

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 46 (1971)

Artikel: On p-equivalences and p-universal spaces
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DOI: <https://doi.org/10.5169/seals-35507>

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On p -equivalences and p -universal spaces

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Introduction

Throughout this paper we work in the category \mathcal{C} of simply connected, finite CW complexes.

Let p be a prime or zero. Denote $Z_p = Z/pZ$ for $p \neq 0$ and $Z_0 = \mathbb{Q}$. A space X is p -equivalent to a space Y if there exists a map $f: X \rightarrow Y$ such that f induces isomorphisms: $H^*(Y; Z_p) \cong H^*(X; Z_p)$. Then f is called a p -equivalence. It is not known if p -equivalence is an equivalence relation, in particular, if it satisfies symmetricity.

Let us recall that a space K is called p -universal [6] if, for any given p -equivalence $k: X \rightarrow Y$ and for an arbitrary map $g: K \rightarrow Y$, there is a map $h: K \rightarrow X$ and there is a p -equivalence $f: K \rightarrow K$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ \uparrow h & & \uparrow g \\ K & \xrightarrow{f} & K \end{array}$$

or equivalently, if, for any given p -equivalence $k: X \rightarrow Z$, and for an arbitrary map $g: X \rightarrow K$, there is a map $h: Y \rightarrow K$ and there is a p -equivalence $f: K \rightarrow K$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ \downarrow g & & \downarrow h \\ K & \xrightarrow{f} & K \end{array}$$

Thus, for a given p -equivalence $f: X \rightarrow Y$, if one of X and Y is p -universal, there exists a converse p -equivalence $Y \rightarrow X$, namely symmetricity holds, and hence p -equivalence is an equivalence relation in the category of p -universal spaces.

The paper is organized as follows: We show that p -universality is “preserved” under a 0-equivalence. More precisely we prove in §1.

THEOREM 1.2. *Let $f: X \rightarrow Y$ be a 0-equivalence. If X is p -universal, so is Y .*

As corollaries of this theorem, we can see that an H -space mod 0 and a co - H -space mod 0 are p -universal for every prime p and for $p=0$. Some sufficient conditions for p -universality are given in §2.

THEOREM 2.1. *Let p be a prime or zero. Let K be p -universal. Suppose $\pi_n(K) \otimes Q = Q$ or 0 . Then $K \bigcup_{\alpha} e^{n+1}$ is p -universal.*

THEOREM 2.5 *Let X satisfy $H^*(X; Q) \cong \bigotimes_i Q[x_i] / \{(x_i)^{n_i+1}\}$. Then X is p -universal for every prime p and $p=0$.*

It is shown in §3 that any 3-cell complex is p -universal for any prime p and $p=0$. The last section is devoted to show that there is a four cell complex which is not p -universal. At the same time we show that “ p -equivalence” is not an equivalence relation in the category \mathcal{C} .

In what follows, a map $f: L \rightarrow K$ is often identified with its homotopy class $\{f\} \in [L, K]$. So “the diagram commutes” reads “the diagram commutes up to homotopy”. \mathcal{C}_0 denotes the class of finite abelian groups.

§ 1. 0-equivalence and p -universality

THEOREM 1.1. *Let p be a prime or zero. Let $K \in \mathcal{C}$ and L its subcomplex with $H_*(K, L; Z)$ finite. If L is p -universal, so is K .*

Proof. The case $p=0$ is obvious by Theorem 3.2 of [6], since the inclusion $L \rightarrow K$ is a 0-equivalence.

Let p be a prime for the rest of the proof. Let $M(G, n)$ be a Moore space of type $(G, n-1)$. Put $M_r^n = M(Z_r, n-1)$ for simplicity. Suppose that $H_r(K, L; Z)$ is trivial except $r = n_1, n_2, \dots, n_k$ with $n_1 < n_2 < \dots < n_k$ and that $H_{n_i}(K, L; Z) \cong G_i$ a finite group. Then by [4] there is a homology decomposition: $L = L_0 \rightarrow L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_k = K$, where $L_{i-1} \rightarrow L_i$ is a cofibration inclusion with a cofibre $M(G_i, n_i)$. So by the mathematical induction it suffices to show the theorem for the case $K = L \bigcup_{\alpha} CM_r^n$ with $(r, p) = 1$ or with $r = p^s$, since $M(A+B, n) = M(A, n) \vee M(B, n)$ for any two abelian groups A and B . Let q be any given prime different from p .

