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A Total Whitehead Torsion Obstruction to Fibering over the Circle

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§ 0. Introduction

We introduce a geometrical construction called *twist-gluing*, and using it obtain a total Whitehead torsion obstruction to fibering a smooth (or PL) manifold over the circle. Thus the two successive obstructions for this problem recently found by Farrell [5] [6] in less familiar groups can be combined as a single element of the Whitehead group of the manifold.

In outline our construction of the total Whitehead torsion obstruction $\mathcal{F}$ to fibering is as follows. Let $f: M \to S^1$ be a continuous map to the circle of a closed smooth manifold M of dimension $\geqslant 6$. One wants to know whether f can be homotoped to a smooth fibration with fiber a closed connected smooth manifold. Let $\bar{M}$ be the infinite cyclic covering of M induced by f from the universal covering $R^1 \to S^1$. Granting the necessary condition that $\bar{M}$ be dominated by a finite connected complex, one finds that $\bar{M}$ is an open 2-ended manifold with such pleasant properties that its ends can be *glued* together by a diffeomorphism that incorporates a *twist* by the covering translation. The result is a closed smooth manifold M'-called the *relaxation* of M-equipped with a natural homotopy equivalence $g: M' \to M$. It proves relatively easy to construct a Whitehead torsion obstruction $\mathcal{F}_0(M') \in \mathrm{Wh}(\pi_1 M')$ to homotoping $fg: M' \to S^1$ to a smooth fibration. The Whitehead torsion obstruction $\mathcal{F}(M)$ to homotoping f to a smooth fibration $M \to S^1$ is $g_* \mathcal{F}_0(M') + \tau(g) \in \mathrm{Wh}(\pi_1 M)$, where $\tau(g)$ denotes the Whitehead torsion of g.

Although it will surely be possible to reprove (and improve) our results with more algebraic and homotopy-theoretic methods (cf. 2.3), our geometrical approach has certain virtues. It is pleasantly conceptual. Indeed it is so formal that it extracts some information about homeomorphisms and topological manifolds. For example it shows (§ 6.3): *Let X be a compactum. If $X \times R^1$ is triangulable as a* PL *manifold of dimension $\geqslant 6$, then so is $X \times S^1$.* This fact was essential in [18]. Thus we have examined the notion of twist-gluing extensively in a separate chapter II.

A last chapter, III, deals algebraically with structure of the Whitehead group $\mathrm{Wh}(\pi_1 M)$ for a candidate M for fibering over the circle. It is the beginning of a purely algebraic 'explanation' of Chapters I and II. For an introduction see § 8.

Here is a list of section headings.

Chapter I. **The invariant $\mathcal{F}$.** § 1 Definition of $\mathcal{F}$; § 2 $\mathcal{F}$ is well defined; § 3 A slight topological invariance; § 4 $\mathcal{F}$ is the obstruction to fibering.

Chapter II. **Gluing and relaxation.** § 5 Topological twist-gluing; § 6 PL and DIFF twist-gluing; § 7 Relaxation.

Chapter III. **Algebraic relaxation.** § 8 Main result; § 9 The exact sequence for a ring endomorphism; § 10 The relaxation projection; § 11 Proof of the main result; § 12 Supplementary remarks.

Chapters II and III are each logically self contained and aim for some generality at the risk of making reading dull. In contrast Chapter I is only a tentative formulation and we do not wish to obscure its heuristic interest with generality. Chapter I relies at many points on facts verified in Chapter II. But is it probably wisest for the reader to explore Chapter I first. He will find this easier if he reflects that these facts are sufficiently obvious in the special (but still nontrivial) case where the ∞-cyclic covering of the candidate for fibering is a smooth product with the line.

In a sequel, I shall give a wider definition of the obstruction $\mathscr{F}$ and establish properties expected of a Whitehead torsion invariant. See [32].

Remarks added in proof:

1) Results of R. C. Kirby and the author [31] permit a topological handlebody theory in dimension $\geqslant 6$ [32] [33] just like the smooth and piecewise linear theories. So the considerations of this paper are valid for topological manifolds, cf. [17, § 2.3]. § 3 is dead.

2) In § 4, the splitting theorem of Farrell and Hsiang [8] [8B] is used to show that $\mathscr{F}(M)=0$ implies M fibers over S^1. Recently S. Cappell [35] has found a new geometric proof (and generalization) of it.

3) F. T. Farrell has pointed out an easy proof that M fibers if $\pi_1 M$ is free abelian (cf. § 7.6 below) based on [15] and the s-cobordism theorem.

4) F. Waldhausen has recently obtained important generalizations of Chap. III and [8] [8A].

