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On the Property P_1 of Locally Compact Groups

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Introduction

A locally compact group G (with a left Haar measure dx and modular function Δ_G) is said to have property P_1 if for every $\varepsilon > 0$ and every compact subset K of G there exists $s \in L^1(G)$ with $\|s\|_1 = 1$ and $\sup_{x \in K} \int_G |s(xy) - s(y)| dy < \varepsilon$. This suggests, for a general locally compact group G , studying the minimum ϱ_1 of all non-negative real numbers λ such that for every $\varepsilon > 0$ and every compact subset K of G there exists $s \geq 0$ with $\|s\|_1 = 1$ and $\sup_{x \in K} \int_G |s(xy) - s(y)| dy < \lambda + \varepsilon$.

We prove (theorem 6) that $\varrho_1 < 1$ implies property P_1 (in fact a stronger result is obtained). In other words from $\varrho_1 \neq 0$ it follows $\varrho_1 \geq 1$. An extension to the case of $L^1(G/H)$, with H satisfying property P_1 , is given in section 2 (theorem 7).

The regular representation of G weakly contains the one dimensional identity representation i_G of G if and only if G has property P_1 . This leads us to consider, for an arbitrary unitary continuous representation π of G acting on a Hilbert space $\mathcal{H}(\pi)$, the $\sup_{K \subset G} \inf_{\|\xi\|=1} \sup_{x \in K} |(\pi(x)\xi, \xi) - 1|$ denoted $d(\pi)$.

We remark that π weakly contains i_G if and only if $d(\pi) = 0$. It is therefore possible to consider $d(\pi)$ as the "distance" from i_G to π .

For a large class of π (including those obtained by inducing the identity from closed subgroups) a stronger result is obtained (theorem 13): $d(\pi) \geq 1$ if and only if π does not weakly contain i_G .

In the last part we prove a property similar to property P_1 but valid for arbitrary G (corollary 16): For every $\varepsilon > 0$ and every compact subset K of G there exists $s \in L^1(G)$ with $s \geq 0$, $\int_G s(x) dx = 1$ and $\sup_{x \in K} \|x \cdot s - s\|_{\Sigma} < \varepsilon$ where $\|s\|_{\Sigma}$ is the norm of s as an element of the full C^∞ -algebra of G .

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The main results of this paper had been announced in the Notices of the Amer. Math. Soc. 17 (1970), p. 822 and 17 (1970), p. 958.

1. Some Results on $L^1(G)$

In what follows, G is a locally compact group with unit element e . We use the following notations:

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\mathcal{F} (resp. \mathcal{K}) is the set of all finite (resp. compact) subsets of G ,

$${}_a\phi_b(x) = \phi(axb) \quad \phi \in C^G, \quad a, b \in G;$$

$$\check{f}(x) = f(x^{-1}), \quad f^*(x) = \overline{f(x^{-1})} \Delta_G(x^{-1}),$$

$$A_x f = \Delta_G(x) f_x \quad \text{where } f \in L^1(G).$$

\mathcal{A} is the convex hull of $\{A_x \mid x \in G\}$. \mathcal{M} is the set of all means on $L^\infty(G)$. Let M be an element of \mathcal{M} , then we set $\alpha(M) = \sup\{|M({}_x\phi) - M(\phi)| \mid x \in G, \|\phi\|_\infty \leq 1\}$.

Finally we define $L^+(G)$ to be the set of all $f \in L^1(G)$ which are non-negative and have L^1 -norm equal to one.

PROPOSITION 1. *If there exists a non-negative λ such that, for every $\varepsilon > 0$ and every $F \in \mathcal{F}$, there is an $s \in L^+(G)$ with $\max_{x \in F} \|s - {}_x s\|_1 < \lambda + \varepsilon$, then the set $\{M \in \mathcal{M} \mid \alpha(M) \leq \lambda\}$ is non empty.*

Proof. It is possible to associate to every $s \in L^+(G)$ a linear functional L_s on $L^\infty(G)$ by setting $L_s(\phi) = \int_G \phi(x) s(x) dx$. We have $\{L_s \mid s \in L^+(G)\} \subset \mathcal{M}$. For $\varepsilon > 0$ and $F \in \mathcal{F}$, the set $\{L_s \mid s \in L^+(G), \sup_{z \in F} \|{}_{z-1} s - s\|_1 < \varepsilon + \lambda\}$ is denoted $A_{F, \varepsilon}$. By assumption $A_{F, \varepsilon} \neq \emptyset$. The inequality $|L_s({}_z\phi) - L_s(\phi)| \leq \|\phi\|_\infty \|{}_{z-1} s - s\|_1$ ($s \in L^+(G)$, $z \in G$ and $\phi \in L^\infty(G)$) implies that for every M in the $\sigma(L^{\infty'}, L^\infty)$ -closure $B_{F, \varepsilon}$ of $A_{F, \varepsilon}$ we have $\max_{x \in F} |M({}_x\phi) - M(\phi)| \leq \|\phi\|_\infty (\lambda + \varepsilon)$ for every $\phi \in L^\infty(G)$.

