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## Toda Brackets in Differential Topology<sup>1)</sup>

A. KOSINSKI

It is well known—and indeed it is a basic link between differential topology and homotopy theory—that every map of a manifold  $W$  into a sphere  $S$  can be obtained by the Thom-Pontriagin construction on a framed submanifold  $W$  of  $M$ . The following two questions arise then:

- A. Given a submanifold  $W \subset M$ , what maps  $M \rightarrow S$  can be obtained by various framings of  $W$ ?
- B. Given a map  $f: M \rightarrow S$ , what manifolds  $W$  can be so framed as to give  $f$ ?

This paper discusses a special case of the second question. More precisely, we deal here with the problem of realizing Toda brackets (“secondary compositions”) by the Thom-Pontriagin construction. It turns out that under certain dimensional restrictions Toda brackets of elements in the image of Hopf-Whitehead homomorphism  $J$  can be realized by framings of sphere bundles over spheres. We determine the characteristic element of bundles which occur; in some of the cases this turns out to be a certain Samelson product. This provides an interesting relation between Toda brackets and Samelson products from which one can derive some results about Toda brackets and some computation of non-zero Samelson products.

The results have also applications in differential topology. Namely, given a framed sphere bundle over a sphere then a framed surgery “on a fibre” leads to a homotopy sphere  $\Sigma$  in the image of the Milnor pairing. Moreover,  $\Sigma$  is certainly non-trivial if the framing we started with yielded an element not in the image  $J$ . Hence the method of passing from Toda brackets not in the image  $J$  to non-trivial elements of  $\theta^n$ .

The organization of the present paper is as follows.

§1. introduces notation and gives methods of constructing reduced join of two maps. A simple general construction of an element in a Toda bracket is also given there. Main theorems are stated and proved in section 2, the proofs however depend on a proposition proved in section 3. In section 4 we derive some corollaries for the stable case and give some applications. Section 5 discusses Samelson products and gives construction of sphere bundles over spheres with Samelson products as characteristic elements.

### 1. Notation and General Constructions

We will, in general, adopt the conventions of [4].  $R^n$  will denote the  $n$ -dimensional

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Euclidean space;  $R_+$ ,  $R_-$  will denote the non-negative, resp. non-positive, part of the real line.  $S^n$  will stand for the unit sphere and  $D^{n+1}$  for the unit ball in  $R^{n+1}$ ,  $z_0 = (-1, 0, \dots, 0)$  will be the base point of  $S^n$ .  $S^n$  will be oriented by the map  $\psi_n: R^n \rightarrow S^n - z_0$  given by

$$\psi_n(x_1, \dots, x_n) = \left( \frac{1-x^2}{1+x^2}, \frac{2x_1}{1+x^2}, \dots, \frac{2x_n}{1+x^2} \right), \quad x^2 = \sum x_i^2.$$

We will also need the map  $\tau_n: (D^n, S^{n-1}) \rightarrow (S^n, z_0)$  given by

$$\tau_n(x_1, \dots, x_n) = \psi_n \left( \frac{x_1}{1-x^2}, \dots, \frac{x_n}{1-x^2} \right) \quad \text{for } x^2 < 1, \quad \tau_n(S^{n-1}) = z_0.$$

The upper (resp. lower) hemisphere of  $S^n$  will be denoted  $S_+^n$  (resp.  $S_-^n$ ). “Upper” will refer to the last coordinate.

If  $\alpha$  and  $\beta$  are  $k \times k$  and  $n \times n$  matrices then  $\|\alpha; \beta\|$  will be the notation for the  $(n+k) \times (n+k)$  matrix  $\|\alpha_0 \beta\|$ .  $I_k$  will denote the unit  $k \times k$  matrix.

If  $W$  is a submanifold of dimension  $n-p$  of  $R^n$  and  $F$  a framing of its normal bundle then the Thom-Pontriagin construction yields a map  $(S^n, z_0) \rightarrow (S^p, z_0)$ , we will denote this map  $t(W, F)$ .

Let  $f: S^n \rightarrow S^p$ ,  $g: S^m \rightarrow S^q$ . Then  $f \times g: S^{n+m} \rightarrow S^{p+q}$  is the reduced join of  $f$  and  $g$  as defined in [1].

Suppose we are given submanifolds  $W, V$  of  $R^n, R^m$  respectively with framings  $F, G$ . The submanifold  $W \times V \subset R^{n+m}$  has then a natural framing  $F \times G$ .

### 1.1. PROPOSITION. $t(W \times V, F \times G) = t(W, F) \times t(V, G)$ .

*Proof.* Let  $N$  be a tubular neighborhood of  $W$  in  $R^n$ ,  $M$  a tubular neighborhood of  $V$  in  $R^m$ , and  $K \subset N \times M$  a tubular neighborhood of  $W \times V$  in  $R^n \times R^m$ . Consider a product structure on  $S^{p+q} - z_0$  given by  $\psi_{p+q}: R^p \times R^q = R^{p+q} \rightarrow S^{p+q}$ . According to [1, 3.3], in terms of this product structure,  $t(W, F) \times t(V, G)$  is given by

$$t(W, F) \times t(V, G)(x, y) = \begin{cases} (t(W, F)(x), t(V, G)(y)) & \text{for } (x, y) \in N \times M \\ z_0 & \text{for } (x, y) \notin N \times M. \end{cases}$$

On the other hand

$$t(W \times V, F \times G)(x, y) = \begin{cases} (t(W, F)(x), t(V, G)(y)) & \text{for } (x, y) \in K \subset N \times M \\ z_0 & \text{for } (x, y) \notin K. \end{cases}$$

Since both definitions yield obviously homotopic maps, the proposition follows.

*Remark.* A similar proposition has been proved by Kervaire in [4, 1.11]. There is however a wrong sign in his paper. See also [8, §6].

