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# Nonlinear Functional Equations and Eigenvalue Problems in Nonseparable Banach Spaces<sup>1)</sup>

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1. Let  $X$  be a real reflexive Banach space and  $A, B$  nonlinear mappings of  $X$  into the conjugate space  $X^*$ , with  $A$  of monotone type and  $B$  compact. In the last years, much interest in nonlinear functional analysis has been concentrated on the problem of determining useful conditions under which the functional equation

$$Au = 0 \quad (1)$$

or the eigenvalue problem

$$Au = tBu \quad \text{for some real } t \quad (2)$$

admit solutions (which possibly satisfy additional restrictions).

For  $A$  satisfying certain asymptotic conditions (such as  $A$  coercive or  $A^{-1}$  bounded), various results on the solvability of equation (1) have been obtained (e.g. Brézis [3], Browder [4, 6, 8, 9], Browder-Hess [13], Leray-Lions [22], Minty [23]). There is an alternative type of hypothesis one may impose on the mapping  $A$  in order to get existence theorems for equation (1), namely the hypothesis that  $A = A_0$  is homotopic to a mapping  $A_1$  which commutes with a group  $\mathcal{G}$  of transformations on the spaces  $X$  and  $X^*$ , with  $\mathcal{G}$  having elements of finite order (in particular  $A_1$  odd). Under the assumption that  $X$  is *separable*, several mathematicians have derived existence theorems involving homotopy arguments, making use of an approximation method of Galerkin type (e.g. Browder [8, 9, 10, 11], Browder-Petryshyn [14]). (For a completely different approach see Hess [19]). Though most of the concrete reflexive Banach spaces occurring in applications are separable, it is necessary for the investigation of certain specific problems to have a similar approach in nonseparable spaces. For that reason, Nečas [24] has recently given a method which works in nonseparable spaces, and which is extended in the writer's papers [17, 18].

One way of attacking the eigenvalue problem (2), is by variational methods (e.g. Browder [5], Hess [16], Krasnoselskii [21], Vainberg [26]). In [7, 8], Browder has developed a theory for nonlinear eigenvalue problems in *separable* spaces based on Galerkin approximations. This latter approach has the advantage that it does not involve the theory of infinite-dimensional manifolds (Lusternik's principle), and that it permits to prove the existence of an infinite number of distinct normalized eigenfunctions (Lusternik-Schnirelman theory) under milder differentiability hypotheses.

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It is our purpose in the present note to describe an easy argument of Galerkin approximation type which allows to prove both existence theorems and results on eigenvalue problems in *nonseparable* Banach spaces. In contrary to the Galerkin approximation method in separable spaces, which is based on an a priori given injective approximation scheme, our method consists in recursively constructing a suitable scheme. The main result is the Proposition proved in Section 2. In Section 3 we apply the conclusions of the Proposition to the functional equation (1), assuming that  $A = A_0$  is homotopic to an odd mapping  $A_1$ . The result is closely related to that of Nečas [24], but it seems that our proof is simpler. A brief discussion follows of how our theory can be used in order to study nonlinear equations of Hammerstein type in nonreflexive Banach spaces. In Section 4 we finally show the applicability of the Proposition to the treatment of nonlinear eigenvalue problems in nonseparable spaces.

2. For  $X$  a real Banach space and  $X^*$  its conjugate space, we let  $(w, u)$  denote the duality pairing between elements  $w \in X^*$  and  $u \in X$ . An operator  $A$  defined on a closed set  $C \subset X$ , with range contained in  $X^*$ , is said to be of *type (S)* if it satisfies the condition: *for any sequence  $\{u_n\} \subset C$  converging weakly to some  $u \in X$ , for which  $\lim(Au_n, u_n - u) = 0$ , its strong convergence follows*. Mappings of type (S) have been introduced by Browder [7] and have shown to form a very useful class of operators of monotone type for homotopy considerations and eigenvalue problems. The mapping  $A$  is further *bounded* if it maps bounded sets onto bounded sets. Let  $\Lambda$  be the set of all finite-dimensional subspaces of  $X$ , ordered by inclusion. For  $F \in \Lambda$ ,  $j_F$  denotes the injection mapping of  $F$  into  $X$ . If the operator  $A$  maps  $C \subset X$  into  $X^*$ , the *Galerkin approximant*  $A_F: C \cap F \rightarrow F^*$  is defined by  $A_F = j_F^* A j_F$ . In the following we use the symbols “ $\rightarrow$ ” and “ $\rightharpoonup$ ” to denote strong and weak convergence, respectively.