Case 1. $(r, p) = 1$. By (b) and (c)' of Theorem 2.1 of [6] there exist p -equivalences f, f' and $f'': L \rightarrow L$ such that $f^*H^*(L; Z_r) = 0$, $f'_*1 = 0$ on $\pi_{n-1}(L) * Z_r$ and $f''_* \otimes 1 = 0$ on $\pi_n(L) \otimes Z_r$. We may assume $f = f' = f''$ by taking their compositions if necessary. From the Puppe exact sequence associated with the cofiberings:

$$S^{n-1} \xrightarrow{r} S^{n-1} \xrightarrow{i} M_r^n \xrightarrow{\pi} S^n,$$

we have the following exact and commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow \pi_n(L) \otimes Z_r & \xrightarrow{\pi^*} & [M_r^n, L] & \xrightarrow{i^*} & \pi_{n-1}(L) * Z_r & \rightarrow & 0 \\ \downarrow f_* \otimes 1 & & \downarrow f_* & & \downarrow f_* * 1 & & \\ 0 \rightarrow \pi_n(L) \otimes Z_r & \xrightarrow{\pi^*} & [M_r^n, L] & \xrightarrow{i^*} & \pi_{n-1}(L) * Z_r & \rightarrow & 0 \end{array} \quad (*)$$

Let $\alpha \in [M_r^n, L]$. Then the relation $i^* f_* \alpha = (f_* * 1) i^* \alpha = 0$ implies $f_* \alpha = \pi^* \alpha'$ for some $\alpha' \in \pi_n(L) \otimes Z_r$. Also $(f^2)_* \alpha = f_* (\pi^* \alpha') = \pi^* (f_* \otimes 1) \alpha = 0$. So we have an extension $g: L \cup_\alpha CM_r^n \rightarrow L$ of f^2 . Since the inclusion $i: L \rightarrow L \cup_\alpha CM_r^n$ is a p -equivalence, ifg is a p -equivalence satisfying $(ifg)^* H^*(L \cup_\alpha CM_r^n; Z_q) = 0$. By Theorem 2.1 of [6], $L \cup_\alpha CM_r^n$ is p -universal.

Case 2. $r = p^s$. Since f is a p -equivalence, $f_* \otimes 1$ and $f_* * 1$ are automorphisms in the diagram (*) with $r = p^s$. Then f_* is an automorphism of a finite group $[M_r^n, L]$ and satisfies $(f^s)_* = (f^s)_* = 1$ for some integer s . So we have a commutative diagram

$$\begin{array}{ccc} M_r^n & \xrightarrow{1_M} & M_r^n \\ \alpha \downarrow & & \downarrow \alpha \\ L & \xrightarrow{f^s} & L \end{array}$$

and this defines an extension $g: L \cup CM_r^n \rightarrow L \cup CM_r^n$ of f^s which is a p -equivalence. Since $i^*: H^*(L \cup_\alpha CM_r^n; Z_q) \cong H^*(L; Z_q)$ and since $g^* H^*(L \cup_\alpha CM_r^n; Z_q) = 0$, $L \cup_\alpha CM_r^n$ is p -universal by Theorem 2.1 of [6]. Q.E.D.

More generally we have

THEOREM 1.2. *Let p be a prime or zero. Let $f: X \rightarrow Y$ be a 0-equivalence. If X is p -universal, so is Y .*

By the mapping cylinder argument one may regard X as a subcomplex Y . Then $H_*(Y, X; Z)$ is finite. Theorem 1.2 follows from the former one.

The converse of Theorems 1.1 and 1.2 seem plausible, but we do not know the proof. However the following is true.

THEOREM 1.3. *Let $f: X \rightarrow Y$ be a 0-equivalence. If Y is p -universal as well as q -universal with $p \neq q$, so is X .*

Proof. *Case 1.* $p = 0$ or $q = 0$. Then there exists a converse 0-equivalence $g: Y \rightarrow X$ by Theorem 3.2 of [6]. So we can apply Theorem 1.2.

Case 2. p and q are different primes. By Proposition 2.10 of [6], Y is 0-universal, and hence there exists a converse 0-equivalence $Y \rightarrow X$. Again we can apply Theorem 1.2. Q.E.D.

According to Theorem of [2] the following four conditions are equivalent:

- (A) X is an H -space mod 0.
- (B) There exists a 0-equivalence $f: \prod_{i=1}^k S^{2n_i+1} \rightarrow X$.
- (C) All k -invariants are of finite order in the Postnikov decomposition of X .
- (D) $H^*(X; \mathbb{Q}) \cong \wedge (x_{2n_1+1}, \dots, x_{2n_k+1})$.

The following three conditions are equivalent by Theorem 2.5 of [1]:

- (A)' X is a *co*- H -space mod 0.

(B)' There exists a 0-equivalence $f: X \rightarrow \bigvee_i^k S^{n_i}$.

(C)' All k' -invariants are of finite order in the homology decomposition of X .

