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# Character Modules

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## 1. Introduction

In this paper we use the Bourbaki [2] conventions for rings and modules: all rings are associative but not necessarily commutative and have a 1; all modules are unital.

For any $A$-module $M$ let $M^* = \mathrm{Hom}_z\,[M, Q/Z] = [M, Q/Z]_z = [M, Q/Z]$ denote the character module, where $Q$ = rationals and $Z$ = integers. Then $*$ is an exact contravariant zero reflecting functor from $\mathbf{A}$ to $\mathbf{A}^{\mathrm{opp}}$ where $\mathbf{A}$ is the category of left (or right) $A$-modules. Details may be found in Lambek [7], who has shown [6] that $M$ is flat iff $M^*$ is injective. Here we extend this result by showing that $\mathrm{wd}\,M = \mathrm{injd}\,M^*$ where wd = weak dimension and injd = injective dimension. For coherent rings we show that $\mathrm{wd}\,M^* \leqslant \mathrm{injd}\,M$ with equality iff the ring is noetherian. Finally we give a new characterization of rings for which pure submodules are always direct summands.

## 2. The Dimension Theorems

THEOREM 2.1. *For all $M$ we have* $\mathrm{wd}\,M = \mathrm{injd}\,M^*$

*Proof.* Since $Q/Z$ is $Z$-injective, we have $\mathrm{Ext}^n(N, M^*) \cong (\mathrm{Tor}^n(N, M))^*$ for all $N$ and all $n \geqslant 0$. (See Cartan-Eilenberg [3] p. 120.) Then

$$\begin{aligned}
\mathrm{wd}\,M \leqslant n &\Leftrightarrow \mathrm{Tor}^{n+1}(N, M) = 0 \text{ for all } N \\
&\Leftrightarrow (\mathrm{Tor}^{n+1}(N, M))^* = 0 \text{ for all } N \\
&\Leftrightarrow \mathrm{Ext}^{n+1}(N, M^*) = 0 \text{ for all } N \\
&\Leftrightarrow \mathrm{injd}\,M^* \leqslant n
\end{aligned}$$

We recall that $A$ is left coherent iff every finitely presented left $A$-module is coherent, which means that all finitely generated submodules are finitely presented. For details see Bourbaki ([2] p. 62–3).

LEMMA. *If $A$ is left coherent then every finitely presented left $A$-module has a resolution by finitely generated free modules.*

*Proof.* Clearly it suffices to show that if $O \to K \to F \to M \to O$ is exact with $F$ finitely generated free and $M$ finitely presented then $K$ is finitely presented. Since $M$ is finitely presented and $F$ is finitely generated we have $K$ finitely generated. But $F$ is finitely presented and hence so is $K$.

THEOREM 2.2, *Let $A$ be left coherent. Then* $\operatorname{wd} M^* \leqslant \operatorname{injd} M$ *for all left $A$-modules $M$. Equality holds for all $M$ iff $A$ is left noetherian.*

*Proof.* Since $A$ is left coherent we have, by Cartan-Eilenberg ([3] p. 120–1), $\operatorname{Tor}^n(M^*, N) \cong (\operatorname{Ext}^n(N, M))^*$ for all $M$ and all finitely presented $N$. Then

$$\begin{aligned}\operatorname{injd} M \leqslant n &\Leftrightarrow \operatorname{Ext}^{n+1}(N, M) = 0 \text{ for all } N\\ &\Leftrightarrow (\operatorname{Ext}^{n+1}(N, M))^* = 0 \text{ for all } N\\ &\Rightarrow \operatorname{Tor}^{n+1}(M^*, N) = 0 \text{ for all finitely presented } N\\ &\Leftrightarrow \operatorname{wd} M^* \leqslant n\end{aligned}$$

whence $\operatorname{wd} M^* \leqslant \operatorname{injd} M$.

Suppose equality holds. If $M = \oplus_I M_i$ over any index set $I$ then it is easy to see that $M^* = \pi_I M_i^*$. If each $M_i$ is injective then each $M_i^*$is flat and $\pi M_i^*$ is flat since $A$ is coherent. Since we have equality $M$ is injective and hence $A$ is noetherian by a criterion of Bass [1]. If $A$ is left noetherian the first isomorphism holds for all finitely generated $N$. Hence:

$$\begin{aligned}\operatorname{wd} M^* \leqslant n &\Rightarrow \operatorname{Ext}^{n+1}(A/I, M) = 0 \text{ for all left ideals } I \text{ of } A\\ &\Rightarrow \operatorname{injd} M \leqslant n\end{aligned}$$

and we have equality.

## 3. Purity

A short exact sequence of left $A$-modules $E: O \to E_1 \to E_2 \to E_3 \to O$ is *pure exact* in the sense of Cohn [4] iff for all (or equivalently for all finitely presented) right $A$-modules $M$ we have $O \to M \otimes E_1 \to M \otimes E_2 \to M \otimes E_3 \to O$ exact. We have shown in (5) that this is equivalent to $[N, E_2] \to [N, E_3]$ being epic for all finitely presented left $A$-modules $N$.

THEOREM 3.1. *For any short exact sequence $E: O \to E_1 \to E_2 \to E_3 \to O$ let $E^*$ denote the corresponding character sequence: $O \leftarrow E_1^* \leftarrow E_2^* \leftarrow E_3^* \leftarrow O$. Then the following are equivalent for any such $E$.*

(1) $E$ is pure exact.

(2) $E^*$ is split exact.

(3) $E^*$ is pure exact.

*Proof.* (1)$\Rightarrow$(2). From Theorem 2.1 we have for $n=0$: $[N, M^*] \cong (N \otimes M)^*$. Since $E$ is pure exact we have $E_1^* \otimes E_1 \to E_1{}^* \otimes E_2$ monic

whence $(E_1^* \otimes E_1)^* \leftarrow (E_1^* \otimes E_2)^*$ epic

i.e. $[E_1^*, E_1^*] \leftarrow [E_1^*, E_2^*]$ epic and $E^*$ splits.

(2)$\Rightarrow$(3) since every split exact sequence is pure exact. (3)$\Rightarrow$(1) For all finitely

presented $N$ we have, by Bourbaki ([2] p. 63 Ex.14) $M^* \otimes N \cong [N, M]^*$ for all $M$ and hence

$E^*$ pure exact

$\Rightarrow: E_2^* \otimes N \leftarrow E_3^* \otimes N$ monic

$\Rightarrow: [N, E_2]^* \leftarrow [N, E_3]^*$ monic

$\Rightarrow: [N, E_2] \rightarrow [N, E_3]$ epic

and $E$ is pure exact.

COROLLARY 1. *$E$ is pure exact iff $E^*$ is split exact.*

COROLLARY 2. *$A$ is a ring for which all pure submodules are direct summands iff $E$ split exact $\Leftrightarrow E^*$ split exact.*

*Remark*: Such rings are called PDS rings, and have been studied in [5].

Recall that a module $M$ is pure simple (resp. indecomposable) iff there are no pure submodules (resp. direct summands) other than $O$ and $M$.

COROLLARY 3. *A module $M$ is pure simple iff $M^*$ is indecomposable.*

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