Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 46 (1971)

Artikel: On p-equivalences and p-universal spaces
Autor: Mimura, Mamoru / Toda, Hirosi
DOI: https://doi.org/10.5169/seals-35507

On p-equivalences and p-universal spaces

MAMORU MIMURA and HIROSI TODA

Introduction

Throughout this paper we work in the category $\mathscr{C}$ of simply connected, finite CW complexes.

Let p be a prime or zero. Denote $Z_p = Z/pZ$ for $p \neq 0$ and $Z_0 = Q$. A space X is *p-equivalent* to a space Y if there exists a map $f\colon X \to Y$ such that f induces isomorphisms: $H^*(Y; Z_p) \cong H^*(X; Z_p)$. Then f is called a *p-equivalence*. It is not known if p-equivalence is an equivalence relation, in particular, if it satisfies symmetricity.

Let us recall that a space K is called *p-universal* [6] if, for any given p-equivalence $k\colon X \to Y$ and for an arbitrary map $g\colon K \to Y$, there is a map $h\colon K \to X$ and there is a p-equivalence $f\colon K \to K$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ {\scriptstyle h}\,\vdots\uparrow & & \uparrow{\scriptstyle g} \\ K & \xrightarrow[\cdots]{f} & K \end{array}$$

or equivalently, if, for any given p-equivalence $k\colon X \to Z$, and for an arbitrary map $g\colon X \to K$, there is a map $h\colon Y \to K$ and there is a p-equivalence $f\colon K \to K$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \vdots\downarrow{\scriptstyle h} \\ K & \xrightarrow[\cdots]{f} & K \end{array}$$

Thus, for a given p-equivalence $f\colon X \to Y$, if one of X and Y is p-universal, there exists a converse p-equivalence $Y \to X$, namely symmetricity holds, and hence p-equivalence is an equivalence relation in the category of p-universal spaces.

The paper is organized as follows: We show that p-universality is "preserved" under a 0-equivalence. More precisely we prove in §1.

THEOREM 1.2. *Let $f\colon X \to Y$ be a 0-equivalence. If X is p-universal, so is Y.*

As corollaries of this theorem, we can see that an H-space mod 0 and a *co-H*-space mod 0 are p-universal for every prime p and for $p = 0$. Some sufficient conditions for p-universality are given in §2.

THEOREM 2.1. *Let p be a prime or zero. Let K be p-universal. Suppose* $\pi_n(K)\otimes Q = Q$ *or* 0. *Then* $K\bigcup_\alpha e^{n+1}$ *is p-universal.*

THEOREM 2.5 *Let X satisfy* $H^*(X;Q)\cong\otimes_i Q[x_i]/\{(x_i)^{n_i+1}\}$. *Then X is p-universal for every prime p and* $p=0$.

It is shown in §3 that any 3-cell complex is p-universal for any prime p and $p=0$. The last section is devoted to show that there is a four cell complex which is not p-universal. At the same time we show that "p-equivalence" is not an equivalence relation in the category $\mathscr{C}$.

In what follows, a map $f: L\to K$ is often identified with its homotopy class $\{f\}\in[L, K]$. So "the diagram commutes" reads "the diagram commutes up to homotopy". $\mathscr{C}_0$ denotes the class of finite abelian groups.

§ 1. 0-equivalence and p-universality

THEOREM 1.1. *Let p be a prime or zero. Let* $K\in\mathscr{C}$ *and L its subcomplex with* $H_*(K, L; Z)$ *finite. If L is p-universal, so is K.*

Proof. The case $p=0$ is obvious by Theorem 3.2 of [6], since the inclusion $L\to K$ is a 0-equivalence.

Let p be a prime for the rest of the proof. Let $M(G, n)$ be a Moore space of type $(G, n-1)$. Put $M_r^n = M(Z_r, n-1)$ for simplicity. Suppose that $H_r(K, L; Z)$ is trivial except $r=n_1, n_2,\dots, n_k$ with $n_1<n_2<\cdots<n_k$ and that $H_{n_i}(K, L; Z)\cong G_i$ a finite group. Then by [4] there is a homology decomposition: $L=L_0\to L_1\to L_2\to\cdots\to L_k=K$, where $L_{i-1}\to L_i$ is a cofibration inclusion with a cofibre $M(G_i, n_i)$. So by the mathematical induction it suffices to show the theorem for the case $K=L\bigcup_\alpha CM_r^n$ with $(r, p)=1$ or with $r=p^s$, since $M(A+B, n)=M(A, n)\vee M(B, n)$ for any two abelian groups A and B. Let q be any given prime different from p.

