Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 46 (1971)

Artikel: Nonlinear Functional Equations and Eigenvalue Problems in Nonseperable Banach Spaces

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DOI: https://doi.org/10.5169/seals-35524

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Nonlinear Functional Equations and Eigenvalue Problems in Nonseparable Banach Spaces[1])

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1. Let X be a real reflexive Banach space and A, B nonlinear mappings of X into the conjugate space X^*, with A of monotone type and B compact. In the last years, much interest in nonlinear functional analysis has been concentrated on the problem of determining useful conditions under which the functional equation

$$Au = 0 \tag{1}$$

or the eigenvalue problem

$$Au = tBu \quad \text{for some real } t \tag{2}$$

admit solutions (which possibly satisfy additional restrictions).

For A satisfying certain asymptotic conditions (such as A coercive or A^{-1} bounded), various results on the solvability of equation (1) have been obtained (e.g. Brézis [3], Browder [4, 6, 8, 9], Browder-Hess [13], Leray-Lions [22], Minty [23]). There is an alternative type of hypothesis one may impose on the mapping A in order to get existence theorems for equation (1), namely the hypothesis that $A = A_0$ is homotopic to a mapping A_1 which commutes with a group $\mathscr{G}$ of transformations on the spaces X and X^*, with $\mathscr{G}$ having elements of finite order (in particular A_1 odd). Under the assumption that X is *separable*, several mathematicians have derived existence theorems involving homotopy arguments, making use of an approximation method of Galerkin type (e.g. Browder [8, 9, 10, 11], Browder-Petryshyn [14]). (For a completely different approach see Hess [19]). Though most of the concrete reflexive Banach spaces occurring in applications are separable, it is necessary for the investigation of certain specific problems to have a similar approach in nonseparable spaces. For that reason, Nečas [24] has recently given a method which works in nonseparable spaces, and which is extended in the writer's papers [17, 18].

One way of attacking the eigenvalue problem (2), is by variational methods (e.g. Browder [5], Hess [16], Krasnoselskii [21], Vainberg [26]). In [7, 8], Browder has developed a theory for nonlinear eigenvalue problems in *separable* spaces based on Galerkin approximations. This latter approach has the advantage that it does not involve the theory of infinite-dimensional manifolds (Lusternik's principle), and that it permits to prove the existence of an infinite number of distinct normalized eigenfunctions (Lusternik-Schnirelman theory) under milder differentiability hypotheses.

[1]) Research supported by NSF grant GP-23563 and by the Schweizerischer Nationalfonds.

It is our purpose in the present note to describe an easy argument of Galerkin approximation type which allows to prove both existence theorems and results on eigenvalue problems in *nonseparable* Banach spaces. In contrary to the Galerkin approximation method in separable spaces, which is based on an a priori given injective approximation scheme, our method consists in recursively constructing a suitable scheme. The main result is the Proposition proved in Section 2. In Section 3 we apply the conclusions of the Proposition to the functional equation (1), assuming that $A=A_0$ is homotopic to an odd mapping A_1. The result is closely related to that of Nečas [24], but it seems that our proof is simpler. A brief discussion follows of how our theory can be used in order to study nonlinear equations of Hammerstein type in nonreflexive Banach spaces. In Section 4 we finally show the applicability of the Proposition to the treatment of nonlinear eigenvalue problems in nonseparable spaces.

2. For X a real Banach space and X^* its conjugate space, we let (w, u) denote the duality pairing between elements $w \in X^*$ and $u \in X$. An operator A defined on a closed set $C \subset X$, with range contained in X^*, is said to be of *type (S)* if it satisfies the condition: *for any sequence* $\{u_n\} \subset C$ *converging weakly to some* $u \in X$, *for which* $\lim(Au_n, u_n - u) = 0$, *its strong convergence follows.* Mappings of type (S) have been introduced by Browder [7] and have shown to form a very useful class of operators of monotone type for homotopy considerations and eigenvalue problems. The mapping A is further *bounded* if it maps bounded sets onto bounded sets. Let Λ be the set of all finite-dimensional subspaces of X, ordered by inclusion. For $F \in \Lambda$, j_F denotes the injection mapping of F into X. If the operator A maps $C \subset X$ into X^*, the *Galerkin approximant* $A_F: C \cap F \to F^*$ is defined by $A_F = j_F^* A j_F$. In the following we use the symbols "$\to$" and "$\rightharpoonup$" to denote strong and weak convergence, respectively.

