$ is a unit normal to M at P. An analogous definition can be made for oriented hypermanifolds of a general Riemannian manifold.

First consider a curve $\mathscr{C}$ in a two-dimensional manifold M. Let $P \in \mathscr{C}$ be a fixed point, and let s be (signed) arc length along $\mathscr{C}$ from P. For $Q \in \mathscr{C}$ near P let $N(Q)$ be the outer unit normal, that is, the normal to the right as s increases. In a sufficiently small neighborhood U of P there exist unique geodesic rays $L(Q)$ from P to points Q in $\mathscr{C} \cap U$, and we can define $\varphi(Q) \in T_P(M)$ to be the parallel transport of $N(Q)$ along $L(Q)$ to P. As s varies near 0, Q varies along $\mathscr{C}$ near P and $\varphi(Q)$ varies in the unit circle in $T_P(M)$ near $N(P)$. Let $\theta(s)$ be the angle from $N(P)$ to $\varphi(Q)$, where Q is the point a distance s from P along $\mathscr{C}$. Then the ratio of element of "area" (that is, length) on $\varphi(\mathscr{C})$ to that on $\mathscr{C}$ is $d\theta/ds(0)$.

Let $\alpha(s)$ be the angle from $L(Q)$ to $N(Q)$ and $\beta(s)$ the angle from $N(P)$ to $L(Q)$. Since parallel transport preserves angles, and $L(Q)$ is a geodesic, $\alpha(s)$ is also the angle from $L(Q)$ to $\varphi(Q)$; thus $\theta(s) = \alpha(s) + \beta(s)$.

Applying Liouville's formula with respect to polar coordinates about P, we have

$$\kappa_g(\mathscr{C}; Q) = \frac{d\alpha}{ds} + \kappa_g(\text{geodesic circle}) \cos\alpha$$

$$= \frac{d\alpha}{ds} + \left[\frac{1}{s} + s(\cdots)\right]\left[s\frac{d\alpha}{ds}(0) + s^2(\cdots)\right];$$

$$\kappa_g(\mathscr{C}; P) = 2\frac{d\alpha}{ds}(0).$$

Next consider normal coordinates (u^1, u^2) near P, related to geodesic polar coordinates by the equations $u^1 = r\sin\beta$, $u^2 = -r\cos\beta$, where r is distance from P and β the angle from $N(P)$. Now

$$\kappa_g(\mathscr{C}; P) = -N(P)\cdot\vec{\kappa}_g(\mathscr{C}; P) = \frac{\partial}{\partial u^2}\nabla_{\partial/\partial u^1}\left(\frac{du^1}{ds}\frac{\partial}{\partial u^1} + \frac{du^2}{ds}\frac{\partial}{\partial u^2}\right)$$

$$= \frac{d^2u^2}{ds^2}(0) = \frac{d^2}{ds^2}(-s\cos\beta(s))(0) = 2\frac{d\beta}{ds}(0).$$

Thus $d\theta/ds(0) = d\alpha/ds(0) + d\beta/ds(0) = \kappa_g(\mathscr{C}; P)$; that is, the geodesic curvature of a curve in a two-dimensional manifold is geometrically characterized as a relative Gauss-Kronecker curvature.

The Gauss-Kronecker curvature of a hypersurface is the product of its principal curvatures, that is, of the extreme values of the normal curvature of a curve on the hypersurface passing through a given point. The principal curvatures of a submanifold of a general Riemannian manifold can be defined in the same fashion ([5]), and the relative Gauss-Kronecker curvature is again the product of the principal curvatures.

Let M be an n-dimensional Riemannian manifold, and Γ a (piece of a) regular, smooth submanifold of codimension 1. Fix $P\in\Gamma$, and let U be a neighborhood of P admitting geodesic spherical coordinates about P. For $Q\in U\cap\Gamma$, let $x(Q)\in T_Q(M)$ be a unit normal to Γ, the choice of orientation being made consistently throughout the neighborhood; and let $\bar{x}(Q)\in T_P(M)$ be the result of translating $x(Q)$ parallelly along the geodesic ray PQ. Let $\mathscr{A}$ designate $(n-1)$-dimensional area, and define the relative Gauss-Kronecker curvature of Γ at P:

$$K(\Gamma; P) = \lim \frac{\mathscr{A}(\bar{x}(U\cap\Gamma))}{\mathscr{A}(U\cap\Gamma)};$$

that is, given $\varepsilon > 0$, there is some neighborhood V of P such that for any subneighborhood $U\subseteq V$, the above ratio of areas is within ε of $K(\Gamma; P)$. Clearly this definition can be extended to include polyhedral submanifolds.