Case 1. $(r,p)=1$. By (b) and (c)′ of Theorem 2.1 of [6] there exist p-equivalences f, f' and $f'': L\to L$ such that $f^*H^*(L; Z_r)=0$, $f'_* * 1=0$ on $\pi_{n-1}(L)*Z_r$ and $f''_*\otimes 1=0$ on $\pi_n(L)\otimes Z_r$. We may assume $f=f'=f''$ by taking their compositions if necessary. From the Puppe exact sequence associated with the cofibering:

$$S^{n-1}\xrightarrow{r\iota} S^{n-1}\xrightarrow{i} M_r^n\xrightarrow{\pi} S^n,$$

we have the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
0\to & \pi_n(L)\otimes Z_r & \xrightarrow{\pi^*} & [M_r^n, L] & \xrightarrow{i^*} & \pi_{n-1}(L)*Z_r & \to 0 \\
 & \downarrow{\scriptstyle f_*\otimes 1} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_* * 1} & \\
0\to & \pi_n(L)\otimes Z_r & \xrightarrow{\pi^*} & [M_r^n, L] & \xrightarrow{i^*} & \pi_{n-1}(L)*Z_r & \to 0
\end{array} \qquad (*)$$

Let $\alpha \in [M_r^n, L]$. Then the relation $i^* f_* \alpha = (f_* * 1)\, i^* \alpha = 0$ implies $f_* \alpha = \pi^* \alpha'$ for some $\alpha' \in \pi_n(L) \otimes Z_r$. Also $(f^2)_* \alpha = f_*(\pi^* \alpha') = \pi^*(f_* \otimes 1)\alpha = 0$. So we have an extension $g: L \bigcup_\alpha CM_r^n \to L$ of f^2. Since the inclusion $i: L \to L \bigcup_\alpha CM_r^n$ is a p-equivalence, ifg is a p-equivalence satisfying $(ifg)^* H^*(L \bigcup_\alpha CM_r^n; Z_q) = 0$. By Theorem 2.1 of [6], $L \bigcup_\alpha CM_r^n$ is p-universal.

Case 2. $r = p^s$. Since f is a p-equivalence, $f_* \otimes 1$ and $f_* * 1$ are automorphisms in the diagram $(*)$ with $r = p^s$. Then f_* is an automorphism of a finite group $[M_r^n, L]$ and satisfies $(f^s)_* = (f)_*^s = 1$ for some integer s. So we have a commutative diagram

$$\begin{array}{ccc} M_r^n & \xrightarrow{1_M} & M_r^n \\ \alpha \big\downarrow & & \big\downarrow \alpha \\ L & \xrightarrow{f^s} & L \end{array}$$

and this defines an extension $g: L \bigcup CM_r^n \to L \bigcup CM_r^n$ of f^s which is a p-equivalence. Since $i^*: H^*(L \bigcup_\alpha CM_r^n; Z_q) \cong H^*(L; Z_q)$ and since $g^* H^*(L \bigcup_\alpha CM_r^n; Z_q)$, $L \bigcup_\alpha CM_r^n$ is p-universal by Theorem 2.1 of [6]. Q.E.D.

More generally we have

THEOREM 1.2. *Let p be a prime or zero. Let $f: X \to Y$ be a* 0*-equivalence. If X is p-universal, so is Y.*

By the mapping cylinder argument one may regard X as a subcomplex Y. Then $H_*(Y, X; Z)$ is finite. Theorem 1.2 follows from the former one.

The converse of Theorems 1.1 and 1.2 seem plausible, but we do not know the proof. However the following is true.

THEOREM 1.3. *Let $f: X \to Y$ be a* 0*-equivalence. If Y is p-universal as well as q-universal with $p \neq q$, so is X.*

Proof. Case 1. $p = 0$ or $q = 0$. Then there exists a converse 0-equivalence $g: Y \to X$ by Theorem 3.2 of [6]. So we can apply Theorem 1.2.

Case 2. p and q are different primes. By Proposition 2.10 of [6], Y is 0-universal, and hence there exists a converse 0-equivalence $Y \to X$. Again we can apply Theorem 1.2. Q.E.D.

According to Theorem of [2] the following four conditions are equivalent:

(A) X is an H-space mod 0.

(B) There exists a 0-equivalence $f: \prod_{i=1}^k S^{2n_i+1} \to X$.

(C) All k-invariants are of finite order in the Postnikov decomposition of X.

(D) $H^*(X; Q) \cong \wedge (x_{2n_i+1}, \dots, x_{2n_k+1})$.

The following three conditions are equivalent by Theorem 2.5 of [1]:

(A)′ X is a *co-H*-space mod 0.

(B)′ There exists a 0-equivalence $f: X \to \bigvee_i^k S^{n_i}$.

(C)′ All k'-invariants are of finite order in the homology decomposition of X.

COROLLARY 1.4. *If X satisfies one of the seven conditions in the above, X is p-universal for every prime p and for $p=0$.*

Proof. (i) If X satisfies (B), by Theorem 1.2 X is p-universal for every prime p and $p=0$. For $\prod_{i=1}^k S^{2n_i+1}$ is p-universal for every prime p and $p=0$ by Theorem 3.8 and Corollary 4.3 of [6].

