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Nonlinear Functional Equations and Eigenvalue Problems in Nonseparable Banach Spaces¹⁾

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1. Let X be a real reflexive Banach space and A, B nonlinear mappings of X into the conjugate space X^* , with A of monotone type and B compact. In the last years, much interest in nonlinear functional analysis has been concentrated on the problem of determining useful conditions under which the functional equation

$$Au = 0 \tag{1}$$

or the eigenvalue problem

$$Au = tBu \quad \text{for some real } t \tag{2}$$

admit solutions (which possibly satisfy additional restrictions).

For A satisfying certain asymptotic conditions (such as A coercive or A^{-1} bounded), various results on the solvability of equation (1) have been obtained (e.g. Brézis [3], Browder [4, 6, 8, 9], Browder-Hess [13], Leray-Lions [22], Minty [23]). There is an alternative type of hypothesis one may impose on the mapping A in order to get existence theorems for equation (1), namely the hypothesis that $A = A_0$ is homotopic to a mapping A_1 which commutes with a group \mathcal{G} of transformations on the spaces X and X^* , with \mathcal{G} having elements of finite order (in particular A_1 odd). Under the assumption that X is *separable*, several mathematicians have derived existence theorems involving homotopy arguments, making use of an approximation method of Galerkin type (e.g. Browder [8, 9, 10, 11], Browder-Petryshyn [14]). (For a completely different approach see Hess [19]). Though most of the concrete reflexive Banach spaces occurring in applications are separable, it is necessary for the investigation of certain specific problems to have a similar approach in nonseparable spaces. For that reason, Nečas [24] has recently given a method which works in nonseparable spaces, and which is extended in the writer's papers [17, 18].

One way of attacking the eigenvalue problem (2), is by variational methods (e.g. Browder [5], Hess [16], Krasnoselskii [21], Vainberg [26]). In [7, 8], Browder has developed a theory for nonlinear eigenvalue problems in *separable* spaces based on Galerkin approximations. This latter approach has the advantage that it does not involve the theory of infinite-dimensional manifolds (Lusternik's principle), and that it permits to prove the existence of an infinite number of distinct normalized eigenfunctions (Lusternik-Schnirelman theory) under milder differentiability hypotheses.

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It is our purpose in the present note to describe an easy argument of Galerkin approximation type which allows to prove both existence theorems and results on eigenvalue problems in *nonseparable* Banach spaces. In contrary to the Galerkin approximation method in separable spaces, which is based on an a priori given injective approximation scheme, our method consists in recursively constructing a suitable scheme. The main result is the Proposition proved in Section 2. In Section 3 we apply the conclusions of the Proposition to the functional equation (1), assuming that $A=A_0$ is homotopic to an odd mapping A_1 . The result is closely related to that of Nečas [24], but it seems that our proof is simpler. A brief discussion follows of how our theory can be used in order to study nonlinear equations of Hammerstein type in nonreflexive Banach spaces. In Section 4 we finally show the applicability of the Proposition to the treatment of nonlinear eigenvalue problems in nonseparable spaces.

2. For X a real Banach space and X^* its conjugate space, we let (w, u) denote the duality pairing between elements $w \in X^*$ and $u \in X$. An operator A defined on a closed set $C \subset X$, with range contained in X^* , is said to be of *type (S)* if it satisfies the condition: for any sequence $\{u_n\} \subset C$ converging weakly to some $u \in X$, for which $\lim(Au_n, u_n - u) = 0$, its strong convergence follows. Mappings of type (S) have been introduced by Browder [7] and have shown to form a very useful class of operators of monotone type for homotopy considerations and eigenvalue problems. The mapping A is further *bounded* if it maps bounded sets onto bounded sets. Let Λ be the set of all finite-dimensional subspaces of X , ordered by inclusion. For $F \in \Lambda$, j_F denotes the injection mapping of F into X . If the operator A maps $C \subset X$ into X^* , the *Galerkin approximant* $A_F: C \cap F \rightarrow F^*$ is defined by $A_F = j_F^* A j_F$. In the following we use the symbols “ \rightarrow ” and “ \rightharpoonup ” to denote strong and weak convergence, respectively.