COROLLARY 1.4. *If X satisfies one of the seven conditions in the above, X is p -universal for every prime p and for $p=0$.*

Proof. (i) If X satisfies (B), by Theorem 1.2 X is p -universal for every prime p and $p=0$. For $\prod_{i=1}^k S^{2n_i+1}$ is p -universal for every prime p and $p=0$ by Theorem 3.8 and Corollary 4.3 of [6].

(ii) Suppose that X satisfies (B)'. $\bigvee_i S^{n_i}$ is p -universal for every prime p and for $p=0$ by Theorem 3.8 and Corollary 4.3 of [6]. So X is p -universal for every prime p and for $p=0$ by Theorem 1.3. Q.E.D.

EXAMPLE 1.5. All complex and symplectic Stiefel manifolds are p -universal for every prime p and for $p=0$.

PROPOSITION 1.6. *Let p be a prime or zero.*

- i) *An H -space mod p is an H -space mod 0.*
- ii) *A co- H -space mod p is a co- H -space mod 0.*

Proof. (i) Let X be an H -space mod p with a multiplication $\mu: X \times X \rightarrow X$ and a p -equivalence $h: X \rightarrow X$ such that $h \simeq \mu i_1 \simeq \mu i_2$ where i_1 and i_2 are the canonical inclusions. Then by Proposition 2.9 of [6] h is a 0-equivalence. Hence X is an H -space mod 0.

ii) will be proved similarly. Q.E.D.

THEOREM 1.7. *Let q be a prime or zero.*

- i) *An H -space mod q is p -universal for every prime p and for $p=0$.*
- ii) *A co- H -space mod q is p -universal for every prime p and for $p=0$.*

This follows from Corollary 1.4 and Proposition 1.6.

§ 2. A sufficient condition for p -universality

THEOREM 2.1. *Let p be a prime or zero. Let K be p -universal. Suppose $\pi_n(K) \otimes Q = Q$ or 0. Then $K \bigcup_{\alpha} e^{n+1}$ is p -universal for any $\alpha \in \pi_n(K)$.*

Proof. (i) The case p is a prime. We decompose $\pi_n(K)$ as follows:

$$\pi_n(K) \cong F \oplus T_p \oplus T,$$

where F is the free part, (hence $F = \mathbb{Z}$ or 0), T_p the p -torsion part, and T the other torsion part of $\pi_n(K)$. There exist integers r and s such that $rT = p^s T_p = 0$. By (C) and (C)' of Theorem 2.1 of [6], for a given prime q different from p , there exists a p -equivalence $g: K \rightarrow K$ such that g^* and $g_* \otimes 1$ are trivial on $H^*(K; \mathbb{Z}_q)$ and on $\pi_n(K) \otimes \mathbb{Z}_{rq}$

respectively. Note that $g_*(T)=0$ and $g_*(F) \subset F \oplus T_p$, since we have $T \otimes Z_{r_q} = T$. As $g_* \otimes 1$ is an automorphism of a finite group $\pi_n(K) \otimes Z_p s$, there exists an integer t such that $(g^t)_* \otimes 1$ is an identity on $\pi_n(K) \otimes Z_p s = F \otimes Z_p s + T_p \otimes Z_p s$, where $T_p \otimes Z_p s \cong \cong T_p$. So we have $g_*^t \mid T_p = \text{identity}$, $g_*^t(T)=0$ and $g_*^t(F) \subset F$. Let u be a generator of F . ($u=0$ if $F=\{0\}$.) Put $g_*^t(u)=ku$ with k an integer. Then k is a multiple of rq and $k \equiv 1 \pmod{p^s}$. An arbitrary element of $\pi_n(K)$ is of the following form: $\alpha = nu + y + z$ with an integer n , $y \in T_p$ and $z \in T$. So we have

$$\begin{aligned} (g^t)_*(\alpha) &= knu + y + z \\ &= knu + ky + kz \\ &= k\alpha \end{aligned}$$

That is, the following diagram is commutative:

$$\begin{array}{ccc} S^n & \xrightarrow{k\iota_n} & S^n \\ \downarrow \alpha & & \downarrow \alpha \\ K & \xrightarrow{g^t} & K \end{array}$$

So the map $\bar{g}: K \bigcup_{\alpha} e^{n+1} \rightarrow K \bigcup_{\alpha} e^{n+1}$ obtained from the commutativity of the diagram is a p -equivalence and satisfies (C) of Theorem 2.1 of [6], and hence $K \bigcup_{\alpha} e^{n+1}$ is p -universal.

(ii) The case $p=0$. The proof is similar and easier, and so omitted Q.E.D.

COROLLARY 2.2. *Any simply connected 2-cell complex is p -universal for every prime p and for $p=0$.*

Let $(S^m)_{\infty}$ be the James' reduced product space of S^m which is homotopy equivalent to ΩS^{m+1} and let $(S^m)_n$ be the nm skeleton of $(S^m)_{\infty}$.