(ii) Suppose that X satisfies (B)′. $\bigvee_i S^{n_i}$ is p-universal for every prime p and for $p=0$ by Theorem 3.8 and Corollary 4.3 of [6]. So X is p-universal for every prime p and for $p=0$ by Theorem 1.3. Q.E.D.

EXAMPLE 1.5. All complex and symplectic Stiefel manifolds are p-universal for every prime p and for $p=0$.

PROPOSITION 1.6. *Let p be a prime or zero.*

i) *An H-space* mod p *is an H-space* mod 0.

ii) *A co-H-space* mod p *is a co-H-space* mod 0.

Proof. (i) Let X be an H-space mod p with a multiplication μ: $X \times X \to X$ and a p-equivalence $h: X \to X$ such that $h \simeq \mu i_1 \simeq \mu i_2$ where i_1 and i_2 are the canonical inclusions. Then by Proposition 2.9 of [6] h is a 0-equivalence. Hence X is an H-space mod 0.

ii) will be proved similarly. Q.E.D.

THEOREM 1.7. *Let q be a prime or zero.*

i) *An H-space* mod q *is p-universal for every prime p and for $p=0$.*

ii) *A co-H-space* mod q *is p-universal for every prime p and for $p=0$.*

This follows from Corollary 1.4 and Proposition 1.6.

§ 2. A sufficient condition for p-universality

THEOREM 2.1. *Let p be a prime or zero. Let K be p-universal. Suppose $\pi_n(K) \otimes Q = Q$ or 0. Then $K \bigcup_\alpha e^{n+1}$ is p-universal for any $\alpha \in \pi_n(K)$.*

Proof. (i) The case p is a prime. We decompose $\pi_n(K)$ as follows:

$$\pi_n(K) \cong F \oplus T_p \oplus T,$$

where F is the free part, (hence $F=Z$ or 0), T_p the p-torsion part, and T the other torsion part of $\pi_n(K)$. There exist integers r and s such that $rT = p^s T_p = 0$. By (C) and (C)′ of Theorem 2.1 of [6], for a given prime q different from p, there exists a p-equivalence $g: K \to K$ such that g^* and $g_* \otimes 1$ are trivial on $H^*(K; Z_q)$ and on $\pi_n(K) \otimes Z_{rq}$

respectively. Note that $g_*(T)=0$ and $g_*(F)\subset F\oplus T_p$, since we have $T\otimes Z_{r_q}=T$. As $g_*\otimes 1$ is an automorphism of a finite group $\pi_n(K)\otimes Z_{p^s}$, there exists an integer t such that $(g^t)_*\otimes 1$ is an identity on $\pi_n(K)\otimes Z_{p^s}=F\otimes Z_{p^s}+T_p\otimes Z_{p^s}$, where $T_p\otimes Z_{p^s}\cong \cong T_p$. So we have $g^t_* \mid T_p=\text{identity}$, $g^t_*(T)=0$ and $g^t_*(F)\subset F$. Let u be a generator of F. ($u=0$ if $F=\{0\}$.) Put $g^t_*(u)=ku$ with k an integer. Then k is a multiple of rq and $k\equiv 1$ (p^s). An arbitrary element of $\pi_n(K)$ is of the following form: $\alpha=nu+y+z$ with an integer n, $y\in T_p$ and $z\in T$. So we have

$$\begin{aligned}(g^t)_*(\alpha) &= knu + y + z\\ &= knu + ky + kz\\ &= k\alpha\end{aligned}$$

That is, the following diagram is commutative:

$$\begin{array}{ccc} S^n & \xrightarrow{k\iota_n} & S^n \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ K & \xrightarrow[g^t]{} & K \end{array}$$

So the map $\bar{g}: K\bigcup_\alpha e^{n+1}\to K\bigcup_\alpha e^{n+1}$ obtained from the commutativity of the diagram is a p-equivalence and satisfies (C) of Theorem 2.1 of [6], and hence $K\bigcup_\alpha e^{n+1}$ is p-universal.

(ii) The case $p=0$. The proof is similar and easier, and so omitted Q.E.D.

COROLLARY 2.2. *Any simply connected 2-cell complex is p-universal for every prime p and for p* $=0$.

Let $(S^m)_\infty$ be the James' reduced product space of S^m which is homotopy equivalent to ΩS^{m+1} and let $(S^m)_n$ be the nm skeleton of $(S^m)_\infty$.