PROPOSITION. *Let X a real reflexive Banach space, C a closed subset of X, I a closed interval in R^1, and $A(u, t)$ a mapping of $C \times I$ into X^* with the following properties:*

(i) *For fixed t, $A(u, t): C \to X^*$, is bounded, continuous, and of type (S);*

(ii) *$A(u, t)$ is uniformly continuous in t with respect to u in bounded subsets of C.*

Let $\{E_n\}_{n=1}^{\infty}$ be a given increasing sequence in Λ with $C \cap E_1 \neq \emptyset$. Suppose to each $F \in \Lambda$ with $F \supset E_1$ there exist elements $u_F \in C \cap F$ and $t_F \in I$ such that $j_F^* A(u_F, t_F) = 0$, and assume said elements are uniformly bounded for $F \supset E_1$.

Then $A(u_0, t_0) = 0$ for some $u_0 \in C$ and $t_0 \in I$. Moreover, there exists an increasing sequence $\{F_n\}$ in Λ with $F_n \supset E_n$ for each n, such that for some subsequence $\{n(k)\}$ of $\{n\}$, $u_{F_{n(k)}} \to u_0$ and $t_{F_{n(k)}} \to t_0$.

Proof. We construct the asserted sequence $\{F_n\}$ in Λ as follows:

(a) We set $F_1 = E_1$.

(b) Suppose we have already constructed $F_1 \subset \cdots \subset F_n$, and let $u_n = u_{F_n} \in C \cap F_n$ and $t_n = t_{F_n} \in I$ denote the described elements corresponding to F_n such that $j^*_{F_n} A(u_n, t_n) = 0$. There exists $v_n \in X$, $\|v_n\| = 1$, such that $|(A(u_n, t_n), v_n)| \geqq \frac{1}{2}\|A(u_n, t_n)\|$. We then choose $F_{n+1} \supset F_n + E_{n+1} + [v_n]$.

By hypothesis, the sequences $\{u_n\}$ and $\{t_n\}$ are bounded. We may pass to infinite subsequences and assure that $u_n \rightharpoonup u_0 \in X$ and $t_n \to t_0 \in I$. It follows from condition (ii) that

$$\|A(u_n, t_n) - A(u_n, t_0)\| \to 0 \quad (n \to \infty). \tag{3}$$

We assert that

$$(A(u_n, t_0), w) \to 0 \quad (n \to \infty) \tag{4}$$

for all $w \in X_0 = \text{closure}\,\{\bigcup_{j=1}^{\infty} F_j\}$. Indeed, if w lies in some F_j and $n \geqq j$, we have

$$|(A(u_n, t_0), w)| \leqq |(A(u_n, t_n), w)| + |(A(u_n, t_0) - A(u_n, t_n), w)|,$$

where the first term on the right side vanishes, while the second term tends to 0 as $n \to \infty$ according to (3). Because of the boundedness of the sequence $\{A(u_n, t_0)\}$, (4) extends to all $w \in X_0$. We now get

$$\begin{aligned} |(A(u_n, t_0), u_n - u_0)| &\leqq |(A(u_n, t_n), u_n)| \\ &\quad + |(A(u_n, t_0) - A(u_n, t_n), u_n)| + |(A(u_n, t_0), u_0)|. \end{aligned}$$