The following proposition is due to Toda [7, proof of 4.6]:

**1.2 PROPOSITION.** *Assume that  $\alpha \in \pi_{p+h}(S^h)$ ,  $\beta \in \pi_{q+k}(S^k)$ ,  $\gamma \in \pi_{r+l}(S^l)$  satisfy  $\alpha \times \beta = 0$ ,  $\beta \times \gamma = 0$  and let  $A_t$  and  $B_t$  be null homotopies of  $\alpha \times \beta$  and  $\beta \times \gamma$  respectively. Then  $\alpha \times B_t$  can be interpreted as a map of the upper hemisphere of  $S^e$ ,  $e = p+h+q+k+r+l+1$ , and  $A_t \times \gamma$  as the map of the lower hemisphere of  $S^e$  so that together they yield a map of  $S^e$  which is an element of  $(-1)^{(p+h)q + (p+h+q+k)r} \{E^{k+l}\alpha, E^{p+h+l}\beta, E^{p+h+q+k}\gamma\}$ .*

Suppose now that  $\alpha = t(W^p, F_p)$ ,  $\beta = t(W^q, F_q)$ ,  $\gamma = t(W^r, F_r)$ ,  $W^p \subset R_{p+h}$ ,  $W^q \subset R_{q+k}$ ,  $W^r \subset R_{r+l}$ . Assume that  $\alpha \times \beta = 0$  and  $\beta \times \gamma = 0$ . By Proposition 1.1 this implies that  $(W^p \times W^q, F_p \times F_q)$  bounds  $(V^{p+q+1}, G)$ ,  $V^{p+q+1} \subset R^{p+h} \times R^{q+k} \times R_+$ , and  $(W^q \times W^r, F_q \times F_r)$  bounds  $(V^{q+r+1}, H)$ ,  $V^{q+r+1} \subset R^{q+k} \times R^{r+1} \times R_+$ . Imbed now  $R^{p+h} \times R^{q+k} \times R_+$  in  $R^{p+h} \times R^{q+k} \times R^{r+1} \times R_+$  by a map  $(x, y, t) \rightarrow (x, y, 0, -t)$  and  $R^{q+k} \times R^{r+1} \times R_+$  in  $R^{p+h} \times R^{q+k} \times R^{r+1} \times R_+$  by a map  $(y, z, t) \rightarrow (0, y, z, t)$ . These imbeddings exhibit  $(V^{p+q+1} \times W^r, G \times F_r)$  and  $(W^p \times V^{q+r+1}, F_p \times H)$  as framed submanifolds with boundary of  $R^{p+h+q+k+r+l} \times R_-$ ,  $R^{p+h+q+k+r+l} \times R_+$  respectively. Moreover, since  $\partial(V^{p+q+1} \times W^r) = \partial(W^p \times V^{q+r+1}) = W^p \times W^q \times W^r$ , and both framings  $G \times F_r$ ,  $F_p \times H$  restrict on the boundary to  $F_p \times F_q \times F_r$ , we obtain naturally a manifold  $W = V^{p+q+1} \times W^r \cup W^p \times V^{q+r+1} \subset R^{p+h+q+k+r+l} \times R$  with a framing  $F$ . By proposition 1.2 we have

**1.3 THEOREM.**  $t(W, F) \in (-1)^\varepsilon \{E^{k+l}\alpha, E^{p+h+l}\beta, E^{p+h+q+k}\gamma\}$ ,  $\varepsilon = (p+h)q + (p+h+q+k)r$ .

## 2. Main Theorems

If  $\alpha, \beta, \gamma$  are in the image of the Hopf-Whitehead homomorphism  $J$  then one can take spheres as  $W_p$ ,  $W_q$ ,  $W_r$  and it turns out that it is possible to determine what can be taken as  $W$  in the Theorem 1.3.

More precisely let  $\alpha \in \pi_p(SO_h)$ ,  $\beta \in \pi_q(SO_k)$ ,  $\gamma \in \pi_r(SO_l)$  and set  $\alpha' = t(S^p, F(\alpha))$ ,  $\beta' = t(S^q, F(\beta))$ ,  $\gamma' = t(S^r, F(\gamma))$  where vectors of  $F(\alpha)$  at a point  $x$  are rows of the matrix  $\alpha(x)$  and  $F(\beta)$ ,  $F(\gamma)$  are defined analogously.

The natural homomorphisms  $\pi_i(SO_j) \rightarrow \pi_i(SO)$  will be denoted here  $(i, j)$  and  $T(\alpha, \beta, \gamma)$  will stand for  $(-1)^\varepsilon \{E^{k+l}\alpha', E^{p+h+l}\beta', E^{p+h+k+q}\gamma'\}$ ,  $\varepsilon = (p+h)q + (p+h+q+k)r$ . If  $\lambda \in \pi_p(SO_{q+1})$ ,  $\mu \in \pi_r(SO_{q+1})$  then  $\langle \lambda, \mu \rangle$  will denote the Samelson product of  $\lambda$  and  $\mu$ . Notice that according to [4, 1.8 and the footnote on p. 346]

$$\begin{aligned} E^{k+l}\alpha' &= (-1)^p E^{k+l}((( -1)^{h-1} \iota_h) \circ J\alpha) = (-1)^{p+h-1} E^{k+l} J\alpha \\ &= (-1)^{p+h+k+l-1} J s_*^{k+l} \alpha \end{aligned}$$

and similarly for suspensions of  $\beta'$  and  $\gamma'$ , where  $s_*^k$  is the  $k$ -times iterated suspension in the homotopy groups of the orthogonal group. Therefore  $T(\alpha, \beta, \gamma)$  is a Toda bracket of certain elements in  $\text{Im } J$ .