LEMMA 2.3. *Let n be even. Then* $\pi_i((S^m)_n)$ *is finite for* $i\neq m, (n+1)m-1$ *and* $\pi_{(n+1)m-1}((S^m)_n)/(\partial\iota)$ *is finite where* $\partial\iota$ *is the attaching class of* $e^{(n+1)m}=(S^m)_{n+1}-(S^m)_n$. *Thus* $(S^m)_n\bigcup_\alpha e^k$ *for any* $\alpha\in\pi_{k-1}((S^m)_n)$ *is p-universal for any prime p and* $p=0$. *In particular* $(S^m)_{n+1}$ *is p-universal for any prime p and* $p=0$.

Proof. Consider a map of $(S^m)_{n+1}$ onto $S^{(n+1)m}$ smashing $(S^m)_n$ to a point and let $h:((S^m)_\infty,(S^m)_n)\to((S^{(n+1)m})_\infty,*)$ be its combinatorial extension. Then $H^*((S^m)_\infty;Q)\cong H^*((S^{(n+1)m})_\infty;Q)\otimes H^*((S^m)_n;Q)$ and by the same argument as in the proof of Theorem 2.4 of [8] we can get that h induces a $\mathscr{C}_0$-isomorphism $h_*:\pi_i((S^m)_\infty,(S^m)_n)\to\pi_i((S^{(n+1)m})_\infty,*)$ for all i. Since $\pi_i((S^m)_\infty)\cong\pi_{i+1}(S^{m+1})$ and $\pi_j((S^{(n+1)m})_\infty)\cong\pi_{j+1}(S^{(n+1)m+1})$ are finite unless $i=m$ and $j=(n+1)m$ respectively, we have easily that $\pi_i((S^m)_n)$ is finite for $i\neq m,(n+1)m-1$ and $\pi_{(n+1)m-1}((S^m)_n)/\mathrm{Im}.h_*$ is finite, where $\mathrm{Im}.h_*$ is generated by $\partial\iota$. Applying Theorem 2.1 inductively we have that $(S^m)_n$ is p-universal and so is $(S^m)_n\bigcup_\alpha e_k$ for any α. Q.E.D.

LEMMA 2.4. *If $i \neq (n+1)m$, there exists a 0-equivalence h of $((S^m)_\infty, (S^m)_n)$ into itself such that $h_* = 0$ on $\pi_i((S^m)_\infty, (S^m)_n)$.*

Proof. Since $\pi_m((S^m)_\infty, (S^m)_n) = 0$, we may assume that i is different from m. Since $(S^m)_\infty$ is a free monoid complex, a map of S^m of degree q is extended over a cellular map $h_q: (S^m)_\infty \to (S^m)_\infty$ such that h_q^* is an endomorphism of $H^{km}((S^m)_\infty; Z)$ with degree q^k. By use of these maps h_q's, $(S^m)_k$ is a 0-universal space by Theorem 1.1 of [6]. For given positive integers j, k and r there exists a 0-equivalence $h_{(k)}$ of $(S^m)_k$ into itself such that $h_{(k)*} \otimes 1 = 0$ on $\pi_j((S^m)_k) \otimes Z_r$. Remark that the map $h_{(k)}$ is given compositions of h_q's as is seen in the proof of the theorem. Thus $h_{(k)}$ is defined on the whole of $(S^m)_\infty$. Let $j = i-1$ for $k = n$. By Lemma 2.3, $\pi_{i-1}((S^m)_n)$ is finite. Let r be the order of $\pi_{i-1}((S^m)_n)$, and then we have $h_{(n)}$ such that $h_{(n)*} \otimes 1 = 0$ and hence $h_{(n)*} = 0$ on $\pi_{i-1}((S^m)_n)$. Similarly, for sufficiently large $N\left(\text{e.g. } N > \frac{i}{m}\right)$ we have $h_{(N)}$ such that $h_{(N)*} = 0$ on $\pi_i((S^m)_N)$, and hence on $\pi_i((S^m)_\infty)$. Put $h' = h_{(n)} h_{(N)}$, and consider the exact sequence:

$$\pi_i((S^m)_\infty) \xrightarrow{i_*} \pi_i((S^m)_\infty, (S^m)_n) \xrightarrow{\partial} \pi_{i-1}((S^m)_n).$$

For an arbitraty element $\alpha \in \pi_i((S^m)_\infty, (S^m)_n)$, the relation $\partial h'_* \alpha = h'_* \partial \alpha = h_{(n)*}(h_{(N)*} \partial \alpha) = 0$ implies the existence of $\beta \in \pi_i((S^m)_\infty)$ such that $i_* \beta = h'_* \alpha$. Then $h'_* h'_*(\alpha) = h'_* i_* \beta = i_*(h'_* \beta) = i_*(0) = 0$. Thus $h = h'h'$ satisfies the required conditions. Q.E.D.