Chapter I. THE INVARIANT $\mathscr{F}$

§ 1. Definition of $\mathscr{F}$

First we establish some conventions that will be used repeatedly. Suppose X is a path connected space and $X\to S^1$ is a map inducing a surjection $\pi_1 X\to\pi_1 S^1=Z$. If $\tilde{X}$ denotes the infinite cyclic covering of X induced from the universal covering $R^1\to S^1$, write $H\subsetneq G\to Z$ for the short exact sequence $1\to\pi_1\tilde{X}\to\pi_1 X\to\pi_1 S^1\to 1$. Let $t\in G$ go to $1\in Z$ and let $\theta: H\to H$ be defined by $\theta(h)=tht^{-1}$ for $h\in H$. If P is finitely generated left projective module (called a projective for short) over the integral group ring $Z[H]$, let tP be the same additive group, with t written before each element, endowed with this $Z[H]$-module structure: $h\cdot tp=t(t^{-1}htp)=t(\theta^{-1}(h)p)$ for $h\in H$ and $tp\in tP$. Note that $tP \cong \theta_{\#} P$ (see § 9) by the map $tp \leftrightarrow 1 \otimes p$ for $p \in P$. Suppose $\varphi: P \to tP$ is a $Z[H]$ isomorphism (assuming one exists). Then φ extends uniquely to a $Z[G]$ automorphism of the projective over $Z[G]$

$$Z[G] \otimes_{Z[H]} P = P[T] = \cdots \oplus t^{-1}P \oplus P \oplus tP \oplus t^2 P \oplus \cdots$$

sending $t^k P$ to $t^{k+1}P$. Its class in Wh G we denote $\tau(P, \varphi)$ (cf. § 9). Note that such torsions form a subgroup of Wh G containing the image of Wh H. We write it $R(G, H)$, and call it the subgroup of *relaxed torsions*. As the notation indicates, it is independent of the choice of t, and even of the isomorphism of G/H with Z (see § 11). If φ is a stable isomorphism – i.e. for some n an isomorphism $\varphi: P \oplus Z[H]^n \to tP \oplus Z[H]^n$ – extend the definition of $\tau(P, \varphi)$ as follows. Compose with the isomorphism to $t(P \oplus Z[H]^n)$, $(tp, x_1, \ldots, x_n) \leftrightarrow (tp, t\theta^{-1}x_1, \ldots, t\theta^{-1}x_n)$, then apply the original construction.

It is convenient to assemble the candidates for fibering into a *category* $\mathscr{C}$ as follows. An object of $\mathscr{C}$ is a smooth compact manifold X of dimension $\geqslant 6$ equipped with a continuous map $p: X \to S^1$ satisfying the following conditions: $p_*: \pi_1 X \to \pi_1 S^1$ is onto; the infinite cyclic covering Y of X corresponding to the kernel of p_* is dominated by a finite complex; the restriction of p to the boundary bX is homotopic to a smooth fibration $bX \to S^1$. These conditions are all necessary that p be homotopic to a smooth fibration. A morphism of $\mathscr{C}$, $f: (X, p) \to (X', p')$ is a smooth map $f: X \to X'$ such that $p'f$ is homotopic to p. From now on we will remain in $\mathscr{C}$ unless warning is given. One could just as well work with the PL analogue of $\mathscr{C}$.

Next we introduce the geometrical device called *twist-gluing*. Let Y be a smooth manifold with boundary (not in $\mathscr{C}$) with two ends $\varepsilon_-, \varepsilon_+$, and equipped with a self-diffeomorphism T that doesn't exchange the ends. Suppose that for any given neighborhood of either end $\varepsilon_\pm$ $(=\varepsilon_+$ or $\varepsilon_-)$ there exists a smaller neighborhood $U_\pm$ of $\varepsilon_\pm$ and a diffeomorphism $f_\pm: U_\pm \to Y$ homotopic to $U_\pm \subsetneq Y$ by a homotopy fixing a still smaller neighborhood of $\varepsilon_\pm$. Then choose f_-, f_+ so that $U_- \cap U_+ = \emptyset$, and identify U_- to U_+ under $f_+^{-1} T f_-$ obtaining a compact smooth manifold $Y_T = Y_T(f_-, f_+)$. It is called a (smooth) *twist-gluing* of Y relative to T and $\varepsilon_-, \varepsilon_+$. Y_T in fact comes equipped with a preferred homotopy class of maps to S^1, for it has a natural ∞-cyclic covering and generating covering translation. If $Y_T(f'_-, f'_+)$ is another such twist-gluing there is a diffeomorphism to $Y_T(f_-, f_+)$ in a preferred homotopy class (see 5.2, 7.8). The preferred class in fact respects a natural homotopy equivalence from each twist gluing to the mapping torus of T (see 5.6).

Now for any element $X \equiv (X, p) \in \mathscr{C}$ the infinite cyclic covering Y of X is induced by p from $R^1 \to S^1$ and so has a negative and a positive end $\varepsilon_-, \varepsilon_+$ corresponding to the ends of R^1. Let T be the covering translation corresponding to $+1 \in Z = \pi_1 S^1$. By

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[1]) Do not confuse the covering space Z with the integers Z.

an engulfing argument (see [21] and 7.4), there exist twist-gluings $Y_T = Y_T(f_-, f_+)$ as described above. Now there is a natural homotopy equivalence $g: Y_T \to X$ (see 7.7). In fact there is a natural homotopy equivalence of each to the mapping torus of T. The pair $(Y_T, pg) \in \mathscr{C}$ is called a *relaxation* of (X, p). If $Y_T(f'_-, f'_+)$ is another twist-gluing, and g' its homotopy equivalence to X, the natural homotopy class of diffeomorphisms $\varphi: Y_T(f_-, f_+) \to Y_T(f'_-, f'_+)$ is such that $g'\varphi$ is homotopic to g (see § 7.7). Hence in particular the relaxation $(Y_T, pg) \in \mathscr{C}$ is well defined up to isomorphism in $\mathscr{C}$.

For an application in § 3, we note that if we forget about differentiability in the definition of morphisms of $\mathscr{C}$, the twist gluing becomes a *topological* twist-gluing operation. By formally the same arguments it is well defined up to isomorphism (=homeomorphism now!) in a preferred homotopy class. (See 7.7 and 6.2 below).