It is easy to verify that $\{A_{F, \varepsilon} \mid F \in \mathcal{F}, \varepsilon > 0\}$ has the finite intersection property; *a fortiori* so does $\{B_{F, \varepsilon} \mid F \in \mathcal{F}, \varepsilon > 0\}$. Then from the $\sigma(L^{\infty'}, L^\infty)$ -compactness of \mathcal{M} it follows that $\bigcap \{B_{F, \varepsilon} \mid F \in \mathcal{F}, \varepsilon > 0\}$ is non empty. Let M be any element of this set and x an arbitrary element of G . We have $M \in B_{x, \varepsilon}$ for every $\varepsilon > 0$. This implies $|M({}_x\phi) - M(\phi)| \leq (\lambda + \varepsilon) \|\phi\|_\infty$ for every $\phi \in L^\infty(G)$. This inequality is satisfied for every $x \in G$ therefore we have $\alpha(M) \leq \lambda + \varepsilon$, i.e. $\alpha(M) \leq \lambda$.

Remark. See ([9] p. 179 and [10]) for the case $\lambda = 0$.

We observe that for $M \in \mathcal{M}$ and $g \in L^+(G)$ the map which associates to every $\phi \in L^\infty(G)$ the number $M(g^{**}\phi)$ is an element M_g of \mathcal{M} .

LEMMA 2. *If $M \in \mathcal{M}$ and $g \in L^+(G)$, then*

$$|M_g(f^{**}\phi) - \int_G f(x) dx M_g(\phi)| \leq \alpha(M) \|\phi\|_\infty \left(\|f\|_1 + \left| \int_G f(x) dx \right| \right).$$

Proof. We can assume that both f and ϕ are different from 0. Choose an arbitrary $\varepsilon > 0$. It is possible to find $h \in L^+(G)$ with $\|h * f * g - f * g\|_1 < \eta$ and

$$\|h * g - g\|_1 < \eta \quad \text{where } \eta = \min \left(\frac{\varepsilon}{3 \|f\|_1 \|\phi\|_\infty}, \frac{\varepsilon}{3 \|\phi\|_\infty} \right).$$

This clearly implies

$$|M_g(\overline{f^* * \phi}) - M((h * f^* * g)^* * \phi)| < \frac{\varepsilon}{3} \quad (1)$$

and

$$\left| \int_G f(x) dx \right| |M_{h * g}(\phi) - M_g(\phi)| \leq \frac{\varepsilon}{3} \quad (2)$$

If we take into account the fact that the mapping $x \mapsto_x (h^* * \phi)$ of G into $C^b(G)$ is continuous, we see that we can find a finite subset $\{x_j\}_{j=0}^n$ of G and disjoint Borel subsets $\{A_j\}_{j=0}^n$ of G such that $\bigcup_{j=0}^n A_j = G$ and

$$\left\| (f^* * g)^* * h^* * \phi - \sum_{j=0}^n a_{j x_j} (h^* * \phi) \right\|_{\infty} < \frac{\varepsilon}{3}$$

where $a_j = \int_{A_j} (f^* * g)^*(x) dx$. We therefore have

$$\left| M_{f^* * g}(h^* * \phi) - \left(\int_G f * g(x) dx \right) M_h(\phi) \right| < \frac{\varepsilon}{3} + \alpha(M) \|f\|_1 \|\phi\|_{\infty} \quad (3)$$

In the same way we get

$$\left| \int_G f(x) dx M_{h * g}(\phi) - \int_G f * g(x) dx M_h(\phi) \right| \leq \alpha(M) \|\phi\|_{\infty} \left| \int_G f(x) dx \right|. \quad (4)$$

From (1), (2), (3) and (4) it finally follows

$$\left| M_g(f^* * \phi) - \int_G f(x) dx M_g(\phi) \right| < \alpha(M) \|\phi\|_{\infty} \left(\left| \int_G f(x) dx \right| + \|f\|_1 \right) + \varepsilon.$$

For $f \in L^1(G)$ we denote by $d(f)$ the infimum of $\{\|Af\|_1 \mid A \in \mathcal{A}\}$.

PROPOSITION 3. For arbitrary $f \in L^1(G)$ and M in \mathcal{M} the following inequality holds:

$$(1 - \alpha(M)) d(f) \leq (1 + \alpha(M)) \left| \int_G f(x) dx \right|.$$

Proof. We can assume $d(f) > 0$. In this case there exists $\phi \in L^{\infty}(G)$ such that $\operatorname{Re} \int_G Af(x) \overline{\phi(x)} dx \geq 1$ for every $A \in \mathcal{A}$ and $\|\phi\|_{\infty} = 1/d(f)$.