THEOREM 2.5. *Let X satisfy $H^*(X; Q) \cong \otimes_i Q[x_i]/\{(x_i)^{n_i+1}\}$. Then X is p-universal for every prime p and $p = 0$.*

Proof. It is sufficient to show that there exists a map $f_i: X \to (S^{m_i})_{n_i}$ such that $f_i^*(u_i) = x_i$ up to non-zero coefficient for the fundamental class u_i of $H^{m_i}(S^{m_i}; Q)$, since the composite of the maps

$$X \xrightarrow{\Delta} X \times \cdots \times X \xrightarrow{\Pi f_i} \prod_i (S^{m_i})_{n_i}$$

is a 0-equivalence, and since $\prod_i (S^{m_i})_{n_i}$ is p-universal as a product of p-universal spaces by Lemma 2.3 and by Theorem 3.8 of [6]. For simplicity we omit the indexes of x_i, u_i, m_i, n_i and f_i.

If m is odd, then by the Serre's theorem [7] there exists a map $f: X \to S^m$ such that $x = f^*(u)$ up to non-zero coefficient. So we suppose that m is even. Consider the suspension SX of X and let $\sigma: H^i(X; Q) \to H^{i+1}(SX; Q)$ be the suspension isomorphism ($i > 0$). Since $m+1$ is odd, there exists a map $F: SX \to S^{m+1}$ such that $\sigma x = F^*(\sigma u)$ up to non-zero coefficient. Consider the adjoint map $f_\infty: X \to (S^m)_\infty = \Omega S^{m+1}$ of F. Then $f_\infty^*(u) = \sigma^{-1} F^*(\sigma u) = x$. Let $g: (X, X^{(mn+m-1)}) \to ((S^m)_{n+N}, (S^m)_n)$ be a cellular approximation of f_∞. Then it is sufficient to prove:

(*) If a map $g: (X, X^{(i-1)}) \to ((S^m)_\infty, (S^m)_n)$ satisfies $g^*(u) = x$ up to non-zero coefficient for $i \geqslant m(n+1)$ and if $x^{n+1}=0$ in $H^*(X; Q)$, then there exists a 0-equivalence h of $((S^m)_\infty, (S^m)_n)$ into itself such that hg is homotopic to a map $g':(X, X^{(i)}) \to ((S^m)_{N+n}, (S^m)_n)$.

The obstruction $\gamma(g)$ to deform g to a map g' belongs to $H^i(X; \pi_{i-1}((S^m)_\infty, (S^m)_n))$ and represented by the following cocycle;

$$c: C_i(X) = H_i(X^{(i)}, X^{(i-1)}; Z) \cong \pi_i(X^{(i)}, X^{(i-1)}) \xrightarrow{g_*} \pi_i((S^m)_\infty, (S^m)_n).$$

Assume that $i=m(n+1)$, then $\pi_i((S^m)_\infty, (S^m)_n) \cong H_i((S^m)_\infty, (S^m)_n)$ and c is equivalent to $g_*: C_i(X) = H_i(X^{(i)}, X^{(i-1)}; Z) \to H_i((S^m)_\infty, (S^m)_n; Z) \cong H_i((S^m)_\infty; Z)$. Thus $\gamma(g) = g^*(\varepsilon)$ for a generator ε of $H^{m(n+1)}((S^m)_{n+N}; Z)$. Up to non-zero coefficient, $\varepsilon = u^{n+1}$ and $g^*(\varepsilon) = g^*(u^{n+1}) = x^{n+1} = 0$ in $H^{m(n+1)}(X; Q)$. Thus $\gamma(g)$ is of finite order, say q. The map h_q in the proof of Lemma 2.4 induces h_{q*} having degree q^{n+1} on the $m(n+1)$-dimensional cohomology group, i.e., $h_{q*}(\varepsilon) = q^{n+1}\varepsilon$. Then $\gamma(h_q g) = h_{q*}\gamma(g) = = h_{q*}g^*(\varepsilon) = g^*(h_{q*}\varepsilon) = q^{n+1}g^*(\varepsilon) = q^{n+1}\gamma(g) = 0$, and hence $(*)$ is proved for $h = h_q$.

Next let $i > m(n+1)$, then by use of a 0-equivalence h in Lemma 2.4, we have that $\gamma(hg) = h_*\gamma(g) = 0$ and $(*)$ is proved. Q.E.D.

COROLLARY 2.6. *$H^*(X; Q) \cong \otimes_i Q[x_i]/\{(x_i)^{n_i+1}\}$ if and only if there is a 0-equivalence $X \to \prod_i (S^{m_i})_{n_i}$, $m_i = \deg x_i$.*

This is a generalization of the result due to Arkowitz-Curjel that (D) implies (B) in §1.

EXAMPLE 2.7. The following spaces are p-universal for every prime p and for $p=0$.

(i) The complex projective space CP^n for any $n \geqslant 1$.

(ii) The quarternionic projective space QP^n for any $n \geqslant 1$.

(iii) The Cayley projective plane Π.