Next we construct an invariant $\mathscr{F}_0$ for the relaxation (Y_T, pg) of (X, p). It will have the form $\tau(P, \varphi)$ in the notation introduced above, where P is a projective representing the positive end invariant in $\tilde{K}_0 Z[H]$ for Y [15] [31] (see 4.6–4.7), and $\varphi: P \to tP$ is a $Z[H]$-isomorphism derived from the generating covering translation $\bar{T}$ of the ∞-cyclic covering[1]) Z for $(Y_T, pg) \in \mathscr{C}$. Now Z is the covering formed from infinitely many copies Y_n of Y, $n = 0, \pm 1, \pm 2, \dots$ by identifying U_+ in Y_n to U_- in Y_{n+1} under $f_- T^{-1} f_+$, and $\bar{T}$ sends $y_n \in Y_n$ to $y_{n+1} \in Y_{n+1}$ for each $y \in Y$ (see 7.7–7.8). We identify $\pi_1 Y_T$ to $\pi_1 X = G$ by $g: Y_T \simeq X$ and hence $\pi_1 Z$ to $\pi_1 Y = H$.

Remark: By an engulfing argument each inclusion $Y_n \subsetneq Z$ is homotopic through embeddings to a diffeomorphism onto Z. In fact if $A \subset Y_n$ is closed in Z the homotopy can fix A pointwise (see 7.9).

Let $c: \tilde{Z} \to Z$ be a universal covering and for any $S \subset Z$ let $\tilde{S}$ denote $c^{-1}(S)$. Choose $U \subset Z$ a closed smooth neighborhood of the negative end of Z disjoint from some neighborhood of the positive end, and such that

(i) *The frontier ∂U of U (which is the submanifold $U - \mathring{U}$ of bU) has fundamental group $\pi_1(\partial U) \cong \pi_1 Z$ by inclusion.*

(ii) *$H_*(Z, \tilde{U})$ is a projective over $Z[H]$ isolated in one dimension, say* 4.

(iii) *$V = \bar{T}U$ contains U in its interior* – i.e. *$\mathring{V} \supset U$, and $H_*(\tilde{V}, \tilde{U}) \to H_*(\tilde{Z}, \tilde{U})$ is onto.*

Such a U can easily be constructed in this situation as follows. Using [15] (see 7.2) one can at least find a neighborhood $U_0 \subset Y$ as described satisfying (i) and (ii) but perhaps not (iii). Let $f: Z \to Y_0 \subset Z$ be a diffeomorphism sending positive end to positive end. Because of the covering translation on $Y = Y_0$ we can asume $f(\partial U_0) \subset U_+ \subset Y = Y_0$. Then $f(\partial U_0)$ separates Z into a neighborhood $U \supset f(U_0)$ of the negative end with $\partial U = f(\partial U_0)$ and a complementary neighborhood of the positive end. This U satisfies (i) and (ii) since $(Z, U, \partial U)$ is diffeomorphic to $(Y_0, f(U_0), f(\partial U_0))$ by the above remark. It also satisfies (iii) since $(U \cup Y_0, U) \subset (V, U)$ and since $(U \cup Y_0, U) \subsetneq (Z, U)$ is homotopic to a diffeomorphism by the same remark.

In the exact sequence for triple $(\tilde{Z}, \tilde{V}, \tilde{U})$ the only nonzero maps are two $Z[H]$-module isomorphisms $H_4(\tilde{V}, \tilde{U}) \to H_4(\tilde{Z}, \tilde{U}) = P$ and $Q = H_4(\tilde{Z}, \tilde{V}) \xrightarrow{\partial} H_3(\tilde{V}, \tilde{U})$.

Let $\mathbf{C}_* = \mathbf{C}_*(\tilde{V}, \tilde{U})$ be the free based $Z[H]$-complex arising from a handle decomposition of V on U. Using the above isomorphisms we can regard the projectives P and Q as summands of $\mathbf{C}_4$ and $\mathbf{C}_3$ respectively. There is a (stable) isomorphism $i: P \to Q$ such that if one formally alters $\partial: \mathbf{C}_4 \to \mathbf{C}_3$ by setting $\partial \mid P = i$ instead of zero, $\mathbf{C}_*$ becomes acyclic with zero torsion.

The last element of the definition of $\mathscr{F}_0$ is an isomorphism $\psi: tP \to Q$ determined as follows by $\bar{T}: (Z, U) \to (Z, V)$. Consider first the general case of a map $f: (A, B) \to (A', B')$ of connected C.W. pairs. Let A, A' have base points a, a' and specify a path γ from $f(a)$ to a' to get a homomorphism $f_*: \pi_1 A \to \pi_1 A$. There is a unique map of (pointed) universal coverings $\tilde{f}: (\tilde{A}, \tilde{B}) \to (\tilde{A}', \tilde{B}')$ (pointed by $\tilde{a}, \tilde{a}'$ say) so that γ lifts to a path from $\tilde{f}(\tilde{a})$ to $\tilde{a}' \in \tilde{A}'$. One can readily check that $\tilde{f}_*: H_*(\tilde{A}, \tilde{B}) \to H_*(\tilde{A}', \tilde{B}')$ satisfies