On the right side of this estimate, the first summand vanishes, the middle term tends to 0 because of (3), and the last approaches 0 according to (4), since the weak limit u_0 of the sequence $\{u_n\} \subset X_0$ lies in X_0. Property (S) of the mapping $A(u, t_0)$ implies that $u_n \to u_0$. Hence $u_0 \in C$, $A(u_n, t_0) \to A(u_0, t_0)$, and

$$A(u_n, t_n) \to A(u_0, t_0) \tag{5}$$

because of the continuity of the mapping $A(u, t_0)$ in u and the estimate (3). We infer that, according to (4),

$$(A(u_0, t_0), w) = 0 \quad \text{for all} \quad w \in X_0. \tag{6}$$

We finally prove that $A(u_0, t_0) = 0$. Suppose to the contrary that $A(u_0, t_0) \neq 0$. Then, by (5), $\|A(u_n, t_n)\| \geqq d > 0$ for some constant d and all $n \geqq n_0$, which implies that

$$|(A(u_n, t_n), v_n)| \geqq d/2 > 0$$

for $n \geqq n_0$. But (5) and the fact that some subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) converges weakly to an element $v_0 \in X_0$ have as a consequence that

$$(A(u_n, t_n), v_n) \to (A(u_0, t_0), v_0),$$

the expression on the right being 0 according to (6). This contradiction shows that $A(u_0, t_0)=0$, q.e.d.

3. We apply the Proposition in order to obtain results on the existence of solutions of the functional equation (1).

THEOREM 1[2]). *Let X a real reflexive Banach space, G an open bounded subset of X containing* 0 *and symmetric about the origin, and $A_t u = A(u, t)$ a mapping of* $\mathrm{cl}(G) \times \times [0, 1]$ into X^* *as follows:*

(i) *For fixed t, A_t is a bounded continuous mapping of type (S);*

(ii) *$A(u, t)$ is uniformly continuous in t with respect to $u \in \mathrm{cl}(G)$;*

(iii) *A_1 is odd on* $\mathrm{bdry}(G)$, *i.e. $A(-u, 1) = -A(u, 1)$ for $u \in \mathrm{bdry}(G)$.*

Assume that $A(u, t) \neq 0$ for all $u \in \mathrm{bdry}(G)$ and all $t \in [0, 1]$. Then the equation $A_0 u = 0$ has a solution u_0 in G.

Theorem 1 follows by the classical Borsuk theorem [2, 15, 21], the invariance of the Brouwer degree under homotopies, and arguments which have become standard in the theory of mappings of monotone type (e.g. [3, 4, 6, 8, 9, 13, 17, 18, 22, 23]) from

LEMMA 1. *Let $E \in \Lambda$ be given. Then under the assumptions of Theorem 1 there exists $F \in \Lambda$, $F \supset E$, such that $j_F^* A(u, t) \neq 0$ for all $u \in \mathrm{bdry}(G) \cap F$ and all $t \in [0, 1]$.*

Proof of Lemma 1. Suppose to each $F \in \Lambda$ with $F \supset E$ we can find elements $u_F \in \mathrm{bdry}(G) \cap F$ and $t_F \in [0, 1]$ such that $j_F^* A(u_F, t_F) = 0$. Applying the Proposition with $C = \mathrm{bdry}(G)$ and $I = [0, 1]$, we are led to a contradiction to the assumptions of Theorem 1, q.e.d.

DEFINITION. *A mapping A from X to X^* is said to be pseudo-monotone if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ and* $\limsup(Au_n, u_n - u) \leqq 0$, *it follows that for all $v \in X$,* $\liminf(Au_n, u_n - v) \geqq (Au, u - v)$.

Pseudo-monotone mappings have been introduced by Brézis [3] and have grown increasingly important in the discussion of nonlinear elliptic boundary value problems [3, 11, 13, 22]. Everywhere defined continuous monotone operators from X to X^* (i.e. mappings A satisfying $(Au - Av, u - v) \geqq 0$ for all u, v in X) are pseudo-monotone.

For pseudo-monotone operators we have the following extension of Theorem 1.

