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## Invertible Knot Cobordisms\*

D. W. SUMNERS

### I. Introduction

In [1], R. H. Fox studies embeddings of  $S^2$  in  $S^4$  by means of 3-dimensional hyperplanes slicing the embedded  $S^2$  in 1-complexes. The connected manifold cross-sections are slice knots; the non-connected manifold cross-sections are (weakly) slice links. Fox poses the following question [2, #39]: Which slice knots and weakly slice links can appear as cross-sections of the *unknotted*  $S^2$  in  $S^4$ ? Many examples of this phenomenon are known [3, 4, 14, 16, 19], and Hosokawa [3] has given sufficient geometric conditions on the cross-sectional sequence of the  $S^2$  in  $S^4$  for the  $S^2$  to unknot.

This paper considers Fox's question in the setting of higher-dimensional knot theory, but from a different viewpoint – that of inverting a knot cobordism. A slice knot is one which is cobordant to the unknot. If the slice knot bounds a cobordism (to the unknot) which is invertible from the knotted end, then the slice knot is said to be doubly-null-cobordant [19]. The doubly-null-cobordant knots are precisely the cross-sections of the unknot. We supply in this paper detailed proofs of the results announced in [18]. Using techniques of Levine [8], we develop necessary algebraic conditions for an odd-dimensional knot to be doubly-null-cobordant. These conditions are shown to be sufficient in a restricted case. We show that the Stevedore's Knot ( $6_1$ ) is not doubly-null-cobordant, and that in fact  $9_{46}$  is the only knot in Reidemeister's table of prime knots [10] which is doubly-null-cobordant.

We then turn to the problem of geometric realization of doubly-null-cobordant knots. We show that all allowable systems of invariants can be geometrically realized.

A development of similar results for higher-dimensional codimension two links will be dealt with in a future paper.

### II. Necessary Conditions for Invertibility

An  $n$ -knot  $K$  is a smooth pair  $(S^{n+2}, k)$  where  $k$  is a smooth oriented submanifold homeomorphic to  $S^n$ . Two  $n$ -knots  $K_1, K_2$  are *cobordant* if there exists a proper smooth oriented submanifold  $w$  of  $S^{n+2} \times I$ , with  $\partial w = (k_1 \times 0 \cup (-k_2) \times 1)$  and  $w$  homeomorphic to  $S^n \times I$ . Let  $(W; K_1, K_2)$  denote  $(S^{n+2} \times I, w)$  the cobordism between  $K_1$  and  $K_2$ . If  $U = (S^{n+2}, S^n)$  denotes the standard (unknotted) sphere pair, then an  $n$ -knot  $K$  is said to be null-cobordant if it is cobordant to  $U$ .

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Following [13], we identify two cobordisms if there is an orientation preserving diffeomorphism (of pairs) between them. If  $(W; K_1, K_2)$  and  $(W'; K_2, K_3)$  are two cobordisms with a common boundary component (oriented oppositely) we can then paste  $W'$  to  $W$  along  $K_2$  to get  $(W \cup W'; K_1, K_3)$ .

**DEFINITION.** A cobordism  $(W; K_1, K_2)$  is said to be *invertible  $K_2$*  if there is a cobordism  $(W'; K_2, K_1)$  such that  $(W \cup W'; K_1, K_1) = (K_1 \times I, K_1, K_1)$  the product cobordism of pairs.

Given the above situation, we say that  $K_1$  *bounds an invertible cobordism and  $K_2$  splits  $K_1 \times I$ .*

**PROPOSITION 2.1.**  $(W; K_1, K_2)$  *invertible from both ends then  $(S^{n+2} \times I - w)$  is an  $h$ -cobordism from  $S^{n+2} - k_1$  to  $S^{n+2} - k_2$ .*

*Proof.* Just as in [7], we have  $(S^{n+2} \times I - w) - (S^{n+2} - k_2) \approx (S^{n+2} - k_1) \times [0, 1)$ ,  $(S^{n+2} \times I - w) - (S^{n+2} - k_1) \approx (S^{n+2} - k_2) \times (0, 1]$  from which the lemma immediately follows.

Because of Proposition 2.1, we will restrict consideration to cobordisms invertible from one end only. Noninvertible cobordisms abound in knot theory, and we are led to the following:

*Question.* Do there exist invertible knot cobordisms (other than product cobordisms)? The answer is provided by

**THEOREM 2.2.** *Every  $n$ -knot  $K$  bounds infinitely many distinct invertible cobordisms.*

*Proof.* As will be shown in §IV, it is possible to split  $U \times I$  by infinitely many distinct knots. If  $(B^{n+2}, B^n)$  denotes the unknotted ball pair, then each splitting of  $U \times I$  induces a splitting of  $(B^{n+2}, B^n) \times I$  which is the trivial product splitting on the boundary pair. Given  $K \times I$ , pick a point  $x \in k \subset S^{n+2}$ . The point  $x$  has an unknotted ball pair neighborhood  $x \in (B_x^{n+2}, B_x^n) \subset (S^{n+2}, k)$ . If  $K'$  splits  $U \times I$ , then we have an induced splitting of  $(B_x^{n+2}, B_x^n) \times I$ , and this allows us to split  $K \times I$  by  $K \# K'$ , where  $\#$  denotes the usual connected sum operation on knots.

*Note.* If  $(M^{n+2}, N^n)$  denotes a smooth codimension two manifold pair, then the above techniques can be used to obtain infinitely many distinct splittings of  $(M^{n+2}, N^n) \times I$ .

*Question.* Which knots split  $U \times I$ ?

**DEFINITION.** An  $n$ -knot  $K$  is *doubly-null-cobordant* if it splits  $U \times I$ . (These knots are sometimes called *cross-sections of the unknot* [1, 4].)

In order to obtain a necessary condition on the invariants of a knot in order for it

to split  $U \times I$ , we recall Levine's formulation [8, 9] of the Seifert matrix of a knot:

Let  $K = (S^{2q+1}, k)$  be a  $(2q - 1)$ -knot.

Then  $k$  bounds an orientable manifold  $V^{2q}$ , called a *Seifert manifold* for  $k$ . We define a pairing

$$\theta: H_q(V) \otimes H_q(V) \rightarrow Z \quad \alpha \otimes \beta \mapsto L(\alpha, i_*(\beta))$$

where  $L$  denotes homological linking number in  $S^{2q+1}$  and  $i_*(\beta)$  denotes the translate of the cycle  $\beta$  off  $V$  in the positive normal direction. A basis for the free part of  $H_q(V)$  determines a matrix  $M$  representing  $\theta$ .  $M$  is called a *Seifert matrix for  $k$ , associated with the manifold  $V$* .

DEFINITION. A square matrix  $M$  (over  $Z$ ) is said to be *null-cobordant* if it is congruent to  $\left( \begin{array}{c|c} 0 & N_1 \\ \hline N_2 & N_3 \end{array} \right)$  where  $N_i$  are square and of the same size.  $M$  is said to be *doubly-null-cobordant* if it is congruent to  $\left( \begin{array}{c|c} 0 & N_1 \\ \hline N_2 & 0 \end{array} \right)$  where  $N_i$  are square and of the same size.

Levine [8] proves the following:

THEOREM (Levine).  $K = (S^{2q+1}, k)$   $q \geq 1$  *null-cobordant* then  $K$  has a *null-cobordant Seifert matrix*. The implication is reversible for  $q \geq 2$ .

A similar relationship can be obtained between doubly-null-cobordant knots and matrices.

THEOREM 2.3.  $K = (S^{2q+1}, k)$   $q \geq 1$  *doubly-null-cobordant* then  $K$  has a *doubly-null-cobordant Seifert matrix*.

*Proof.* We have cobordisms  $(W; U, K)$  and  $(W'; K, U)$  such that  $(W \cup W'; U, U) = (U \times I; U, U)$ . We convert the trivial cobordism  $(U \times I; U, U)$  to the unknotted sphere pair  $(S^{2q+2}, S^{2q})$  by adding two copies of the unknotted ball pair  $(B^{2q+2}, B^{2q})$ , one to each end of the cobordism. Now  $S^{2q}$  is unknotted hence bounds a disc  $D^{2q+1}$  in  $S^{2q+2}$ . By relative transversality arguments [11] we can assume that  $D^{2q+1}$  hits the hypersphere  $S^{2q+1}$  defining  $K$  transversely; that is  $D^{2q+1} \cap S^{2q+1} = V^{2q}$  and  $\partial V = k$ .  $V$  is a Seifert manifold (not necessarily connected) for  $k$ .  $S^{2q+1}$  splits  $D^{2q+1}$  along  $V$  into two parts,  $W_1$  and  $W_2$ ,  $W_1$  lying on one side of  $S^{2q+1}$  in  $S^{2q+2}$ ,  $W_2$  on the other.