PROPOSITION. *Let X a real reflexive Banach space, C a closed subset of X , I a closed interval in R^1 , and $A(u, t)$ a mapping of $C \times I$ into X^* with the following properties:*

- (i) *For fixed t , $A(u, t): C \rightarrow X^*$, is bounded, continuous, and of type (S);*
- (ii) *$A(u, t)$ is uniformly continuous in t with respect to u in bounded subsets of C .*

Let $\{E_n\}_{n=1}^\infty$ be a given increasing sequence in Λ with $C \cap E_1 \neq \emptyset$. Suppose to each $F \in \Lambda$ with $F \supset E_1$ there exist elements $u_F \in C \cap F$ and $t_F \in I$ such that $j_F^ A(u_F, t_F) = 0$, and assume said elements are uniformly bounded for $F \supset E_1$.*

Then $A(u_0, t_0) = 0$ for some $u_0 \in C$ and $t_0 \in I$. Moreover, there exists an increasing sequence $\{F_n\}$ in Λ with $F_n \supset E_n$ for each n , such that for some subsequence $\{n(k)\}$ of $\{n\}$, $u_{F_{n(k)}} \rightarrow u_0$ and $t_{F_{n(k)}} \rightarrow t_0$.

Proof. We construct the asserted sequence $\{F_n\}$ in Λ as follows:

- (a) We set $F_1 = E_1$.

(b) Suppose we have already constructed $F_1 \subset \dots \subset F_n$, and let $u_n = u_{F_n} \in C \cap F_n$ and $t_n = t_{F_n} \in I$ denote the described elements corresponding to F_n such that $j_{F_n}^* A(u_n, t_n) = 0$. There exists $v_n \in X$, $\|v_n\| = 1$, such that $|(A(u_n, t_n), v_n)| \geq \frac{1}{2} \|A(u_n, t_n)\|$. We then choose $F_{n+1} \supset F_n + E_{n+1} + [v_n]$.

By hypothesis, the sequences $\{u_n\}$ and $\{t_n\}$ are bounded. We may pass to infinite subsequences and assure that $u_n \rightarrow u_0 \in X$ and $t_n \rightarrow t_0 \in I$. It follows from condition (ii) that

$$\|A(u_n, t_n) - A(u_n, t_0)\| \rightarrow 0 \quad (n \rightarrow \infty). \quad (3)$$

We assert that

$$(A(u_n, t_0), w) \rightarrow 0 \quad (n \rightarrow \infty) \quad (4)$$

for all $w \in X_0 = \text{closure} \left\{ \bigcup_{j=1}^{\infty} F_j \right\}$. Indeed, if w lies in some F_j and $n \geq j$, we have

$$|(A(u_n, t_0), w)| \leq |(A(u_n, t_n), w)| + |(A(u_n, t_0) - A(u_n, t_n), w)|,$$

where the first term on the right side vanishes, while the second term tends to 0 as $n \rightarrow \infty$ according to (3). Because of the boundedness of the sequence $\{A(u_n, t_0)\}$, (4) extends to all $w \in X_0$. We now get

$$\begin{aligned} |(A(u_n, t_0), u_n - u_0)| &\leq |(A(u_n, t_n), u_n)| \\ &\quad + |(A(u_n, t_0) - A(u_n, t_n), u_n)| + |(A(u_n, t_0), u_0)|. \end{aligned}$$