THEOREM 2. *Let G a convex open bounded subset of the real reflexive Banach space X, with $0 \in G$ and G symmetric about* 0. *Suppose the mapping $A_t u = A(u, t): X \times \times [0, 1] \to X^*$ satisfies the conditions:*

(i) *For fixed t, A_t is bounded, continuous, and pseudo-monotone;*

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[2]) For G a subset of a Banach space, $\mathrm{cl}(G)$ denotes its closure and $\mathrm{bdry}(G)$ its boundary.

(ii) *$A(u, t)$ is continuous in t, uniformly with respect to $u \in \mathrm{cl}(G)$;*

(iii) *A_1 is odd on* $\mathrm{bdry}(G)$.

If there exists $\varepsilon > 0$ such that $\|A(u, t)\| \geqq \varepsilon$ for all $u \in \mathrm{bdry}(G)$ and $t \in [0, 1]$, then the equation $A_0 u = 0$ is solvable in G.

Proof. By a recent result of Troyanski [25] we can assume without loss of generality that both X and X^* are locally uniformly convex spaces. Let J denote the (single-valued) normalized duality mapping from X to X^* given by

$$Ju = \{q \in X^* : (q, u) = \|q\| \, \|u\|, \, \|q\| = \|u\|\}.$$

For each $\lambda > 0$ and $t \in [0, 1]$, the mapping $B_t^{(\lambda)} = A_t + \lambda J$ is then continuous and of type (S). By the boundedness of G, there exists $\varepsilon_0 > 0$ such that $B_t^{(\lambda)} u \neq 0$ for all $u \in \mathrm{bdry}(G)$, $t \in [0, 1]$, and $0 \leqq \lambda < \varepsilon_0$. Hence for fixed $\lambda \in (0, \varepsilon_0)$, the mapping $B_t^{(\lambda)} u$ satisfies the assumptions of Theorem 1, and there exists an element $u_\lambda \in G$ with $(A_0 + \lambda J) u_\lambda = 0$. Taking a sequence $\{\lambda_n\} \to 0^+$ and assuming that $u_n = u_{\lambda_n} \rightharpoonup u_0 \in \mathrm{cl}(G)$, we obtain $A_0 u_n = -\lambda_n J u_n \to 0$ and $\lim (A_0 u_n, u_n - u_0) = 0$. By the pseudo-monotonicity of A_0,

$$0 = \lim (A_0 u_n, u_n - v) \geqq (A_0 u_0, u_0 - v)$$

for all $v \in X$. This implies that $A_0 u_0 = 0$ and $u_0 \in G$, q.e.d.

We show now how our theory can be applied to the investigation of nonlinear equations of Hammerstein type

$$u + TFu = f$$

in a nonreflexive Banach space X. Here F denotes a (nonlinear) mapping of X to X^*, T a linear operator of X^* to X, and $f \in X$ a given element. Without assuming that T is compact (which case leads back to the now-classical theory of compact operators in Banach spaces), it seems to be the first time that Hammerstein equations are considered by methods of operators of monotone type in a nonreflexive space X. Former investigations were restricted to equations in a reflexive space X, or in the conjugate space X^* of some Banach space X (e.g. [1, 3, 12, 18, 20]).

DEFINITION. *A bounded linear monotone operator T of X^* into X is said to be angle-bounded if there exists a constant $\gamma \geqq 0$ such that for all v, w in X^*,*

$$|(v, Tw) - (w, Tv)| \leqq \gamma (v, Tv)^{1/2} (w, Tw)^{1/2}.$$

LEMMA 2. *Let X an arbitrary real Banach space, F a pseudo-monotone mapping of X to X^*, and T an angle-bounded linear operator of X^* to X. Then the equation $u + TFu = f$ in X can be reduced to an equivalent equation $Av = 0$ in a Hilbert space H, with A a pseudo-monotone mapping of H into itself. If X^* is nonseparable, then H has the same property in general.*

Proof. By the natural imbedding, we identify X with a subspace of X^{**} and consider T as an (angle-bounded) mapping of X^* to X^{**}. By a result of Browder-Gupta [12] (cf. also Amann [1], Hess [20]), there exist a Hilbert space H (whose norm and inner product we denote by $\|.\|_H$ and $(.,.)_H$, respectively), a continuous linear mapping S of X^* to H with range dense in H, and a monotone linear bijective mapping C of H onto H, such that $T=S^*CS$ and $(C^{-1}v, v)_H \geqq d\|v\|_H^2$ for all $v \in H$, with $d>0$. Since T has range contained in X and $CS(X^*)$ is dense in H, it follows that the range of S^* is contained in $X \subset X^{**}$.