LEMMA 2.3. *Let n be even. Then $\pi_i((S^m)_n)$ is finite for $i \neq m, (n+1)m-1$ and $\pi_{(n+1)m-1}((S^m)_n)/(\partial\iota)$ is finite where $\partial\iota$ is the attaching class of $e^{(n+1)m} = (S^m)_{n+1} - (S^m)_n$. Thus $(S^m)_n \bigcup_{\alpha} e^k$ for any $\alpha \in \pi_{k-1}((S^m)_n)$ is p -universal for any prime p and $p=0$. In particular $(S^m)_{n+1}$ is p -universal for any prime p and $p=0$.*

Proof. Consider a map of $(S^m)_{n+1}$ onto $S^{(n+1)m}$ smashing $(S^m)_n$ to a point and let $h: ((S^m)_{\infty}, (S^m)_n) \rightarrow ((S^{(n+1)m})_{\infty}, *)$ be its combinatorial extension. Then $H^*((S^m)_{\infty}; Q) \cong H^*((S^{(n+1)m})_{\infty}; Q) \otimes H^*((S^m)_n; Q)$ and by the same argument as in the proof of Theorem 2.4 of [8] we can get that h induces a \mathcal{C}_0 -isomorphism $h_*: \pi_i((S^m)_{\infty}, (S^m)_n) \rightarrow \pi_i((S^{(n+1)m})_{\infty}, *)$ for all i . Since $\pi_i((S^m)_{\infty}) \cong \pi_{i+1}(S^{m+1})$ and $\pi_j((S^{(n+1)m})_{\infty}) \cong \pi_{j+1}(S^{(n+1)m+1})$ are finite unless $i=m$ and $j=(n+1)m$ respectively, we have easily that $\pi_i((S^m)_n)$ is finite for $i \neq m, (n+1)m-1$ and $\pi_{(n+1)m-1}((S^m)_n)/\text{Im}.h_*$ is finite, where $\text{Im}.h_*$ is generated by $\partial\iota$. Applying Theorem 2.1 inductively we have that $(S^m)_n$ is p -universal and so is $(S^m)_n \bigcup_{\alpha} e_k$ for any α . Q.E.D.

LEMMA 2.4. *If $i \neq (n+1)m$, there exists a 0-equivalence h of $((S^m)_\infty, (S^m)_n)$ into itself such that $h_* = 0$ on $\pi_i((S^m)_\infty, (S^m)_n)$.*

Proof. Since $\pi_m((S^m)_\infty, (S^m)_n) = 0$, we may assume that i is different from m . Since $(S^m)_\infty$ is a free monoid complex, a map of S^m of degree q is extended over a cellular map $h_q: (S^m)_\infty \rightarrow (S^m)_\infty$ such that h_q^* is an endomorphism of $H^{km}((S^m)_\infty; Z)$ with degree q^k . By use of these maps h_q 's, $(S^m)_k$ is a 0-universal space by Theorem 1.1 of [6]. For given positive integers j, k and r there exists a 0-equivalence $h_{(k)}$ of $(S^m)_k$ into itself such that $h_{(k)*} \otimes 1 = 0$ on $\pi_j((S^m)_k) \otimes Z_r$. Remark that the map $h_{(k)}$ is given compositions of h_q 's as is seen in the proof of the theorem. Thus $h_{(k)}$ is defined on the whole of $(S^m)_\infty$. Let $j = i - 1$ for $k = n$. By Lemma 2.3, $\pi_{i-1}((S^m)_n)$ is finite. Let r be the order of $\pi_{i-1}((S^m)_n)$, and then we have $h_{(n)}$ such that $h_{(n)*} \otimes 1 = 0$ and hence $h_{(n)*} = 0$ on $\pi_{i-1}((S^m)_n)$. Similarly, for sufficiently large N (e.g. $N > \frac{i}{m}$) we have $h_{(N)}$ such that $h_{(N)*} = 0$ on $\pi_i((S^m)_N)$, and hence on $\pi_i((S^m)_\infty)$. Put $h' = h_{(n)}h_{(N)}$, and consider the exact sequence:

$$\pi_i((S^m)_\infty) \xrightarrow{i_*} \pi_i((S^m)_\infty, (S^m)_n) \xrightarrow{\partial} \pi_{i-1}((S^m)_n).$$

For an arbitrary element $\alpha \in \pi_i((S^m)_\infty, (S^m)_n)$, the relation $\partial h'_* \alpha = h'_* \partial \alpha = h_{(n)*} (h_{(N)*} \partial \alpha) = 0$ implies the existence of $\beta \in \pi_i((S^m)_\infty)$ such that $i_* \beta = h'_* \alpha$. Then $h'_* h'_* (\alpha) = h'_* i_* \beta = i_* (h'_* \beta) = i_* (0) = 0$. Thus $h = h' h'$ satisfies the required conditions. Q.E.D.