On the right side of this estimate, the first summand vanishes, the middle term tends to 0 because of (3), and the last approaches 0 according to (4), since the weak limit u_0 of the sequence $\{u_n\} \subset X_0$ lies in X_0 . Property (S) of the mapping $A(u, t_0)$ implies that $u_n \rightarrow u_0$. Hence $u_0 \in C$, $A(u_n, t_0) \rightarrow A(u_0, t_0)$, and

$$A(u_n, t_n) \rightarrow A(u_0, t_0) \quad (5)$$

because of the continuity of the mapping $A(u, t_0)$ in u and the estimate (3). We infer that, according to (4),

$$(A(u_0, t_0), w) = 0 \quad \text{for all } w \in X_0. \quad (6)$$

We finally prove that $A(u_0, t_0) = 0$. Suppose to the contrary that $A(u_0, t_0) \neq 0$. Then, by (5), $\|A(u_n, t_n)\| \geq d > 0$ for some constant d and all $n \geq n_0$, which implies that

$$|(A(u_n, t_n), v_n)| \geq d/2 > 0$$

for $n \geq n_0$. But (5) and the fact that some subsequence of $\{v_n\}$ (denoted again by $\{v_n\}$) converges weakly to an element $v_0 \in X_0$ have as a consequence that

$$(A(u_n, t_n), v_n) \rightarrow (A(u_0, t_0), v_0),$$

the expression on the right being 0 according to (6). This contradiction shows that $A(u_0, t_0) = 0$, q.e.d.

3. We apply the Proposition in order to obtain results on the existence of solutions of the functional equation (1).

THEOREM 1²⁾. *Let X a real reflexive Banach space, G an open bounded subset of X containing 0 and symmetric about the origin, and $A_t u = A(u, t)$ a mapping of $\text{cl}(G) \times [0, 1]$ into X^* as follows:*

- (i) *For fixed t , A_t is a bounded continuous mapping of type (S);*
- (ii) *$A(u, t)$ is uniformly continuous in t with respect to $u \in \text{cl}(G)$;*
- (iii) *A_t is odd on $\text{bdry}(G)$, i.e. $A(-u, 1) = -A(u, 1)$ for $u \in \text{bdry}(G)$.*

Assume that $A(u, t) \neq 0$ for all $u \in \text{bdry}(G)$ and all $t \in [0, 1]$. Then the equation $A_0 u = 0$ has a solution u_0 in G .

Theorem 1 follows by the classical Borsuk theorem [2, 15, 21], the invariance of the Brouwer degree under homotopies, and arguments which have become standard in the theory of mappings of monotone type (e.g. [3, 4, 6, 8, 9, 13, 17, 18, 22, 23]) from

LEMMA 1. *Let $E \in \Lambda$ be given. Then under the assumptions of Theorem 1 there exists $F \in \Lambda$, $F \supset E$, such that $j_F^* A(u, t) \neq 0$ for all $u \in \text{bdry}(G) \cap F$ and all $t \in [0, 1]$.*

Proof of Lemma 1. Suppose to each $F \in \Lambda$ with $F \supset E$ we can find elements $u_F \in \text{bdry}(G) \cap F$ and $t_F \in [0, 1]$ such that $j_F^* A(u_F, t_F) = 0$. Applying the Proposition with $C = \text{bdry}(G)$ and $I = [0, 1]$, we are led to a contradiction to the assumptions of Theorem 1, q.e.d.

DEFINITION. *A mapping A from X to X^* is said to be pseudo-monotone if for any sequence $\{u_n\}$ in X with $u_n \rightarrow u$ and $\limsup (Au_n, u_n - u) \leq 0$, it follows that for all $v \in X$, $\liminf (Au_n, u_n - v) \geq (Au, u - v)$.*

Pseudo-monotone mappings have been introduced by Brézis [3] and have grown increasingly important in the discussion of nonlinear elliptic boundary value problems [3, 11, 13, 22]. Everywhere defined continuous monotone operators from X to X^* (i.e. mappings A satisfying $(Au - Av, u - v) \geq 0$ for all u, v in X) are pseudo-monotone.

For pseudo-monotone operators we have the following extension of Theorem 1.

THEOREM 2. *Let G a convex open bounded subset of the real reflexive Banach space X , with $0 \in G$ and G symmetric about 0. Suppose the mapping $A_t u = A(u, t): X \times [0, 1] \rightarrow X^*$ satisfies the conditions:*

- (i) *For fixed t , A_t is bounded, continuous, and pseudo-monotone;*

²⁾ For G a subset of a Banach space, $\text{cl}(G)$ denotes its closure and $\text{bdry}(G)$ its boundary.