In the following three theorems it will be assumed that

$$2.1 \quad h+k \geq p+q+2, \quad k+l \geq q+r+2$$

$$2.2 \quad \alpha' \times \beta' = 0, \quad \beta' \times \gamma' = 0. \quad (\text{i.e. } J\alpha \times J\beta = 0, \quad J\beta \times J\gamma = 0)$$

$E(\xi)$  will denote the total space of the sphere bundle with characteristic map  $\xi$ .

2.3 THEOREM. *If  $(p, k), (p, q), (r, k), (r, q)$  are surjective then there exist elements  $\lambda' \in \pi_p(SO_q)$ ,  $\mu' \in \pi_r(SO_q)$  satisfying*

$$s_*^{q+k}\alpha + s_*^{h+k}\lambda' = 0, \quad s_*^{q+k}\gamma + s_*^{l+k}\mu' = 0, \quad 2.31$$

and such that for a certain framing  $F$

$$t(E(\langle \mu, \lambda \rangle), F) \in T(\alpha, \beta, \gamma), \text{ where } \lambda = s_*\lambda', \mu = s_*\mu'.$$

2.4 THEOREM. *If  $(q, h), (q, p), (r, k), (r, q)$  are surjective then there exist elements  $\lambda' \in \pi_q(SO_p)$ ,  $\mu' \in \pi_r(SO_q)$  satisfying*

$$s_*^{p+h}\beta + s_*^{k+h}\lambda' = 0, \quad s_*^{q+k}\gamma + s_*^{l+k}\mu' = 0 \quad 2.41$$

and such that for a certain framing  $F$

$$t(E(\lambda^{-1} \circ J\mu'), F) \in T(\alpha, \beta, \gamma).$$

2.5 THEOREM. *If  $(q, p), (q, r), (q, l), (q, h), (p, r), (p, l)$  are surjective and  $J\alpha \times J\gamma = 0$  then there are elements  $\lambda' \in \pi_q(SO_r)$ ,  $\mu' \in \pi_p(SO_r)$  satisfying*

$$s_*^{r+l}\beta + s_*^{k+l}\lambda' = 0, \quad s_*^{r+l}\alpha + s_*^{h+l}\mu' = 0 \quad 2.51$$

and such that for a certain framing  $F$

$$t(E(\langle \lambda, \mu \rangle), F) \in T(\alpha, \beta, \gamma), \text{ where } \lambda = s_*\lambda', \mu = s_*\mu'.$$

*Proof of 2.3* By Theorem 1.3 an element of  $T(\alpha, \beta, \gamma)$  can be realized by a framing of  $W = V^{p+q+1} \times S^r \cup S^p \times V^{q+r+1}$ . By 3.1 we can take as  $V^{p+q+1}$  the manifold  $S^p \times S^q \cup_{f_\lambda} D^{p+1} \times S^q$  where  $\lambda$  satisfies 2.31. Similarly, as  $V^{q+r+1}$  we can take  $S^q \times S^r \cup_{f_\mu} S^q \times D^{r+1}$ . Recall that  $f_\lambda(x, y) = (x, \lambda(x) \cdot y)$  and  $f_\mu(y, z) = (\mu(z) \cdot y, z)$ . Therefore  $W = D^{p+1} \times S^q \times S^r \cup_{f_{\lambda, \mu}} S^p \times S^q \times D^{r+1}$  where  $f_{\lambda, \mu}(x, y, z) = (x, (\mu^{-1}(z) \cdot \lambda(x)) \cdot y, z)$ .

Applying 5.2 we see that  $W$  is an  $S^q$ -bundle over  $S^{p+r+1}$  with characteristic element  $\langle \mu, \lambda \rangle$ .

*Proof of 2.4* We proceed as in the proof above. As  $V^{p+q+1}$  we take  $S^p \times S^q \cup_{f_\lambda} S^p \times D^{q+1}$  and as  $V^{q+r+1}$  we take  $S^q \times S^r \cup_{f_\mu} S^q \times D^{r+1}$  where  $\lambda, \mu$  satisfy 2.41 and  $f_\lambda(x, y) = (\lambda(y) \cdot x, y)$ ,  $f_\mu(y, z) = (\mu(z) \cdot y, z)$ . Thus  $W' = S^p \times D^{q+1} \times S^r \cup_{f'_{\lambda, \mu}} S^p \times S^q \times D^{r+1}$  where  $f'_{\lambda, \mu}(x, y, z) = (x, (\mu^{-1}(z) \cdot \lambda(y)) \cdot x, z)$ .

$\times D^{r+1}$  where  $f'_{\lambda, \mu}(x, y, z) = (\lambda^{-1}(\mu(z) \cdot y) \cdot x, \mu(z) \cdot y, z)$ . Set  $W = S^p \times D^{q+1} \times S^r \cup_{f_{\lambda, \mu}} S^p \times S^q \times D^{r+1}$  where  $f_{\lambda, \mu}(x, y, z) = (\lambda^{-1}(\mu(z) \cdot y) \cdot x, y, z)$ .

We claim that  $W$  and  $W'$  are diffeomorphic. For consider the identity map of the “right hand half”,  $S^p \times S^q \times D^{r+1}$ , of  $W$  onto the right hand half of  $W'$ . This map induces on the boundary of  $S^p \times D^{q+1} \times S^r$  the diffeomorphism  $f'_{\lambda, \mu} \circ f_{\lambda, \mu}^{-1}$ . This diffeomorphism however sends  $(x, y, z)$  to  $(x, \mu(z) \cdot y, z)$  and therefore extends over  $S^p \times D^{q+1} \times S^r$ . Hence  $W$  and  $W'$  are diffeomorphic.