$$\tilde{f}_*(g \cdot x) = f_*(g) \cdot \tilde{f}_*(x)$$

for $g \in \pi_1 A$ and $x \in H_*(\tilde{A}, \tilde{B})$. Dot indicates the left action of fundamental group by covering translations. Hence $\tilde{f}_*$ determines a unique map of $Z[\pi_1 A']$ modules

$$(f_*)_\# \, H_*(\tilde{A}, \tilde{B}) = Z[\pi_1 A'] \otimes_{Z[\pi_1 A]} H_*(\tilde{A}, \tilde{B}) \longrightarrow H_*(\tilde{A}', \tilde{B}') \qquad (\S\S)$$

by the rule $r \otimes x \mapsto r \cdot \tilde{f}_*(x)$ (cf. § 9). It is bijective if f_* and $\tilde{f}_*$ are. Applying these observations to $\bar{T}: (Z, U) \to (Z, V)$ we can choose a path γ to the base point z_0 of Z from $\bar{T}(z_0)$, such that γ projects to class $t^{-1} \in \pi_1(Y_T) = \pi_1 X = G$. Then $f_* = \bar{T}_*: \pi_1 Z \to \pi_1 Z = H$ is precisely $\theta: g \mapsto tgt^{-1}$! Hence (§§) becomes the $Z[H]$-isomorphism $\psi: = \theta_\# P = tP \to Q$ required.

Now define $\mathscr{F}_0(Y_T)$ to be $\tau(P, \varphi)$ where φ is the composed (stable) isomorphism $\varphi: P \xrightarrow{i} Q \xrightarrow{\psi^{-1}} tP$.

Finally, we define the total Whitehead torsion obstruction to fibering to be $\mathscr{F}(X) = \mathscr{F}_0(Y_T) + \tau(g) \in \mathrm{Wh}(\pi_1 X)$, for X in C.

Recall that $g: Y_T \to X$ denotes the natural homotopy equivalence with the relaxation Y_T of X.

Remark: $\mathscr{F}$ is designed to behave under h-cobordism formally like a Reidemeister torsion. The components $\tau(g)$ and $\mathscr{F}_0(Y_T)$ are also diffeomorphism invariants that vanish when X fibers.

§ 2. $\mathscr{F}$ is well-defined

If $Y'_T = Y_T(f'_-, f'_+)$ is another relaxation of X and $g': Y'_T \to X$ is the natural homotopy equivalence, there is (by 7.7 below) a diffeomorphism $\Psi: Y_T \to Y'_T$ so that $g'\psi$ is homotopic to g. Thus $\tau(g') = \tau(g)$. Now it remains to show that $\mathscr{F}_0(Y_T)$ is an invariant of diffeomorphism in $\mathscr{C}$. This amounts to showing successively that

(a) $\mathscr{F}_0(Y_T)$ is independent of the choice of $t \in G$ mapping onto $+1 \in \pi_1 S^1$.

(b) $\mathscr{F}_0(Y_T)$ is independent of the choice of handle decomposition of $V = \tilde{T}U$ on U.

(c) $\mathscr{F}_0(Y_T)$ is independent even of the choice of U.

Proof of (a): If $t' \in G$ also maps to $+1 \in \pi_1 S^1$ then $t' = tu$ where $u \in H$. By inspecting the construction of $\mathscr{F}_0(Y_T)$ one sees that use of t' in place of t (with no other changes) replaces $\varphi: P \to tP$ by an isomorphism $\varphi' = \psi'^{-1} i: P \to (tu) P = t'P$ (i is unaffected), such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi'} & (tu)P = t'P \\ \| & \cong & \downarrow \text{nat.} \\ P & \xrightarrow{\varphi} & tP \end{array}$$

is commutative, where the vertical isomorphism is $(tu)p \mapsto t(up)$, $p \in P$. It follows easily that the relaxed torsions derived from φ and φ' using respectively t and t' are equal (cf. § 11). This proves (a).

The proof of (b) and (c) requires some preliminary work.

A stable isomorphism $M \to N$ of left $Z[H] = \Lambda$-projectives is represented by an isomorphism $f: M \oplus \Lambda^a \to N \oplus \Lambda^b$. Another isomorphism $f': M \oplus \Lambda^{a'} \to N \oplus \Lambda^{b'}$ (with $a' \geqslant a$) represents the same stable isomorphism if and only if when one writes $a' = a + t$, $b' = b + t$ one has $f' = f \oplus 1_{\Lambda^t}$. Two stable isomorphisms $\varphi, \varphi': M \to N$ are said to be *homotopic* if when represented by isomorphisms $\psi, \psi': M \oplus \Lambda^a \to N \oplus \Lambda^b$, which is always possible for large a, $\psi^{-1}\psi'$ represents zero in $\mathrm{Wh}\, H$. The homotopy classes of stable isomorphisms of projectives will be here called *h-isomorphisms*. They clearly form a category.

In this section $\mathbf{C}_*$ will stand for a finitely generated based free left $Z[H]$ complex, not a fixed one, but always one such that $H_i(\mathbf{C}) = 0$ for $i \neq k$, $k-1$ (k fixed), and such that $H_k(\mathbf{C}) = P$, $H_{k-1}(\mathbf{C}) = Q$ are both projectives over $Z[H]$. Then P and Q are necessarily stably isomorphic [28].