**PROPOSITION.** *Let  $X$  a real reflexive Banach space,  $C$  a closed subset of  $X$ ,  $I$  a closed interval in  $R^1$ , and  $A(u, t)$  a mapping of  $C \times I$  into  $X^*$  with the following properties:*

- (i) *For fixed  $t$ ,  $A(u, t): C \rightarrow X^*$ , is bounded, continuous, and of type (S);*
- (ii)  *$A(u, t)$  is uniformly continuous in  $t$  with respect to  $u$  in bounded subsets of  $C$ .*

*Let  $\{E_n\}_{n=1}^{\infty}$  be a given increasing sequence in  $\Lambda$  with  $C \cap E_1 \neq \emptyset$ . Suppose to each  $F \in \Lambda$  with  $F \supset E_1$  there exist elements  $u_F \in C \cap F$  and  $t_F \in I$  such that  $j_F^* A(u_F, t_F) = 0$ , and assume said elements are uniformly bounded for  $F \supset E_1$ .*

*Then  $A(u_0, t_0) = 0$  for some  $u_0 \in C$  and  $t_0 \in I$ . Moreover, there exists an increasing sequence  $\{F_n\}$  in  $\Lambda$  with  $F_n \supset E_n$  for each  $n$ , such that for some subsequence  $\{n(k)\}$  of  $\{n\}$ ,  $u_{F_{n(k)}} \rightarrow u_0$  and  $t_{F_{n(k)}} \rightarrow t_0$ .*

*Proof.* We construct the asserted sequence  $\{F_n\}$  in  $\Lambda$  as follows:

- (a) *We set  $F_1 = E_1$ .*

(b) Suppose we have already constructed  $F_1 \subset \dots \subset F_n$ , and let  $u_n = u_{F_n} \in C \cap F_n$  and  $t_n = t_{F_n} \in I$  denote the described elements corresponding to  $F_n$  such that  $j_{F_n}^* A(u_n, t_n) = 0$ . There exists  $v_n \in X$ ,  $\|v_n\| = 1$ , such that  $|A(u_n, t_n), v_n| \geq \frac{1}{2} \|A(u_n, t_n)\|$ . We then choose  $F_{n+1} \supset F_n + E_{n+1} + [v_n]$ .

By hypothesis, the sequences  $\{u_n\}$  and  $\{t_n\}$  are bounded. We may pass to infinite subsequences and assure that  $u_n \rightarrow u_0 \in X$  and  $t_n \rightarrow t_0 \in I$ . It follows from condition (ii) that

$$\|A(u_n, t_n) - A(u_n, t_0)\| \rightarrow 0 \quad (n \rightarrow \infty). \quad (3)$$

We assert that

$$(A(u_n, t_0), w) \rightarrow 0 \quad (n \rightarrow \infty) \quad (4)$$

for all  $w \in X_0 = \text{closure } \{\bigcup_{j=1}^{\infty} F_j\}$ . Indeed, if  $w$  lies in some  $F_j$  and  $n \geq j$ , we have

$$|(A(u_n, t_0), w)| \leq |(A(u_n, t_n), w)| + |(A(u_n, t_0) - A(u_n, t_n), w)|,$$

where the first term on the right side vanishes, while the second term tends to 0 as  $n \rightarrow \infty$  according to (3). Because of the boundedness of the sequence  $\{A(u_n, t_0)\}$ , (4) extends to all  $w \in X_0$ . We now get

$$\begin{aligned} |(A(u_n, t_0), u_n - u_0)| &\leq |(A(u_n, t_n), u_n)| \\ &\quad + |(A(u_n, t_0) - A(u_n, t_n), u_n)| + |(A(u_n, t_0), u_0)|. \end{aligned}$$