By the above result, the equation

$$u + TFu = f \tag{7}$$

is equivalent to the equation

$$u - f + S^*CSFu = 0.$$

Since S^* is injective, there exists a uniquely determined v in H with $u-f=S^*v$, and the initial equation (7) and

$$v + CSF(S^*v + f) = 0 \tag{8}$$

are equivalent. By the bijectiveness of C, (8) holds if and only if

$$C^{-1}v + SF(S^*v + f) = 0.$$

It is readily seen that the operator A:

$$Av = C^{-1}v + SF(S^*v + f) \quad (v \in H)$$

is a pseudo-monotone mapping of H into itself. Finally, if X^* is nonseparable, the same is true in general for H as the completion of a factorspace X^* modulo some subspace (cf. the construction of H in [12]), q.e.d.

An application of Theorem 2 gives the following existence theorem of Fredholm alternative type for asymptotically homogeneous and odd Hammerstein equations.

THEOREM 3. *Let X a separable real Banach space, B a bounded continuous pseudo-monotone mapping of X to X^* which is odd and homogeneous (i.e. $B(\lambda u)=\lambda Bu$ for $\lambda \in R^1$), and $N: X \to X^*$ a bounded continuous operator with $\lim_{\|u\| \to \infty} \|u\|^{-1}\|Nu\| = 0$, and such that $B+N$ is pseudo-monotone. Let further T a linear angle-bounded operator of X^* to X. Then the range of $I+T(B+N)$ is all of X, provided $u+TBu=0$ implies that $u=0$.*

Proof[3]). In order to show that the mapping $I+T(B+N)$ is surjective, it suffices

[3]) Here we denote by "$\rightharpoonup$" weak convergence in X or H, by "$\overset{*}{\rightharpoonup}$" weak* convergence in X^*.

by Lemma 2 to prove the solvability of the equation

$$C^{-1}v + S(B+N)(S^*v+f) = 0 \tag{9}$$

in H for arbitrarily given $f \in X$. We observe that if $u + TBu = 0$ only for $u = 0$, then the equation $C^{-1}v + SBS^*v = 0$ implies that $v = 0$.

In the following let $f \in X$ be fixed. For $t \in [0, 1]$ and $v \in H$ we let

$$A_t v = C^{-1}v + (1 - \tfrac{1}{2}t)\, S(B+N)(S^*v+f) - \tfrac{1}{2}tS(B+N)(S^*(-v)+f).$$

It is readily seen that the homotopy $A_t v$ has the following properties:

(i) For fixed t, A_t is pseudo-monotone, bounded and continuous;

(ii) $A_t v$ is continuous in t, uniformly for v in bounded sets;

(iii) $A_0 v = C^{-1}v + S(B+N)(S^*v+f)$, while A_1 is odd.

The desired result on the solvability of the equation (9) follows from Theorem 2, if we prove that, assuming $C^{-1}v + SBS^*v = 0$ only for $v = 0$, there exists $R > 0$ such that $\|A_t v\|_H \geqq 1$ for all $t \in [0, 1]$ and all $v \in H$ with $\|v\|_H \geqq R$.