For the sake of argument suppose a genuine isomorphism $\delta: P \to Q$ exists. $\mathbf{C}_*$ admits a splitting Σ,

$$\mathbf{C}_*: \cdots \xrightarrow{\partial} B_{k+1} \oplus B_k \xrightarrow{\partial} B_k \oplus P \oplus B_{k-1} \xrightarrow{\partial} B_{k-1} \oplus Q \oplus B_{k-2} \xrightarrow{\partial} B_{k-2} \oplus B_{k-3} \xrightarrow{\partial} \cdots$$

where ∂ maps $B_j \subset \mathbf{C}_{j+1}$ by the identity to $\partial C_{j+1} = B_j \subset \mathbf{C}_j$ and is zero on other summands. Make $\mathbf{C}_*$ acyclic by setting $\partial \mid P = \delta$ instead of zero. The new complex $\bar{\mathbf{C}}_*$ then has a torsion $\tau(\mathbf{C}_*, \delta, \Sigma) \in \mathrm{Wh}\, H$. First note that $\tau(\mathbf{C}_*, \delta, \Sigma)$ is independent of Σ. In fact only the choice of the two injections $B_{k-1} \hookrightarrow \mathbf{C}_k$ and $Q \hookrightarrow \ker(\partial \mid C_{k-1}) \subset C_{k-1}$ affect $\bar{\mathbf{C}}_*$ at all, and changing these does not change the simple isomorphism type of $\bar{\mathbf{C}}_*$ [14, p. 41]. So write $\tau(\mathbf{C}_*, \delta)$. The following properties of the construction of $\tau(\mathbf{C}_*, \delta)$ are trivial to verify:

1) If $\sigma: P \to P$ is an automorphism with class $[\sigma] \in \mathrm{Wh}\, G$ (cf. [3, § 2.4]) then

$$\tau(\mathbf{C}_*, \delta\sigma) = \tau(\mathbf{C}_*, \delta) + (-1)^{k-1} [\sigma]$$

(We use Milnor's sign convention's [13] which are the opposite of Maumary's [14]).

2) Form $\mathbf{C}''_*$ by adding $Z[G]^n$ to each of $\mathbf{C}_k$ and $\mathbf{C}_{k-1}$ and making ∂ zero on both copies. Replace δ by $\delta\oplus 1: P\oplus Z[G]^n\to Q\oplus Z[G]^n$. Then $\tau(\mathbf{C}'', \delta\oplus 1)=\tau(\mathbf{C}_*, \delta)$.

3) If E_* is an acyclic based complex over $Z[G]$ with torsion zero, then $\tau(\mathbf{C}_*\oplus E_*, \delta)=\tau(\mathbf{C}_*, \delta)$.

4) If $\mathbf{C}'_*$ is $\mathbf{C}_*$ with a new basis related to the old by a simple isomorphism, then $\tau(\mathbf{C}'_*, \delta)=\tau(\mathbf{C}_*, \delta)$.

Let $\varrho(\mathbf{C}_*)$ be the h-isomorphism $\delta\alpha: P\to Q$ where $\alpha: P\to P$ is the (unique) h-isomorphism with torsion $(-1)^k\tau(\mathbf{C}_*, \delta)$. 1) shows that $\varrho(\mathbf{C}_*)$ is independent of δ. 1) and 2) together show that the definitions of $\tau(\mathbf{C}_*, \delta)$ and $\varrho(\mathbf{C}_*)$ extend to the case where $\delta: P\to Q$ is any h-isomorphism. Then 3) and 4) continue to hold and $\varrho(\mathbf{C}_*)$ is the unique h-isomorphism such that $\tau(\mathbf{C}_*, \varrho(\mathbf{C}_*))=0$.

To sum up, let $\mathbf{C}_*$ be as introduced, without any $\delta: P\to Q$ being given.

CONCLUSION I: $\mathbf{C}_*$ *determines a unique h-isomorphism* $\varrho(\mathbf{C}_*): H_k(\mathbf{C}_*)\to H_{k-1}(\mathbf{C}_*)$.

If we combine properties 3) and 4), then by [14, Theorem 2, p. 52] we get

CONCLUSION II: *If* $f: \mathbf{C}_*\to\bar{\mathbf{C}}_*$ *is a simple homotopy equivalence of free based* $Z[H]$ *complexes and we identity homology under f, then* $\varrho(\mathbf{C}_*)=\varrho(\bar{\mathbf{C}}_*)$.

Proof of (b): Any two handle decompositions of V on U give based $Z[H]$-complexes that are simple homotopy equivalent by a homotopy equivalence respecting the identifications of homologies with the singular homology of $(\tilde{V}, \tilde{U})$, see [13, § 9]. Hence II gives the required conclusion.

Proof of (c): Let U_1, U_2, U_3 be three neighborhoods of the negative end of Z satisfying conditions (i) and (ii) of § 1. Suppose $U_i\subset\mathring{U}_{i+1}$ and $H_*(\tilde{U}_{i+1}, \tilde{U}_i)\to H_*(\tilde{Z}, \tilde{U}_i)=P_i$ is onto for $i=1, 2$. Then for $j>i$ the integral homology sequence of $(\tilde{Z}, \tilde{U}_j, \tilde{U}_i)$ is

$$\begin{array}{ccccccccc} 0\to & H_4(\tilde{U}_j, \tilde{U}_i) & \xrightarrow{\cong} & H_4(\tilde{Z}, \tilde{U}_i) & \to & H_4(\tilde{Z}, \tilde{U}_j) & \xrightarrow{\cong} & H_3(\tilde{U}_j, \tilde{U}_i) & \to 0 \\ & \cong & & \| & & \| & & \cong & \\ & P_i & & P_i & & P_j & & P_j & \end{array}$$

From $\mathbf{C}_*(\tilde{U}_j, \tilde{U}_i)$ we get (by I and II) a well-defined h-isomorphism

$$\delta_{ij}: P_i\to P_j$$

LEMMA 2.1. $\varrho_{23}\varrho_{12}=-\varrho_{13}: P_1\to P_3$.