On the right side of this estimate, the first summand vanishes, the middle term tends to 0 because of (3), and the last approaches 0 according to (4), since the weak limit  $u_0$  of the sequence  $\{u_n\} \subset X_0$  lies in  $X_0$ . Property (S) of the mapping  $A(u, t_0)$  implies that  $u_n \rightarrow u_0$ . Hence  $u_0 \in C$ ,  $A(u_n, t_0) \rightarrow A(u_0, t_0)$ , and

$$A(u_n, t_n) \rightarrow A(u_0, t_0) \quad (5)$$

because of the continuity of the mapping  $A(u, t_0)$  in  $u$  and the estimate (3). We infer that, according to (4),

$$(A(u_0, t_0), w) = 0 \quad \text{for all } w \in X_0. \quad (6)$$

We finally prove that  $A(u_0, t_0) = 0$ . Suppose to the contrary that  $A(u_0, t_0) \neq 0$ . Then, by (5),  $\|A(u_n, t_n)\| \geq d > 0$  for some constant  $d$  and all  $n \geq n_0$ , which implies that

$$|(A(u_n, t_n), v_n)| \geq d/2 > 0$$

for  $n \geq n_0$ . But (5) and the fact that some subsequence of  $\{v_n\}$  (denoted again by  $\{v_n\}$ ) converges weakly to an element  $v_0 \in X_0$  have as a consequence that

$$(A(u_n, t_n), v_n) \rightarrow (A(u_0, t_0), v_0),$$

the expression on the right being 0 according to (6). This contradiction shows that  $A(u_0, t_0) = 0$ , q.e.d.

**3.** We apply the Proposition in order to obtain results on the existence of solutions of the functional equation (1).

**THEOREM 1**<sup>2)</sup>. *Let  $X$  a real reflexive Banach space,  $G$  an open bounded subset of  $X$  containing 0 and symmetric about the origin, and  $A_t u = A(u, t)$  a mapping of  $\text{cl}(G) \times [0, 1]$  into  $X^*$  as follows:*

- (i) *For fixed  $t$ ,  $A_t$  is a bounded continuous mapping of type (S);*
- (ii)  *$A(u, t)$  is uniformly continuous in  $t$  with respect to  $u \in \text{cl}(G)$ ;*
- (iii)  *$A_1$  is odd on  $\text{bdry}(G)$ , i.e.  $A(-u, 1) = -A(u, 1)$  for  $u \in \text{bdry}(G)$ .*

*Assume that  $A(u, t) \neq 0$  for all  $u \in \text{bdry}(G)$  and all  $t \in [0, 1]$ . Then the equation  $A_0 u = 0$  has a solution  $u_0$  in  $G$ .*

Theorem 1 follows by the classical Borsuk theorem [2, 15, 21], the invariance of the Brouwer degree under homotopies, and arguments which have become standard in the theory of mappings of monotone type (e.g. [3, 4, 6, 8, 9, 13, 17, 18, 22, 23]) from

**LEMMA 1.** *Let  $E \in \Lambda$  be given. Then under the assumptions of Theorem 1 there exists  $F \in \Lambda$ ,  $F \supset E$ , such that  $j_F^* A(u, t) \neq 0$  for all  $u \in \text{bdry}(G) \cap F$  and all  $t \in [0, 1]$ .*

*Proof of Lemma 1.* Suppose to each  $F \in \Lambda$  with  $F \supset E$  we can find elements  $u_F \in \text{bdry}(G) \cap F$  and  $t_F \in [0, 1]$  such that  $j_F^* A(u_F, t_F) = 0$ . Applying the Proposition with  $C = \text{bdry}(G)$  and  $I = [0, 1]$ , we are led to a contradiction to the assumptions of Theorem 1, q.e.d.