Suppose that to each n we can find elements $v_n \in H$ with $\|v_n\|_H \geqq n$, $t_n \in [0, 1]$, and $e_n \in H$ with $\|e_n\|_H < 1$, such that $A_{t_n} v_n = e_n$. We may assume that $t_n \to t \in [0, 1]$. Setting $w_n = \|v_n\|_H^{-1} v_n$, we then obtain

$$\begin{aligned} &C^{-1}w_n + (1 - \tfrac{1}{2}t)\, SB(S^*w_n + \|v_n\|_H^{-1} f) + \tfrac{1}{2}tSB(S^*w_n - \|v_n\|_H^{-1} f) \\ &\quad = \tfrac{1}{2}(t_n - t)\{SB(S^*w_n + \|v_n\|_H^{-1} f) - SB(S^*w_n - \|v_n\|_H^{-1} f)\} \\ &\qquad - (1 - \tfrac{1}{2}t_n)\|v_n\|_H^{-1} SN(S^*v_n + f) + \tfrac{1}{2}t_n\|v_n\|_H^{-1} SN(S^*(-v_n) + f) \\ &\qquad + \|v_n\|_H^{-1} e_n \to 0 \quad (n \to \infty). \end{aligned}$$

Because of the separability of X, the weak* topology on closed balls in X^* is metrizable, and balls in X^* are thus weak* sequentially compact. By passing to infinite subsequences, we may assure that $w_n \rightharpoonup w$ in H, $B(S^*w_n + \|v_n\|_H^{-1} f) \overset{*}{\rightharpoonup} a$ and $B(S^*w_n - \|v_n\|_H^{-1} f) \overset{*}{\rightharpoonup} b$ in X^*. It follows that $S^*w_n \pm \|v_n\|_H^{-1} f \rightharpoonup S^*w$ in X, $C^{-1}w_n \rightharpoonup C^{-1}w$ in H, and $C^{-1}w + (1 - \frac{1}{2}t)\, Sa + \frac{1}{2}tSb = 0$. We further infer that

$$\begin{aligned} &(C^{-1}w_n, w_n - w)_H + (1 - \tfrac{1}{2}t)\,(B(S^*w_n + \|v_n\|_H^{-1} f), (S^*w_n + \|v_n\|_H^{-1} f) - S^*w) \\ &\quad + \tfrac{1}{2}t(B(S^*w_n - \|v_n\|_H^{-1} f), (S^*w_n - \|v_n\|_H^{-1} f) - S^*w) \to 0. \end{aligned}$$

We assume that $0 < t \leqq 1$ (the case $t = 0$ is treated similarly) and choose further infinite subsequences such that the three limits $\lim (C^{-1}w_n, w_n - w)_H$, $\lim (B(S^*w_n + \|v_n\|_H^{-1} f), (S^*w_n + \|v_n\|_H^{-1} f) - S^*w)$, and $\lim (B(S^*w_n - \|v_n\|_H^{-1} f), (S^*w_n - \|v_n\|_H^{-1} f) - S^*w)$ exist. By the pseudo-monotonicity property of the mappings C^{-1} and B, all of the three limits are 0. Hence, again by pseudo-monotonicity, $a = b = BS^*w$, and consequently $C^{-1}w + SBS^*w = 0$.

Since $(C^{-1}v_n, v_n)_H \geqq d\|v_n\|_H^2$, we conclude that $(C^{-1}w_n, w_n)_H \geqq d > 0$. Moreover $(C^{-1}w_n, w_n)_H \to (C^{-1}w, w)_H$. Thus $w \neq 0$, q.e.d.

Remark. Theorem 3 remains true for X nonseparable, but reflexive.

4. Our principal methodological result on nonlinear eigenvalue problems which extends the corresponding Theorem 1 of Browder [7] to mappings in nonseparable spaces is

THEOREM 4. *Let X a real reflexive Banach space, C a closed subset of X, and A, B continuous mappigs of C into X^*, with A bounded and of type (S) and B compact. Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence in Λ with $C \cap E_1 \neq \emptyset$. Suppose to each $F \in \Lambda$ with $F \supset E_1$ there exist elements $u_F \in C \cap F$ and $t_F \in R^1$ such that $j_F^* A u_F = t_F j_F^* B u_F$, and assume u_F and t_F remain uniformly bounded for $F \supset E_1$.*

Then there exists a sequence $\{F_n\}$ in Λ with $F_n \supset E_n$ for each n, such that for some subsequence $\{n(k)\}$ of $\{n\}$, $u_{F_{n(k)}} \to u_0 \in C$, $t_{F_{n(k)}} \to t_0 \in R^1$, and $Au_0 = t_0 B u_0$.