The proof is a calculation in $\mathbf{C}_*(\tilde{U}_3, \tilde{U}_1)$ which we suppress. It is not difficult if one arranges that only handles of dimension 3 and 4 appear [14]. The sign arises from an interchange of two copies of one of the projectives.

Now let U' be a second choice of U, and set $V'=\tilde{T}U'$. It is no loss of generality to choose U' so that $U\subset \mathring{U}'$ and $H_*(\tilde{U}', \tilde{V})\to H_*(\tilde{Z}, \tilde{V})$ is onto. Then there arises the following diagram of h-isomorphisms.

$$\begin{array}{ccccl}
 & tP & \xrightarrow{tp} & tP' & = tH_4(\tilde{Z}, \tilde{U}') \\
\tilde{T}_*^{-1} & \uparrow & ① & \uparrow & \tilde{T}_*^{-1} \\
 & Q & \xrightarrow{q} & Q' & = H_4(\tilde{Z}, \tilde{V}') \\
i & \uparrow & ② & \uparrow & i' \\
 & P & \xrightarrow{p} & P' & = H_4(\tilde{Z}, \tilde{U}')
\end{array}$$

By definition, the square ② arises from the square of inclusions

$$\begin{array}{ccc}
V & \to & V' \\
\uparrow & & \uparrow \\
U & \to & U'
\end{array}$$

Hence the lemma shows that ② is commutative. Observe that ① also commutes. This is a translation of the fact that $\tilde{T}$ gives a diffeomorphism from (U', U) which defines p, to (V', V) which defines q (see construction of $\psi=\tilde{T}_*: tP\to Q$ in § 1). Thus

$$\begin{array}{ccc}
tP & \xrightarrow{tp} & tP' \\
{\scriptstyle \varphi=\tilde{T}_*^{-1}i}\uparrow & & \uparrow{\scriptstyle \varphi'=\tilde{T}_*^{-1}i'} \\
P & \xrightarrow{p} & P'
\end{array}$$

is commutative. It follows easily that $\tau(P, \varphi)=\tau(P', \varphi')$, which proves (c).

Remark 2.2. The reader will want to check that $\mathscr{F}_0$ doesn't depend on the rather arbitrary choice of dimension 4 for the homology $H_*(\tilde{Z}, \tilde{U})$. As one would expect, dimension k yields $(-1)^k\,\mathscr{F}_0$.

Remark 2.3. From the constructions of this section it should be clear that the relaxed torsion invariant $\mathscr{F}_0$ could be defined directly for any $X\in\mathscr{C}$. It seems likely that $\mathscr{F}(X)$ can be defined as $\mathscr{F}_0(X)$ plus another directly defined invariant $\mathscr{F}_1(X)$ obtained by Farrell's methods in the summand of $\mathrm{Wh}\,G$ complementary to $R(G, H)$ (see § 12).

§ 3. A slight topological invariance

Suppose $f\colon Y\to X$ is a homeomorphism of compact connected PL manifolds (for example C^1-triangulated smooth manifolds). Suppose that $p\colon X\to S^1$ induces a surjection $p_*\colon \pi_1X\to\pi_1S^1\cong Z$. Write $H\lhd G\to Z$ for $\ker(p_*)\lhd\pi_1X\to\pi_1S^1$ and adopt the notations introduced in § 1.

PROPOSITION 3.1. *The Whitehead torsion* $\tau(f)\in\mathrm{Wh}\,G$ *is relaxed – i.e. it is of the form* $\tau(P, \varphi)\in R(G, H)\subset\mathrm{Wh}\,G$.

Proof: Multiplying by a high dimensional simply connected manifold with euler characteristic 1 does not alter torsion – see [12]. Hence we can assume $\dim(X) \geqslant 6$. Let $S^0 = \{p^-, p^+\}$ split S^1 as $D_1 \cup D_2$ and as in [14] deform $X \to S^1$ to a smooth map s transverse to S^0 so that $X_1 = s^{-1}D_1$, $X_2 = s^{-1}D_2$, $X_0^- = s^{-1}(p^+)$, $X_0^+ = s^{-1}(p^+)$ are all connected. For X, X_1, X_2 and X_0^- choose a common base point x_0 in X_0^-. Then choose base paths γ_1, γ_2 from x_0 to X_0^+ in X_1, X_2 respectively so that γ_1 followed by γ_2^{-1} represents $t \in G = \pi_1 X$.