We will show that the Seifert matrix associated with  $V$  is doubly-null-cobordant. It suffices to find a free basis  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$  for the free part of  $H_q(V; Z)$  such that  $\theta(\alpha_i, \alpha_j) = 0 = \theta(\beta_i, \beta_j)$   $1 \leq i, j \leq r$ . Consider now the Mayer-Vietoris sequence for the triad  $(D^{2q+1}, W_1, W_2)$ :

$$0 \rightarrow H_q(V) \xrightarrow[\cong]{i_{1*} \oplus (-i_{2*})} H_q(W_1) \oplus H_q(W_2) \rightarrow 0$$

where  $i_{1*}$  and  $i_{2*}$  are inclusion homomorphisms. We have that  $H_q(V) = \text{Ker } i_{1*} \oplus \text{Ker } i_{2*}$ . Levine [8, Lemma 2] proves that  $\text{rank}(\text{Ker } i_{1*}) = \frac{1}{2} \text{rank}(H_n(V)) = \text{rank}(\text{Ker } i_{2*})$ , so let  $\alpha_1, \dots, \alpha_r$  freely generate the free part of  $\text{Ker } i_{1*}$ , and  $\beta_1, \dots, \beta_r$  freely generate the free part of  $\text{Ker } i_{2*}$ . Now  $\alpha_i, \alpha_j \in \text{Ker } i_{1*}$  so  $\alpha_i, i_*(\alpha_j)$  bound disjoint chains in  $B^{2q+2}$ , hence  $\theta(\alpha_i, \alpha_j) = 0$ . This completes the proof of Theorem 2.3.

Let  $M$  be a Seifert matrix for  $K = (S^{2q+1}, k)$ , and  $\tilde{K}$  denote the infinite cyclic cover of  $S^{2q+1} - k$ . Let  $J(t)$  be the infinite cyclic group of covering translations of  $\tilde{K}$ , and  $\Lambda$  denote the integral group ring of  $J(t)$ . Let  $\Gamma = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ , the rational group ring of  $J(t)$ .  $\Gamma$  is a principal ideal domain, and  $\Lambda$  is a Noetherian domain.  $H_i(\tilde{K}; \mathbb{Z})$  is a finitely presented  $\Lambda$ -module [9] for all  $i$ ; likewise for  $H_i(\tilde{K}; \mathbb{Q})$  as a  $\Gamma$ -module. In fact  $[tM + (-1)^q M']$  presents  $H_q(\tilde{K}; \mathbb{Q})$  as a  $\Gamma$ -module, where  $M'$  denotes the transpose of  $M$ . (See [9].) If  $K$  is a simple knot ( $\pi_i(S^{2q+1} - k) \cong \pi_i(S^1)$   $i \leq q - 1$ ) then  $[tM + (-1)^q M']$  presents  $H_q(\tilde{K}; \mathbb{Z})$  as a  $\Lambda$ -module. It also turns out [12] that  $[M + (-1)^{q+1} M']$  presents  $H_q(K_2; \mathbb{Z})$  as an abelian group, where  $K_2$  denotes the 2-fold branched cyclic covering of the knot  $K$ .

We have the following corollary to Theorem 2.3:

**COROLLARY 2.4.** *If a  $(2q - 1)$  knot  $K$  ( $q \geq 1$ ) is doubly-null-cobordant, then  $H_q(\tilde{K}; \mathbb{Q})$  is presented as a  $\Gamma$ -module by a diagonal matrix of the form  $\text{diag}(\theta_1^q(t), \dots, \theta_m^q(t), \theta_1^q(t^{-1}), \dots, \theta_m^q(t^{-1}))$  where  $\theta_i^q \in \Lambda$ ,  $\theta_{i+1}^q | \theta_i^q$  in  $\Lambda$ , and  $\theta_i^q(1) = \pm 1$  for all  $i$ .*

*Proof.* If  $A$  and  $B$  are matrices over a ring  $R$ , we say that  $A \sim B$  iff  $A$  and  $B$  present isomorphic  $R$ -modules. Now we have by Theorem 2.3 a doubly-null-cobordant Seifert

matrix  $M = \left( \begin{array}{c|c} 0 & N_1 \\ \hline N_2 & 0 \end{array} \right)$ . Hence

$$[tM + (-1)^q M'] = \left( \begin{array}{c|c} 0 & tN_1 + (-1)^q N_2' \\ \hline tN_2 + (-1)^q N_1' & 0 \end{array} \right) \sim \left( \begin{array}{c|c} 0 & L(t) \\ \hline L'(t^{-1}) & 0 \end{array} \right)$$

where  $L(t) = tN_1 + (-1)^q N_2'$ . Since  $\Gamma$  is a principal ideal domain, there is a diagonal matrix  $\text{diag}(\theta_i^q(t), \dots, \theta_m^q(t))$ ,  $\theta_i^q \in \Lambda$ ,  $\theta_{i+1}^q | \theta_i^q$  in  $\Lambda$  which presents the same  $\Gamma$ -module as  $L(t)$ . The fact that  $\theta_i^q(1) = \pm 1$  follows from the fact that the knot complement is a homology circle. (See [12].)

The following two corollaries provide an answer to question 39(a) of Fox [2].

**COROLLARY 2.5.** *The Stevedore's knot  $(6_1)$  is not doubly-null-cobordant.*

*Proof.* Using the methods of Fox [1] one obtains as a presentation matrix for

$H_1(\tilde{K}; \mathbb{Z})$  the  $2 \times 2$  matrix  $\begin{pmatrix} 2-t & 0 \\ 1 & 2t-1 \end{pmatrix}$ , and this has elementary ideals

$$\varepsilon_1 = \{(2 - t)(2t - 1)\}$$

$$\varepsilon_i = \{1\} \quad i \geq 2$$

where  $\{\lambda\}$  denotes the  $\Lambda$ -ideal generated by  $\lambda \in \Lambda$ .

Now if the Stevedore's knot were a cross-section of the unknot, then  $H_1(\tilde{K}; Z)$  would have a presentation matrix of the form  $M = \begin{bmatrix} 0 & L(t) \\ L'(t^{-1}) & 0 \end{bmatrix}$  where  $L$  is an  $r \times r$  matrix over  $\Lambda$ . Furthermore, the determinant  $|L(t)| = (2-t)$  or  $(2t-1)$  because the Alexander polynomial of  $K$  satisfies  $\Delta_1(t) = (2-t)(2t-1) = |L(t)| |L'(t^{-1})|$ . Now the matrix  $M$  has the following elementary ideals:  $\varepsilon_1 = \{(2-t)(2t-1)\}$ ,  $\varepsilon_2 \neq \{1\}$ .  $\varepsilon_2$  is generated as a  $\Lambda$ -ideal by the determinants of the  $(2r-1) \times (2r-1)$  submatrices of  $M$ . These determinants will be either zero, divisible by  $(2-t)$  or divisible by  $(2t-1)$ . The ideal they generate will be non-trivial, which is easily seen by considering the mapping  $\begin{matrix} \Lambda \xrightarrow{\phi} Z \\ t \mapsto -1 \end{matrix}$  induced by sending  $t$  to  $(-1)$ . The image of  $\varepsilon_2$  under  $\phi$  is an ideal of  $Z$  generated by some collection of multiples of 3, hence non-trivial.

The loss of information in passing from  $Z$  to  $Q$  coefficients is evident from the fact that  $\begin{pmatrix} (2-t) & 0 \\ 1 & (2t-1) \end{pmatrix}$  and  $\begin{pmatrix} (2-t) & 0 \\ 0 & (2t-1) \end{pmatrix}$  present the same  $\Gamma$ -module, but different  $\Lambda$ -modules. In fact, there is a cross-section of the unknot with  $\begin{pmatrix} (2-t) & 0 \\ 0 & (2t-1) \end{pmatrix}$  as a presentation matrix for  $H_1(\tilde{K}; Z)$ , namely  $9_{46}$ . This will be dealt with in detail in section III.