Proof. Follows immediately from the Proposition, with $I = R^1$ and $A(u, t) = Au - tBu$.

As an application to the "selfadjoint" case where A and B are the derivatives of two functions, we get the following extension of Theorem 3 in [7] and Theorem 14 in [8]:

THEOREM 5. *Let f, h continuously differentiable real-valued functions defined on the (not necessarily separable) real reflexive Banach space X, with f' bounded and of type (S) and h' compact. Suppose that for a given constant c the level set $M_c(f) = \{u \in X : f(u) = c\}$ is nonempty and bounded, and that for $u \in M_c(f)$, $(f'u, u) \neq 0$. Suppose further that there exists a point $v_0 \in M_c(f)$ and a constant $d > 0$ such that for all $u \in M_c(f)$ for which $h(u) \geqq h(v_0)$, $(h'u, u) \geqq d$.*

Then h assumes its maximum on $M_c(f)$ at a point u_0 which is a solution of the equation $f'u_0 = t_0 h' u_0$ for some real number t_0.

Proof. By the continuity of f, the level set $M_c(f)$ is closed in X. Let F an arbitrary element of Λ with $M_c(f) \cap F \neq \emptyset$, and let f_F, h_F denote the restrictions of f and h to F. The functions f_F and h_F are continuously differentiable on F, with $(f_F)' = j_F^* f' j_F$, $(h_F)' = j_F^* h' j_F$. We set $M_{c,F}(f) = M_c(f) \cap F$. Since $((f_F)'u, u) = (f'u, u) \neq 0$ for all $u \in M_{c,F}(f)$, $M_{c,F}(f)$ is a compact manifold of codimension 1 in F. Thus there exists $u_F \in M_{c,F}(f)$ such that $h(u_F) = \sup_{u \in M_{c,F}(f)} h(u)$. By the Lagrange multiplier method,

$$(h_F)' u_F = \lambda_F (f_F)' u_F \tag{10}$$

for some real λ_F.

Let $\{w_n\}$ be a sequence in $M_c(f)$ with $h(w_n) \to m = \sup_{u \in M_c(f)} h(u)$. We choose an increasing sequence $\{E_n\}_{n=1}^{\infty}$ in Λ such that $E_1 \supset \{v_0, w_1\}$, while $w_n \in E_n$ for $n \geqq 2$.

In order to prove the applicability of Theorem 4 with $C=M_c(f)$, we show that for $F\in\Lambda$, $F\supset E_1$, the corresponding numbers $(\lambda_F)^{-1}$ of (10) are uniformly bounded. Indeed, it follows from (10) that $|(h'u_F, u_F)|=|\lambda_F|\,|(f'u_F, u_F)|$, where $|(h'u_F, u_F)|\geqq d>0$ and $|(f'u_F, u_F)|\leqq k_0$ for each $F\in\Lambda$ with $F\supset E_1$. Thus $|\lambda_F|\geqq k_1>0$, and we can write $(f_F)'u_F=t_F(h_F)'u_F$, with $t_F=(\lambda_F)^{-1}$ uniformly bounded.

By Theorem 4 there exists a sequence $\{F_n\}$ in Λ with $F_n\supset E_n$ for each n, such that $u_{F_{n(k)}}\to u_0\in M_c(f)$, $t_{F_{n(k)}}\to t_0\in R^1$, and $f'u_0=t_0h'u_0$. Since $w_n\in E_n\subset F_n$, $h(w_n)\leqq$ $\leqq h(u_{F_n})$. In this last relation the left side converges to m, while $h(u_{F_{n(k)}})\to h(u_0)$ by continuity of h. Hence $h(u_0)=\sup_{u\in M_c(f)}h(u)$, q.e.d.

In a similar way one generalizes Theorem 15 of [8].

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Received December 20, 1970