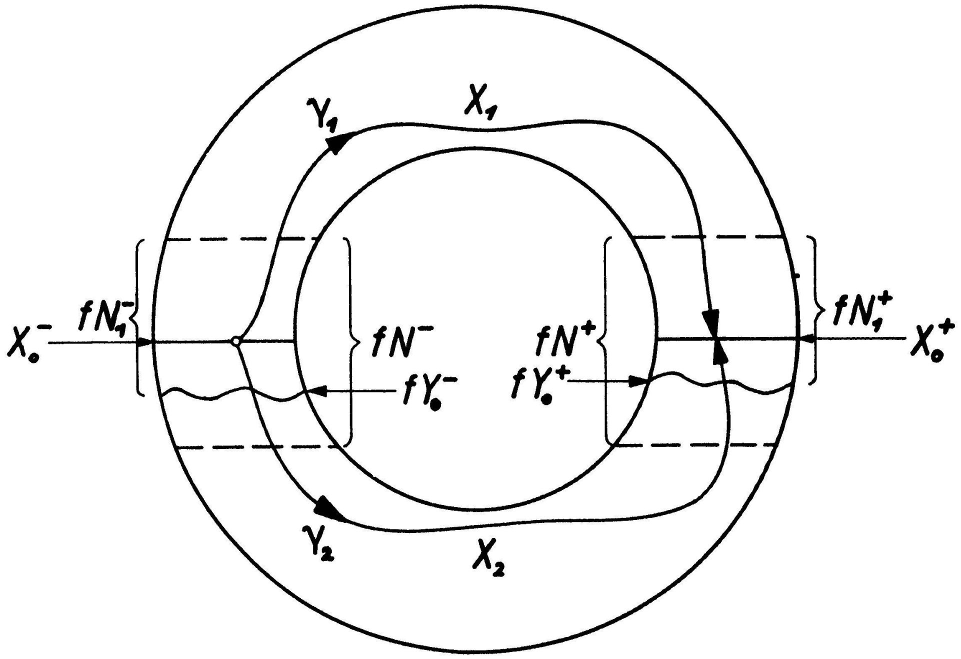

Figure 3.1.1.

Let $f(N)$ be an open bicollar neighborhood of X_0. There may be an obstruction to splitting $N \subset Y$ as a PL product, but in any case one can as in [15] [31] find a smooth small closed neighborhood N_1 of $N \cap f^{-1}X_1$ in N that is a PL submanifold such that (N, N_1) has, with universal coefficients[1]), (i.e. twisted integral group ring coefficients), a projective as homology, isolated in one dimension. More precisely there is one projective for each of the two components (fN^-, fN_1^-), (fN^+, fN_1^+) of (fY, fN_1) corresponding to X_0^-, X_0^+. We pass to twisted $Z[H]$ coefficients, [25, p. 58], using for fN^+ the base path γ_1, and call the resulting $Z[H]$-projectives P and Q respectively. Now write $Y_1 = f^{-1}(X_1) \cup N_1$, $Y_2 = \overline{Y - Y_1}$ and $Y_0 = bN_1 = Y_1 \cap Y_2$. Then up to homotopy there is a natural map of triads

$$g: (Y; Y_1, Y_2) \to (X; X_1, X_2)$$

[1]) This is nothing more than the integral homology of the universal covering with the action of the fundamental group by covering translations. But the terminology is convenient when we pass to new coefficients, i.e. consider a new covering.

giving maps $g_i: Y_i \to X_i$, $i=0, 1, 2$. This g is homotopic to f, and to calculate torsion we can make g simplicial with respect to triangulations of $(X; X_1, X_2)$ and $(Y; Y_1, Y_2)$.

Consider the inclusion sequence

$$(M(g_1) \cup Y, Y) \to (M(g), Y) \to (M(g), M(g_1) \cup Y) \qquad (\S)$$

where $M(\cdot)$ is the mapping cylinder of a map·

After some preparation we will deduce the proposition from Milnor's sum theorem [13, § 3.2] applied to based $Z[G]$-complexes arising from (§). The left hand pair is related by excision to $(M(g_1), Y_1)$, which with $Z[H]$ coefficients has homology naturally isomorphic to that of $f(N, N_1)$, namely to $P \oplus Q$. The right hand pair is related by excision to $(M(g_2), M(g_0) \cup Y_2)$ whose homology with $Z[H]$ coefficients is again naturally isomorphic to that of $f(N, N_1)$ provided the base path to fN^+ is γ_2 through X_2. This means $H_*(M(g_2), M(g_0) \cup Y_2; Z[H]) = P \oplus tQ$. Further one verifies that under these excisions and isomorphisms

$$\partial : H_*(M(g), M(g_1) \cup Y; Z[G]) \to H_*(M(g_1) \cup Y, Y; Z[G])$$

corresponds to the identity map

$$P[T] \oplus tQ[T] = P[T] \oplus Q[T] \to P[T] \oplus Q[T].$$

Now $P \oplus Q$ and $P \oplus tQ$ are both stably free over $Z[H]$, for each is the homology of a finite simplicial pair [28]. It follows that there is some stable isomorphism $\psi: Q \to tQ$. Now we can easily add free summands to Q by suitably changing g (c.f. [15] or [28]). So we can and do arrange that $P \oplus Q$, $P \oplus tQ$ are free and that ψ is an isomorphism. Pick any basis b for $P \oplus Q$, and as basis for $P \oplus tQ$ choose $b' = (1_P \oplus \psi)\, b$. Tensoring with $Z[G]$ we get bases $1 \otimes b'$ and $1 \oplus b$ respectively for the $Z[G]$ homologies of the left and right hand pairs of (§). Then the torsions τ_L, τ_R of these pairs are defined and clearly lie in $i_* \operatorname{Wh} H \subset \operatorname{Wh} G$. The torsion of the middle pair is $\tau(g) = \tau(f)$. On the other hand Milnor's sum theorem [13, § 3.2] says that the torsion of the middle is $\tau_L + \tau_R + \tau(\mathscr{H})$ where $\tau(\mathscr{H})$ is the torsion of the homology sequence of (§). Since τ_L, τ_R have the wanted form, it remains to check the same for $\tau(\mathscr{H})$. In view of our choice of bases, $\tau(\mathscr{H})$ is represented (up to sign) by the composed isomorphism