THEOREM 2.5. *Let X satisfy $H^*(X; Q) \cong \otimes_i Q[x_i] / \{(x_i)^{n_i+1}\}$. Then X is p -universal for every prime p and $p = 0$.*

Proof. It is sufficient to show that there exists a map $f_i: X \rightarrow (S^{m_i})_{n_i}$ such that $f_i^*(u_i) = x_i$ up to non-zero coefficient for the fundamental class u_i of $H^{m_i}(S^{m_i}; Q)$, since the composite of the maps

$$X \xrightarrow{\Delta} X \times \cdots \times X \xrightarrow{\prod f_i} \prod_i (S^{m_i})_{n_i}$$

is a 0-equivalence, and since $\prod_i (S^{m_i})_{n_i}$ is p -universal as a product of p -universal spaces by Lemma 2.3 and by Theorem 3.8 of [6]. For simplicity we omit the indexes of x_i, u_i, m_i, n_i and f_i .

If m is odd, then by the Serre's theorem [7] there exists a map $f: X \rightarrow S^m$ such that $x = f^*(u)$ up to non-zero coefficient. So we suppose that m is even. Consider the suspension SX of X and let $\sigma: H^i(X; Q) \rightarrow H^{i+1}(SX; Q)$ be the suspension isomorphism ($i > 0$). Since $m+1$ is odd, there exists a map $F: SX \rightarrow S^{m+1}$ such that $\sigma x = F^*(\sigma u)$ up to non-zero coefficient. Consider the adjoint map $f_\infty: X \rightarrow (S^m)_\infty = \Omega S^{m+1}$ of F . Then $f_\infty^*(u) = \sigma^{-1} F^*(\sigma u) = x$. Let $g: (X, X^{(mn+m-1)}) \rightarrow ((S^m)_{n+N}, (S^m)_n)$ be a cellular approximation of f_∞ . Then it is sufficient to prove:

(*) If a map $g: (X, X^{(i-1)}) \rightarrow ((S^m)_\infty, (S^m)_n)$ satisfies $g^*(u) = x$ up to non-zero coefficient

for $i \geq m(n+1)$ and if $x^{n+1} = 0$ in $H^*(X; Q)$, then there exists a 0-equivalence h of $((S^m)_\infty, (S^m)_n)$ into itself such that hg is homotopic to a map $g': (X, X^{(i)}) \rightarrow ((S^m)_{N+n}, (S^m)_n)$.

The obstruction $\gamma(g)$ to deform g to a map g' belongs to $H^i(X; \pi_{i-1}((S^m)_\infty, (S^m)_n))$ and represented by the following cocycle;

$$c: C_i(X) = H_i(X^{(i)}, X^{(i-1)}; Z) \cong \pi_i(X^{(i)}, X^{(i-1)}) \xrightarrow{g_*} \pi_i((S^m)_\infty, (S^m)_n).$$

Assume that $i = m(n+1)$, then $\pi_i((S^m)_\infty, (S^m)_n) \cong H_i((S^m)_\infty, (S^m)_n)$ and c is equivalent to $g_*: C_i(X) = H_i(X^{(i)}, X^{(i-1)}; Z) \rightarrow H_i((S^m)_\infty, (S^m)_n; Z) \cong H_i((S^m)_\infty; Z)$. Thus $\gamma(g) = g^*(\varepsilon)$ for a generator ε of $H^{m(n+1)}((S^m)_{N+n}; Z)$. Up to non-zero coefficient, $\varepsilon = u^{n+1}$ and $g^*(\varepsilon) = g^*(u^{n+1}) = x^{n+1} = 0$ in $H^{m(n+1)}(X; Q)$. Thus $\gamma(g)$ is of finite order, say q . The map h_q in the proof of Lemma 2.4 induces h_{q*} having degree q^{n+1} on the $m(n+1)$ -dimensional cohomology group, i.e., $h_{q*}(\varepsilon) = q^{n+1}\varepsilon$. Then $\gamma(h_q g) = h_{q*}\gamma(g) = h_{q*}g^*(\varepsilon) = g^*(h_{q*}\varepsilon) = q^{n+1}g^*(\varepsilon) = q^{n+1}\gamma(g) = 0$, and hence (*) is proved for $h = h_q$.

Next let $i > m(n+1)$, then by use of a 0-equivalence h in Lemma 2.4, we have that $\gamma(hg) = h_*\gamma(g) = 0$ and (*) is proved. Q.E.D.

COROLLARY 2.6. $H^*(X; Q) \cong \otimes_i Q[x_i]/\{(x_i)^{n_i+1}\}$ if and only if there is a 0-equivalence $X \rightarrow \prod_i (S^{m_i})_{n_i}$, $m_i = \deg x_i$.