§ 3. Some further examples of a p-universal space

THEOREM 3.1. *Let A and B be co-H-spaces* mod 0. *Let $f: A \to B$ be any map. If there is a 0-equivalence from the mapping cone C_f of f into Y, then Y is p-universal for any prime p and $p=0$.*

Proof. As in §1, there exist 0-equivalences $h_A: A \to \vee_j^t S^{m_j}$ and $h_B: B \to \vee_i^r S^{n_i}$. Since $\vee_j S^{m_i}$ and $\vee_i S^{n_i}$ are both 0-universal, there are converse 0-equivalences $k_A: \vee_j S^{m_j} \to A$ and $k_B: \vee_i S^{n_i} \to B$. Consider the composite $a_j: S^{m_j} \subset \vee_j S^{m_j} \to A$ and the homotopy class $f_*\{a_j\}$ of $\pi_{n_j}(B)$. The cokernel of the homomorphism $k_{B*}: \pi_{m_j}(\vee_i S^{n_i}) \to \pi_{m_j}(B)$ is finite, since k_B is 0-equivalence.

Hence there exists a number N_j such that $f_* N_j\{a_j\}$ is in the image of k_{B*}. Put

$g=\vee_j a_j N\iota: \vee_j S^{m_j}\to A$. Then we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k_A}\uparrow & & \uparrow{\scriptstyle k_B} \\ \bigvee_j S^{m_j} & \xrightarrow[g]{} & \bigvee_i S^{n_i} \end{array}$$

It follows from the commutativity that there exists a map $C_g\to C_f$ which is a 0-equivalence. So, if C_g is p-universal, so is C_f, and X is p-universal too. Thus it suffices to show that C_g is p-universal for any prime p and $p=0$. We put $\alpha_j=\{g \mid S^{m_j}\}$. By the well known theorem of Hilton there exists an integer N_j such that $N_j\alpha_j$ is a linear combination of some higher Whitehead product $[\iota_1,\cdots,\iota_s]$, where ι_j is the homotopy class of the inclusion $S^{k_j}\to \vee_i S_{n_i}$ and $\{k_1,\dots,k_s\}$ is a subset of $\{n_1,\dots,n_r\}$. Note that $m_j=\sum_{e=1}^{s}(k_e-1)+1$. Then there exists a 0-equivalence from $M=\vee_{i=1}^{r}S^{n_i}\cup_\beta C(\vee_{j=1}^{s}S^{m_j})$ with $\beta=\vee N_j\alpha_j$ to $K=\vee_{i=1}^{r}S^{n_i}\cup_g C(\vee_j S^{m_j})$. By Theorem 1.1, K is p-universal if so is M. So we will show that M is p-universal. By the linearity of the higher Whitehead product, we have a commutative diagram:

$$\begin{array}{ccc} \bigvee_{j=1}^{s} S^{m_j} & \xrightarrow{\vee q^{b_j}\iota_j} & \bigvee_{j=1}^{s} S^{m_j} \\ \downarrow & & \downarrow \\ \bigvee_{i=1}^{r} S^{n_i} & \xrightarrow[\vee q^{a_i}\iota_i]{} & \bigvee_{i=1}^{r} S^{n_i} \end{array}$$

where $a_i=n_i-1$ and $b_j=m_j-1$. By choosing a prime q different from p, we can see that the map $f: M\to M$ derived from the above commutative diagram is a p-equivalence and satisfies Theorem 2.1, (b) of [6] Q.E.D.

THEOREM 3.2. *Every simply connected three cell complex $K=S^l\cup e^m\cup e^n$, $l<m<n$, is p-universal for any prime p and $p=0$.*

Proof. Let $B=S^l\cup e^m$ and let $f\in\pi_{n-1}(B)$ and $\beta\in\pi_{m-1}(S^l)$ be the attaching elements of the cells e^n and e^m respectively. We devide into three cases:

Case 1: The order of β is finite. Let t be the order of β, then there exists a map $S^m\to B$ which is of degree t by smashing S^l. This map and the inclusion of S^l define a 0-equivalence $S^l\vee S^m\to B$. Then by Theorem 3.1, K is p-universal as a mapping cone of f.

Case 2: $m=l+1$. We may assume that β is non-trivial in $\pi_{m-1}(S^l)\cong Z$. Then $H^*(K;Q)\cong H^*(S^n;Q)=\wedge(x_n)$. Thus A is p-universal by Corollary 1.4.