A more tractable invariant to consider when dealing with doubly-null-cobordant knots is the set of torsion numbers of the 2-fold branched cyclic cover  $\tilde{K}_2$ .

**COROLLARY 2.6.**  *$K$  a simple  $(2q-1)$  knot which is doubly-null-cobordant ( $q \geq 1$ ), then the torsion numbers of  $\tilde{K}_2$  appear in pairs.*

*Proof.* In order to obtain a presentation matrix for  $H_q(\tilde{K}_2; Z)$  as an abelian group, one simply sets  $t = -1$  in a  $\Lambda$ -presentation matrix for  $H_q(\tilde{K}; Z)$ . Hence  $H_q(\tilde{K}_2; Z)$  is presented by a matrix of the form  $\left( \begin{array}{c|c} 0 & L(-1) \\ \hline L'(-1) & 0 \end{array} \right)$  from which the result immediately follows.

**THEOREM 2.7.** *The only doubly-null-cobordant knot included in Reidemeister's table of prime knots is  $9_{46}$*

*Proof.* The proof that  $9_{46}$  is doubly-null-cobordant will be postponed to section IV. For another proof, see [19]. By considering torsion numbers of  $\tilde{K}_2$ , signature, etc., one rules out all other possible candidates (including  $6_1$ ), save  $9_{41}$ .  $9_{41}$  can be ruled out as follows, by arguments on the second elementary ideal of the presentation matrix  $H_1(\tilde{K}; Z)$ , similar to the argument in Corollary 2.5. The following argument was developed at the 1969 Georgia Topology Conference during conversation with a number of people, and I would like to thank them for their observations. For  $9_{41}$ , the torsion numbers of  $\tilde{K}_2$  are  $7, 7$ ;  $\sigma = 0, C_p = +1$ . A presentation matrix for  $H_1(\tilde{K}; Z)$

is the  $2 \times 2$  matrix

$$\begin{pmatrix} 3 - 3t + t^2 & 0 \\ t + 1 & 3t^2 - 3t + 1 \end{pmatrix}$$

$$\varepsilon_1 \neq \{1\}, \quad \varepsilon_2 = \{(3 - 3t + t^2), (3t^2 - 3t + 1), (t + 1)\}.$$

As in Corollary 2.5, we have that if  $9_{41}$  is doubly-null-cobordant, then  $\varepsilon_2 = \{(3 - 3t + t^2), (3t^2 - 3t + 1)\}$ . Consider the ring homomorphism  $\Lambda \xrightarrow{\phi} \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_2 = \Lambda_2$ . Then  $\phi(3 - 3t + t^2) = \phi(3t^2 - 3t + 1) = t^2 + t + 1 \in \Lambda_2$ ,  $\phi(t + 1) = t + 1$ . By degree considerations,  $(t + 1) \notin \{(t^2 + t + 1)\}$  in  $\Lambda_2$ .

If one desires non-prime doubly-null-cobordant knots, consider the following surprising theorem of Zeeman [21].

**THEOREM 2.8 (Zeeman).** *Every 1-twist-spun knot is unknotted.*

Zeeman proves the theorem by showing that a  $p$ -twist-spun knot fibers with fiber whose closure is the bounded punctured  $p$ -fold branched cyclic covering space of the knot you are spinning. When  $p = 1$ , the closed fiber is a disc. Let  $(-K)$  denote the cobordism inverse of  $K$ , that is,  $(-K)$  denotes the knot obtained by taking the image of the submanifold  $k$ , with reversed orientation, under a reflection of  $S^{n+2}$ . Then we have the following corollary to Theorem 2.8:

**COROLLARY 2.9.**  *$K \# (-K)$  is doubly-null-cobordant, for every knot  $K$ .*

This shows that the square knot is doubly-null-cobordant, a fact first proved by Stallings [14].

### III. Geometric Realization of Invertible Cobordisms

One approach to the geometric realization problem would be to prove a converse to Theorem 2.3, then appeal to the geometric realization theorem of Kervaire [5, Chapter II, §6]. We have the following weak converse to Theorem 2.3:

**THEOREM 3.1.**  *$K = (S^{2q+1}, k)$   $q \geq 2$  a simple knot, and  $K$  has a  $(q - 1)$ -connected Seifert manifold whose associated Seifert matrix is doubly-null-cobordant, then  $K$  is doubly-null-cobordant.*

*Proof.* Case 1:  $q \geq 3$ .

We will construct using surgery a cobordism  $(W; U, K)$  invertible from  $K$ . The knot  $k$  has a  $(q - 1)$ -connected Seifert manifold  $V^{2q}$ , and  $H_q(V^{2q})$  is free of rank  $2r$  on generators  $\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r$  such that  $\theta(\alpha_i, \alpha_j) = 0 = \theta(\beta_i, \beta_j)$   $1 \leq i, j \leq r$ . Consider  $S^{2q+2}$  as the union of 2 balls  $B_1^{2q+2}$  and  $B_2^{2q+2}$  identified along  $S^{2q+1}$ , the equator of  $S^{2q+2}$ . The knot  $k$  and its Seifert manifold  $V^{2q}$  sit in  $S^{2q+1}$ . Push  $V$  out into  $B_1^{2q+2}$

and  $B_2^{2q+2}$  obtaining an embedding of  $V \times I, I = [-1, 1]$ , such that  $V \times \{0\} = V$ . Levine [8, Lemma 5] describes a method for doing surgery on  $V \times \{1\}$  inside  $B_1^{2q+2}$ , producing a cobordism  $W_1$  between  $V$  and  $D^{2q}$  by adding  $(q+1)$ -handles  $h_{\alpha_i}^{q+1}$  to  $V \times [0, 1]$  along  $V \times \{1\}$  via the  $\{\alpha_i\}$ . Likewise, we can do surgery on the  $\beta_j$  inside  $B_2^{2q+2}$  to produce a cobordism  $W_2$  between  $V$  and  $D^{2q}$  inside  $B_2^{2q+2}$ . Let  $\Delta = (W_1 \cup W_2)/V$ . Then  $W_1 \approx V \times [0, 1] \cup_{i=1}^r h_i^{q+1}$  and  $W_2 \approx V \times [0, 1] \cup_{j=1}^r h_j^{q+1}$ .  $V$  is  $(q-1)$ -connected, so each of  $W_1$  and  $W_2$  is  $(q-1)$ -connected, hence  $\Delta$  is  $(q-1)$ -connected. The Mayer-Vietoris sequence for the triad  $(\Delta, W_1, W_2)$  is

$$H_q(V) \xrightarrow{i_{1*} \oplus -i_{2*}} H_q(W_1) \oplus H_q(W_2) \rightarrow H_q(\Delta) \rightarrow 0.$$

Now  $H_q(V)$  is free on the generators  $\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r$  and by construction  $\text{Ker } i_{1*}$  is the summand of  $H_q(V)$  generated by the  $\{\alpha_i\}$ , and  $\text{Ker } i_{2*}$  is the summand generated by the  $\{\beta_i\}$ , so  $i_{1*} \oplus -i_{2*}$  is an isomorphism and  $H_q(\Delta) = 0$ . Since  $\partial\Delta = S^{2q}$ , then by the  $h$ -cobordism theorem  $\Delta \approx D^{2q+1}$  and  $\partial\Delta$  is unknotted in  $S^{2q+2}$ . So  $k$  is a cross-section of the unknot, and clearly there exists a cobordism  $(W; U, K)$  invertible from  $K$ .

Case. 2:  $q=2$ .

As in Levine [8, p. 235], we have two difficulties: (a) representing the  $\{\alpha_i\}$  and  $\{\beta_j\}$  by embedded 2-spheres (b) asserting that surgery by either  $\{\alpha_i\}$  or  $\{\beta_j\}$  produces  $D^4$ .

Levine's argument using the results of Wall go through as in [8], making one extra observation.