From $(\overline{Af})^* * \overline{\phi}(z) = \int_G A_{z^{-1}} Af(y) \overline{\phi(y)} dy$ it follows that $1 \leq \operatorname{Re} M_g((\overline{Af})^* * \overline{\phi})$

for arbitrary $g \in L^+(G)$ and every $A \in \mathcal{A}$. By lemma 2 we therefore have

$$1 \leq \left| \int_G f(x) dx \right| |M_g(\bar{\phi})| + \alpha(M) \|\phi\|_\infty \left(\|Af\|_1 + \left| \int_G f(x) dx \right| \right)$$

for every $A \in \mathcal{A}$, i.e.

$$d(f) \leq \left| \int_G f(x) dx \right| + \alpha(M) d(f) + \alpha(M) \left| \int_G f(x) dx \right|.$$

LEMMA 4. *If for $\{f_n\}_{n=1}^M \subset L^1(G)$ we have*

$$\int_G f_n(x) dx = 0, \quad 1 \leq n \leq M,$$

then

$$\inf_{A \in \mathcal{A}} \max_{1 \leq n \leq M} \|A f_n\|_1 \leq \sup \left\{ d(g) \mid \int_G g(x) dx = 0, \|g\|_1 \leq \max_{1 \leq n \leq M} \|f_n\|_1 \right\}.$$

Proof. We denote by L_M the right hand side of the above inequality. For $M=1$ there is nothing to prove. Assume that for arbitrary $\varepsilon > 0$ there exists $A' \in \mathcal{A}$ with $\max_{1 \leq m \leq M-1} \|A' f_m\|_1 < \varepsilon + L_{M-1}$. We can find $A'' \in \mathcal{A}$ such that $\|A'' A' f_M\|_1 < d(A' f_M) + \varepsilon$. $\int_G A' f_M dx = 0$ and $\|A' f_M\|_1 \leq \|f_M\|_1 \leq L_M$ imply $\|A'' A' f_M\|_1 < \varepsilon + L_M$.

For $1 \leq n \leq M-1$ we have $\|A'' A' f_n\|_1 \leq \|A' f_n\|_1 < \varepsilon + L_{M-1} \leq \varepsilon + L_M$. To conclude the proof of the lemma it is enough to take $A = A'' A'$.

PROPOSITION 5. *Let G be an arbitrary locally compact group. For every $f \in L^1(G)$ and every $K \in \mathcal{K}$ we have*

$$\inf_{A \in \mathcal{A}} \sup_{x \in K} \|x(Af) - Af\|_1 \leq 2 \sup \left\{ d(g) \mid \int_G g(x) dx = 0, \|g\|_1 \leq \|f\|_1 \right\}.$$

Proof. Let ε be an arbitrary positive real number. We can find U an open neighborhood of e such that $y \in U$ implies $\|y f - f\|_1 < \varepsilon/2$. On the other hand we can choose a finite set $\{a_n\}_{n=1}^M \subset K$ with $\bigcup_{n=1}^M U a_n \supset K$. By lemma 4 there exists $A \in \mathcal{A}$ such that

$$\begin{aligned} & \max_{1 \leq n \leq M} \|A(a_n f - f)\|_1 < \varepsilon/2 \\ & + \sup \left\{ d(g) \mid \int_G g(x) dx = 0, \|g\|_1 \leq \max_{1 \leq n \leq M} \|a_n f - f\|_1 \right\} \\ & \leq \varepsilon/2 + \sup \left\{ d(g) \mid \int_G g(x) dx = 0, \|g\|_1 \leq 2 \|f\|_1 \right\}. \end{aligned}$$

Therefore we obtain

$$\sup_{y \in K} \|_y(Af) - Af\|_1 < \varepsilon + 2 \sup \left\{ d(g) \mid \|g\|_1 \leq \|f\|_1, \int_G g(x) dx = 0 \right\}.$$

Remarks.

- 1) The idea of this proof comes from [9] p. 176–177. However our formulation is more general.
- 2) It follows from prop. 5 that if G is a locally compact group such that $f \in L^1(G)$ with $\int_G f(x) dx = 0$ implies $d(f) = 0$, then G has property P_1 . In fact, only functions f with $\int_G f(x) dx = 0$ are used in the proof given in [9] p. 176–177.

THEOREM 6. *If there exists λ with $0 \leq \lambda < 1$, such that for every $\varepsilon > 0$ and every $F \in \mathcal{F}$ one can find $s \in L^+(G)$ with $\max_{y \in F} \|_y s - s\|_1 < \lambda + \varepsilon$, then G has property P_1 .*

Proof. By proposition 1 there exists $M \in \mathcal{M}$ with $\alpha(M) \leq \lambda$; from proposition 3 it follows that for every $f \in L^1(G)$ $(1 - \lambda) d(f) \leq (1 + \lambda) |\int_G f(x) dx|$. Finally proposition 5 and the assumption $0 \leq \lambda < 1$ imply that G has property P_1 .

Remarks.