Case 3: The order of β is infinite and $m\neq l+1$. By [7], if $m\neq l+1$ and $\pi_{m-1}(S^l)$ is infinite, then $l=2m$ and we have an exact sequence:

$$\pi_{2l-2}(S^{l-1})\to\pi_{2l-1}(S^l)\xrightarrow{H} Z.$$

Since $\pi_{2l-2}(S^{l-1})$ is finite and $H([\iota_l, \iota_l])=\pm 2$, there exist non-zero integers r and s such that $r\beta=s[\iota_l, \iota_l]$. Let $C=S^l \cup e^{2l}$ be a mapping cone of $s[\iota_l, \iota_l]$, and then there exists a 0-equivalence $B\to C$ which extends the identity of S^l and is of degree r on S^{2l}. Similarly we have a 0-equivalence $(S^l)_2\to C$; where $(S^l)_2$ is a mapping cone of $[\iota_l, \iota_l]$. Thus $\pi_{n-1}(B)$ and $\pi_{n-1}((S^l)_2)$ are $\mathscr{C}_0$-isomorphic. By Lemma 2.3 $\pi_{n-1}((S^l)_2)$ is finite or it has only one free summand Z. The same is true for $\pi_{n-1}(B)$. Thus K is p-universal by Corollary 2.2 and Theorem 2.1. Q.E.D.

COROLLARY 3.3. *Any simply connected sphere bundle over sphere is p-universal for every prime p and* $p=0$.

§ 4. A counter example for p-universality and for the symmetricity of p-equivalence

Consider a complex $L=S^3\vee CP^2$. Let $(L, 3)$ be the 2-connective fibre space over L. We have a fibering: $S^1\to(L, 3)\xrightarrow{p}L$, and hence we have the Gysin exact sequence: $\cdots\to H^{s-2}(L; Z)\xrightarrow{\Psi}H^s(L; Z)\xrightarrow{p^*}H^s((L, 3); Z)\to H^{s-1}(L; Z)\to\cdots$ where $\Psi(u)=uv$ for a generator $v\in H^2(L; Z)$. Thus we have

(4.1) $H^i((L, 3); Z)\cong Z$ *for* $i=3, 4, 5$
$=0$ *other wise.*

Consider the inclusions $S^3\subset L$ and $S^2\subset L$ and denote by ι_3 and ι_2 their homotopy classes. Let $h: S^5\to CP^2\subset L$ be the composite of the Hopf map and the inclusion. Put $\omega=[\iota_3, \iota_2]\in\pi_4(L)$ the Whitehead product of ι_2 and ι_3. By G. W. Whitehead [9], $\pi_i(S^3\vee CP^2)\cong\pi_i(S^3)\oplus\pi_i(CP^2)\oplus\partial\pi_{i+1}(S^3\times CP^2, S^3\vee CP^2)$, and we see that the map $g_0=\iota_3\vee\omega: S^3\vee S^4\to S^3\vee CP^2$ induces isomorphisms of π_i for $i=3, 4$. Consider a lift $g: S^3\vee S^4\to(L, 3)$ of g_0, then g induces isomorphisms of π_i and H_i for $i\leqslant 4$. Since $h: S^5\to CP^2$ is a 2-connective fibering, the inclusion $i: CP^2\to L$ and the projection $\pi: L\to CP^2$ induce $\bar{\imath}: S^5\to(L, 3)$ and $\bar{\pi}:(L, 3)\to S^5$ such that $\bar{\pi}\bar{\imath}=$identity. Thus $\bar{\imath}_*: H_i(S^5; Z)\to H_i((L, 3); Z)$ is a split monomorphism. Then $f=g\vee\bar{\imath}$ induces an isomorphism $H_*(S^3\vee S^4\vee S^5; Z)\to H_*((L, 3); Z)$ and f is a homotopy equivalence. Obviously $\bar{\imath}$ is a lift of h. Going back to L, we have isomorphisms:

(4.2) $(\iota_3\vee\omega\vee h)_*: \pi_i(S^3\vee S^4\vee S^5)\cong\pi_i(S^3\vee CP^2)=\pi_i(L)$ *for* $i\neq 2$.

For example, $\pi_2(L)\cong Z$, $\pi_3(L)\cong Z$, $\pi_4(L)\cong Z\oplus Z_2$ and $\pi_5(L)\cong Z\oplus Z_2\oplus Z_2$, and generators of each free part are ι_2, ι_3, ω and h respectively.

Consider a map $f': L\to L$ and put $f'_*\iota_2=r\iota_2$ and $f'_*\iota_3=s\iota_3$. By the linearity of the Whitehead product, we have that $f'_*\omega=rs\omega$. Put $f'_*h\equiv th$ modulo 2-torsion. We consider ι_2 and h in $\pi_5(CP^2)$ and replace f' by $f''=\pi f' i: CP^2\subset L\xrightarrow{f'}L\xrightarrow{\pi}CP^2$, then $f''_*\iota_2=r\iota_2$ and $f''_*h=th$. Since $CP^3=CP^2\cup_h e^6$, f'' has an extension $F: CP^3\to CP^3$ such that the degree of F^* on $H^6(CP^3; Z)$ is t. By the linearity of the cup product we have $t=r^3$. Thus we get