LEMMA 3. *$K(\Gamma; P)$ is the product of the principal curvatures of Γ at P.*

Proof. Choose coordinates in U so that $U \cap \Gamma = \{u^n = 0\}$; $g_{in} \equiv 0$ for $i < n$, $g_{nn} \equiv 1$, and on $U \cap \Gamma$, $\partial/\partial u^n = x$. Define

$$L_{ij} = \frac{\partial \bar{x}}{\partial u^i} \cdot \frac{\partial \bar{x}}{\partial u^j}, \qquad i, j < n.$$

Clearly $K(\Gamma; P) = 1\sqrt{\det L_{ij}(P)}/\sqrt{g}$. Since $\bar{x}^n(P) = 1$ is an absolute maximum, $\partial \bar{x}^n/\partial u^i(P) = 0$; thus

$$L_{kl}(P) = \sum_{i,j=1}^{n-1} g_{ij} \frac{\partial \bar{x}^i}{\partial u^k} \frac{\partial \bar{x}^j}{\partial u^l}$$

and

$$K(\Gamma; P) = \det \frac{\partial \bar{x}^i}{\partial u^j}(P).$$

For a moment fix $Q \in \Gamma$, and define the vector function $\sum_{i=1}^n f^i(\varrho)\, \partial/\partial u^i$ which is obtained by transporting $x(Q)$ along PQ to P; ϱ here is the distance from P. The equations of parallel transport imply

$$\frac{df^i}{d\varrho} = -f^k \frac{du^l}{d\varrho} \Gamma^i_{kl}.$$

Let $r = \varrho(Q)$, the distance from P to Q. Then

$$f^i(0) = f^i(r) - r\dot{f}^i(r) + \frac{r^2}{2}\ddot{f}^i(r) - r^3(\cdots), \qquad = \delta^i_n + r\frac{du^k}{d\varrho}\Gamma^i_{nk} + r^2(\ldots).$$

Therefore, as Q varies,

$$\frac{\partial \bar{x}^i}{\partial u^j} = \frac{\partial f^i(0, Q)}{\partial u^j(Q)} = \frac{\partial r}{\partial u^j}\frac{du^k}{d\varrho}\Gamma^i_{nk} + r(\cdots)$$

which tends to $\Gamma^i_{nj}(P)$ as $Q \to P$. Therefore

$$K(\Gamma; P) = \det \Gamma^i_{nj}(P) \qquad (i, j = 1, \ldots, n-1),$$

which is the product of principal curvatures of Γ at P (see [5], Chapter IV, sections 44 and 45.)

This gives a geometric characterization of the quantity previously denoted by Ψ_0, which is a generalization of geodesic curvature of a curve in a two-dimensional manifold.

5. The Spaces $\mathscr{L}^p(M^{n-1})$

Since M^{n-1} is compact, it can be written as the union of finitely many compact sets E_i, each homeomorphic, by a map $\varphi_i: E_i \to K_i$, to a compact subset of $\mathbf{R}^{n-1}$. We say that a function f on M^{n-1} is in $\mathscr{L}^p(M^{n-1})$ if $f \circ \varphi_i^{-1} \in \mathscr{L}^p(K_i)$ for each i. This notion is invariant under admissible changes of coordinates, which are continuously differentiable and thus have Jacobians in $\mathscr{L}^\infty(K_i) \subseteq \mathscr{L}^p(K_i)$. Convergence in $\mathscr{L}^p(M^{n-1})$ is similarly defined, and also invariant under change of coordinates. The statement "$f_n \to f$ a.e." will mean that $f_n \circ \varphi_i^{-1} \to f \circ \varphi_i^{-1}$ a.e. with respect to Lebesgue measure in each K_i. For quantities like γ which depend on coordinates, the above concepts are defined using the coordinate maps φ_i with respect to which the quantities are defined.

The following facts follow directly from the above definitions and well-known facts of real analysis.