**DEFINITION.** *A mapping  $A$  from  $X$  to  $X^*$  is said to be pseudo-monotone if for any sequence  $\{u_n\}$  in  $X$  with  $u_n \rightarrow u$  and  $\limsup (Au_n, u_n - u) \leq 0$ , it follows that for all  $v \in X$ ,  $\liminf (Au_n, u_n - v) \geq (Au, u - v)$ .*

Pseudo-monotone mappings have been introduced by Brézis [3] and have grown increasingly important in the discussion of nonlinear elliptic boundary value problems [3, 11, 13, 22]. Everywhere defined continuous monotone operators from  $X$  to  $X^*$  (i.e. mappings  $A$  satisfying  $(Au - Av, u - v) \geq 0$  for all  $u, v$  in  $X$ ) are pseudo-monotone.

For pseudo-monotone operators we have the following extension of Theorem 1.

**THEOREM 2.** *Let  $G$  a convex open bounded subset of the real reflexive Banach space  $X$ , with  $0 \in G$  and  $G$  symmetric about 0. Suppose the mapping  $A_t u = A(u, t) : X \times [0, 1] \rightarrow X^*$  satisfies the conditions:*

- (i) *For fixed  $t$ ,  $A_t$  is bounded, continuous, and pseudo-monotone;*

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<sup>2)</sup> For  $G$  a subset of a Banach space,  $\text{cl}(G)$  denotes its closure and  $\text{bdry}(G)$  its boundary.

- (ii)  $A(u, t)$  is continuous in  $t$ , uniformly with respect to  $u \in \text{cl}(G)$ ;
- (iii)  $A_1$  is odd on  $\text{bdry}(G)$ .

If there exists  $\varepsilon > 0$  such that  $\|A(u, t)\| \geq \varepsilon$  for all  $u \in \text{bdry}(G)$  and  $t \in [0, 1]$ , then the equation  $A_0 u = 0$  is solvable in  $G$ .

*Proof.* By a recent result of Troyanski [25] we can assume without loss of generality that both  $X$  and  $X^*$  are locally uniformly convex spaces. Let  $J$  denote the (single-valued) normalized duality mapping from  $X$  to  $X^*$  given by

$$Ju = \{q \in X^* : (q, u) = \|q\| \|u\|, \|q\| = \|u\|\}.$$

For each  $\lambda > 0$  and  $t \in [0, 1]$ , the mapping  $B_t^{(\lambda)} = A_t + \lambda J$  is then continuous and of type  $(S)$ . By the boundedness of  $G$ , there exists  $\varepsilon_0 > 0$  such that  $B_t^{(\lambda)} u \neq 0$  for all  $u \in \text{bdry}(G)$ ,  $t \in [0, 1]$ , and  $0 \leq \lambda < \varepsilon_0$ . Hence for fixed  $\lambda \in (0, \varepsilon_0)$ , the mapping  $B_t^{(\lambda)} u$  satisfies the assumptions of Theorem 1, and there exists an element  $u_\lambda \in G$  with  $(A_0 + \lambda J) u_\lambda = 0$ . Taking a sequence  $\{\lambda_n\} \rightarrow 0^+$  and assuming that  $u_n = u_{\lambda_n} \rightarrow u_0 \in \text{cl}(G)$ , we obtain  $A_0 u_n = -\lambda_n J u_n \rightarrow 0$  and  $\lim (A_0 u_n, u_n - u_0) = 0$ . By the pseudo-monotonicity of  $A_0$ ,

$$0 = \lim (A_0 u_n, u_n - v) \geq (A_0 u_0, u_0 - v)$$

for all  $v \in X$ . This implies that  $A_0 u_0 = 0$  and  $u_0 \in G$ , q.e.d.