(4.3) $f'_*\iota_2=r\iota_2$ *and* $f'_*\iota_3=s\iota_3$ *imply that* $f'_*\omega=rs\omega$ *and* $f'_*h\equiv r^3h$ modulo 2-*torsion.*

Let us consider the element
$\alpha=[[[\omega, \iota_3], \omega], \iota_3]+[[\iota_3, h], h]+[[\omega, h], \omega]$ of $\pi_{11}(S^3 \vee CP^2)$. For simplicity we put $\alpha_1=[[[\omega, \iota_3], \omega], \iota_3]$, $\alpha_2=[[\iota_3, h], h]$ and $\alpha_3=[[\omega, h], \omega]$. The mapping cone of $q\alpha$ is denoted by K_q.

LEMMA 4.1. *Let $f: K_1 \to K_1$ be a p-equivalence. Then $f_*: H_*(K_1; Z) \to H_*(K_1; Z)$ is the identity.*

Proof. Denote by $f'=f \mid L: L \to L$. By (4.3) and by the linearity of the Whitehead product we have

(4.4) $f'_*(\alpha) \equiv r^2 s^4 \alpha_1 + r^6 s \alpha_2 + r^5 s^2 \alpha_3$ modulo 2-*torsion*.

Let $f_*: H_{12}(K_1; Z) \to H_{12}(K_1; Z)$ be of degree t and consider the following diagram

$$\begin{array}{ccc}
\pi_{11}(L) & \xrightarrow{f_*'} & \pi_{11}(L) \\
\uparrow \partial & & \uparrow \partial \\
\pi_{12}(K_1, L) & \xrightarrow{f_*} & \pi_{12}(K_1, L) \\
\downarrow \mathscr{H} & & \downarrow \mathscr{H} \\
H_{12}(K_1, L; Z) & \xrightarrow{f_*} & H_{12}(K_1, L; Z) \\
\uparrow j_* & & \uparrow j_* \\
H_{12}(K_1; Z) & \xrightarrow{f_*} & H_{12}(K_1; Z)
\end{array}$$

where j_* and the Hurewicz homomorphism $\mathscr{H}$ are isomorphisms. Hence $f'_*(\alpha)=t\alpha$. It follows that $r^2 s^4 = r^6 s = r^5 s^2 = t$. If r and s are non-zero, we get $r=s=t=1$. Thus $f_*: H_i(K_1; Z) \to H_i(K_1; Z)$ is the identity unless $i \neq 4$. For $i=4$ this is shown by use of the ring structure of $H^*(K_1; Z)$. Q.E.D.

Let q be a positive integer with $(q, p)=1$. Then the following diagram is commutative:

$$\begin{array}{ccc}
S^{11} & \xrightarrow{q\iota_{11}} & S^{11} \\
\downarrow q\alpha & & \downarrow \alpha \\
X & \xrightarrow[1_X]{} & X
\end{array}$$

So we obtain a p-equivalence $h: K_q \to K_1$. Suppose that K_1 is p-universal. Then by Theorem 3.2 of [6], there exists a converse p-equivalence $k: K_1 \to K_q$. Then $f=hk: K_1 \to \to K_q \to K_1$ is a p-equivalence and the degree of f_* on $H_{12}(K_1; Z)$ is a multiple of q. On the other hand, it follows from the above lemma that f_* is of degree 1. This is a contradiction. We have proved:

THEOREM 4.2. *There exists a* 4-*cell complex which is not p-universal.*

THEOREM 4.3. *The p-equivalence is not an equivalence relation in the category of simply connected finite CW-complexes.*

REFERENCES

[1] M. ARKOWITZ and C. R. CURJEL, *The Hurewicz homomorphism and finite homotopy invariants*, Trans. Amer. Math. Soc. *110* (1964), 538–551.
[2] ——, *Zum Begriff des H-Raumes mod $\mathscr{F}$*, Archiv der Math. *XVI* (1965), 186–190.
[3] P. J. HILTON, *On the homotopy groups of the union of spheres*, J. London Math. Soc. (1955), 154–172.
[4] ——, *Homotopy theory and duality* (Gordon-Breach, 1965).
[5] I. M. JAMES, *Reduced product spaces*, Ann. of Math. *62* (1955), 170–197.
[6] M. MIMURA, R. C. O'NEILL and H. TODA, *On p-equivalence in the sense of Serre* Japanese J. Math. (to appear).
[7] J-P. SERRE, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. *58* (1953), 258–294.
[8] H. TODA, *Composition methods in homotopy groups of spheres*, Ann. Math. Studies No. 49 (1962).
[9] G. W. WHITEHEAD, *A generalization of the Hopf invariant*, Ann. of Math. *51* (1950), 192–237.

Kyoto University and Michigan State University;
Kyoto Uuiversity and Northwestern University.

Received June 10, 1970