- 1) In fact we have proved a stronger result. Namely, for every $f \in L^1(G)$ and for every $K \in \mathcal{K}$, $\inf_{A \in \mathcal{A}} \sup_{x \in K} \|_x(Af) - Af\|_1 = 0$.
- 2) Let ϱ_1^* be the least non-negative real number λ such that for every $\varepsilon > 0$ and every $F \in \mathcal{F}$ there exists an $s \in L^+(G)$ with $\max_{x \in F} \|_x s - s\|_1 < \lambda + \varepsilon$. Replacing \mathcal{F} by \mathcal{K} we define ϱ_1 in the same way. By prop. 1, 3 and 5 we have

$$\begin{aligned} \sup \left\{ d(g) \mid \int_G g(x) dx = 0, \|g\|_1 \leq 1 \right\} &\leq \varrho_1^* \leq \varrho_1 \\ &\leq 2 \sup \left\{ d(g) \mid \int_G g(x) dx = 0, \|g\|_1 \leq 1 \right\}. \end{aligned}$$

Theorem 6 is equivalent to the following assertion: $\varrho_1^* < 1$ implies $\varrho_1 = 0$.

2. Extension to the Case of $L^1(G/H)$

Let H be a closed subgroup of G with a left Haar measure $d\xi$, the modular function Δ_H and let q be a strictly positive continuous solution of the functional equation $q(x\xi) = q(x) \Delta_H(\xi) \Delta_G(\xi)^{-1}$ for $x \in G$ and $\xi \in H$; $d\dot{x}$ is the corresponding quasi-invariant measure on G/H . We set $\chi(y, \dot{x}) = q(yx) q(x)^{-1}$ where $x, y \in G$ and $\dot{x} = xH = \pi_H(x)$. Define a map of $L^1(G)$ onto $L^1(G/H)$ by

$$T_H f(\dot{x}) = \int_H \frac{f(x\xi)}{q(x\xi)} d\xi$$

and denote by $L^+(G/H)$ the set $\{s \in L^1(G/H) \mid \|s\|_1 = 1, s \geq 0\}$.

THEOREM 7. *If H has property P_1 and if there exists $0 \leq \lambda < 1$ such that, for every $\varepsilon > 0$ and $F \in \mathcal{F}$ there is some $s \in L^+(G/H)$ with $\max_{y \in F} \int_{G/H} |\chi(y^{-1}, \dot{x}) s(y^{-1}\dot{x}) - s(\dot{x})| d\dot{x} < \varepsilon + \lambda$, then G has property P_1 .*

Proof. It is enough to prove that the assumptions of theorem 6 are satisfied. Choose $F \in \mathcal{F}$ and $\varepsilon > 0$ arbitrarily. Clearly there is some $s' \in L^+(G/H)$ which is continuous, has compact support, and satisfies

$$\int_{G/H} |\chi(y^{-1}, \dot{x}) s'(y^{-1}, \dot{x}) - s'(\dot{x})| d\dot{x} < \lambda + \varepsilon/2 \quad \text{for every } y \in F.$$

Let β be a Bruhat function for the closed subgroup H . Then by the definition of β , $f_1 = s' \pi_H \beta q$ is continuous and has compact support on G , and $\int_G f_1(x) dx = 1$. We verify that

$$\text{supp}(y^{-1}f_1 - f_1) \subset (F \text{ supp } f_1) \cup \text{supp } f_1 \quad \text{for every } y \in F.$$

Then if we use a slight modification of the argument given in [9] p. 116, taking into account the definition of $d\dot{x}$, we can conclude that there exists an $s_1 \in L^+(H)$, continuous and with compact support on H , such that

$$\begin{aligned} \int_G \left| \int_H (y^{-1}f_1 - f_1)(x\xi^{-1}) \Delta_G(\xi^{-1}) s_1(\xi) d\xi \right| dx \\ < \|T_H(y^{-1}f_1 - f_1)\|_1 + \varepsilon/2 \quad \text{for every } y \in F. \end{aligned}$$

Defining

$$s(x) = \int_H f_1(x\xi^{-1}) \Delta_G(\xi^{-1}) s_1(\xi) d\xi,$$

we have $s \geq 0$ and

$$\begin{aligned} \int_G s(x) dx &= \int_G \left(\int_H f_1(x\xi^{-1}) dx \right) \Delta_G(\xi^{-1}) s_1(\xi) d\xi \\ &= \int_G f_1(x) dx \int_H s_1(\xi) d\xi = 1, \quad \text{i.e. } s \in L^+(G). \end{aligned}$$

Observe that

$$T_H(y^{-1}f_1) = \chi(y^{-1}, \cdot)_{y^{-1}}(T_H f).$$

Therefore we have

$$\int_{G_1} |s(y^{-1}x) - s(x)| dx < \int_{G/H} |\chi(y^{-1}, \dot{x}) s'(y^{-1}\dot{x}) - s'(\dot{x})| d\dot{x} + \varepsilon/2$$

for every $y \in F$ i.e.

$$\max_{y \in F} \|y^{-1}s - s\|_1 < \lambda + \varepsilon.$$

Remarks.

- 1) This proof is a modification of the one for the case where H is normal and where P_1 holds for G/H and H ([9] p. 169).
- 2) For $\lambda=0$ a different proof of theorem 7 has already been obtained ([3]).