Now,  $W$  is an  $S^p$ -sphere bundle over  $S^{p+r+1}$  and by 3.2 its characteristic element is  $\lambda^{-1} \circ J\mu'$ .

*Proof of 2.5* Again, proceeding as above we obtain

$$W = S^p \times D^{q+1} \times S^r \cup_{f_{\kappa, \lambda}} S^p \times D^{q+1} \times S^r,$$

where  $f_{\kappa, \lambda}(x, y, z) = (\kappa(y) \cdot x, y, \lambda^{-1}(y) \cdot z)$  and  $s_*^{r+l} \beta + s_*^{k+l} \lambda' = 0$ ,  $s_*^{p+h} \beta + s_*^{k+h} \kappa = 0$

However, now  $W$  is not a sphere bundle over a sphere. We will thus construct a framed cobordism between  $W$  and a desired bundle.

We can consider  $W$  as one of the boundaries of  $W \times [0, 1]$  and we attach to  $S^p \times D^{q+1} \times S^r$  the product  $D^{p+1} \times D^{q+1} \times S^r$  by a map  $(x, y, z) \rightarrow (x, y, \mu(x) \cdot z)$ , where  $s_*^{r+l} \alpha + s_*^{h+l} \mu' = 0$ ,  $\mu = s_* \mu'$ . It is not difficult to check that our assumptions allow an application of 3.1 to conclude that the framing of  $S^p \times 0 \times S^r$  can be extended over  $D^{p+1} \times 0 \times S^r$ . This last set is a deformation retract of  $D^{p+1} \times D^{q+1} \times S^r$ , hence we eventually obtain a framed cobordism between  $W$  and the other boundary of the cobordism. This other boundary is  $E = D^{p+1} \times S^q \times S^r \cup_{g'} S^p \times D^{q+1} \times S^r$  where  $g'(x, y, z) = (\kappa(y) \cdot x, y, \lambda^{-1}(y) \cdot \mu(x) \cdot z)$ . The same construction as in the proof of 2.4 shows that  $E$  is diffeomorphic to  $D^{p+1} \times S^q \times S^r \cup_g S^p \times D^{q+1} \times S^r$  where  $g(x, y, z) = (x, y, (\lambda^{-1}(y) \cdot \mu(x)) \cdot z)$ . By 5.2  $E$  is an  $r$ -sphere bundle over  $S^{p+q+1}$  with characteristic element  $\langle \lambda, \mu \rangle$ . This concludes the proof of 2.5.

### 3. Framings of products of spheres

Let  $S^p \subset R^{p+1} \subset R^{p+h}$ ,  $S^q \subset R^{q+1} \subset R^{q+k}$  be the standard imbeddings. We will coordinatize the normal bundle to  $S^p$  by vectors  $v, v_2, \dots, v_h$  where  $v$  is the unit outwards normal to  $S^p$  in  $R^{p+1}$  and  $v_2, \dots, v_h$  are coordinate vectors in  $R^{h-1}$ ,  $R^{p+h} = R^{p+1} \times R^{h-1}$ . Similarly, the coordinates in the normal bundle to  $S^q$  will be given by  $w, w_2, \dots, w_k$ .

Let  $\alpha \in \pi_p(SO_h)$ ,  $\beta \in \pi_q(SO_k)$ , we have a framing  $F(\alpha, \beta)$  of  $S^p \times S^q \subset R^{p+h} \times R^{q+k}$  given in terms of  $v - w$  coordinates by rows of the matrix  $\|\alpha; \beta\|$ .

Let  $\gamma' \in \pi_q(SO_p)$  and  $f_\gamma: \partial(S^p \times D^{q+1}) \rightarrow S^p \times S^q$  be given by  $f_\gamma(x, y) = (\gamma(y) \cdot x, y)$  where  $\gamma = s_* \gamma'$ . If  $p+q+2 \leq h+k$  then  $f_\gamma$  extends to an imbedding of  $S^p \times D^{q+1}$  in  $R^{p+h} \times R^{q+k} \times R_+$ . Let  $V_\gamma$  be the resulting sub-manifold, i.e.  $V_\gamma = S^p \times S^q \cup_{f_\gamma} S^p \times \times D^{q+1} \subset R^{p+h} \times R^{q+k} \times R_+$ . Can we choose  $\gamma'$  so that  $F(\alpha, \beta)$  extends over  $V_\gamma$ ? The purpose of this section is to prove the following

### 3.1 PROPOSITION. Assume that

- (a)  $h+k \geq p+q+2$ ,
- (b)  $\pi_q(SO_h) \rightarrow \pi_q(SO)$  and  $\pi_q(SO_p) \rightarrow \pi_q(SO)$  are surjective,
- (c)  $J\alpha \times J\beta = 0$ .

Then there exists  $\gamma' \in \pi_q(SO_p)$  satisfying

- (d)  $s_*^{p+h}\beta + s_*^{k+h}\gamma' = 0$

and if  $\gamma'$  is so chosen then  $F(\alpha, \beta)$  extends over  $V_\gamma$ .

*Remarks.*

3.11 By [1, 3.2]  $J\alpha \times J\beta = \pm (Js_*^k\alpha) \circ (Js_*^{p+h}\beta)$ . Therefore assuming, say,  $p \geq q$  the condition (c) is certainly satisfied unless either  $q=1$  and  $p=8s$  or  $8s-1$ , or  $p=q=1, 3, 7$  [6, 7.5].

3.12 Assume  $p \geq q$ , but not  $p=q=1, 3, 7$ , and  $h \geq p+1, k \geq q+1$ . This implies that both (a) and (b) are satisfied and (d) simplifies to  $s_*^p\beta + s_*^k\gamma' = 0$ .

If, in addition,  $q > 1$  then by 3.11 the condition (c) will be satisfied too.