We are given  $V^4, \partial V^4 \approx S^3$ . Form the smooth closed manifold  $\bar{V}$  by putting a disc on  $\partial V$ . Just as in Levine, we get that  $\bar{V} \# (S^2 \times S^2) \# \dots \# (S^2 \times S^2) \approx \partial W$  where  $W$  is a handlebody with handles of index 2 only. The handlebody decomposition for  $W$  gives us  $\{S'_i\}$  a family of disjoint embedded 2-spheres in  $\partial W$ , the boundaries of the transverse 3-discs of the 2-handles. Let  $\alpha'_i \in H_2(\partial W)$  denote the homology class of  $S'_i$ , then  $\{\alpha'_i\}$  is half a basis for  $H_2(\partial W)$ . We can embed disjoint copies of  $S^2 \times S^2$  in  $S^5$ , and take the connected sum of these with  $V$ . This has the effect of adjoining the

matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  to the existing Seifert matrix, thereby preserving double-null-co-

bordance. We can thus take  $V = \partial W - \text{Int}(D^4)$ .  $H_2(V)$  has as a free basis  $\{\alpha_i\}_{i=1}^q \cup \{\beta_i\}_{i=1}^q$  with  $\theta(\alpha_i, \alpha_j) = \theta(\beta_i, \beta_j) = 0 \ 1 \leq i, j \leq q$ . Now  $\alpha'_i \cdot \alpha'_j = 0 = \alpha_i \cdot \alpha_j$  all  $i, j$ . Since the intersection pairing on  $V$  is unimodular, both  $\{\alpha_i\}$  and  $\{\alpha'_i\}$  extend to bases  $\{\alpha_i, \gamma_i\}, \{\alpha'_i, \gamma'_i\}$  satisfying  $\alpha_i \cdot \gamma_j = \alpha'_i \cdot \gamma'_j = \delta_{ij}, \gamma_i \cdot \gamma_j = \gamma'_i \cdot \gamma'_j = 0$ . By Wall [20, Theorem 2]  $\exists$  a diffeomorphism  $h_1: \partial W \rightarrow \partial W$  such that  $h_{1*}(\alpha'_i) = \alpha_i$ . Likewise  $\exists$  a diffeomorphism  $h_2$  such that  $h_{2*}(\alpha'_i) = \beta_i$ . Hence we can take  $\{h_1(S'_i)\}, \{h_2(S'_i)\}$  as embedded representatives for  $\{\alpha_i\}, \{\beta_i\}$ . Doing surgery by either set is equivalent to removing the handles of  $W$ , so we get  $D^4$  at either end. Adding the trace of both surgeries as before yields a smooth  $S^4$  which bounds a contractible submanifold in  $S^6$ . By the  $h$ -cobordism theorem, the contractible submanifold is a smooth 5-disc, so the  $S^4$  is unknotted. This completes the proof of Theorem 3.1.

Ideally, one would like to prove that if  $K=(S^{2q+1}, k)$   $q \geq 2$  is a simple knot with a doubly-null-cobordant matrix, then  $K$  is doubly-null-cobordant. One approach which I have tried, but failed so far to succeed in doing, is to do surgery on the Seifert manifold to make it  $(q-1)$ -connected, preserving double-null-cobordance of the Seifert matrix at each stage. For an analysis of the effects of such surgery on the Seifert matrix, see Levine [7].

We will now describe the Alexander invariants that are to be realized. If  $K^k$  is a finite simplicial complex with the homology of  $S^1$ , let  $\tilde{K}$  denote the infinite cyclic cover of  $K$ . With  $\Lambda$  and  $\Gamma$  as in section II, we have that  $H_q(\tilde{K}; Q)$  is presented as a  $\Gamma$ -module by a diagonal matrix of the form  $\text{diag}(\lambda_1^q, \dots, \lambda_m^q)$  where  $\lambda_i^q \in \Lambda$ ,  $\lambda_{i+1}^q | \lambda_i^q$  in  $\Lambda$ , and  $\lambda_i^q(1) = \pm 1$ . (See [22].) If  $K$  happens to be a knot complement, the  $\{\lambda_i^q\}$  are called the Alexander invariants of  $K$ .

Combining a theorem of Levine [9, Theorem 1] and Corollary 2.4, we have the following:

**THEOREM 3.2.** *Let  $\{\lambda_i^q\}$   $0 < q < n+1$ ,  $0 < i < m$  be the family of Alexander invariants of a doubly-null-cobordant  $n$ -knot,  $n \geq 1$ . Then the following are satisfied:*

- (i)  $\lambda_{i+1}^q | \lambda_i^q$  in  $\Lambda$
- (ii)  $\lambda_i^q(1) = \pm 1 \quad \forall q, i$
- (iii)  $\lambda_i^q(t) = \epsilon t^a \lambda_i^{n-q+1}(t^{-1})$   
 $\epsilon = \pm 1 \quad a$  an integer

(iv)  $n = 2q - 1$  then  $\exists$  a collection  $\{\theta_j^q\}_{j=1}^r$  of elements of  $\Lambda$  satisfying the conclusion of Corollary 2.4 such that each of the diagonal matrices  $\text{diag}(\lambda_1^q(t), \dots, \lambda_k^q(t))$  and  $\text{diag}(\theta_1^q(t), \dots, \theta_r^q(t), \theta_1^q(t^{-1}), \dots, \theta_r^q(t^{-1}))$  presents  $H_q(\tilde{K}; Q)$  as a  $\Gamma$ -module.

We will prove the following

**THEOREM 3.3.** *Let  $\{\lambda_i^q\}$   $0 < q < n+1$ ,  $0 < i < m$  be a set of elements of  $\Lambda$  satisfying (i)–(iv) of Theorem 3.2. Then for  $n \geq 1$  there exists a doubly-null-cobordant  $n$ -knot with the  $\{\lambda_i^q\}$  as its Alexander invariants.*

*Proof.* The proof is by construction, an extension of that in [4, 15, 17].

*The Construction*

*Case 1.* Suppose that we are given  $\lambda \in \Lambda$ ,  $\lambda(1) = \pm 1$ , and  $n \geq 2$ ,  $1 \leq q \leq \frac{n}{2}$ . We will

construct a doubly-null-cobordant knot  $K=(S^{n+2}, k)$  such that if  $\tilde{K}$  is the infinite cyclic covering space of  $S^{n+2} - k$ , then

$$H_i(\tilde{K}; Z) = \begin{cases} \Lambda & i = q \\ \frac{\Lambda}{\lambda(t)} & i = q \\ \Lambda & i = n + 1 - q \\ \frac{\Lambda}{\lambda(t^{-1})} & i = n + 1 - q \\ 0 & 0 < i < n + 1 \quad i \neq q, n + 1 - q. \end{cases}$$

This means that the Alexander invariants of  $K$  are

$$\lambda_i^p = \begin{cases} \lambda(t) & p = q, i = 1 \\ \lambda(t^{-1}) & p = n + 1 - q, i = 1 \\ 1 & 0 < p < n + 1 \quad 0 < i < m \quad (p, i) \neq (q, 1) \text{ or } (n + 1 - q, 1). \end{cases}$$

Let the unknotted sphere pair  $(S^{n+3}, S^{n+1})$  be decomposed into the union of two unknotted ball pairs  $\{(B_1^{n+3}, B_1^{n+1}), (B_2^{n+3}, B_2^{n+1})\}$  glued together along their common unknotted boundary  $(S^{n+2}, S^n)$ . We will exchange handles across  $(S^{n+2}, S^n)$  to obtain a new decomposition of  $(S^{n+3}, S^{n+1})$  as the union of two knotted ball pairs glued together along a common knotted boundary sphere pair.