This is a generalization of the result due to Arkowitz-Curjel that (D) implies (B) in §1.

EXAMPLE 2.7. The following spaces are p -universal for every prime p and for $p=0$.

- (i) The complex projective space CP^n for any $n \geq 1$.
- (ii) The quaternionic projective space QP^n for any $n \geq 1$.
- (iii) The Cayley projective plane Π .

§ 3. Some further examples of a p -universal space

THEOREM 3.1. Let A and B be co- H -spaces mod 0. Let $f: A \rightarrow B$ be any map. If there is a 0-equivalence from the mapping cone C_f of f into Y , then Y is p -universal for any prime p and $p=0$.

Proof. As in §1, there exist 0-equivalences $h_A: A \rightarrow \bigvee_j S^{m_j}$ and $h_B: B \rightarrow \bigvee_i S^{n_i}$. Since $\bigvee_j S^{m_j}$ and $\bigvee_i S^{n_i}$ are both 0-universal, there are converse 0-equivalences $k_A: \bigvee_j S^{m_j} \rightarrow A$ and $k_B: \bigvee_i S^{n_i} \rightarrow B$. Consider the composite $a_j: S^{m_j} \subset \bigvee_j S^{m_j} \rightarrow A$ and the homotopy class $f_*\{a_j\}$ of $\pi_{m_j}(B)$. The cokernel of the homomorphism $k_{B*}: \pi_{m_j}(\bigvee_i S^{n_i}) \rightarrow \pi_{m_j}(B)$ is finite, since k_B is 0-equivalence.

Hence there exists a number N_j such that $f_*N_j\{a_j\}$ is in the image of k_{B*} . Put

$g = \bigvee_j a_j N_l : \bigvee_j S^{m_j} \rightarrow A$. Then we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ k_A \uparrow & & \uparrow k_B \\ \bigvee_j S^{m_j} & \xrightarrow{g} & \bigvee_i S^{n_i} \end{array}$$

It follows from the commutativity that there exists a map $C_g \rightarrow C_f$ which is a 0-equivalence. So, if C_g is p -universal, so is C_f , and X is p -universal too. Thus it suffices to show that C_g is p -universal for any prime p and $p=0$. We put $\alpha_j = \{g \mid S^{m_j}\}$. By the well known theorem of Hilton there exists an integer N_j such that $N_j \alpha_j$ is a linear combination of some higher Whitehead product $[\iota_1, \dots, \iota_s]$, where ι_j is the homotopy class of the inclusion $S^{k_j} \rightarrow \bigvee_i S^{n_i}$ and $\{k_1, \dots, k_s\}$ is a subset of $\{n_1, \dots, n_r\}$. Note that $m_j = \sum_{e=1}^s (k_e - 1) + 1$. Then there exists a 0-equivalence from $M = \bigvee_{i=1}^r S^{n_i} \bigcup_{\beta} C(\bigvee_{j=1}^s S^{m_j})$ with $\beta = \bigvee N_j \alpha_j$ to $K = \bigvee_{i=1}^r S^{n_i} \bigcup_g C(\bigvee_j S^{m_j})$. By Theorem 1.1, K is p -universal if so is M . So we will show that M is p -universal. By the linearity of the higher Whitehead product, we have a commutative diagram:

$$\begin{array}{ccc} \bigvee_{j=1}^s S^{m_j} & \xrightarrow{\bigvee q^{b_j} \iota_j} & \bigvee_{j=1}^s S^{m_j} \\ \downarrow r & & \downarrow r \\ \bigvee_{i=1}^r S^{n_i} & \xrightarrow{\bigvee q^{a_i} \iota_i} & \bigvee_{i=1}^r S^{n_i} \end{array}$$

where $a_i = n_i - 1$ and $b_j = m_j - 1$. By choosing a prime q different from p , we can see that the map $f: M \rightarrow M$ derived from the above commutative diagram is a p -equivalence and satisfies Theorem 2.1, (b) of [6] Q.E.D.

THEOREM 3.2. *Every simply connected three cell complex $K = S^l \cup e^m \cup e^n$, $l < m < n$, is p -universal for any prime p and $p=0$.*

Proof. Let $B = S^l \cup e^m$ and let $f \in \pi_{n-1}(B)$ and $\beta \in \pi_{m-1}(S^l)$ be the attaching elements of the cells e^n and e^m respectively. We devide into three cases:

Case 1: The order of β is finite. Let t be the order of β , then there exists a map $S^m \rightarrow B$ which is of degree t by smashing S^l . This map and the inclusion of S^l define a 0-equivalence $S^l \vee S^m \rightarrow B$. Then by Theorem 3.1, K is p -universal as a mapping cone of f .