*Proof.* Consider first the framing  $F(\bar{\gamma})$  of  $S^p \times S^q$  given at  $(x, y)$  by the rows of the matrix  $\|\bar{\gamma}(y), I_k\|$  where  $\bar{\gamma} \in \pi_q(SO_h)$  is such that

$$(i) \quad s_*^{p+k}\bar{\gamma} + s_*^{h+k}\gamma' = 0.$$

We claim that  $F(\bar{\gamma})$  extends over  $V_\gamma$ . An extension is found in two steps. First we extend  $F(\bar{\gamma})$  over a neighborhood of  $(x_0) \times D^{q+1}$ . To achieve this, notice that a tangent framing of  $S^p \times S^q$  given by twisting the standard framing by  $\|\gamma(y), I_{q+1}\|$  extends over  $V_\gamma$ : the extension is precisely the image of the standard framing of  $S^p \times D^{q+1}$  under the imbedding map. Now, notice that the normal framing  $F(\bar{\gamma})$  together with the restriction of the just extended tangential framing yields a map  $w: S^q \rightarrow SO_{p+h+k+q+1}$  which is null homotopic by (i). The tangential framing above gives a map  $S^q \rightarrow V_{p+h+q+k+1, p+q+1}$  which equals  $\pi \circ w$  where  $\pi: SO_{p+h+k+q+1} \rightarrow V_{p+h+q+k+1, p+q+1}$  is the projection of the fibration  $SO_{p+h+k+q+1}/SO_{h+k}$ . To extend  $F(\bar{\gamma})$  over a neighborhood of  $x_0 \times D^{q+1}$  we have then to find a null-homotopy of  $w$  which covers a null homotopy of  $\pi \circ w$  given by the extension of the tangential framing over  $x_0 \times D^{q+1}$ . But this is possible since, by (a),  $q < h+k$  and so  $\pi_q(SO_{p+h+q+k+1}, SO_{h+k}) = 0$ .

The second step consists in completing the extension over the complement of  $S^p \times S^q \cup_{f_\gamma} (x_0) \times D^{q+1}$  in  $V_\gamma$ . To achieve this notice first that  $S^p \times S^q \subset R^{p+h} \times R^{q+k}$  is a boundary of a standardly imbedded  $D^{p+1} \times S^q$  in  $R^{p+h} \times R^{q+k} \times R_-$  and that  $F(\bar{\gamma})$  certainly extends over  $D^{p+1} \times S^q$ . However  $V_\gamma \cup D^{p+1} \times S^q$  is an imbedded sphere and by [5, proof of 3.1] every framing of a complement of an open disc in an imbedded sphere extends over that disc, provided that  $h+k \geq p+q+2$ , which we assumed in (a). This concludes the extension.

Now, at the point  $(x, y) \in S^p \times \partial D^{q+1}$  which corresponds to  $(\gamma(y) \cdot x, y) \in S^p \times S^q$  we have also the frame of  $F(\alpha, \beta)$  given by rows of  $\|\alpha(\gamma(y) \cdot x); \beta(y)\|$ . Therefore

extending  $F(\alpha, \beta)$  amounts to extending the map  $\phi: \partial(S^p \times D^{q+1}) \rightarrow SO_{h+k}$  given by

$$(x, y) \mapsto \|\bar{\gamma}(y); I_k\|^{-1} \|\alpha(\gamma(y) \cdot x); \beta(y)\| = \|\bar{\gamma}^{-1}(y); \beta(y)\| \cdot \|\alpha(\gamma(y) \cdot x); I_k\|.$$

over  $S^p \times D^{q+1}$ .

First, define maps  $\phi_1, \phi_2: \partial(S^p \times D^{q+1}) \rightarrow SO_{h+k}$  by  $\phi_1(x, y) = \|\bar{\gamma}^{-1}(y); \beta(y)\|$ ,  $\phi_2(x, y) = \|\alpha(\gamma(y) \cdot x); I_k\|$ . Then  $\phi_1$  will extend over  $S^p \times D^{q+1}$  if  $\bar{\gamma}$  satisfies

$$(ii) \quad s_*^k \bar{\gamma} = s_*^h \beta.$$

On the other hand since  $\gamma$  is a suspension there is  $x_0 \in S^p$  such that for all  $y \in S^q$   $\gamma(y) \cdot x_0 = x_0$ . Thus  $\phi_2(x_0, y) = \|\alpha(x_0), I_k\| = I_{k+h}$  for we can assume  $\alpha(x_0) = I_h$ . Hence  $\phi_2$  extends trivially over  $x_0 \times D^{q+1}$  and to complete the extension of  $\phi$  we have to extend  $\phi_2$  over the complement of  $\partial(S^p \times D^{q+1}) \cup x_0 \times D^{q+1}$ .

Let  $g: D^p \times D^{q+1} \rightarrow S^p \times D^{q+1}$  be given by  $g(x, y) = (\tau_p(x), y)$  where  $\tau_p: D^p \rightarrow S^p$  is a relative homeomorphism shrinking the boundary of  $D^p$  to a point. Let  $\omega: S^{p+q} \rightarrow SO_{h+k}$  be a composition of  $g|_{\partial(D^p \times D^{q+1})}$  with  $\phi_2$ . Then  $\phi_2$  will extend if  $\omega$  is null-homotopic.