We show now how our theory can be applied to the investigation of nonlinear equations of Hammerstein type

$$u + TFu = f$$

in a nonreflexive Banach space  $X$ . Here  $F$  denotes a (nonlinear) mapping of  $X$  to  $X^*$ ,  $T$  a linear operator of  $X^*$  to  $X$ , and  $f \in X$  a given element. Without assuming that  $T$  is compact (which case leads back to the now-classical theory of compact operators in Banach spaces), it seems to be the first time that Hammerstein equations are considered by methods of operators of monotone type in a nonreflexive space  $X$ . Former investigations were restricted to equations in a reflexive space  $X$ , or in the conjugate space  $X^*$  of some Banach space  $X$  (e.g. [1, 3, 12, 18, 20]).

**DEFINITION.** A bounded linear monotone operator  $T$  of  $X^*$  into  $X$  is said to be angle-bounded if there exists a constant  $\gamma \geq 0$  such that for all  $v, w$  in  $X^*$ ,

$$|(v, Tw) - (w, Tv)| \leq \gamma (v, Tv)^{1/2} (w, Tw)^{1/2}.$$

**LEMMA 2.** Let  $X$  an arbitrary real Banach space,  $F$  a pseudo-monotone mapping of  $X$  to  $X^*$ , and  $T$  an angle-bounded linear operator of  $X^*$  to  $X$ . Then the equation  $u + TFu = f$  in  $X$  can be reduced to an equivalent equation  $Av = 0$  in a Hilbert space  $H$ , with  $A$  a pseudo-monotone mapping of  $H$  into itself. If  $X^*$  is nonseparable, then  $H$  has the same property in general.

*Proof.* By the natural imbedding, we identify  $X$  with a subspace of  $X^{**}$  and consider  $T$  as an (angle-bounded) mapping of  $X^*$  to  $X^{**}$ . By a result of Browder-Gupta [12] (cf. also Amann [1], Hess [20]), there exist a Hilbert space  $H$  (whose norm and inner product we denote by  $\|\cdot\|_H$  and  $(\cdot, \cdot)_H$ , respectively), a continuous linear mapping  $S$  of  $X^*$  to  $H$  with range dense in  $H$ , and a monotone linear bijective mapping  $C$  of  $H$  onto  $H$ , such that  $T = S^*CS$  and  $(C^{-1}v, v)_H \geq d\|v\|_H^2$  for all  $v \in H$ , with  $d > 0$ . Since  $T$  has range contained in  $X$  and  $CS(X^*)$  is dense in  $H$ , it follows that the range of  $S^*$  is contained in  $X \subset X^{**}$ .

By the above result, the equation

$$u + TFu = f \quad (7)$$

is equivalent to the equation

$$u - f + S^*CSFu = 0.$$

Since  $S^*$  is injective, there exists a uniquely determined  $v$  in  $H$  with  $u - f = S^*v$ , and the initial equation (7) and

$$v + CSF(S^*v + f) = 0 \quad (8)$$

are equivalent. By the bijectiveness of  $C$ , (8) holds if and only if

$$C^{-1}v + SF(S^*v + f) = 0.$$

It is readily seen that the operator  $A$ :

$$Av = C^{-1}v + SF(S^*v + f) \quad (v \in H)$$

is a pseudo-monotone mapping of  $H$  into itself. Finally, if  $X^*$  is nonseparable, the same is true in general for  $H$  as the completion of a factorspace  $X^*$  modulo some subspace (cf. the construction of  $H$  in [12]), q.e.d.

An application of Theorem 2 gives the following existence theorem of Fredholm alternative type for asymptotically homogeneous and odd Hammerstein equations.

