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**Autor:** Fieldhouse, D.J.  
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## Character Modules

by D. J. FIELDHOUSE (University of Guelph, Guelph, Ontario, Canada)

### 1. Introduction

In this paper we use the Bourbaki [2] conventions for rings and modules: all rings are associative but not necessarily commutative and have a 1; all modules are unital.

For any  $A$ -module  $M$  let  $M^* = \text{Hom}_z[M, Q/Z] = [M, Q/Z]_z = [M, Q/Z]$  denote the character module, where  $Q$ =rationals and  $Z$ =integers. Then  $*$  is an exact contravariant zero reflecting functor from  $\mathbf{A}$  to  $\mathbf{A}^{\text{opp}}$  where  $\mathbf{A}$  is the category of left (or right)  $A$ -modules. Details may be found in Lambek [7], who has shown [6] that  $M$  is flat iff  $M^*$  is injective. Here we extend this result by showing that  $\text{wd } M = \text{injd } M^*$  where  $\text{wd}$ =weak dimension and  $\text{injd}$ =injective dimension. For coherent rings we show that  $\text{wd } M^* \leq \text{injd } M$  with equality iff the ring is noetherian. Finally we give a new characterization of rings for which pure submodules are always direct summands.

### 2. The Dimension Theorems

**THEOREM 2.1.** *For all  $M$  we have  $\text{wd } M = \text{injd } M^*$*

*Proof.* Since  $Q/Z$  is  $Z$ -injective, we have  $\text{Ext}^n(N, M^*) \cong (\text{Tor}^n(N, M))^*$  for all  $N$  and all  $n \geq 0$ . (See Cartan-Eilenberg [3] p. 120.) Then

$$\begin{aligned} \text{wd } M \leq n &\Leftrightarrow \text{Tor}^{n+1}(N, M) = 0 \text{ for all } N \\ &\Leftrightarrow (\text{Tor}^{n+1}(N, M))^* = 0 \text{ for all } N \\ &\Leftrightarrow \text{Ext}^{n+1}(N, M^*) = 0 \text{ for all } N \\ &\Leftrightarrow \text{injd } M^* \leq n \end{aligned}$$

We recall that  $A$  is left coherent iff every finitely presented left  $A$ -module is coherent, which means that all finitely generated submodules are finitely presented. For details see Bourbaki ([2] p. 62-3).

**LEMMA.** *If  $A$  is left coherent then every finitely presented left  $A$ -module has a resolution by finitely generated free modules.*

*Proof.* Clearly it suffices to show that if  $O \rightarrow K \rightarrow F \rightarrow M \rightarrow O$  is exact with  $F$  finitely generated free and  $M$  finitely presented then  $K$  is finitely presented. Since  $M$  is finitely presented and  $F$  is finitely generated we have  $K$  finitely generated. But  $F$  is finitely presented and hence so is  $K$ .

**THEOREM 2.2.** *Let  $A$  be left coherent. Then  $\text{wd } M^* \leq \text{injd } M$  for all left  $A$ -modules  $M$ . Equality holds for all  $M$  iff  $A$  is left noetherian.*

*Proof.* Since  $A$  is left coherent we have, by Cartan-Eilenberg ([3] p. 120–1),  $\text{Tor}^n(M^*, N) \cong (\text{Ext}^n(N, M))^*$  for all  $M$  and all finitely presented  $N$ . Then

$$\begin{aligned} \text{injd } M \leq n &\Leftrightarrow \text{Ext}^{n+1}(N, M) = 0 \text{ for all } N \\ &\Leftrightarrow (\text{Ext}^{n+1}(N, M))^* = 0 \text{ for all } N \\ &\Rightarrow \text{Tor}^{n+1}(M^*, N) = 0 \text{ for all finitely presented } N \\ &\Leftrightarrow \text{wd } M^* \leq n \end{aligned}$$

whence  $\text{wd } M^* \leq \text{injd } M$ .

Suppose equality holds. If  $M = \bigoplus_I M_i$  over any index set  $I$  then it is easy to see that  $M^* = \pi_I M_i^*$ . If each  $M_i$  is injective then each  $M_i^*$  is flat and  $\pi M_i^*$  is flat since  $A$  is coherent. Since we have equality  $M$  is injective and hence  $A$  is noetherian by a criterion of Bass [1]. If  $A$  is left noetherian the first isomorphism holds for all finitely generated  $N$ . Hence:

$$\begin{aligned} \text{wd } M^* \leq n &\Rightarrow \text{Ext}^{n+1}(A/I, M) = 0 \text{ for all left ideals } I \text{ of } A \\ &\Rightarrow \text{injd } M \leq n \end{aligned}$$

and we have equality.

### 3. Purity

A short exact sequence of left  $A$ -modules  $E: O \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow O$  is *pure exact* in the sense of Cohn [4] iff for all (or equivalently for all finitely presented) right  $A$ -modules  $M$  we have  $O \rightarrow M \otimes E_1 \rightarrow M \otimes E_2 \rightarrow M \otimes E_3 \rightarrow O$  exact. We have shown in (5) that this is equivalent to  $[N, E_2] \rightarrow [N, E_3]$  being epic for all finitely presented left  $A$ -modules  $N$ .

**THEOREM 3.1.** *For any short exact sequence  $E: O \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow O$  let  $E^*$  denote the corresponding character sequence:  $O \leftarrow E_1^* \leftarrow E_2^* \leftarrow E_3^* \leftarrow O$ . Then the following are equivalent for any such  $E$ .*

- (1)  $E$  is pure exact.
- (2)  $E^*$  is split exact.
- (3)  $E^*$  is pure exact.

*Proof.* (1)  $\Rightarrow$  (2). From Theorem 2.1 we have for  $n=0$ :  $[N, M^*] \cong (N \otimes M)^*$ . Since  $E$  is pure exact we have  $E_1^* \otimes E_1 \rightarrow E_1^* \otimes E_2$  monic

whence  $(E_1^* \otimes E_1)^* \leftarrow (E_1^* \otimes E_2)^*$  epic

i.e.  $[E_1^*, E_1^*] \leftarrow [E_1^*, E_2^*]$  epic and  $E^*$  splits.

(2)  $\Rightarrow$  (3) since every split exact sequence is pure exact. (3)  $\Rightarrow$  (1) For all finitely

presented  $N$  we have, by Bourbaki ([2] p. 63 Ex.14)  $M^* \otimes N \cong [N, M]^*$  for all  $M$  and hence

$$\begin{aligned} E^* &\text{ pure exact} \\ \Rightarrow : E_2^* \otimes N &\leftarrow E_3^* \otimes N \text{ monic} \\ \Rightarrow : [N, E_2]^* &\leftarrow [N, E_3]^* \text{ monic} \\ \Rightarrow : [N, E_2] &\rightarrow [N, E_3] \text{ epic} \end{aligned}$$

and  $E$  is pure exact.

**COROLLARY 1.**  *$E$  is pure exact iff  $E^*$  is split exact.*

**COROLLARY 2.** *A is a ring for which all pure submodules are direct summands iff  $E$  split exact  $\Leftrightarrow E^*$  split exact.*

*Remark:* Such rings are called PDS rings, and have been studied in [5].

Recall that a module  $M$  is pure simple (resp. indecomposable) iff there are no pure submodules (resp. direct summands) other than  $O$  and  $M$ .

**COROLLARY 3.** *A module  $M$  is pure simple iff  $M^*$  is indecomposable.*

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