1. If $f_j \in \mathscr{L}^{kq}$ for $j = 1, \ldots, k$, then $f_1 f_2 \cdots f_k \in \mathscr{L}^q$ $(q \geqslant 1)$.
2. If $f_n \to f$ in $\mathscr{L}^q$, then for each s such that $1 \leqslant s \leqslant q$, $f_n \to f$ in $\mathscr{L}^s$.
3. For p and q conjugate $(1/p + 1/q = 1)$, if $f_n \to f$ in $\mathscr{L}^q$ and $g_n \to g$ in $\mathscr{L}^p$, then $f_n g_n \to fg$ in $\mathscr{L}^1$.
4. If $p \geqslant 1$ and $k \geqslant 1/p$, then if $f_n \to f$ in $\mathscr{L}^{kp}$, it follows that $|f_n|^k \to |f|^k$ in $\mathscr{L}^p$.
5. Let $q \geqslant 1$. If $f_n^{(i)} \to f^{(i)}$ in $\mathscr{L}^{kq}$ for $i = 1, \ldots, k$, then $f_n^{(1)} \ldots f_n^{(k)} \to f^{(1)} \ldots f^{(k)}$ in $\mathscr{L}^q$.

6. Asymptotic Behavior of $\int \Psi_0 \, dA$

In this section a geometric interpretation is given for the asymptotic behavior of the relative Gauss-Kronecker curvature, analogous to the formula of Finn [6]. While the conditions in the n-dimensional case are more restrictive than in the two-dimensional case, they are not unreasonable, as is demonstrated in Section 7 by the fact that most of the manifolds of Section 3 satisfy the hypotheses of the theorem in this section. It seems possible that these or similar hypotheses may figure in an attempt to construct, on a given tube region, a semi-geodesic net, with respect to which the tube region will be orthogonal.

THEOREM: *Let M be a complete, orthogonal tube region, with cross-section Γ having principal curvatures λ_i. Suppose that $\sqrt{\gamma}/\mathscr{A} \to G$ a.e. and in $\mathscr{L}^p$, and that $\mathscr{A}^{1/n-1}\lambda_i \to l_i$ a.e. and in $\mathscr{L}^{(n-1)q}$ for each i, $1/p + 1/q = 1$. Then both* $\lim \int \Psi_0 \, dA$ *and* $\lim \mathscr{A}^n/(n\mathscr{V})^{n-1}$ *exist, and $\sum l_i$ is a constant a.e. with respect to the limit measure $G\,dx$ (that is, the hypermanifolds Γ are asymptotically submanifolds of constant mean curvature). Furthermore:*

I) *If for each i, $l_i = l$ a.e. with respect to the limit measure (that is, the hypermani-*

folds Γ are asymptotically umbilic), then

$$\lim \int \Psi_0 \, dA = (-1)^{n-1} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}};$$

II) *If $n=2, 3$, or if for each i, $l_i \geq 0$ a.e. with respect to the limit measure, then*

$$(-1)^{n-1} \lim \int \Psi_0 \, dA \leq \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}},$$

with equality iff $l_i = l$ a.e. for each i (and so always for $n=2$).

Proof.

$$\frac{d}{ds} \mathscr{A}^{1/n-1} = \frac{1}{n-1} \mathscr{A}^{2-n/n-1} \int \operatorname{tr} \Gamma^i_{jn} \, dA = \frac{1}{n-1} \int (\sum \mathscr{A}^{1/n-1} \lambda_i) \frac{\sqrt{\gamma}}{\mathscr{A}} \, dx.$$

Since $\mathscr{A}^{1/n-1}\lambda_i \to l_i$ in $\mathscr{L}^{(n-1)q}$ and thus also in $\mathscr{L}^q$, and $\sqrt{\gamma}/\mathscr{A} \to G$ in $\mathscr{L}^p$, it follows that $(\sum \mathscr{A}^{1/n-1}\lambda_i)\sqrt{\gamma}/\mathscr{A} \to (\sum l_i)\, G$ in $\mathscr{L}^1$, and thus

$$\lim \frac{d}{ds} \mathscr{A}^{1/n-1} = \frac{1}{n-1} \int (\sum l_i) \, G \, dx. \tag{6.1}$$

By Lemma 1,

$$\lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}} = \left[\frac{1}{n-1} \int (\sum l_i) \, G \, dx \right]^{n-1}.$$

Similarly,

$$(-1)^{n-1} \int \Psi_0 \, dA = \int \det \Gamma^i_{jn} \, dA = \int (\Pi \mathscr{A}^{1/n-1} \lambda_i) \frac{\sqrt{\gamma}}{\mathscr{A}} \, dx,$$

thus

$$(-1)^{n-1} \lim \int \Psi_0 \, dA = \int l_1 \cdots l_{n-1} \, G \, dx. \tag{6.2}$$