Case 2: $m = l + 1$. We may assume that β is non-trivial in $\pi_{m-1}(S^l) \cong \mathbb{Z}$. Then $H^*(K; \mathbb{Q}) \cong H^*(S^n; \mathbb{Q}) = \wedge(x_n)$. Thus A is p -universal by Corollary 1.4.

Case 3: The order of β is infinite and $m \neq l + 1$. By [7], if $m \neq l + 1$ and $\pi_{m-1}(S^l)$ is infinite, then $l = 2m$ and we have an exact sequence:

$$\pi_{2l-2}(S^{l-1}) \rightarrow \pi_{2l-1}(S^l) \xrightarrow{H} \mathbb{Z}.$$

Since $\pi_{2l-2}(S^{l-1})$ is finite and $H([l_l, l_l]) = \pm 2$, there exist non-zero integers r and s such that $r\beta = s[l_l, l_l]$. Let $C = S^l \cup e^{2l}$ be a mapping cone of $s[l_l, l_l]$, and then there exists a 0-equivalence $B \rightarrow C$ which extends the identity of S^l and is of degree r on S^{2l} . Similarly we have a 0-equivalence $(S^l)_2 \rightarrow C$; where $(S^l)_2$ is a mapping cone of $[l_l, l_l]$. Thus $\pi_{n-1}(B)$ and $\pi_{n-1}((S^l)_2)$ are \mathcal{C}_0 -isomorphic. By Lemma 2.3 $\pi_{n-1}((S^l)_2)$ is finite or it has only one free summand Z . The same is true for $\pi_{n-1}(B)$. Thus K is p -universal by Corollary 2.2 and Theorem 2.1. Q.E.D.

COROLLARY 3.3. *Any simply connected sphere bundle over sphere is p -universal for every prime p and $p=0$.*

§ 4. A counter example for p -universality and for the symmetricity of p -equivalence

Consider a complex $L = S^3 \vee CP^2$. Let $(L, 3)$ be the 2-connective fibre space over L . We have a fibering: $S^1 \rightarrow (L, 3) \xrightarrow{p} L$, and hence we have the Gysin exact sequence: $\cdots \rightarrow H^{s-2}(L; Z) \xrightarrow{\Psi} H^s(L; Z) \xrightarrow{p^*} H^s((L, 3); Z) \rightarrow H^{s-1}(L; Z) \rightarrow \cdots$ where $\Psi(u) = uv$ for a generator $v \in H^2(L; Z)$. Thus we have

$$(4.1) \quad H^i((L, 3); Z) \cong Z \text{ for } i=3, 4, 5 \\ = 0 \text{ other wise.}$$

Consider the inclusions $S^3 \subset L$ and $S^2 \subset L$ and denote by ι_3 and ι_2 their homotopy classes. Let $h: S^5 \rightarrow CP^2 \subset L$ be the composite of the Hopf map and the inclusion. Put $\omega = [\iota_3, \iota_2] \in \pi_4(L)$ the Whitehead product of ι_2 and ι_3 . By G. W. Whitehead [9], $\pi_i(S^3 \vee CP^2) \cong \pi_i(S^3) \oplus \pi_i(CP^2) \oplus \partial\pi_{i+1}(S^3 \times CP^2, S^3 \vee CP^2)$, and we see that the map $g_0 = \iota_3 \vee \omega: S^3 \vee S^4 \rightarrow S^3 \vee CP^2$ induces isomorphisms of π_i for $i=3, 4$. Consider a lift $g: S^3 \vee S^4 \rightarrow (L, 3)$ of g_0 , then g induces isomorphisms of π_i and H_i for $i \leq 4$. Since $h: S^5 \rightarrow CP^2$ is a 2-connective fibering, the inclusion $i: CP^2 \rightarrow L$ and the projection $\pi: L \rightarrow CP^2$ induce $l: S^5 \rightarrow (L, 3)$ and $\bar{\pi}: (L, 3) \rightarrow S^5$ such that $\bar{\pi}l = \text{identity}$. Thus $l_*: H_i(S^5; Z) \rightarrow H_i((L, 3); Z)$ is a split monomorphism. Then $f = g \vee l$ induces an isomorphism $H_*(S^3 \vee S^4 \vee S^5; Z) \rightarrow H_*((L, 3); Z)$ and f is a homotopy equivalence. Obviously l is a lift of h . Going back to L , we have isomorphisms:

$$(4.2) \quad (\iota_3 \vee \omega \vee h)_*: \pi_i(S^3 \vee S^4 \vee S^5) \cong \pi_i(S^3 \vee CP^2) = \pi_i(L) \text{ for } i \neq 2.$$

For example, $\pi_2(L) \cong Z$, $\pi_3(L) \cong Z$, $\pi_4(L) \cong Z \oplus Z_2$ and $\pi_5(L) \cong Z \oplus Z_2 \oplus Z_2$, and generators of each free part are ι_2 , ι_3 , ω and h respectively.