As above, define $\mathcal{M}(G/H)$ as the set of all means on $L^\infty(G/H)$; and for $M \in \mathcal{M}(G/H)$, set

$$\alpha(M) = \sup \{ |M(x\phi) - M(\phi)| \mid \|\phi\|_\infty \leq 1, x \in G \}.$$

It is also possible ([3]) to formulate a version of property P_1 , for G/H :

G/H is said to have property P_1 , if for every $K \in \mathcal{K}$ and $\varepsilon > 0$ there is some $s \in L^+(G/H)$ with

$$\sup_{x \in K} \|\chi(x^{-1}, \cdot)_{x^{-1}} s - s\|_1 < \varepsilon.$$

PROPOSITION 8. *Let $\lambda \geq 0$. If for every $\varepsilon > 0$ and every $F \in \mathcal{F}$ there exists some $s \in L^+(G/H)$ with*

$$\max_{x \in F} \|\chi(x^{-1}, \cdot)_{x^{-1}} s - s\|_1 < \lambda + \varepsilon,$$

then the set $\{M \in \mathcal{M}(G/H) \mid \alpha(M) \leq \lambda\}$ is non-empty.

The proof is exactly the same as for proposition 1.

$L^1(G)$ acts on $L^\infty(G/H)$ in the following way: if $f \in L^1(G)$ and $\phi \in L^\infty(G/H)$, then the function $i \mapsto \int_G f(x) \phi(xi) dx$ is an element $\overline{f} * \phi \in L^\infty(G/H)$ (see [7] and [8]). For arbitrary $g \in L^+(G)$ and $M \in \mathcal{M}(G/H)$ the map which associates to every $\phi \in L^\infty(G/H)$ the number $M(g * \phi)$ is an element M_g of $\mathcal{M}(G/H)$. Similarly to lemma 2 one can prove that for $M \in \mathcal{M}(G/H)$, $f \in L^1(G)$, $\phi \in L^\infty(G/H)$ and $g \in L^+(G)$ the inequality

$$\left| M_g(\overline{f} * \phi) - \left(\int_G f(x) dx \right) M_g(\phi) \right| \leq \alpha(M) \|\phi\|_\infty \left(\left| \int_G f(x) dx \right| + \|T_H f\|_1 \right)$$

holds, provided that H is compact and $hg = g$ for every $h \in H$ ²⁾. We were not able to drop the assumption on the compactness of H . In the case $\alpha(M) = 0$, but for an arbitrary closed subgroup H and arbitrary $g \in L^+(G)$, the preceding result is due to F. P. Greenleaf ([7] p. 303–304).

²⁾ The compactness of H is used only to assert the existence of such a g .

PROPOSITION 9. For every $M \in \mathcal{M}(G/H)$ and every finite subset $\{f_1, \dots, f_p\}$ of $L^1(G)$ we have

(a) if $\alpha(M) = 0$ then

$$\inf_{A \in \mathcal{A}} \max_{1 \leq j \leq p} \|T_H A f_j\|_1 = \max_{1 \leq j \leq p} \left| \int_G f_j(x) dx \right|,$$

(b) if H is compact then

$$(1 - \alpha(M)) \inf_{A \in \mathcal{A}} \max_{1 \leq j \leq p} \|T_H A f_j\|_1 \leq (1 + \alpha(M)) \max_{1 \leq j \leq p} \left| \int_G f_j(x) dx \right|.$$

Proof. We denote by E the cartesian product of p copies of $L^1(G/H)$ and define on E a norm topology as follows: $\|v\| = \max_{1 \leq j \leq p} \|v_j\|_1$ where $v = (v_1, \dots, v_p)$. We can assume that $d = \inf_{A \in \mathcal{A}} \max_{1 \leq j \leq p} \|T_H A f_j\|_1$ is positive. Then we can find a continuous linear functional ϕ on E such that $\operatorname{Re} \phi((T_H A f_1, \dots, T_H A f_p)) \geq 1$ for every $A \in \mathcal{A}$ and $\|\phi\| = 1/d$. Clearly $\|\phi\| = \sum_{j=1}^p \|\phi_j\|_\infty$ where $\phi = (\phi_1, \dots, \phi_p)$ and $\phi_j \in L^\infty(G/H)$. For $z \in G$ and every $A \in \mathcal{A}$ we have

$$\phi((T_H A_{z^{-1}} f_1, \dots, T_H A_{z^{-1}} f_p)) = \sum_{j=1}^p \overline{(A f_j)^*} * \bar{\phi}_j(z).$$

In case (a) we choose an arbitrary $g \in L^+(G)$. We have

$$M_g(\overline{(A f_j)^*} * \bar{\phi}_j) = \left(\int_G f_j(x) dx \right) M_g(\bar{\phi}_j)$$

for every $A \in \mathcal{A}$ and $1 \leq j \leq p$. We therefore obtain

$$1 \leq \sum_{j=1}^p \left| \int_G f_j(x) dx \right| \|\phi_j\|_\infty \leq \max_j \left| \int_G f_j(x) dx \right| 1/d.$$

This inequality implies (a).