- (ii) $A(u, t)$ is continuous in t , uniformly with respect to $u \in \text{cl}(G)$;
 (iii) A_1 is odd on $\text{bdry}(G)$.

If there exists $\varepsilon > 0$ such that $\|A(u, t)\| \geq \varepsilon$ for all $u \in \text{bdry}(G)$ and $t \in [0, 1]$, then the equation $A_0 u = 0$ is solvable in G .

Proof. By a recent result of Troyanski [25] we can assume without loss of generality that both X and X^* are locally uniformly convex spaces. Let J denote the (single-valued) normalized duality mapping from X to X^* given by

$$Ju = \{q \in X^* : (q, u) = \|q\| \|u\|, \|q\| = \|u\|\}.$$

For each $\lambda > 0$ and $t \in [0, 1]$, the mapping $B_t^{(\lambda)} = A_t + \lambda J$ is then continuous and of type (S). By the boundedness of G , there exists $\varepsilon_0 > 0$ such that $B_t^{(\lambda)} u \neq 0$ for all $u \in \text{bdry}(G)$, $t \in [0, 1]$, and $0 \leq \lambda < \varepsilon_0$. Hence for fixed $\lambda \in (0, \varepsilon_0)$, the mapping $B_t^{(\lambda)} u$ satisfies the assumptions of Theorem 1, and there exists an element $u_\lambda \in G$ with $(A_0 + \lambda J) u_\lambda = 0$. Taking a sequence $\{\lambda_n\} \rightarrow 0^+$ and assuming that $u_n = u_{\lambda_n} \rightarrow u_0 \in \text{cl}(G)$, we obtain $A_0 u_n = -\lambda_n J u_n \rightarrow 0$ and $\lim (A_0 u_n, u_n - u_0) = 0$. By the pseudo-monotonicity of A_0 ,

$$0 = \lim (A_0 u_n, u_n - v) \geq (A_0 u_0, u_0 - v)$$

for all $v \in X$. This implies that $A_0 u_0 = 0$ and $u_0 \in G$, q.e.d.

We show now how our theory can be applied to the investigation of nonlinear equations of Hammerstein type

$$u + TFu = f$$

in a nonreflexive Banach space X . Here F denotes a (nonlinear) mapping of X to X^* , T a linear operator of X^* to X , and $f \in X$ a given element. Without assuming that T is compact (which case leads back to the now-classical theory of compact operators in Banach spaces), it seems to be the first time that Hammerstein equations are considered by methods of operators of monotone type in a nonreflexive space X . Former investigations were restricted to equations in a reflexive space X , or in the conjugate space X^* of some Banach space X (e.g. [1, 3, 12, 18, 20]).

DEFINITION. A bounded linear monotone operator T of X^* into X is said to be angle-bounded if there exists a constant $\gamma \geq 0$ such that for all v, w in X^* ,

$$|(v, Tw) - (w, Tv)| \leq \gamma (v, Tv)^{1/2} (w, Tw)^{1/2}.$$

LEMMA 2. Let X an arbitrary real Banach space, F a pseudo-monotone mapping of X to X^* , and T an angle-bounded linear operator of X^* to X . Then the equation $u + TFu = f$ in X can be reduced to an equivalent equation $Av = 0$ in a Hilbert space H , with A a pseudo-monotone mapping of H into itself. If X^* is nonseparable, then H has the same property in general.

Proof. By the natural imbedding, we identify X with a subspace of X^{**} and consider T as an (angle-bounded) mapping of X^* to X^{**} . By a result of Browder-Gupta [12] (cf. also Amann [1], Hess [20]), there exist a Hilbert space H (whose norm and inner product we denote by $\|\cdot\|_H$ and $(\cdot, \cdot)_H$, respectively), a continuous linear mapping S of X^* to H with range dense in H , and a monotone linear bijective mapping C of H onto H , such that $T = S^*CS$ and $(C^{-1}v, v)_H \geq d\|v\|_H^2$ for all $v \in H$, with $d > 0$. Since T has range contained in X and $CS(X^*)$ is dense in H , it follows that the range of S^* is contained in $X \subset X^{**}$.