Consider a map $f': L \rightarrow L$ and put $f'_*\iota_2 = r\iota_2$ and $f'_*\iota_3 = s\iota_3$. By the linearity of the Whitehead product, we have that $f'_*\omega = rs\omega$. Put $f'_*h \equiv th$ modulo 2-torsion. We consider ι_2 and h in $\pi_5(CP^2)$ and replace f' by $f'' = \pi f' i: CP^2 \subset L \xrightarrow{f'} L \xrightarrow{\pi} CP^2$, then $f''_*\iota_2 = r\iota_2$ and $f''_*h = th$. Since $CP^3 = CP^2 \bigcup_h e^6$, f'' has an extension $F: CP^3 \rightarrow CP^3$ such that the degree of F^* on $H^6(CP^3; Z)$ is t . By the linearity of the cup product we have $t = r^3$. Thus we get

$$(4.3) \quad f'_*\iota_2 = r\iota_2 \text{ and } f'_*\iota_3 = s\iota_3 \text{ imply that } f'_*\omega = rs\omega \text{ and } f'_*h \equiv r^3h \text{ modulo 2-torsion.}$$

Let us consider the element

$\alpha = [[\omega, \iota_3], \omega], \iota_3] + [[\iota_3, h], h] + [[\omega, h], \omega]$ of $\pi_{11}(S^3 \vee CP^2)$. For simplicity we put $\alpha_1 = [[\omega, \iota_3], \omega], \iota_3]$, $\alpha_2 = [[\iota_3, h], h]$ and $\alpha_3 = [[\omega, h], \omega]$. The mapping cone of $q\alpha$ is denoted by K_q .

LEMMA 4.1. *Let $f: K_1 \rightarrow K_1$ be a p -equivalence. Then $f_*: H_*(K_1; Z) \rightarrow H_*(K_1; Z)$ is the identity.*

Proof. Denote by $f' = f|L: L \rightarrow L$. By (4.3) and by the linearity of the Whitehead product we have

$$(4.4) \quad f'_*(\alpha) \equiv r^2 s^4 \alpha_1 + r^6 s \alpha_2 + r^5 s^2 \alpha_3 \text{ modulo 2-torsion.}$$

Let $f_*: H_{12}(K_1; Z) \rightarrow H_{12}(K_1; Z)$ be of degree t and consider the following diagram

$$\begin{array}{ccc} \pi_{11}(L) & \xrightarrow{f_*'} & \pi_{11}(L) \\ \uparrow \partial & & \uparrow \partial \\ \pi_{12}(K_1, L) & \xrightarrow{f_*} & \pi_{12}(K_1, L) \\ \downarrow \mathcal{H} & & \downarrow \mathcal{H} \\ H_{12}(K_1, L; Z) & \xrightarrow{f_*} & H_{12}(K_1, L; Z) \\ \uparrow j_* & & \uparrow j_* \\ H_{12}(K_1; Z) & \xrightarrow{f_*} & H_{12}(K_1; Z) \end{array}$$

where j_* and the Hurewicz homomorphism \mathcal{H} are isomorphisms. Hence $f'_*(\alpha) = t\alpha$. It follows that $r^2 s^4 = r^6 s = r^5 s^2 = t$. If r and s are non-zero, we get $r = s = t = 1$. Thus $f_*: H_i(K_1; Z) \rightarrow H_i(K_1; Z)$ is the identity unless $i \neq 4$. For $i = 4$ this is shown by use of the ring structure of $H^*(K_1; Z)$. Q.E.D.

Let q be a positive integer with $(q, p) = 1$. Then the following diagram is commutative:

$$\begin{array}{ccc} S^{11} & \xrightarrow{q\iota_{11}} & S^{11} \\ \downarrow q\alpha & & \downarrow \alpha \\ X & \xrightarrow{1_X} & X \end{array}$$

So we obtain a p -equivalence $h: K_q \rightarrow K_1$. Suppose that K_1 is p -universal. Then by Theorem 3.2 of [6], there exists a converse p -equivalence $k: K_1 \rightarrow K_q$. Then $f = hk: K_1 \rightarrow K_q \rightarrow K_1$ is a p -equivalence and the degree of f_* on $H_{12}(K_1; Z)$ is a multiple of q . On the other hand, it follows from the above lemma that f_* is of degree 1. This is a contradiction. We have proved:

THEOREM 4.2. *There exists a 4-cell complex which is not p -universal.*

THEOREM 4.3. *The p -equivalence is not an equivalence relation in the category of simply connected finite CW-complexes.*

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Received June 10, 1970