It remains to prove case (b). We can find $g \in L^+(G)$ such that $hg = g$ for every $h \in H$. This implies (see above comment)

$$1 \leq \left| M_g \left(\sum_{j=1}^p \overline{(A f_j)^*} * \bar{\phi}_j \right) \right| \leq \|\phi\| \max_j \left| \int_G f_j(x) dx \right| + \alpha(M) \|\phi\| \left\{ \max_j \|T_H A f_j\|_1 + \max_j \left| \int_G f_j(x) dx \right| \right\}$$

for each $A \in \mathcal{A}$. We therefore have

$$d \leq \max_j \left| \int_G f_j(x) dx \right| + \alpha(M) \left(d + \max_j \left| \int_G f_j(x) dx \right| \right).$$

Remarks.

- 1) For $\alpha(M)=0$ and $\int_G f_j dx=0$ ($1 \leq j \leq p$) prop. 9 is due to P. Eymard ([3] p. 8–9). Using it, he proves an analogue of theorem 6 for G/H in the case $\lambda=0$. Except for H compact, which is then a special case of theorem 7, we were not able to obtain a complete analogue of theorem 6 for G/H .
- 2) If H has property P_1 we have (by [9] p. 174) $\inf_{A \in \mathcal{A}} \|T_H A f\|_1 = d(f)$ for every $f \in L^1(G)$.

3. Other Generalizations and Applications to the Study of $P(G)$

Let π be an arbitrary unitary continuous representation acting on the Hilbert space $\mathcal{H}(\pi)$. Directly related to $d(\pi)$ (defined in the introduction) is

$$\varrho(\pi) = \sup_{K \in \mathcal{K}} \inf_{\|\xi\|=1} \sup_{x \in K} \|\pi(x)\xi - \xi\|.$$

We have in fact $\frac{1}{2}\varrho(\pi)^2 \leq d(\pi) \leq \varrho(\pi)$. If we replace \mathcal{K} by \mathcal{F} we define $\varrho^*(\pi)$ and $d^*(\pi)$, which satisfy the same types of inequalities.

Let $A(\pi)$ be the set of all continuous positive definite functions associated to π and $\sum A(\pi)$ the set of all finite sums of elements of $A(\pi)$. We recall ([4] p. 371) that π weakly contains π' if and only if $A(\pi')$ lies in the compact-open closure of $\sum A(\pi)$. The following proposition is just slightly different from theorem 1.5 ([4] p. 374) and lemma 2.2. ([5] p. 246). Nevertheless, we indicate a direct proof avoiding Banach-algebra techniques.

PROPOSITION 10. *π weakly contains an irreducible representation π' if and only if $A(\pi')$ is in the compact-open closure of $A(\pi)$.*

Proof. Let p be an arbitrary element of $A(\pi')$. We have to show that if p lies in the compact-open closure of $\sum A(\pi)$ then p is already contained in the compact-open closure of $A(\pi)$. It is easy to verify that $p/p(e)$ is in the compact-open closure of $\{q/q(e) \mid q \in \sum A(\pi), q(e) > 0\}$ and that $\{q/q(e) \mid q \in \sum A(\pi), q(e) > 0\}$ is contained in $P_0 \cap \sum A(\pi)$, where P_0 denotes the set $\{u \in P(G) \mid u(e) = 1\}$. The relation $P_0 \cap \sum A(\pi) \subset \text{co}(P_0 \cap A(\pi))$ implies that $p(e)^{-1}p$ is in the compact-open closure of $\text{co}(P_0 \cap A(\pi))$. A fortiori $p(e)^{-1}p$ lies in the $\sigma(P(G), L^1(G))$ -closed convex hull $\overline{\text{co}}(P_0 \cap A(\pi))$ of $P_0 \cap A(\pi)$. The $\sigma(P(G), L^1(G))$ -compactness of $\overline{\text{co}}(P_0 \cap A(\pi))$ and the irreducibility of π' imply ([2] p. 440) that $p(e)^{-1}p$ (an extremal point of $\overline{\text{co}}(P_0 \cap A(\pi))$) is in the $\sigma(P(G), L^1(G))$ -closure of $P_0 \cap A(\pi)$. By D. A. Raikov ([1] p. 260) for every $K \in \mathcal{K}$ and $\varepsilon > 0$ there exists an $\eta > 0$ and a finite set $\{f_j\}_{j=1}^n \subset L^1(G)$ such that $u \in P_0$ and $|\int_G f_j(x)(p(e)^{-1}p(x) - u(x)) dx| < \eta$ ($1 \leq j \leq n$) imply $\sup_{x \in K} |p(e)^{-1}p(x) - u(x)| < \varepsilon$. If we choose $q \in P_0 \cap A(\pi)$ such that

$$\left| \int_G f_j(x)(p(e)^{-1}p(x) - q(x)) dx \right| < \eta \quad 1 \leq j \leq n,$$

we finally get

$$\sup_{x \in K} |p(x) - q'(x)| < \varepsilon \quad \text{where} \quad q' = p(e) q \in A(\pi).$$

COROLLARY 11. *A continuous unitary representation π of G weakly contains the one-dimensional identity representation i_G of G if and only if $d(\pi) = 0$.*

PROPOSITION 12. *Let π be an unitary continuous representation of G such that $d(\pi) < 1$. Then for every $f \in L^1(G)$ we have*

$$2 \left| \int_G f(x) dx \right| \leq \|\pi(f)\| + \|\pi(\bar{f})\|.$$

Proof. a) For every $f \in C_{00}(G)$ (set of all complex-valued continuous functions with compact support) with $f \geq 0$ we have $(1 - d(\pi)) \int_G f(x) dx \leq \|\pi(f)\|$.