**THEOREM 3.** *Let  $X$  a separable real Banach space,  $B$  a bounded continuous pseudo-monotone mapping of  $X$  to  $X^*$  which is odd and homogeneous (i.e.  $B(\lambda u) = \lambda Bu$  for  $\lambda \in \mathbb{R}^1$ ), and  $N: X \rightarrow X^*$  a bounded continuous operator with  $\lim_{\|u\| \rightarrow \infty} \|u\|^{-1} \|Nu\| = 0$ , and such that  $B + N$  is pseudo-monotone. Let further  $T$  a linear angle-bounded operator of  $X^*$  to  $X$ . Then the range of  $I + T(B + N)$  is all of  $X$ , provided  $u + TBu = 0$  implies that  $u = 0$ .*

*Proof<sup>3)</sup>.* In order to show that the mapping  $I + T(B + N)$  is surjective, it suffices

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<sup>3)</sup> Here we denote by “ $\rightharpoonup$ ” weak convergence in  $X$  or  $H$ , by “ $\stackrel{*}{\rightharpoonup}$ ” weak\* convergence in  $X^*$ .

by Lemma 2 to prove the solvability of the equation

$$C^{-1}v + S(B + N)(S^*v + f) = 0 \quad (9)$$

in  $H$  for arbitrarily given  $f \in X$ . We observe that if  $u + TBu = 0$  only for  $u = 0$ , then the equation  $C^{-1}v + SBS^*v = 0$  implies that  $v = 0$ .

In the following let  $f \in X$  be fixed. For  $t \in [0, 1]$  and  $v \in H$  we let

$$A_t v = C^{-1}v + (1 - \frac{1}{2}t) S(B + N)(S^*v + f) - \frac{1}{2}tS(B + N)(S^*(-v) + f).$$

It is readily seen that the homotopy  $A_t v$  has the following properties:

- (i) For fixed  $t$ ,  $A_t$  is pseudo-monotone, bounded and continuous;
- (ii)  $A_t v$  is continuous in  $t$ , uniformly for  $v$  in bounded sets;
- (iii)  $A_0 v = C^{-1}v + S(B + N)(S^*v + f)$ , while  $A_1$  is odd.

The desired result on the solvability of the equation (9) follows from Theorem 2, if we prove that, assuming  $C^{-1}v + SBS^*v = 0$  only for  $v = 0$ , there exists  $R > 0$  such that  $\|A_t v\|_H \geq 1$  for all  $t \in [0, 1]$  and all  $v \in H$  with  $\|v\|_H \geq R$ .

Suppose that to each  $n$  we can find elements  $v_n \in H$  with  $\|v_n\|_H \geq n$ ,  $t_n \in [0, 1]$ , and  $e_n \in H$  with  $\|e_n\|_H < 1$ , such that  $A_{t_n} v_n = e_n$ . We may assume that  $t_n \rightarrow t \in [0, 1]$ . Setting  $w_n = \|v_n\|_H^{-1} v_n$ , we then obtain

$$\begin{aligned} & C^{-1}w_n + (1 - \frac{1}{2}t) SB(S^*w_n + \|v_n\|_H^{-1} f) + \frac{1}{2}tSB(S^*w_n - \|v_n\|_H^{-1} f) \\ &= \frac{1}{2}(t_n - t) \{SB(S^*w_n + \|v_n\|_H^{-1} f) - SB(S^*w_n - \|v_n\|_H^{-1} f)\} \\ & \quad - (1 - \frac{1}{2}t_n) \|v_n\|_H^{-1} SN(S^*v_n + f) + \frac{1}{2}t_n \|v_n\|_H^{-1} SN(S^*(-v_n) + f) \\ & \quad + \|v_n\|_H^{-1} e_n \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Because of the separability of  $X$ , the weak\* topology on closed balls in  $X^*$  is metrizable, and balls in  $X^*$  are thus weak\* sequentially compact. By passing to infinite subsequences, we may assure that  $w_n \rightarrow w$  in  $H$ ,  $B(S^*w_n + \|v_n\|_H^{-1} f) \rightharpoonup a$  and  $B(S^*w_n - \|v_n\|_H^{-1} f) \rightharpoonup b$  in  $X^*$ . It follows that  $S^*w_n \pm \|v_n\|_H^{-1} f \rightarrow S^*w$  in  $X$ ,  $C^{-1}w_n \rightarrow C^{-1}w$  in  $H$ , and  $C^{-1}w + (1 - \frac{1}{2}t) Sa + \frac{1}{2}tSb = 0$ . We further infer that