By the above result, the equation

$$u + TFu = f \tag{7}$$

is equivalent to the equation

$$u - f + S^*CSFu = 0.$$

Since S^* is injective, there exists a uniquely determined v in H with $u - f = S^*v$, and the initial equation (7) and

$$v + CSF(S^*v + f) = 0 \tag{8}$$

are equivalent. By the bijectiveness of C , (8) holds if and only if

$$C^{-1}v + SF(S^*v + f) = 0.$$

It is readily seen that the operator A :

$$Av = C^{-1}v + SF(S^*v + f) \quad (v \in H)$$

is a pseudo-monotone mapping of H into itself. Finally, if X^* is nonseparable, the same is true in general for H as the completion of a factorspace X^* modulo some subspace (cf. the construction of H in [12]), q.e.d.

An application of Theorem 2 gives the following existence theorem of Fredholm alternative type for asymptotically homogeneous and odd Hammerstein equations.

THEOREM 3. *Let X a separable real Banach space, B a bounded continuous pseudo-monotone mapping of X to X^* which is odd and homogeneous (i.e. $B(\lambda u) = \lambda Bu$ for $\lambda \in \mathbb{R}^1$), and $N: X \rightarrow X^*$ a bounded continuous operator with $\lim_{\|u\| \rightarrow \infty} \|u\|^{-1} \|Nu\| = 0$, and such that $B + N$ is pseudo-monotone. Let further T a linear angle-bounded operator of X^* to X . Then the range of $I + T(B + N)$ is all of X , provided $u + TBu = 0$ implies that $u = 0$.*

*Proof*³⁾. In order to show that the mapping $I + T(B + N)$ is surjective, it suffices

³⁾ Here we denote by “ \rightarrow ” weak convergence in X or H , by “ $\overset{*}{\rightarrow}$ ” weak* convergence in X^* .

by Lemma 2 to prove the solvability of the equation

$$C^{-1}v + S(B + N)(S^*v + f) = 0 \quad (9)$$

in H for arbitrarily given $f \in X$. We observe that if $u + TBu = 0$ only for $u = 0$, then the equation $C^{-1}v + SBS^*v = 0$ implies that $v = 0$.

In the following let $f \in X$ be fixed. For $t \in [0, 1]$ and $v \in H$ we let

$$A_t v = C^{-1}v + (1 - \frac{1}{2}t)S(B + N)(S^*v + f) - \frac{1}{2}tS(B + N)(S^*(-v) + f).$$

It is readily seen that the homotopy $A_t v$ has the following properties:

- (i) For fixed t , A_t is pseudo-monotone, bounded and continuous;
- (ii) $A_t v$ is continuous in t , uniformly for v in bounded sets;
- (iii) $A_0 v = C^{-1}v + S(B + N)(S^*v + f)$, while A_1 is odd.

The desired result on the solvability of the equation (9) follows from Theorem 2, if we prove that, assuming $C^{-1}v + SBS^*v = 0$ only for $v = 0$, there exists $R > 0$ such that $\|A_t v\|_H \geq 1$ for all $t \in [0, 1]$ and all $v \in H$ with $\|v\|_H \geq R$.