This is certainly the case if  $p+q=2, 4, 5, 6$ , mod 8. To deal with the case  $p+q=3$  mod 4 notice that  $\omega = \omega_1 \circ \omega_2$  where  $\omega_1: S^p \rightarrow SO_{h+k}$  equals  $s_*^k \alpha$  and  $\omega_2: S^{p+q} \rightarrow S^p$  is given by

$$(iii) \quad \omega_2(x, y) = \begin{cases} x_0 & \text{if } (x, y) \in \partial D^p \times D^{q+1} \\ \gamma(y) \circ \tau_p(x) & \text{if } (x, y) \in D^p \times \partial D^{q+1} \end{cases}$$

Now,  $\omega_2$  is of finite order except, possibly, when  $p$  is even. However for  $p$  even  $\omega_1$  is in the (stable) group  $\pi_p(SO_{h+k})$  of finite order. Therefore  $\omega$  is always of finite order which implies that it is null homotopic since  $\pi_{p+q}(SO_{h+k})$  is cyclic infinite for  $p+q=3$  mod 4.

There remains the case  $p+q=0, 1$  mod 8. One sees easily using (iii) and [4, 1.8] that  $\omega_2 = \pm J\gamma'$  where  $\gamma' \in \pi_q(SO_p)$  is such that  $s_* \gamma' = \gamma$ . Hence  $\omega = \pm s_*^k \alpha \circ J\gamma'$  and, by [6, 7.5]

$$J\omega = + (J s_*^k \alpha) \circ E^{h+k} J\gamma' = \pm (J s_*^k \alpha) \circ (J s_*^{h+k} \gamma').$$

However, by (i) and (ii)  $s_*^{h+k} \gamma' = s_*^{p+h} \beta$  and we have

$$J\omega = + (J s_*^k \alpha) \circ (J s_*^{p+h} \beta) = \pm J\alpha \times J\beta$$

by [1, 3.2]. Thus  $J\omega = 0$  and, since  $J$  is a monomorphism in the case under consideration, we have  $\omega = 0$ . This shows that if  $\gamma', \bar{\gamma}$  satisfy (i) and (ii) then  $F(\alpha, \beta)$  can be extended over  $V_\gamma$ .

To show that they can be so found notice, first, that since  $\pi_q(SO_p) \rightarrow \pi_q(SO_{p+k+h})$  is surjective, in view of (a) and (b), it is possible to find  $\gamma' \in \pi_q(SO_p)$  so that (d) is satisfied. Now, it follows from (a) and (b) that it is possible to find  $\bar{\gamma}$  satisfying (i). Then

(i) and (d) together imply

$$s_*^{p+h}\beta = s_*^{p+k}\bar{\gamma}$$

which is equivalent to (ii) by (a). This concludes the proof of 3.1.

The following useful remark can be extracted from the above proof.

Let  $\psi: \partial(S^p \times D^{q+1}) \rightarrow SO_h$  be given by  $\psi(x, y) = \|\alpha(\gamma(y) \cdot x)\|$ , where  $\alpha: S^p \rightarrow SO_h$ ,  $\gamma: S^q \rightarrow SO_{p+1}$ ,  $\gamma = s_* \gamma'$ ,  $x \in S^p$ ,  $y \in S^q$ . We have then

3.2 *The only obstruction to extending  $\psi$  over  $S^p \times D^{q+1}$  is the homotopy class of  $\alpha \circ J\gamma' \in \pi_{p+q}(SO_h)$ .*

#### 4. Applications

First, we will restate theorems 2.3–2.5 in a less general but much simpler form.

Let  $\alpha \in \pi_p(SO_{p+h})$ ,  $\beta \in \pi_q(SO_{q+k})$ ,  $\gamma \in \pi_r(SO_{r+l})$  and assume

4.1  $h, k, l \geq 1$ , it is neither  $p=q=1, 3, 7$  nor  $r=q=1, 3, 7$ .

4.2  $J\alpha \times J\beta = 0 = J\beta \times J\gamma$ .

Under those assumptions we have

4.3 THEOREM. *An element of  $\{J\alpha, J\beta, J\gamma\}$  can be realized by a framing of a sphere bundle over a sphere where*

- (a) *if  $p \leq q \geq r$  then we have a  $q$ -sphere bundle over  $S^{p+r+1}$  with characteristic element  $\langle s_*^{q-r-l+1}\gamma, s_*^{q-p-h+1}\alpha \rangle$*
- (b) *if  $p \geq q \geq r$  then we have a  $p$ -sphere bundle over  $S^{q+r+1}$  with characteristic element  $\langle s_*^{p-q-k+1}\beta, J s_*^{q-r-l+1}\gamma \rangle$*
- (c) *if  $q \leq p \leq r$  but not  $p=r=1, 3, 7$  then we have an  $r$ -sphere bundle over  $S^{p+q+1}$  with characteristic element  $\langle s_*^{r-p-h+1}\alpha, s_*^{r-q-k+1}\beta \rangle$ .*

This leaves out cases  $p \leq q \leq r$  and  $q \leq r \leq p$  which can however be obtained from (b) and (c), respectively, by interchanging  $p$  and  $r$ .

The theorem follows immediately from 2.3–2.5; for case (c) use also 3.11. (Notice that by [2]  $(i, j)$  is surjective if  $i < 2j - 2$  provided  $j \geq 13$ . This fact allows to use 2.3–2.5 to derive information about Toda brackets of elements in the meta-stable range).

Now, let  $x = \max(p, q, r)$  and let  $y, z$  be the remaining two numbers.

4.4 THEOREM. *If  $x > y + z + 2$  and  $y, z > 7$  then  $\{J\alpha, J\beta, J\gamma\} \equiv 0 \pmod{\text{indeterminacy}}$*

*Proof.* The proof is based on the following facts:

- (i) Every framing of  $S^p \times S^q$  yields an element of  $\text{Im } J$  except, perhaps, when  $p=q=1, 3, 7$ . (This is due to Novikov.)
- (ii)  $\{\alpha, \beta, \gamma \circ \delta\} \supset \{\alpha, \beta, \gamma\} \circ \delta$  and similarly for compositions with other elements in the bracket (Toda).