$$\begin{aligned} & (C^{-1}w_n, w_n - w)_H + (1 - \frac{1}{2}t) (B(S^*w_n + \|v_n\|_H^{-1} f), (S^*w_n + \|v_n\|_H^{-1} f) - S^*w) \\ & \quad + \frac{1}{2}t(B(S^*w_n - \|v_n\|_H^{-1} f), (S^*w_n - \|v_n\|_H^{-1} f) - S^*w) \rightarrow 0. \end{aligned}$$

We assume that  $0 < t \leq 1$  (the case  $t = 0$  is treated similarly) and choose further infinite subsequences such that the three limits  $\lim(C^{-1}w_n, w_n - w)_H$ ,  $\lim(B(S^*w_n + \|v_n\|_H^{-1} f), (S^*w_n + \|v_n\|_H^{-1} f) - S^*w)$ , and  $\lim(B(S^*w_n - \|v_n\|_H^{-1} f), (S^*w_n - \|v_n\|_H^{-1} f) - S^*w)$  exist. By the pseudo-monotonicity property of the mappings  $C^{-1}$  and  $B$ , all of the three limits are 0. Hence, again by pseudo-monotonicity,  $a = b = BS^*w$ , and consequently  $C^{-1}w + SBS^*w = 0$ .

Since  $(C^{-1}v_n, v_n)_H \geq d\|v_n\|_H^2$ , we conclude that  $(C^{-1}w_n, w_n)_H \geq d > 0$ . Moreover  $(C^{-1}w_n, w_n)_H \rightarrow (C^{-1}w, w)_H$ . Thus  $w \neq 0$ , q.e.d.

*Remark.* Theorem 3 remains true for  $X$  nonseparable, but reflexive.

**4.** Our principal methodological result on nonlinear eigenvalue problems which extends the corresponding Theorem 1 of Browder [7] to mappings in nonseparable spaces is

**THEOREM 4.** *Let  $X$  a real reflexive Banach space,  $C$  a closed subset of  $X$ , and  $A$ ,  $B$  continuous mappings of  $C$  into  $X^*$ , with  $A$  bounded and of type (S) and  $B$  compact. Let  $\{E_n\}_{n=1}^\infty$  be an increasing sequence in  $\Lambda$  with  $C \cap E_1 \neq \emptyset$ . Suppose to each  $F \in \Lambda$  with  $F \supset E_1$  there exist elements  $u_F \in C \cap F$  and  $t_F \in R^1$  such that  $j_F^*Au_F = t_F j_F^*Bu_F$ , and assume  $u_F$  and  $t_F$  remain uniformly bounded for  $F \supset E_1$ .*

*Then there exists a sequence  $\{F_n\}$  in  $\Lambda$  with  $F_n \supset E_n$  for each  $n$ , such that for some subsequence  $\{n(k)\}$  of  $\{n\}$ ,  $u_{F_{n(k)}} \rightarrow u_0 \in C$ ,  $t_{F_{n(k)}} \rightarrow t_0 \in R^1$ , and  $Au_0 = t_0 Bu_0$ .*

*Proof.* Follows immediately from the Proposition, with  $I = R^1$  and  $A(u, t) = Au - tBu$ .