Begin by choosing a smooth proper embedding  $i: D^q \rightarrow (B_2^{n+3} - B_2^{n+1})$ . See [4, Figure 1]. A tubular neighborhood of  $i(D^q)$  in  $(B_2^{n+3} - B_2^{n+1})$  is diffeomorphic to  $D^q \times D^{n+3-q}$ , a  $q$ -handle, and adding this to  $B_1^{n+3}$  produces a manifold  $M = B_1^{n+3} \cup_i h^q$ . Let  $L = \overline{S^{n+3} - M}$  denote the complementary solid torus. Then if  $\approx$  denotes diffeomorphism,  $\simeq$  denotes homotopy equivalence and  $\vee$  denotes wedge product, we have

$$\begin{aligned} M &\approx S^q \times D^{n+3-q} & L &\approx D^{q+1} \times S^{n+2-q} \\ M' = M - B_1^{n+1} &\simeq S^1 \vee S^q & L' = L - B_2^{n+2} &\simeq S^1 \vee S^{n+2-q} \\ \partial M' = \partial L' &\approx S^q \times S^{n+2-q} - S^{n+1} & &\simeq S^1 \vee S^q \vee S^{n+2-q}. \end{aligned}$$

Now if  $q > 1$  then over  $\Lambda \cong \mathbb{Z}\pi_1$  we have that  $\pi_q(\partial L') \cong \Lambda(\alpha)$  a free  $\Lambda$ -module of rank 1 generated by  $\alpha$ , which we can take to be an embedded  $q$ -sphere going once around the handle  $h^q$ . By general position we can choose a smooth embedding  $g: S^q \rightarrow \partial L'$  in the homotopy class of  $\lambda \cdot \alpha \in \pi_q(\partial L')$ . Since  $\pi_q(L) = 0$  then  $g(S^q) \sim 0$  in  $L'$ , and we can be general position again extend  $g$  to a proper embedding  $g: D^{q+1} \rightarrow L'$ . We then add a tubular neighborhood of  $g(D^{q+1})$  in  $L'$  to  $K$ , obtaining  $B = K \cup_g h^{q+1}$ . Now in  $\partial L' \cup S^n$ ,  $\lambda \cdot \alpha$  is homotopic to  $\alpha$ , hence diffeotopic to  $\alpha$ . So  $B \cup B_1^{n+1} = K \cup_g h^{q+1} \approx B^{n+3}$ , hence  $B$  is a knot complement, that is  $B = B^{n+3} - k' B^{n+1}$  for some smooth proper embedding  $k': B^{n+1} \rightarrow B^{n+3}$ . Furthermore  $\partial B = S^{n+2} - k S^n$  is the complement of a doubly-null-cobordant knot ( $k = k'|S^n$ ), and the calculations of [17] show that this knot has the desired invariants.

If  $q = 1$ , then  $\pi_1(\partial L')$  is the free group on generators  $\alpha$  and  $\beta$ , where  $\alpha$  can be taken as an embedded  $S^1$  going once around  $h^1$ , and  $\beta$  as an embedded  $S^1$  homologically linking the submanifold  $S^n$  once in  $\partial L$ . If  $\lambda(t) = \sum_{i=0}^m a_i t_i$ , then there is an embedding  $g: S^1 \rightarrow \partial L'$  in the homotopy class of

$$\alpha^{a_0} \beta^{a_1} \alpha \beta \dots \alpha^{a_m} \beta^{-m} \in \pi_1(\partial L').$$

The construction proceeds exactly as above, and the calculations of [17] show that the resulting doubly-null-cobordant knot has the desired invariants.

Case 2. The middle dimension.

We are left with the case  $n=2q-1$ ,  $q \geq 1$ , and  $q$ -dimensional invariants to realize. Given  $\lambda \in A$  such that  $\lambda(1)=1$ , we will construct a doubly-null-cobordant knot  $K=(S^{2q+1}, k)$  such that if  $\tilde{K}$  is the infinite cyclic cover of  $S^{2q+1}-k$ , then

$$H_i(\tilde{K}; Z) = \begin{cases} \frac{A}{\lambda(t)} \oplus \frac{A}{\lambda(t^{-1})} & i = q \\ 0 & 0 < i < 2q \quad i \neq q. \end{cases}$$

The proof begins exactly as before, producing

$$\begin{aligned} M &\approx S^q \times D^{q+2} & L &\approx D^{q+1} \times S^{q+1} \\ M' = M - B_1^{2q+2} &\simeq S^1 \vee S^q & L' = L - B_2^{2q+2} &\simeq S^1 \vee S^{q+1} \\ \partial M' = \partial L' &\approx S^q \times S^{q+1} - S^{2q+1} &&\simeq S^1 \vee S^q \vee S^{q+1}. \end{aligned}$$

If  $q > 1$ , we can as before obtain an embedding representing  $\lambda \cdot \alpha$  in  $\pi_q(\partial L')$ , but the fact that  $L'$  is not simply connected prevents us from removing double points to extend the embedded  $S^q$  in  $\partial L'$  to an embedded  $D^{q+1}$  in  $L'$ . In fact, there are choices of embedded representatives of  $\lambda \cdot \alpha$  which do not span embedded discs in  $L'$ . However, we can construct an embedded representative which will span an embedded disc in  $L'$ .

Let  $\lambda(t) = \sum_{i=0}^m a_i t^i$ . Choose  $\sum_{i=0}^m |a_i|$  distinct points in  $S^{q+1}$ ; label them  $\{x_{i,j}\} 1 \leq j \leq |a_i|$ . Consider the embedded spheres  $S_{i,j}^q = S^q \times \{x_{i,j}\}$  in the product structure  $\partial L \approx S^q \times S^{q+1}$ . Each  $S_{i,j}^q$  spans the embedded disc  $D_{i,j}^{q+1} = D^{q+1} \times \{x_{i,j}\}$  in the product structure of  $L \approx D^{q+1} \times S^{q+1}$ . We can assume  $B_2^{2q} \cap (\bigcup_{i,j} D_{i,j}^{q+1}) = \emptyset$ , altering our choice of  $\{x_{i,j}\}$  if necessary. We will now pipe the  $\{S_{i,j}^q\}$  together in  $\partial L'$  to produce an embedding in the homotopy class of  $\lambda \cdot \alpha$  ( $q > 1$ ) or  $\alpha^{a_0} \beta^{a_1} \beta \dots \alpha^{a_m} \beta^{-m}$  ( $q = 1$ ) where  $\alpha = S^q \times \{*\}$ ,  $* \in S^{q+1}$  the basepoint. We can without loss of generality take  $a_0 > 0$ ; and we choose an orientation for  $\alpha$ . Orient  $S_{i,j}^q$  with orientation corresponding to sign  $a_i$ ; that is, if sign  $a_i = 1$  we orient  $S_{i,j}^q$  compatibly with  $\alpha$ , and sign  $a_i = -1$  we orient  $S_{i,j}^q$  the opposite of  $\alpha$ .

$S^{2q-1}$  spans a disc  $D^{2q}$  in  $\partial L$ . Choose a mutually disjoint family of embedded arcs  $f_{j,j+1}^i: I \rightarrow \partial L'$  satisfying the following:

- (i)  $f_{j,j+1}^i(0) \in S_{i,j}^q, \quad f_{j,j+1}^i(1) \in S_{i,j+1}^q$
- (ii)  $f_{j,j+1}^i((0,1)) \cap (\bigcup_{i,j} S_{i,j}^q) = \emptyset, \quad f_{j,j+1}^i(I) \cap D^{2q} = \emptyset.$

Thicken  $f_{j,j+1}^i(I)$  to a tube  $T_{j,j+1}^i \approx f_{j,j+1}^i(I) \times D^q$  such that

$$\begin{aligned} T_{j,j+1}^i \cap S_{i,j}^q &= f_{j,j+1}^i(0) \times D^q = D_{i,j}^q, \\ T_{j,j+1}^i \cap S_{i,j+1}^q &= f_{j,j+1}^i(1) \times D^q = D_{i,j+1}^q. \end{aligned}$$

Otherwise  $T_{j,j+1}^i$  misses  $\bigcup_{i,j} S_{i,j}^q$ , and  $T_{j,j+1}^i$  is chosen so that the orientations of

$D_{i,j}^q$ , and  $D_{i,j+1}^q$  match up with the orientations of  $S_{i,j}^q$  and  $S_{i,j+1}^q$  correctly. By the words “pipe  $S_{i,j}^q$  to  $S_{i,j+1}^q$  via  $f_{i,j+1}^i(I)$ ” I mean take the sphere

$$(S_{i,j}^q - \text{Int } D_{i,j}^q) \cup (S_{i,j+1}^q - \text{Int } D_{i,j+1}^q) \cup (f_{i,j+1}^i(I) \times \partial D^q).$$