(iii) If  $\varepsilon_i$  is a generator of  $\pi_i(SO)$  and  $\eta_j$  generates  $\pi_{j+1}(S^j)$  then  $\varepsilon_{8s-1} \circ \eta_{8s-1} = \varepsilon_{8s}$ ,  $\varepsilon_{8s-1} \circ \eta_{8s-1} \circ \eta_{8s} = \varepsilon_{8s+1}$  (Kervaire).

We can restrict ourselves to consideration of Toda brackets of generators of  $\text{Im } J$  and to the case when  $p, q, r$  are congruent to 0, 1, 3, 7 mod 8. Now (ii) and (iii) show that  $\{J\alpha, J\beta, J\gamma\} = T$  contains a composition with  $\{J\alpha', J\beta', J\gamma'\} = T'$  where dimensions of  $\alpha', \beta', \gamma'$  are 3 or 7 mod 8. Moreover,  $T'$  will satisfy the relation  $x > y + z$ . By 4.3 an element of  $T'$  can be realized by a framing of an  $x$ -sphere bundle over an  $(y+z+1)$ -sphere, hence of a stable bundle. But a stable bundle which is a  $\pi$ -manifold must be a product. We conclude that an element of  $T'$  can be realized by a framing of  $S^x \times S^{y+z+1}$ . Since  $y+z > 10$  (i) implies that an element of  $T'$  is in  $\text{Im } J$ , more precisely in  $J_{x+y+z+1}$  where the numbers  $x, y, z$  are 3 or 7 mod 8. However  $x+y+z+1 = 2, 6$  mod 8 in such a case and  $J_{8s+2} = J_{8s+6} = 0$ . Hence  $T'$  will always contain zero and so will  $T$  which either equals  $T'$  or contains a composition of  $T'$  with either  $\eta$  or  $\eta^2$ . This concludes the proof of 4.4.

*Remark.* The dimensional restrictions in 4.4 can be somewhat relaxed, at the expense of more detailed analysis, however things do go wrong in low dimensions:  $\{\eta, v, \eta\} = v^2 \neq 0$ .

To give an example of computations we begin with the following

**4.5 COROLLARY.** *Let  $\alpha, \beta, \gamma$  be as in 4.3(a) and assume that it is not  $q = p+r+1 = 3, 7$ . If  $\{J\alpha, J\beta, J\gamma\} \cap J_{p+q+r+1} = \emptyset$  then  $\langle s_*^{q-r-l+1} \gamma, s_*^{q-p-h+1} \alpha \rangle \neq 0$ .*

*Proof.* It follows from 4.3(a) that an element  $\zeta$  of  $\{J\alpha, J\beta, J\gamma\}$  can be represented by a framing of a  $q$ -sphere bundle over  $S^{p+r+1}$  with the above product as the characteristic element. If this bundle is trivial then by 4.4(i)  $\zeta \in J_{p+q+r+1}$

**4.6 EXAMPLE.** Let  $\alpha \in \pi_7(SO_9)$ ,  $\beta \in \pi_8(SO_{10})$ ,  $\gamma \in \pi_3(SO_5)$  be chosen so that  $J\alpha = \sigma$ ,  $J\beta = \sigma\eta$ ,  $J\gamma = \eta$ . Since  $\{\sigma, \sigma\eta, v\}$  is a single element and not in  $J_{19}$ , 4.5 implies that  $\langle s_*^4 \gamma, \alpha \rangle \neq 0$ .

Set  $E = E(\langle s_*^4 \gamma, \alpha \rangle)$ ,  $E$  is an 8-sphere bundle over  $S^{11}$ . Let  $t(E) \subset \Pi_{19}$  denote the set of elements obtainable by all possible framings of  $E$ . Then  $t(E) = \Pi_{19}$ . For  $\Pi_{19}$  is the direct sum of  $J_{19}$  and of the subgroup generated by  $\{\sigma, \sigma\eta, v\}$ . Since  $E$  bounds a parallelizable manifold,  $t(E) \supset J_{19}$ . But 4.3(a) implies that  $\{\sigma, \sigma\eta, v\}$  can be realized by a framing of  $E$  and the remark follows. I do not know whether there does always exist such a “universal manifold” (for the part of  $\Pi_k$  with trivial Kervaire invariant).

Let  $E$  be so framed as to realize  $\{\sigma, \sigma\eta, v\}$ . By a framed surgery on a fibre we obtain a homotopy sphere  $\Sigma$  and it is not difficult to show that  $\Sigma \in I(E)$ , the inertia group of  $E$ . Since  $\{\sigma, \sigma\eta, v\}$  is not in  $J_{19}$ ,  $\Sigma$  does not bound a parallelizable manifold. Hence we have another counter-example to a conjecture of Novikov (that  $t(I(W)) \subset \text{Im } J$  if  $W$  is a  $\pi$ -manifold). A theorem of W. Browder implies that  $I(E) = \Sigma$ .

## 5. Samelson products

We give here, first, a representation of Samelson-James product [3] of two maps  $\alpha:S^p \rightarrow G$ ,  $\beta:S^q \rightarrow G$ , where  $G$  is a group, as a certain map  $\langle\alpha, \beta\rangle:S^{p+q} \rightarrow G$  and then use it to give a simple construction of sphere bundles over spheres with the characteristic element a Samelson product.

Let  $\alpha:(S^p, z_0) \rightarrow (G, e)$ ,  $\beta:(D^q, \partial D^q) \rightarrow (G, e)$  be two maps. Define a map  $\phi_{\alpha\beta}:S^p \times D^q \rightarrow G$  by  $\phi_{\alpha\beta}(x, y) = \alpha^{-1}(x)\beta(y)\alpha(x)$ . Notice that  $\phi_{\alpha\beta}(x, y) = e$  if  $(x, y) \in \partial(S^p \times D^q)$ . Therefore for an arbitrary imbedding  $S^p \times D^q \rightarrow S^{p+q}$  the map  $\phi_{\alpha\beta}$  induces a map  $S^{p+q} \rightarrow G$ . We choose the standard imbedding and denote the induced map  $\langle\alpha, \beta\rangle$ . Clearly, the homotopy class of  $\langle\alpha, \beta\rangle$  depends only on the homotopy classes of  $\alpha$  and  $\beta$ .