Suppose that to each n we can find elements $v_n \in H$ with $\|v_n\|_H \geq n$, $t_n \in [0, 1]$, and $e_n \in H$ with $\|e_n\|_H < 1$, such that $A_{t_n} v_n = e_n$. We may assume that $t_n \rightarrow t \in [0, 1]$. Setting $w_n = \|v_n\|_H^{-1} v_n$, we then obtain

$$\begin{aligned} & C^{-1}w_n + (1 - \frac{1}{2}t)SB(S^*w_n + \|v_n\|_H^{-1}f) + \frac{1}{2}tSB(S^*w_n - \|v_n\|_H^{-1}f) \\ &= \frac{1}{2}(t_n - t)\{SB(S^*w_n + \|v_n\|_H^{-1}f) - SB(S^*w_n - \|v_n\|_H^{-1}f)\} \\ &\quad - (1 - \frac{1}{2}t_n)\|v_n\|_H^{-1}SN(S^*v_n + f) + \frac{1}{2}t_n\|v_n\|_H^{-1}SN(S^*(-v_n) + f) \\ &\quad + \|v_n\|_H^{-1}e_n \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Because of the separability of X , the weak* topology on closed balls in X^* is metrizable, and balls in X^* are thus weak* sequentially compact. By passing to infinite subsequences, we may assure that $w_n \rightarrow w$ in H , $B(S^*w_n + \|v_n\|_H^{-1}f) \xrightarrow{*} a$ and $B(S^*w_n - \|v_n\|_H^{-1}f) \xrightarrow{*} b$ in X^* . It follows that $S^*w_n \pm \|v_n\|_H^{-1}f \rightarrow S^*w$ in X , $C^{-1}w_n \rightarrow C^{-1}w$ in H , and $C^{-1}w + (1 - \frac{1}{2}t)Sa + \frac{1}{2}tSb = 0$. We further infer that

$$\begin{aligned} & (C^{-1}w_n, w_n - w)_H + (1 - \frac{1}{2}t)(B(S^*w_n + \|v_n\|_H^{-1}f), (S^*w_n + \|v_n\|_H^{-1}f) - S^*w) \\ & + \frac{1}{2}t(B(S^*w_n - \|v_n\|_H^{-1}f), (S^*w_n - \|v_n\|_H^{-1}f) - S^*w) \rightarrow 0. \end{aligned}$$

We assume that $0 < t \leq 1$ (the case $t = 0$ is treated similarly) and choose further infinite subsequences such that the three limits $\lim (C^{-1}w_n, w_n - w)_H$, $\lim (B(S^*w_n + \|v_n\|_H^{-1}f), (S^*w_n + \|v_n\|_H^{-1}f) - S^*w)$, and $\lim (B(S^*w_n - \|v_n\|_H^{-1}f), (S^*w_n - \|v_n\|_H^{-1}f) - S^*w)$ exist. By the pseudo-monotonicity property of the mappings C^{-1} and B , all of the three limits are 0. Hence, again by pseudo-monotonicity, $a = b = BS^*w$, and consequently $C^{-1}w + SBS^*w = 0$.

Since $(C^{-1}v_n, v_n)_H \geq d\|v_n\|_H^2$, we conclude that $(C^{-1}w_n, w_n)_H \geq d > 0$. Moreover $(C^{-1}w_n, w_n)_H \rightarrow (C^{-1}w, w)_H$. Thus $w \neq 0$, q.e.d.

Remark. Theorem 3 remains true for X nonseparable, but reflexive.

4. Our principal methodological result on nonlinear eigenvalue problems which extends the corresponding Theorem 1 of Browder [7] to mappings in nonseparable spaces is

THEOREM 4. *Let X a real reflexive Banach space, C a closed subset of X , and A, B continuous mappings of C into X^* , with A bounded and of type (S) and B compact. Let $\{E_n\}_{n=1}^\infty$ be an increasing sequence in Λ with $C \cap E_1 \neq \emptyset$. Suppose to each $F \in \Lambda$ with $F \supset E_1$ there exist elements $u_F \in C \cap F$ and $t_F \in \mathbb{R}^1$ such that $j_F^* Au_F = t_F j_F^* Bu_F$, and assume u_F and t_F remain uniformly bounded for $F \supset E_1$.*

Then there exists a sequence $\{F_n\}$ in Λ with $F_n \supset E_n$ for each n , such that for some subsequence $\{n(k)\}$ of $\{n\}$, $u_{F_{n(k)}} \rightarrow u_0 \in C$, $t_{F_{n(k)}} \rightarrow t_0 \in \mathbb{R}^1$, and $Au_0 = t_0 Bu_0$.