Note that

$$\frac{d}{ds} \gamma^{1/2(n-1)} = \frac{1}{n-1} \gamma^{1/2(n-1)} \sum \lambda_i = \frac{1}{n-1} \left(\frac{\sqrt{\gamma}}{\mathscr{A}} \right)^{1/n-1} (\sum \mathscr{A}^{1/n-1} \lambda_i)$$

converges in $\mathscr{L}^{n-1}$ to $G^{1/n-1}(\sum l_i)/n-1$. It follows that $\sum l_i$ is nonnegative a.e. with respect to the limit measure $G\,dx$. For suppose there were a set $B \subseteq K_j$ of positive Lebesgue measure where $G>0$ and $\sum l_i < 0$. Then

$$0 > \int_B G^{1/n-1} (\sum l_i) \, dx = \lim \int_B \left(\frac{\sqrt{\gamma}}{\mathscr{A}} \right)^{1/n-1} (\sum \mathscr{A}^{1/n-1} \lambda_i) \, dx$$

$$= (n-1) \lim \int_B \frac{d}{ds} \gamma^{1/2(n-1)} \, dx = (n-1) \lim \frac{d}{ds} \int_B \gamma^{1/2(n-1)} \, dx.$$

Since $\int_B \gamma^{1/2(n-1)}\,dx > 0$ for all s, its derivative cannot tend to a negative limit as $s \to \infty$.

Now if $d/ds\, \mathscr{A}^{1/n-1} \to 0$, then by (6.1), together with the fact that $\sum l_i \geqslant 0$ a.e. with respect to $G\,dx$, it follows that $\sum l_i = 0$ a.e. with respect to $G\,dx$. Now suppose $\lim d/ds\, \mathscr{A}^{1/n-1} > 0$; then $\mathscr{A} \to \infty$. Let $G > 0$ on B_1, $\sqrt{\gamma}/\mathscr{A} \to G$ on B_2 and $\sum \mathscr{A}^{1/n-1}\lambda_i \to \sum l_i$ on B_3. Then $d/ds\, \gamma^{1/2(n-1)} \to G^{1/n-1} \sum l_i/n-1$ on $B_2 \cap B_3$. For $x \in B_1 \cap B_2$, $\sqrt{\gamma(x,s)}/\mathscr{A}(s) \to G(x) > 0$ implies $\sqrt{\gamma(x,s)} \to \infty$ as $s \to \infty$; that is, $\sqrt{\gamma} \to \infty$ a.e. with respect to $G\,dx$. Furthermore, for $x \in B_1 \cap B_2 \cap B_3$,

$$0 < G(x) = \lim \frac{\sqrt{\gamma}}{\mathscr{A}} = \left[\frac{\lim \dfrac{d}{ds}\gamma^{1/2(n-1)}}{\lim \dfrac{d}{ds}\mathscr{A}^{1/n-1}}\right]^{n-1},$$

$$= \left(\frac{1}{n-1}\right)^{n-1} G(x) \left[\frac{\sum l_i}{\lim \dfrac{d}{ds}\mathscr{A}^{1/n-1}}\right]^{n-1},$$

whence

$$\sum l_i = (n-1) \lim \frac{d}{ds}\mathscr{A}^{1/n-1} \quad \text{on} \quad B_1 \cap B_2 \cap B_3,$$

that is, a.e. with respect to $G\,dx$. Therefore $\sum l_i$ is a constant a.e. with respect to $G\,dx$.

Now if for each i, $l_i = l$ a.e. with respect to the limit measure,

$$(-1)^{n-1} \lim \int \Psi_0\, dA = \int l_1 \cdots l_{n-1}\, G\,dx = l^{n-1} \int G\,dx = l^{n-1},$$

whereas

$$\lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}} = \left[\frac{1}{n-1}\int (n-1)\, lG\,dx\right]^{n-1} = l^{n-1}.$$

For $n = 2, 3$, or if $l_i \geqslant 0$ for each i, we have the inequality

$$l_1 \cdots l_{n-1} \leqslant \left(\frac{1}{n-1}\sum_{i=1}^{n-1} l_i\right)^{n-1}$$

with equality iff $l_i = l$; this yields (II) directly, since $\sum l_i$ is a constant:

$$\lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}} = \left[\frac{1}{n-1}\int \left(\sum l_i\right) G\,dx\right]^{n-1} = \left(\frac{1}{n-1}\sum l_i\right)^{n-1}$$

$$(-1)^{n-1} \lim \int \Psi_0\, dA = \int l_1 \cdots l_{n-1}\, G\,dx \leqslant \left(\frac{1}{n-1}\sum l_i\right)^{n-1} = \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

