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Character Modules

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1. Introduction

In this paper we use the Bourbaki [2] conventions for rings and modules: all rings are associative but not necessarily commutative and have a 1; all modules are unital.

For any A-module M let $M^* = \mathrm{Hom}_z\,[M, Q/Z] = [M, Q/Z]_z = [M, Q/Z]$ denote the character module, where Q=rationals and Z=integers. Then * is an exact contravariant zero reflecting functor from **A** to $\mathbf{A}^{\mathrm{opp}}$ where **A** is the category of left (or right) A-modules. Details may be found in Lambek [7], who has shown [6] that M is flat iff M^* is injective. Here we extend this result by showing that $\mathrm{wd}\,M = \mathrm{injd}\,M^*$ where wd=weak dimension and injd=injective dimension. For coherent rings we show that $\mathrm{wd}\,M^* \leqslant \mathrm{injd}\,M$ with equality iff the ring is noetherian. Finally we give a new characterization of rings for which pure submodules are always direct summands.

2. The Dimension Theorems

THEOREM 2.1. *For all M we have* $\mathrm{wd}\,M = \mathrm{injd}\,M^*$

Proof. Since Q/Z is Z-injective, we have $\mathrm{Ext}^n(N, M^*) \cong (\mathrm{Tor}^n(N, M))^*$ for all N and all $n \geqslant 0$. (See Cartan-Eilenberg [3] p. 120.) Then

$$\begin{aligned} \mathrm{wd}\, M \leqslant n &\Leftrightarrow \mathrm{Tor}^{n+1}(N, M) = 0 \text{ for all } N \\ &\Leftrightarrow (\mathrm{Tor}^{n+1}(N, M))^* = 0 \text{ for all } N \\ &\Leftrightarrow \mathrm{Ext}^{n+1}(N, M^*) = 0 \text{ for all } N \\ &\Leftrightarrow \mathrm{injd}\, M^* \leqslant n \end{aligned}$$

We recall that A is left coherent iff every finitely presented left A-module is coherent, which means that all finitely generated submodules are finitely presented. For details see Bourbaki ([2] p. 62–3).

LEMMA. *If A is left coherent then every finitely presented left A-module has a resolution by finitely generated free modules.*

Proof. Clearly it suffices to show that if $O \to K \to F \to M \to O$ is exact with F finitely generated free and M finitely presented then K is finitely presented. Since M is finitely presented and F is finitely generated we have K finitely generated. But F is finitely presented and hence so is K.

THEOREM 2.2, *Let A be left coherent. Then* $\mathrm{wd}\, M^* \leqslant \mathrm{injd}\, M$ *for all left A-modules M. Equality holds for all M iff A is left noetherian.*

Proof. Since A is left coherent we have, by Cartan-Eilenberg ([3] p. 120–1), $\mathrm{Tor}^n(M^*, N) \cong (\mathrm{Ext}^n(N, M))^*$ for all M and all finitely presented N. Then

$$\begin{aligned} \mathrm{injd}\, M \leqslant n &\Leftrightarrow \mathrm{Ext}^{n+1}(N, M) = 0 \text{ for all } N \\ &\Leftrightarrow (\mathrm{Ext}^{n+1}(N, M))^* = 0 \text{ for all } N \\ &\Rightarrow \mathrm{Tor}^{n+1}(M^*, N) = 0 \text{ for all finitely presented } N \\ &\Leftrightarrow \mathrm{wd}\, M^* \leqslant n \end{aligned}$$

whence $\mathrm{wd}\, M^* \leqslant \mathrm{injd}\, M$.

Suppose equality holds. If $M = \oplus_I M_i$ over any index set I then it is easy to see that $M^* = \pi_I M_i^*$. If each M_i is injective then each M_i^* is flat and πM_i^* is flat since A is coherent. Since we have equality M is injective and hence A is noetherian by a criterion of Bass [1]. If A is left noetherian the first isomorphism holds for all finitely generated N. Hence:

$$\begin{aligned} \mathrm{wd}\, M^* \leqslant n &\Rightarrow \mathrm{Ext}^{n+1}(A/I, M) = 0 \text{ for all left ideals } I \text{ of } A \\ &\Rightarrow \mathrm{injd}\, M \leqslant n \end{aligned}$$

and we have equality.

3. Purity

A short exact sequence of left A-modules $E: O \to E_1 \to E_2 \to E_3 \to O$ is *pure exact* in the sense of Cohn [4] iff for all (or equivalently for all finitely presented) right A-modules M we have $O \to M \otimes E_1 \to M \otimes E_2 \to M \otimes E_3 \to O$ exact. We have shown in (5) that this is equivalent to $[N, E_2] \to [N, E_3]$ being epic for all finitely presented left A-modules N.

THEOREM 3.1. *For any short exact sequence $E: O \to E_1 \to E_2 \to E_3 \to O$ let E^* denote the corresponding character sequence: $O \leftarrow E_1^* \leftarrow E_2^* \leftarrow E_3^* \leftarrow O$. Then the following are equivalent for any such E.*

(1) E is pure exact.

(2) E^* is split exact.

(3) E^* is pure exact.

Proof. (1)$\Rightarrow$(2). From Theorem 2.1 we have for $n=0$: $[N, M^*] \cong (N \otimes M)^*$. Since E is pure exact we have $E_1^* \otimes E_1 \to E_1{}^* \otimes E_2$ monic

whence $(E_1^* \otimes E_1)^* \leftarrow (E_1^* \otimes E_2)^*$ epic

i.e. $[E_1^*, E_1^*] \leftarrow [E_1^*, E_2^*]$ epic and E^* splits.

(2)$\Rightarrow$(3) since every split exact sequence is pure exact. (3)$\Rightarrow$(1) For all finitely presented N we have, by Bourbaki ([2] p. 63 Ex.14) $M^* \otimes N \cong [N, M]^*$ for all M and hence

$$E^* \text{ pure exact}$$
$$\Rightarrow: E_2^* \otimes N \leftarrow E_3^* \otimes N \text{ monic}$$
$$\Rightarrow: [N, E_2]^* \leftarrow [N, E_3]^* \text{ monic}$$
$$\Rightarrow: [N, E_2] \rightarrow [N, E_3] \text{ epic}$$

and E is pure exact.

COROLLARY 1. *E is pure exact iff E^* is split exact.*

COROLLARY 2. *A is a ring for which all pure submodules are direct summands iff E split exact $\Leftrightarrow E^*$ split exact.*

Remark: Such rings are called PDS rings, and have been studied in [5].

Recall that a module M is pure simple (resp. indecomposable) iff there are no pure submodules (resp. direct summands) other than O and M.

COROLLARY 3. *A module M is pure simple iff M^* is indecomposable.*

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