Proof. Follows immediately from the Proposition, with $I = \mathbb{R}^1$ and $A(u, t) = Au - tBu$.

As an application to the ‘selfadjoint’ case where A and B are the derivatives of two functions, we get the following extension of Theorem 3 in [7] and Theorem 14 in [8]:

THEOREM 5. *Let f, h continuously differentiable real-valued functions defined on the (not necessarily separable) real reflexive Banach space X , with f' bounded and of type (S) and h' compact. Suppose that for a given constant c the level set $M_c(f) = \{u \in X : f(u) = c\}$ is nonempty and bounded, and that for $u \in M_c(f)$, $(f'u, u) \neq 0$. Suppose further that there exists a point $v_0 \in M_c(f)$ and a constant $d > 0$ such that for all $u \in M_c(f)$ for which $h(u) \geq h(v_0)$, $(h'u, u) \geq d$.*

Then h assumes its maximum on $M_c(f)$ at a point u_0 which is a solution of the equation $f'u_0 = t_0 h'u_0$ for some real number t_0 .

Proof. By the continuity of f , the level set $M_c(f)$ is closed in X . Let F an arbitrary element of Λ with $M_c(f) \cap F \neq \emptyset$, and let f_F, h_F denote the restrictions of f and h to F . The functions f_F and h_F are continuously differentiable on F , with $(f_F)' = j_F^* f' j_F$, $(h_F)' = j_F^* h' j_F$. We set $M_{c,F}(f) = M_c(f) \cap F$. Since $((f_F)'u, u) = (f'u, u) \neq 0$ for all $u \in M_{c,F}(f)$, $M_{c,F}(f)$ is a compact manifold of codimension 1 in F . Thus there exists $u_F \in M_{c,F}(f)$ such that $h(u_F) = \sup_{u \in M_{c,F}(f)} h(u)$. By the Lagrange multiplier method,

$$(h_F)'u_F = \lambda_F (f_F)'u_F \tag{10}$$

for some real λ_F .

Let $\{w_n\}$ be a sequence in $M_c(f)$ with $h(w_n) \rightarrow m = \sup_{u \in M_c(f)} h(u)$. We choose an increasing sequence $\{E_n\}_{n=1}^\infty$ in Λ such that $E_1 \supset \{v_0, w_1\}$, while $w_n \in E_n$ for $n \geq 2$.

In order to prove the applicability of Theorem 4 with $C = M_c(f)$, we show that for $F \in \Lambda$, $F \supset E_1$, the corresponding numbers $(\lambda_F)^{-1}$ of (10) are uniformly bounded. Indeed, it follows from (10) that $|(h'u_F, u_F)| = |\lambda_F| |(f'u_F, u_F)|$, where $|(h'u_F, u_F)| \geq d > 0$ and $|(f'u_F, u_F)| \leq k_0$ for each $F \in \Lambda$ with $F \supset E_1$. Thus $|\lambda_F| \geq k_1 > 0$, and we can write $(f_F)'u_F = t_F(h_F)'u_F$, with $t_F = (\lambda_F)^{-1}$ uniformly bounded.

By Theorem 4 there exists a sequence $\{F_n\}$ in Λ with $F_n \supset E_n$ for each n , such that $u_{F_n(k)} \rightarrow u_0 \in M_c(f)$, $t_{F_n(k)} \rightarrow t_0 \in \mathbb{R}^1$, and $f'u_0 = t_0 h'u_0$. Since $w_n \in E_n \subset F_n$, $h(w_n) \leq h(u_{F_n})$. In this last relation the left side converges to m , while $h(u_{F_n(k)}) \rightarrow h(u_0)$ by continuity of h . Hence $h(u_0) = \sup_{u \in M_c(f)} h(u)$, q.e.d.

In a similar way one generalizes Theorem 15 of [8].

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