7. Examples

1. Spherical symmetry (see Section 3, case II). Since

$$\gamma = r^{2n-2} \det\left(I + \frac{u^i u^j}{y^2}\right)$$

and

$$\mathscr{A} = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1},$$

$$\frac{\sqrt{\gamma}}{\mathscr{A}} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \left[\det\left(I + \frac{u^i u^j}{y^2}\right)\right]^{1/2}$$

is independent of u^n and therefore converges trivially. Furthermore, since $\Gamma^j_{in} = r'/r\ \delta_{ij}$, it follows that $\lambda_i = r'/r\sqrt{g_{nn}}$ for each i, and

$$\mathscr{A}^{1/n-1}\lambda_i = \frac{r'}{\sqrt{g_{nn}}}\left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)^{1/n-1}.$$

Since this quantity is independent of $u^1, \ldots, u^{n-1}$, its convergence depends only on the convergence of $r'/\sqrt{g_{nn}} = dr/ds$. If dr/ds converges, then

$$(-1)^{n-1} \lim \int \Psi_0\, dA = \frac{2\pi^{n/2}}{\Gamma(n/2)} \lim \left(\frac{r'}{\sqrt{g_{nn}}}\right)^{n-1} = \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

2. Similar cross-sections (see Section 3, Case V). Again $\sqrt{\gamma}/\mathscr{A}$ is independent of u^n and $\Gamma^j_{in} = r'/r\ \delta_{ij}$, whence $\mathscr{A}^{1/n-1}\lambda_i = r'\ \mathscr{A}^{1/n-1}(0)/\sqrt{g_{nn}}$. Thus if dr/ds converges,

$$\lim \int \Psi_0\, dA = \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}} \int_{\Gamma(0)} \Theta_0\, dA;$$

but $\Theta_0 = (-1)^{n-1}/\mathscr{A}(0)$, and thus

$$\lim \int \Psi_0\, dA = (-1)^{n-1} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

3. Flat tube regions (see Section 3, Case IV). The equation

$$\Gamma^i_{nk,n} = -\Gamma^j_{nk}\Gamma^i_{jn}$$

implies the matrix equation

$$[\Gamma^i_{nk}(s)] = [\Gamma^i_{nj}(0)]\,[I + s\Gamma^j_{nk}(0)]^{-1};$$

thus the eigen-vectors of $[\Gamma^i_{nk}(s)]$ are the same as the eigen-vectors of $[\Gamma^i_{nk}(0)]$, and the eigen-values (that is, the principal curvatures) are related by the equation

$$\lambda_i(s) = \frac{\lambda_i(0)}{1 + s\lambda_i(0)},$$

which tends to zero as $s \to \infty$. However, $\mathscr{A} \to \infty$ in general (unless $\lambda_i(0)=0$ for each i), and thus

$$\lim \mathscr{A}^{1/n-1}\lambda_i = \lim \frac{\mathscr{A}^{1/n-1}}{(1+s\lambda_i(0))/\lambda_i(0)} = \lim \frac{d}{ds}\mathscr{A}^{1/n-1}$$

(applying l'Hospital's rule). Thus, if $\lim d/ds\, \mathscr{A}^{1/n-1}$ exists, it follows that $\lim \mathscr{A}^{1/n-1}\lambda_i$ exists and is independent of i and of $u^1, \ldots, u^{n-1}$; then

$$\lim \int \Psi_0\, dA = (-1)^{n-1} \lim \frac{\mathscr{A}^n}{(n\mathscr{V})^{n-1}}.$$

But the discussion in Section 3 implies that $d/ds\, \mathscr{A}^{1/n-1}$ must converge: for s sufficiently large,

$$\mathscr{A}(s) = b_0 + b_1 s + \cdots + b_{n-1}s^{n-1},$$

$$b_{n-1} = \frac{2\pi^p}{(p-1)!} \int\limits_{\Gamma(s)} \Pi\, dA \quad (= \text{const.}).$$

Thus

$$\lim \frac{d}{ds}\mathscr{A}^{1/n-1} = (b_{n-1})^{1/n-1}.$$

4. A manifold whose cross-section is a torus (see Section 3, Case III). $\sqrt{\gamma}/\mathscr{A} = (1/2\pi)^{n-1}$. Since $\Gamma^j_{in} = r_i'/r_i\, \delta_{ij}$, it follows that the principal curvatures

$$\lambda_i = \frac{1}{r_i}\frac{r_i'}{\sqrt{g_{nn}}} \quad \text{and} \quad \mathscr{A}^{1/n-1}\lambda_i = 2\pi \frac{1}{r_i}(r_1 \cdots r_{n-1})^{1/n-1}\frac{dr_i}{ds}.$$