We can assume $\int_G f(x) dx > 0$. For every $\varepsilon \in (0, (1 - d(\pi)) \int_G f(x) dx)$ we can find $\xi \in \mathcal{H}(\pi)$ such that $\|\xi\| = 1$ and

$$\sup_{x \in \text{supp } f} |(\pi(x) \xi, \xi) - 1| < d(\pi) + \frac{\varepsilon}{1 + \int_G f(x) dx}.$$

This implies clearly that

$$\left| \int_G f(x) (\pi(x) \xi, \xi) dx - \int_G f dx \right| < \varepsilon + d(\pi) \int_G f(x) dx.$$

From

$$\left| \int_G (\pi(x) \xi, \xi) f(x) dx \right| = |(\pi(f) \xi, \xi)| \leq \|\pi(f)\|$$

it follows that

$$\|\pi(f)\| > (1 - d(\pi)) \int_G f(x) dx - \varepsilon$$

for every $\varepsilon \in (0, (1 - d(\pi)) \int_G f(x) dx)$. This proves a).

b) For every $f \in C_{00}(G)$ with $f \geq 0$ we have $\|\pi(f)\| = \int_G f dx$. Let us assume that there exists $f_0 \in C_{00}(G)$ such that $f_0 \geq 0$ and $\|\pi(f_0)\| \neq \int_G f_0 dx$. We clearly have $\|\pi(f_0)\| < \int_G f_0 dx$ and therefore (by a)) $\|\pi(f_0)\| > 0$. Consider $f_1 = \|\pi(f_0)\|^{-1} f_0$. For arbitrary $n \in \mathbb{N}$ we have $\int_G f_1^{(n)} dx = (\int_G f_1 dx)^n$ and $\|\pi(f_1^{(n)})\| \leq 1$ where $f_1^{(n)} = f_1 * \dots * f_1$ (n -times). Assertion a) implies that $(1 - d(\pi)) (\int_G f_1 dx)^n \leq 1$ for every $n \in \mathbb{N}$. But on the other hand this inequality is not satisfied for $n > -\log(1 - d(\pi)) / \log \int_G f_1 dx$.

c) From b) it follows that for every real-valued $f \in C_{00}(G)$ we have $|\int_G f dx| \leq \|\pi(f)\|$. Let f be an arbitrary function in $C_{00}(G)$. We can write $\int_G f dx = |\int_G f dx| e^{i\theta}$. It follows that

$$\left| \int_G f dx \right| = \int_G \operatorname{Re}(e^{-i\theta} f) dx \leq \|\pi(\operatorname{Re} e^{-i\theta} f)\|.$$

Finally

$$\|\pi(\operatorname{Re} e^{-i\theta} f)\| = \left\| \pi \left(\frac{e^{-i\theta} f + e^{i\theta} \bar{f}}{2} \right) \right\|$$

implies

$$2 \left| \int_G f dx \right| \leq \|\pi(f)\| + \|\pi(\bar{f})\|.$$

By continuity this inequality extends to $L^1(G)$.

THEOREM 13. *Assume that G acts continuously on a locally compact space X and that X admits a quasi-invariant Radon measure μ with modular function χ . Let π be the representation of G in $L^2(X, \mu)$ defined by $\pi(x)\varphi = \chi(x^{-1}, \cdot)^{1/2} \varphi$. If $d(\pi) < 1$, then π weakly contains i_G .*

Proof. By definition of π , $\|\pi(\bar{f})\| = \|\pi(f)\|$ for every $f \in L^1(G)$. Then by prop. 12 we have $|\int_G f dx| \leq \|\pi(f)\|$.

This inequality permits us to finish the proof (by [11] theoreme 1).

Remark. An important exemple of a representation of the above kind is the unitary representation U^H induced on G by the one dimensional identity representation i_H of an arbitrary closed subgroup H of G .