With  $i$  fixed, piping  $S_{i,j}^q$  to  $S_{i,j+1}^q$  for all  $1 \leq j \leq |a_i| - 1$ , we obtain  $S_i^q$ . Suppose that  $a_k$  is the “next” non-vanishing coefficient in  $\lambda(t)$ ; that is in the expansion  $\lambda(t) = \sum_{i=0}^m a_i t^i$ ,  $a_j = 0$   $i < j < k$ . We then pipe  $S_i^q$  to  $S_k^q$  by an arc that loops around  $S^{2q-1}$  precisely  $(k-i)$  times. That is, the arc has intersection number  $(k-i)$  with  $D^{2q}$ . We now have an embedded sphere in the homotopy class of  $\lambda \cdot \alpha$ . See [Figure 1.] Furthermore, this sphere spans an embedded  $D^{q+1}$  in  $L'$ , obtained by adding the interiors of the pipes to  $(\cup_{i,j} D_{i,j}^{q+1})$  and then pushing the pipe interiors out into  $L'$ . Furthermore,

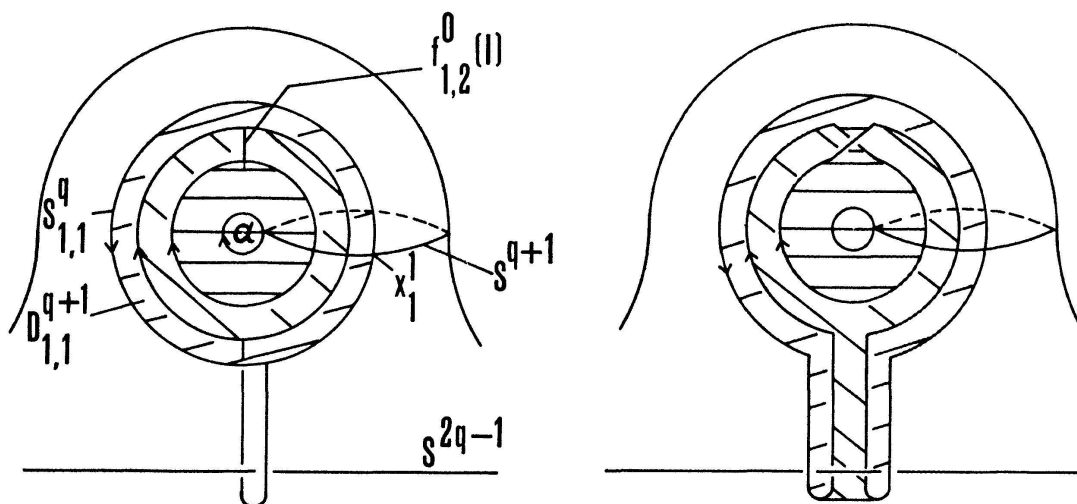


Fig. 1

$\lambda \cdot \alpha$  is isotopic to  $\pm \alpha$  in  $\partial L$ , the sign depending on the sign of  $\lambda(1)$ . Some care must be taken in the case  $q=1$  to insure that the arcs chosen to pipe along do not link each other, but clearly this can be done.

Continuing as before, we obtain a doubly-null-equivalent knot. It remains to calculate the homology invariants of this knot.

*Calculation of the Invariants*

We proceed as in [17]. From the handlebody structure for the complement  $B = B^{2q+2} - k'(B^{2q})$  we will have induced a handlebody structure for  $\tilde{B}$ , the infinite cyclic cover of  $B$ . Or equivalently we can consider constructing the covering as follows: The unknotted open complement  $B^* = B^{2q+2} - B^{2q}$  fibers over  $S^1$  with fiber  $\hat{D}^{2q+1}$ , a halfclosed  $D^{2q+1}$ . That is,  $D^{2q+1}$  has a closed  $B^{2q}$  (corresponding to the unknotted sub-ball) removed from its boundary. The infinite cyclic cover of  $B^*$  is  $\tilde{B}^* \approx \approx R^1 \times \hat{D}^{2q+1}$ . We thereupon add  $Z$  copies  $\{h_i^q\}$  of  $h^q$  to  $\tilde{B}^*$  to obtain  $\tilde{M}'$ , the infinite

cyclic cover of  $M'$ . We then add  $Z$  copies  $\{h_i^{q+1}\}$  of  $h^{q+1}$  to  $\partial\tilde{M}'$  to get  $\tilde{B}$ . The attaching spheres for the  $\{h_i^{q+1}\}$  are  $\{g_i\}$ , lifts of  $g: S^q \rightarrow \partial\tilde{M}$ .

The framings of the normal bundle of  $g(S^q)$  in  $\partial M'$  are in 1-1 correspondence with  $\pi_q(SO_{q+1})$ , and  $0 \in \pi_q(SO_{q+1})$  corresponds to the canonical product framing of the normal bundle of  $\alpha$ . Clearly from the construction the fact that  $h^{q+1}$  is the tubular neighborhood of a  $D^{q+1}$  in  $L'$ , we have that the framing must correspond to  $0 \in \pi_q(SO_{q+1})$ , since the framing on the attaching sphere extends over the core of the handle. However, we can consider the possibility of constructing arbitrary, possibly non-doubly-null-cobordant slice knots by altering choice of framing, and we will do the calculation in this more general setting.

Choose a framing  $f \in \pi_q(SO_{q+1})$ ; let  $G: S^q \times D^{q+1} \rightarrow \partial M'$  be an embedding representing this framing;  $G|_{S^q \times \{0\}} = g$ . Let  $\tilde{S}_0 = \partial\tilde{M}'$ . Then  $G$  lifts to a family  $\{G_i\}$  of trivializations for the  $\{g_i(S^q)\}$  in  $\tilde{S}_0$ . Let  $\tilde{S}_1 = \overline{\tilde{S}_0 - \bigcup_i [G_i(S^q \times D^{q+1})]}$  where upper bar denotes topological closure. If  $\tilde{S} = \partial\tilde{B}$ , then we have

$$H_i(\tilde{S}_0) \cong \begin{cases} \Lambda & i = q, q + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(\tilde{S}_0, \tilde{S}_1) \cong \begin{cases} \Lambda & i = q + 1, 2q + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(\tilde{S}, \tilde{S}_1) \cong \begin{cases} \Lambda & i = q + 1, 2q + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Following Kervaire-Milnor [6], we have the following exact sequences of  $\Lambda$ -modules:

$$H_{q+1}(\tilde{S}_0) \xrightarrow{\circ\lambda} H_{q+1}(\tilde{S}_0, \tilde{S}_1) \xrightarrow{\partial'} H_q(\tilde{S}_1) \xrightarrow{i'} H_q(\tilde{S}_0) \rightarrow 0 \tag{1}$$

$$H_{q+1}(\tilde{S}) \xrightarrow{\circ\lambda'} H_{q+1}(\tilde{S}, \tilde{S}_1) \xrightarrow{\partial} H_q(\tilde{S}_1) \xrightarrow{i} H_q(\tilde{S}) \rightarrow 0. \tag{2}$$

The homomorphisms in (1) and (2) are as follows: Let  $\lambda_0$  denote the generator of  $H_q(\tilde{S}_0) \cong \Lambda(\lambda_0)$  corresponding to  $G_0(S^q \times 0)$ , the attaching sphere of  $h_0^{q+1}$ . Let  $\gamma_0$  be the free generator of  $H_{q+1}(\tilde{S}_0)$ , corresponding to the lift at level 0 of the belt sphere of  $h^q$ .  $H_{q+1}(\tilde{S}_0, \tilde{S}_1) \cong \Lambda(\beta_0)$  where  $\beta_0$  is the class of  $G_0(x_0 \times D^{q+1})$   $x_0 \in S^q$  the base point. Then  $H_{q+1}(\tilde{S}_0) \xrightarrow{\circ\lambda} H_{q+1}(\tilde{S}_0, \tilde{S}_1)$  is the module homomorphism induced by taking

$$\gamma_0 \mapsto \left[ \sum_{t^i \in J(t)} t^i \langle \gamma_0, t^i \lambda_0 \rangle \right] \beta_0$$

where  $J(t)$  is the infinite cyclic multiplicative group of covering transformations, and  $\langle \gamma_0, t^i \lambda_0 \rangle$  is the intersection number of  $\gamma_0$  and the image of  $\lambda_0$  under the translation  $t^i$ . The map  $\circ\lambda$  is equal to the usual map of the homology exact sequence of the pair  $(\tilde{S}_0, \tilde{S}_1)$  because  $\langle \beta_0, \lambda_0 \rangle = 1$  and  $\langle \beta_0, t^i \lambda_0 \rangle = 0$   $i \neq 0$ .  $\circ\lambda'$  is defined analogously.