**5.1 PROPOSITION.**  $\langle\alpha, \beta\rangle$  is the Samelson product of  $\alpha$  and  $\beta$ .

*Proof.* We can assume that  $\alpha(x) = e$  for  $x \in S^p_-$ . Let  $|_p:S^p \rightarrow G$  be the trivial map:  $|_p(x) = e$ . Obviously  $\langle|_p, \beta^{-1}\rangle$  is null-homotopic. On the other hand  $\langle|_p, \beta^{-1}\rangle + \langle\alpha, \beta\rangle$  is the homotopy class of the map  $\phi:S^{p+q} \rightarrow G$  given by

$$\phi(z) = \begin{cases} \beta^{-1}(y)\alpha^{-1}(x)\beta(y)\alpha(x) & \text{if } z = (x, y) \in S^p_+ \times D^q \\ e & \text{if } z \notin S^p_+ \times D^q \end{cases}$$

Since  $\phi$  is the Samelson product of  $\alpha$  and  $\beta$ , the proof is complete.

*Remark.* Let  $i:S^p \times D^q \rightarrow S^{p+q}$  and let  $\langle\alpha, \beta\rangle_i:S^{p+q} \rightarrow G$  be the map constructed as above using  $i$ . One can show that

$$\langle\alpha, \beta\rangle_i = \langle\alpha, \beta\rangle + \beta^{-1} \circ t(i)$$

where  $t(i):S^{p+q} \rightarrow S^q$  is the map corresponding to  $i$  via the Thom-Pontriagin construction.

In particular, if  $i|S^p \times 0$  is the standard imbedding then

$$\langle\alpha, \beta\rangle_i = \langle\alpha, \beta\rangle + \beta^{-1} \circ (\pm J\gamma)$$

where  $\gamma \in \pi_p(SO_q)$  gives the twist of the normal bundle of  $i(S^p \times 0)$ .

Now, let  $S^n = S^p \times D^{q+1} \cup D^{p+1} \times S^q$ ,  $p+q+1 = n$  be the standard decomposition of  $S^n$ .

Let  $\alpha:S^p \rightarrow SO_{r+1}$ ,  $\beta:S^q \rightarrow SO_{r+1}$ . We will construct an  $S^r$  bundle over  $S^n$  by glueing  $S^p \times D^{q+1} \times S^r$  to  $S^p \times S^q \times S^r$  by the map

$$(x, y, z) \mapsto (x, y, \beta(y) \cdot z), (x, y) \in \partial(S^p \times D^{q+1}).$$

and  $D^{p+1} \times S^q \times S^r$  to  $S^p \times S^q \times S^r$  by the map

$$(x, y, z) \mapsto (x, y, \alpha(x) \cdot z), (x, y) \in \partial(D^{p+1} \times S^q).$$

The same bundle  $E$  is also obtained by glueing

$S^p \times D^{q+1} \times S^r$  to  $D^{p+1} \times S^q \times S^r$  by the map of the boundaries

$$(x, y, z) \mapsto (x, y, \alpha^{-1}(x) \cdot \beta(y) \cdot z).$$

The projection  $\pi: E \rightarrow S^n$  is given by  $\pi(x, y, z) = (x, y)$ .

### 5.2 PROPOSITION. *The characteristic element of the bundle $E$ is $\langle \alpha, \beta \rangle$ .*

*Proof.* We will assume that in the coordinates in  $S^n - z_0$  given by  $\psi_n$ ,  $S^p \times D^{q+1}$  corresponds to a tubular neighborhood of the unit  $p$ -sphere in the first  $p+1$  coordinates. Let  $A = S^n_- \cup S^p \times D^{q+1}$ .  $A$  is then an  $n$ -disc and the boundary of  $A$  contains “one-half” of the boundary of  $S^p \times D^{q+1}$ , precisely  $S^p \times S^q_+$ .

We will assume that  $\beta$  is concentrated on  $S^q_+$ , i.e.  $\beta(y) = e$ ,  $y \in S^q_+$ . We can then trivialize  $\pi^{-1}(A)$  by the map  $t_1: A \times S^r \rightarrow \pi^{-1}(A)$  given by

$$t_1(u, z) = \begin{cases} (u, z) & \text{if } u \in A - S^p \times D^{q+1}, \\ (x, y, \alpha(x) \cdot z) & \text{if } u = (x, y) \in S^p \times D^{q+1}. \end{cases}$$

Let  $B = Cl(S^n - A)$ . Then  $B \subset D^{p+1} \times S^q$  and  $\pi^{-1}(B)$  is trivialized by  $t_2: B \times S^r \rightarrow \pi^{-1}(B)$ ,  $t_2(x, y, z) = (x, y, z)$ . Hence  $t_2^{-1}t_1: \partial A \times S^r \rightarrow \partial B \times S^r$  is given by

$$t_2^{-1}t_1(u, z) = \begin{cases} (u, z) & \text{if } u \in \partial A - \partial(S^p \times D^{q+1}) \\ (u, \alpha^{-1}(x) \beta(y) \alpha(x) \cdot z) & \text{if } u = (x, y) \in \partial(S^p \times D^{q+1}) \cap \partial A \\ & = S^p \times S^q_+. \end{cases}$$

Using 5.1 we see that the characteristic element of  $E$  is  $\langle \alpha, \beta \rangle$ .

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