A sufficient, but not necessary, condition for convergence of $\mathscr{A}^{1/n-1}\lambda_i$ is $\lim dr_i/ds > 0$; that condition yields

$$\lim \mathscr{A}^{1/n-1}\lambda_i = 2\pi \left(\frac{dr_1}{ds} \cdots \frac{dr_{n-1}}{ds}\right)^{1/n-1},$$

independent of i and of $u^1, \ldots, u^{n-1}$.

Other examples can be obtained by setting

$$r_j(s) = s^{\alpha_j} \exp \int\limits_1^s \frac{\varepsilon_j(y)}{y}\, dy$$

where $\varepsilon_j(y) \to 0$ as $y \to \infty$. In this case

$$\mathscr{A} = (2\pi)^{n-1} s^{\Sigma \alpha_j} \exp \int_1^s \frac{\sum \varepsilon_j(y)}{y} dy, \qquad \frac{\dot{r}_j}{r_j} = \frac{\alpha_j + \varepsilon_j(s)}{s}$$

and thus

$$\mathscr{A}^{1/n-1} \lambda_j = 2\pi s^{\bar{\alpha}-1} (\alpha_j + \varepsilon_j(s)) \exp \int_1^s \frac{\bar{\varepsilon}(y)}{y} dy,$$

where $\bar{x} = 1/n-1 \sum x_i$, the arithmetic mean. Now if $\sum \alpha_i < n-1$, then $\mathscr{A}^{1/n-1}\lambda_j \to 0$ for each j, while if $\sum \alpha_i > n-1$, $\mathscr{A}^{1/n-1}\lambda_j$ diverges unless $\alpha_j = 0$ and ε_j, $\bar{\varepsilon} \to 0$ rapidly enough. If $\sum \alpha_i = n-1$, convergence depends on the behavior of $\sum \varepsilon_i$; if this sum tends to zero quickly enough so that $\exp \int_1^s \bar{\varepsilon}(y)/y\, dy$ converges, then $\lim \mathscr{A}^{1/n-1}\lambda_j$ will generally depend on j, unless $\alpha_j = 1$ for each j.

The situation is quite different if not all the r_i vary regularly. For example, let $n=3$, $r_1 = b + \sin s$ $(b > 1)$, and $r_2 = s^{2\alpha}$. Then $\mathscr{A}^{1/2}\lambda_1 = 2\pi s^\alpha \cos s\, (b + \sin s)^{-1/2}$, which tends to zero if $\alpha < 0$, oscillates between finite values if $\alpha = 0$, and oscillates without bound if $\alpha > 0$. Similarly, $\mathscr{A}^{1/2}\lambda_2 = 4\pi\alpha s^{\alpha-1} \sqrt{b + \sin s}$ converges or oscillates depending on whether $\alpha < 1$, $\alpha = 1$ or $\alpha > 1$. Thus $\mathscr{A}^{1/2}\lambda_1$, $\mathscr{A}^{1/2}\lambda_2$ can both converge only if $\alpha < 0$, in which case both limits, and indeed the limit of $\mathscr{A}$, are zero.

REFERENCES

[1] C. B. ALLENDOERFER and A. WEIL, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc. *53*, 101–129 (1943).

[2] S. S. CHERN, *On the curvatura integra in a Riemannian manifold*, Ann. Math. *46*, 674–684 (1945).

[3] ——, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. Math. *45*, 747–752 (1944).

[4] ——, *Topics in Differential Geometry*, Mimeographed notes, Institute for Advanced Studies, Princeton, 1951.

[5] L. P. EISENHART, *Riemannian Geometry*, Princeton University Press, 1949.

[6] R. FINN, *On a class of conformal metrics, with application to differential geometry in the large*, Comment. Math. Helv. *40*, 1–30 (1965).

[7] A. HUBER, *Métriques conformes complètes et singularités isolées de fonctions sous-harmoniques*, Comptes Rendus, Acad. des Sci., Paris, *260*, 6267–6268 (1965).

[8] ——, *On the isoperimetric inequality on surfaces of variable Gaussian curvature*, Ann. Math. *60*, 237–247 (1954).

Lowell State College,

Lowell, Massachusetts

Received June 16, 1970