Suppose that  $\lambda(t) = \sum_{i=0}^m a_i t^i$ , and that by construction  $\lambda_0 \in H_q(\tilde{S}_0)$  is homologous to  $\lambda(t) \circ \alpha_0$ , where  $\alpha_0$  is the free generator of  $H_q(\tilde{S}_0) \cong \Lambda(\alpha_0)$ . Now

$$\begin{array}{ccc} \circ\lambda: H_{q+1}(\tilde{S}_0) & \rightarrow & H_{q+1}(\tilde{S}_0, \tilde{S}_1) \\ \parallel \wr & & \parallel \wr \\ \Lambda(\gamma_0) & & \Lambda(\beta_0) \end{array}$$

sends

$$\gamma_0 \xrightarrow{\circ\lambda} \sum_{t^i} \left( t^i \sum_{j=0}^m a_j \langle \gamma_0, t^{i+j} \alpha_0 \rangle \right) \beta_0 = \left[ \sum_{k=0}^m t^{-k} a_k \right] \beta_0 = \lambda(t^{-1}) \beta_0$$

since  $\langle \gamma_0, t^k \alpha_0 \rangle = \delta_{0,k}$ . This shows that  $\circ\lambda$  is injective.

We want a  $\Lambda$ -presentation matrix for  $H_q(\tilde{S})$ . We first obtain a presentation for  $H_q(\tilde{S}_1)$ , and by adjoining one relation, obtain one for  $H_q(\tilde{S})$ . We have the following exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & H_{q+1}(\tilde{S}_0) & \xrightarrow{\circ\lambda} & H_{q+1}(\tilde{S}_0, \tilde{S}_1) & \xrightarrow{\partial'} & H_q(\tilde{S}_1) \xrightarrow{i'} H_q(\tilde{S}_0) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \Lambda(\gamma_0) & & \Lambda(\beta_0) & & \Lambda(\alpha_0) \end{array}$$

So  $H_q(\tilde{S}_1) \cong \text{Im } \partial' \oplus \Lambda(\alpha_0)$  and  $\text{Im } \partial'$  is presented by the  $1 \times 1$  matrix  $(\lambda(t^{-1}))$ . Hence  $H_q(\tilde{S}_1)$  is presented by the  $1 \times 2$  matrix  $\begin{pmatrix} \alpha_0 & \lambda'_0 \\ 0 & \lambda(t^{-1}) \end{pmatrix}$  where  $\lambda'_0 = \partial\beta_0$ .

Consider now the exact sequence

$$\begin{array}{cccc} H_{q+1}(\tilde{S}, \tilde{S}_1) & \xrightarrow{\partial} & H_q(\tilde{S}_1) & \rightarrow H_q(\tilde{S}) \rightarrow 0 \\ \parallel & & & \\ \Lambda(c_0) & & & \end{array}$$

where  $c_0 = G_0(D^{q+1} \times x_0)$ ,  $x_0$  the base point of  $S^q$ .  $c_0$  is the core of  $h_0^{q+1}$  pushed out to the boundary of the handle via the first vector in the framing. The above sequence tells us that  $H_q(\tilde{S})$  is obtained from  $H_q(\tilde{S}_1)$  by adjoining the relation given by  $\partial(c_0)$ . Now by construction,  $\partial(c_0) = \lambda(t) \alpha_0 + \xi \beta_0$ , where  $\xi$  depends on our choice of both attaching sphere  $g: S^q \rightarrow \partial M'$  and framing  $G: S^q \times D^{q+1} \rightarrow \partial K'$ .  $H_q(\tilde{S})$  is presented by

the  $2 \times 2$  matrix  $\begin{pmatrix} \alpha_0 & \lambda'_0 \\ 0 & \lambda(t^{-1}) \\ \lambda(t) & \xi \end{pmatrix}$ .

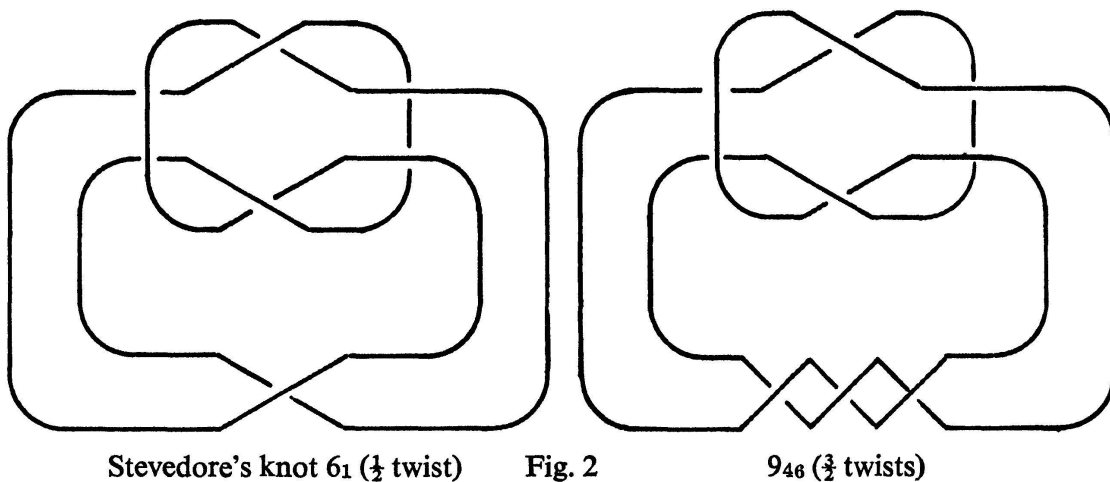
Let  $f \in \pi_q(SO_{q+1})$  be the chosen framing, and let  $(\lambda_0)_f = \partial(c_0) = G_0(S^q \times x_0) \in H_q(\tilde{S}_1)$  be the longitude of the boundary of the tubular neighborhood  $G_0(S^q \times D^{q+1})$ , and  $\lambda'_0 = \partial'(\beta_0) = G_0(x_0 \times S^q)$  be the meridian of the tubular neighborhood. Clearly  $\lambda'_0 \in H_q(\tilde{S}_1)$  is independent of the framing, but  $(\lambda_0)_f$  depends on the framing. In fact, if  $\lambda_0$  corresponds to the zero framing, then  $(\lambda_0)_f = \lambda_0 \oplus j_*(f)(\lambda'_0)$  as elements of  $H_q(\tilde{S}_1)$  where  $j_*: \pi_q(SO_{q+1}) \rightarrow \pi_q(S^q) = Z$  is the homomorphism in the homotopy

exact sequence of the fibration  $SO_{q+1}/SO_q = S^q$  [6].  $j_*(f)$  measures a sort of “self-linking” in the attaching spheres of the  $\{h_i^{q+1}\}$  in  $\tilde{S}_1$ . For the zero framing in our situation,  $j_*(f) = 0$ . Even with the zero framing, it is still possible to have  $\xi \neq 0$ . This is due to a kind of “linking” that occurs among the  $\{g_i(S^q)\}$  in  $\tilde{S}_0$ . (See Fig. 5.) In our construction, if we were to cancel the generator  $a_0$  of  $H_q(\tilde{S}_0)$  by a family of trivially attached  $\{h_i^{q+1}\}$ , the  $\{g_i(S^q)\}$  would simultaneously bound discs in the new manifold; so  $\xi = 0$  and hence  $\lambda_0 = \lambda(t) \alpha_0 \in H_q(\tilde{S}_1)$ . By a suitably bad choice of attaching sphere  $g: S^q \rightarrow \partial M'$ , this situation can be changed, and one could arrange to have  $\xi \neq 0$ . See [16], and section IV, [Figure 5].

Hence  $H_q(\tilde{S})$  is presented by the diagonal presentation matrix  $\text{diag}(\lambda(t), \lambda(t^{-1}))$ . This completes the proof of Case 2. The proof of Theorem 3.3 is completed by taking connected sums of the knots produced above, exactly as in Levine [9].