As an application to the “selfadjoint” case where  $A$  and  $B$  are the derivatives of two functions, we get the following extension of Theorem 3 in [7] and Theorem 14 in [8]:

**THEOREM 5.** *Let  $f$ ,  $h$  continuously differentiable real-valued functions defined on the (not necessarily separable) real reflexive Banach space  $X$ , with  $f'$  bounded and of type (S) and  $h'$  compact. Suppose that for a given constant  $c$  the level set  $M_c(f) = \{u \in X : f(u) = c\}$  is nonempty and bounded, and that for  $u \in M_c(f)$ ,  $(f'u, u) \neq 0$ . Suppose further that there exists a point  $v_0 \in M_c(f)$  and a constant  $d > 0$  such that for all  $u \in M_c(f)$  for which  $h(u) \geq h(v_0)$ ,  $(h'u, u) \geq d$ .*

*Then  $h$  assumes its maximum on  $M_c(f)$  at a point  $u_0$  which is a solution of the equation  $f'u_0 = t_0 h'u_0$  for some real number  $t_0$ .*

*Proof.* By the continuity of  $f$ , the level set  $M_c(f)$  is closed in  $X$ . Let  $F$  an arbitrary element of  $\Lambda$  with  $M_c(f) \cap F \neq \emptyset$ , and let  $f_F$ ,  $h_F$  denote the restrictions of  $f$  and  $h$  to  $F$ . The functions  $f_F$  and  $h_F$  are continuously differentiable on  $F$ , with  $(f_F)' = j_F^* f' j_F$ ,  $(h_F)' = j_F^* h' j_F$ . We set  $M_{c,F}(f) = M_c(f) \cap F$ . Since  $((f_F)'u, u) = (f'u, u) \neq 0$  for all  $u \in M_{c,F}(f)$ ,  $M_{c,F}(f)$  is a compact manifold of codimension 1 in  $F$ . Thus there exists  $u_F \in M_{c,F}(f)$  such that  $h(u_F) = \sup_{u \in M_{c,F}(f)} h(u)$ . By the Lagrange multiplier method,

$$(h_F)'u_F = \lambda_F (f_F)'u_F \quad (10)$$

for some real  $\lambda_F$ .

Let  $\{w_n\}$  be a sequence in  $M_c(f)$  with  $h(w_n) \rightarrow m = \sup_{u \in M_c(f)} h(u)$ . We choose an increasing sequence  $\{E_n\}_{n=1}^\infty$  in  $\Lambda$  such that  $E_1 \supset \{v_0, w_1\}$ , while  $w_n \in E_n$  for  $n \geq 2$ .

In order to prove the applicability of Theorem 4 with  $C = M_c(f)$ , we show that for  $F \in \Lambda$ ,  $F \supset E_1$ , the corresponding numbers  $(\lambda_F)^{-1}$  of (10) are uniformly bounded. Indeed, it follows from (10) that  $|(h'u_F, u_F)| = |\lambda_F| |(f'u_F, u_F)|$ , where  $|(h'u_F, u_F)| \geq d > 0$  and  $|(f'u_F, u_F)| \leq k_0$  for each  $F \in \Lambda$  with  $F \supset E_1$ . Thus  $|\lambda_F| \geq k_1 > 0$ , and we can write  $(f_F)' u_F = t_F (h_F)' u_F$ , with  $t_F = (\lambda_F)^{-1}$  uniformly bounded.

By Theorem 4 there exists a sequence  $\{F_n\}$  in  $\Lambda$  with  $F_n \supset E_n$  for each  $n$ , such that  $u_{F_{n(k)}} \rightarrow u_0 \in M_c(f)$ ,  $t_{F_{n(k)}} \rightarrow t_0 \in R^1$ , and  $f'u_0 = t_0 h'u_0$ . Since  $w_n \in E_n \subset F_n$ ,  $h(w_n) \leq h(u_{F_n})$ . In this last relation the left side converges to  $m$ , while  $h(u_{F_{n(k)}}) \rightarrow h(u_0)$  by continuity of  $h$ . Hence  $h(u_0) = \sup_{u \in M_c(f)} h(u)$ , q.e.d.

In a similar way one generalizes Theorem 15 of [8].

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