Let π be a representation of G of the type described in theorem 13. It makes sense to define

$$\varrho_1(\pi) = \sup_{K \in \mathcal{X}} \inf_{s \in L^1(X, \mu), s \geq 0} \sup_{x \in K} \|\chi(x^{-1}, \cdot)_{x^{-1}} s - s\|_1$$

and $\varrho_1^*(\pi)$. It is straightforward to verify that

$$\varrho(\pi)^2 \leq \varrho_1(\pi) \leq 4\varrho(\pi)$$

$$\varrho^*(\pi)^2 \leq \varrho_1^*(\pi) \leq 4\varrho^*(\pi).$$

Taking into account these inequalities, theorem 6, corollary 11 and theorem 13, we can deduce the following:

COROLLARY 14. *Let π be an unitary representation of G obtained as in theorem 13. Then the following statements are equivalent:*

- (i) π does not weakly contain i_G ,
- (ii) $d(\pi) \geq 1$
- (iii) $\varrho(\pi) \geq 1$
- (iv) $\varrho_1(\pi) \geq 1$

Moreover for $\pi = U^{i(e)}$ the preceding assertions are equivalent to:

- (v) $\varrho_1^*(\pi) \geq 1$
- (vi) $d^*(\pi) \geq \frac{1}{3^2}$.

Remark. H. Leptin introduced (see Bull. Amer. Math. Soc. 72 (1966), p. 870 and Proc. Math. Soc. 19 (1968), p. 489) the following invariant:

$$I(G) = \sup_{K \in \mathcal{K}} \inf \left\{ \frac{m(KU)}{m(U)} \mid U \in \mathcal{K}, m(U) > 0 \right\}.$$

He proved that

$$I(G) = \begin{cases} 1 & \text{if } G \text{ has property } P_1, \\ +\infty & \text{if not.} \end{cases}$$

We were not able to relate directly $I(G)$ with $\varrho_1, \varrho_1^*, d(U^{i(e)})$ and $d^*(U^{i(e)})$.

4. On the C^* -Algebra of G

Let $B(G)$ be the complex linear span of $P(G)$. The supremum norm closure of the convex hull of the left (or right) translates of an arbitrary $u \in B(G)$ contains a unique constant function, denoted $M(u)$. M defines ([6] p. 59–61) a linear functional on $B(G)$ satisfying the following conditions: (i) $M({}_a u_b) = M(u)$, (ii) $M(\bar{u}) = \overline{M(u)}$ and (iii) $|M(u)| \leq \|u\|_\infty$.

PROPOSITION 15. *For every finite subset $\{f_1, \dots, f_n\}$ of $L^1(G)$ we have*

$$\inf_{A \in \mathcal{A}} \max_{1 \leq j \leq n} \|A f_j\|_\Sigma = \max_{1 \leq j \leq n} \left| \int_G f_j(x) dx \right| \tag{*}$$

Proof. We first remark that for $f, g \in L^1(G)$ and $u \in B(G)$ we have

$$\left| \int_G f * u(x) g(x) dx \right| = \left| \int_G u(x) f^* * g(x) dx \right| \leq \|u\| \|f\|_\Sigma \|g\|_\Sigma$$

where $\|u\|$ denotes the norm of u as element of the dual of the C^* -algebra of G . We therefore have $f * u \in B(G)$. From the uniform continuity of u it follows that for every

$\varepsilon > 0$ we can find disjoint Borel subsets $\{A_j\}_{j=1}^m$ of G and $\{x_j\}_{j=1}^m \subset G$ such that

$$\left\| f * u - \sum_{j=1}^m c_j x_j u \right\|_{\infty} < \varepsilon \quad \text{where} \quad c_j = \int_{A_j} f(x) dx.$$

Using (i) and (iii) we obtain

$$\left| M(f * u) - \int_G f dx M(u) \right| < \varepsilon, \quad \text{i.e.} \quad M(f * u) = \int_G f dx M(u).$$

For every $A \in \mathcal{A}$ and $f \in L^1(G)$ we have

$$\left| \int_G f dx \right| = \left| \int_G Af dx \right| \leq \|Af\|_{\Sigma}.$$

This implies that the l.h.s. in (*) is not smaller than the r.h.s. To prove the last part of the theorem we can proceed as in prop. 9. Let d be the l.h.s. We can assume $d > 0$. Then there exists a continuous linear functional ϕ on the product of n copies of $B(G)$ such that $\|\phi\| = 1/d$ and $\text{Re} \phi(Af) \geq 1$ for every $A \in \mathcal{A}$ (where $Af = (Af_1, \dots, Af_n)$). Clearly $\|\phi\| = \sum_{j=1}^n \|u_j\|$ where $\phi = (u_1, \dots, u_n)$ and $u_j \in B(G)$. From

$$\phi(A_{x^{-1}} Af) = \sum_{j=1}^n \overline{(Af_j)^*} * u_j(x)$$

it follows that

$$1 \leq \sum_{j=1}^n |M(\overline{(Af_j)^*} * u_j)| = \sum_{j=1}^n \left| \int_G f_j(x) dx \right| |M(u_j)|$$

i.e.

$$1 \leq \frac{1}{d} \text{Max}_{1 \leq j \leq n} \left| \int_G f_j(x) dx \right|.$$

COROLLARY 16. For every $f \in L^1(G)$ and every $K \in \mathcal{K}$ we have $\inf_{A \in \mathcal{A}} \sup_{x \in K} \|_x(Af) - Af\|_{\Sigma} = 0$.

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