#### IV. Some Geometrical Considerations

It was erroneously claimed in [4] that the Stevedore’s knot ( $6_1$ ) was doubly-null-cobordant. In fact,  $9_{46}$  is the doubly-null-cobordant knot produced by the surgery techniques used in [4], [Figure 2].



Consider the situation of Theorem 3.3, the middle-dimensional-case, where  $n = 1 = q$ . We will look at the case of the doubly-null-cobordant knot with  $\text{diag}((2-t), (2t-1))$  as a presentation matrix for  $H_1(\tilde{K}; \mathbb{Z})$ . Figure 1 shows the attaching sphere for  $h^2$ . The problem is to decide exactly which classical slice knot is produced by the surgery. To do this, we straighten out the attaching curve  $g$ , ambient isotoping it until it goes around the  $S^1$  in  $S^1 \times S^2$  exactly once, taking care to drag the submanifold  $k = S^1$  along in the isotopy [Figure 3]. As far as the boundary situation is concerned, adding the handle  $h^2$  corresponds to performing surgery by the attaching sphere. First, remove an open tubular neighborhood of the attaching sphere, leaving  $k$  embedded

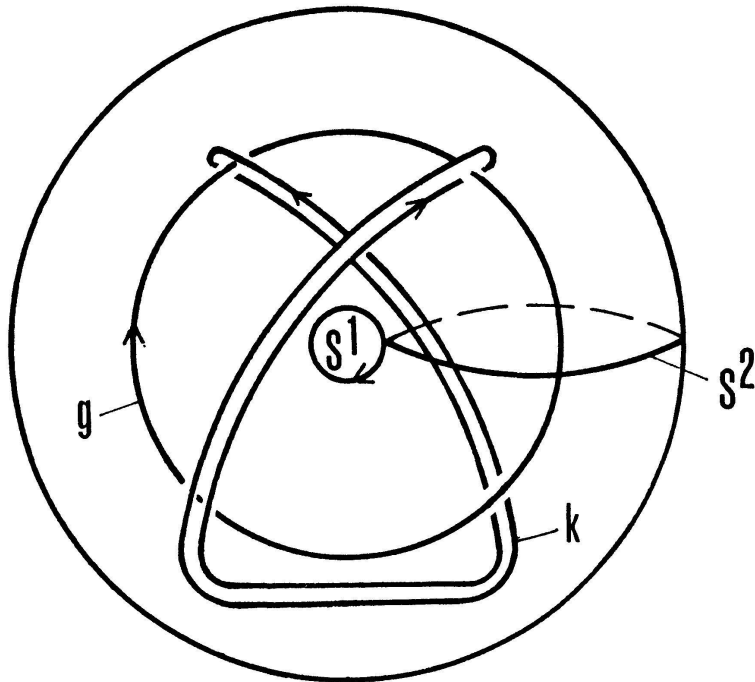


Fig. 3

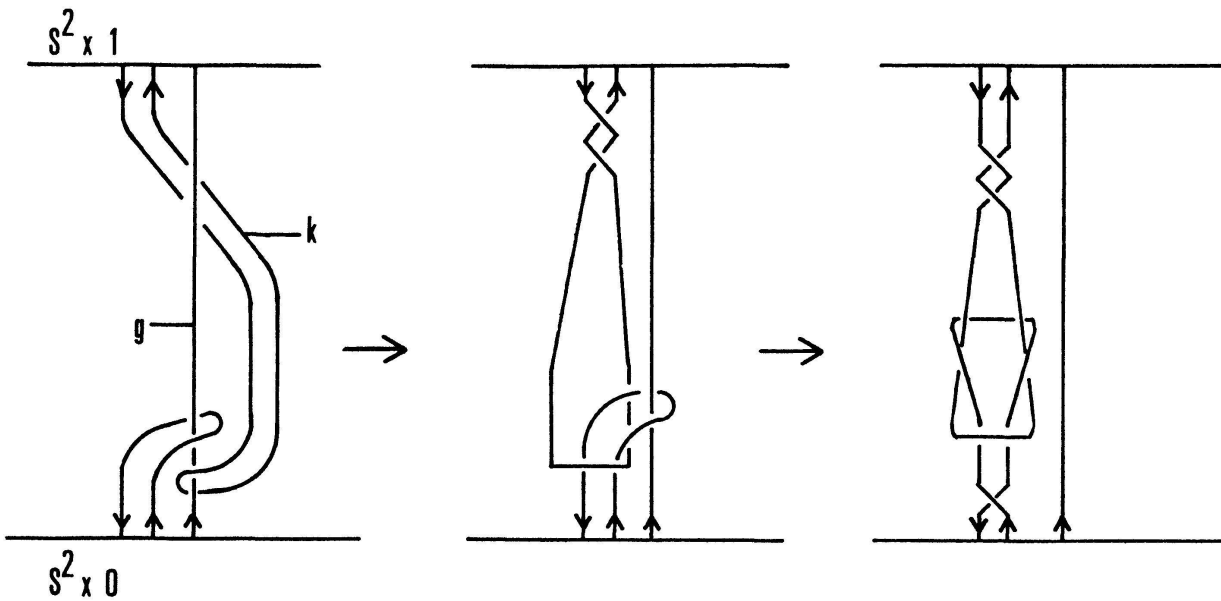


Fig. 4

in  $S^1 \times D^2$ . The attaching sphere has the product framing induced from the product structure of  $S^1 \times S^2$ , and performing the surgery amounts to adding  $D^2 \times S^1$  to  $S^1 \times D$  by the identity map on the boundary. The problem is to visualize  $k$  in the complementary  $S^1 \times D^2$  obtained by removing the open tubular neighborhood of the attaching sphere. When I first did it [4], I missed out one complete twist in the band of the knot, obtaining the Stevedore's knot  $6_1$  ( $\frac{1}{2}$  twist) instead of  $9_{46}$  ( $\frac{3}{2}$  twists). The following simplified method of dealing with the geometry is due to W. B. R. Lickorish. Redraw  $S^1 \times S^2$  as  $S^2 \times I$  with  $S^2 \times \partial I$  to be identified, as in Figure 4. In order to

look at  $k$  in the complimentary torus  $S^1 \times D^2$ , we simply have to pull  $k$  around the  $S^2$  until it is well away from the attaching sphere. Figure 4 outlines the process.

If one desires to produce a noninvertible cobordism, one can obtain a cobordism between the unknot and  $6_1$  (hence noninvertible) by either choosing a “self-linking” attaching curve for  $h^2$  or by choosing a different trivialization of the normal bundle to the attaching curve of Figure 1. Figure 5 depicts the “self-linking” attaching curve,

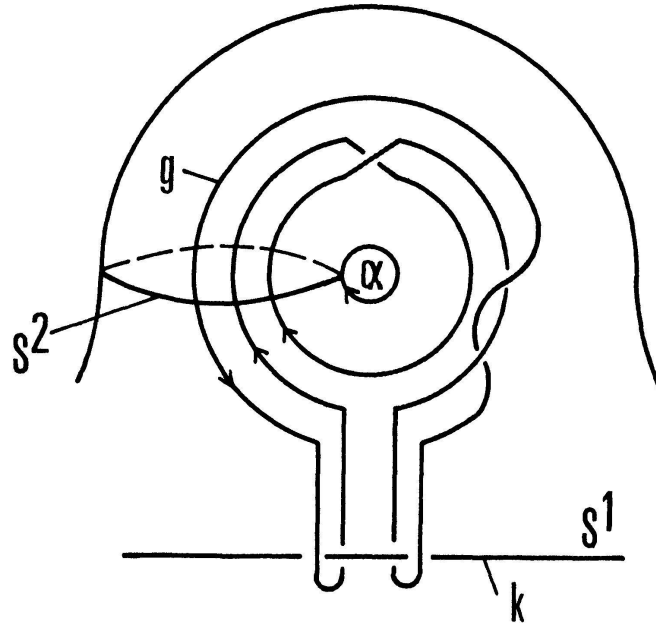


Fig. 5

and checking through the process of Figure 4 reveals that  $6_1$  is the resulting knot. Likewise, a twist in the framing of the tubular neighborhood of the attaching curve of Figure 1 can be seen from the final picture of Figure 4 (before identification) to have the effect of putting one full twist in the band of the knot, hence can be arranged to cancel two of the existing crossovers, producing  $6_1$ .

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