$$[T] \oplus Q[T] \xrightarrow{1 \otimes (1 \oplus \psi)} P[T] \oplus tQ[T] = P[T] \oplus Q[T] \xrightarrow{\partial = 1} P[T] \oplus Q[T]$$

hence by the isomorphism of $Q[T]$ extending $\psi: Q \to tQ$. Thus $\pm \tau(\mathscr{H}) = \tau(Q, \psi)$. This completes the proof of 3.1.

Let $R^{\perp}$ be the quotient $\operatorname{Wh} G / R(G, H)$ (cf. § 12).

THEOREM 3.2. *Let $h: X \to X'$ be a homeomorphism in $\mathscr{C}$, and identify* $\operatorname{Wh} G = \operatorname{Wh} \pi_1 X$ *to* $\operatorname{Wh} \pi_1 X'$ *under* f_*. *Then* $\mathscr{F}(X)$ *and* $\mathscr{F}(X')$ *have identical projections on* $R^{\perp}$.

Proof of 3.2: Primes will distinguish objects formed for X' as those for X. Identify

X and X' under h as topological manifolds. The well-definition of relaxation of X as topological manifold says that, if Y_T and $Y'_{T'}$ are relaxations of $X=X'$, there is a homeomorphism h_0 such that, if g, g' denote natural homotopy equivalences, then

$$\begin{array}{ccc} Y_T & \xrightarrow{h_0} & Y'_{T'} \\ & {}_g\searrow \swarrow_{g'} & \\ & X=X' & \end{array}$$

is homotopy commutative. We can of course let Y_T and $Y'_{T'}$ be smooth relaxations of X, X' respectively so that $\mathscr{F}(X)=\mathscr{F}_0(Y_T)+\tau(g)$ and $\mathscr{F}(X')=\mathscr{F}_0(Y'_{T'})+\tau(g')$. But by 3.1 $\tau(h_0)$ has zero projection on $R^{\perp}$. Thus $\tau(g), \tau(g')$ have equal projection. Since $\mathscr{F}_0$ takes values in $R(G, H)$, the result follows.

§ 4. $\mathscr{F}$ is the total obstruction to fibration

From the splitting theorem of Farrell and Hsiang [8, Theorem 4] which involves an obstruction in an exotic quotient of the Whitehead group we deduce:

THEOREM 4.1. *If* $X\equiv(X, p)\in\mathscr{C}$, *and* $\mathscr{F}(X)=0$, *then* p *is homotopic to a smooth fibration. In fact if* $p \mid bX$ *is a smooth fibration,* p *is homotopic fixing* $p \mid bX$ *to a smooth fibration.*

Thus in particular the splitting theorem of Farrell and Hsiang includes the fibering theorem of Farrell's thesis [5] [6], at least in the presence of [15].

Consider $(X^n, p)\in\mathscr{C}$. Let $F^{n-1}\subset X^n$ be a compact submanifold of X_n equipped with a normal vector field. We assume F^{n-1} meets bX^n transversely, in bF^{n-1}. The Thom-Pontrjagin construction [4] yields a map $p_1: X^n\to S^1$ so that $p_1^{-1}(*)=F(*=\text{base point of } S^1)$ and p_1 is transverse regular at $*$ sending positive normal vectors to F onto positive normal vectors to $*$ in S^1. The homotopy class of p_1 is well defined by these conditions. We assume it is that of p. Cutting X apart along F produces a cobordism (with boundary) $c=(W; V, V')$ where V, V' are copies of F. Thus there is a diffeomorphism $\varphi: V\to V'$ such that when V and V' are identified by φ one retrieves X. We can assume that the inward normal direction along V is carried to the positive normal direction along F.

DEFINITION 4.2: Suppose that $c=(W; V, V')$ is a relative h-cobordism. By this we mean that $V \hookrightarrow W$, $V' \hookrightarrow W$ are homotopy equivalences, and $bW-(\text{int}\, V\cup \text{int}\, V')$ is diffeomorphic to $bV\times[0, 1]$. Then we say that F is a *pseudo-fiber*. We say $X\in\mathscr{C}$ *almost fibers*[1]) in case there exists some pseudo-fibers $F\subset X$.

[1]) One can readily show that (X, p) *almost fibers* if and only if p can be homotoped transverse to $*$ in such a way that $p^{-1}(*)\hookrightarrow X$ is in the sense of homotopy theory the fiber of p and $p \mid bX$ is a smooth fibration.

Whether F is a pseudo-fiber or not, the ∞-cyclic covering Y of X can be described as the union of copies W_n, $n \in Z$, of W under identification of V'_n to V_{n+1} by φ^{-1}. A generating covering translation T sends x_n to x_{n+1} for all $x \in W$. This T corresponds to $+1 \in Z = \pi_1 S^1$.

DEFINITION 4.3. $X \in \mathscr{C}$ is *relaxed* if it admits a diffeomorphism to its relaxation (in the preferred homotopy class).

LEMMA 4.4. *Suppose $X \in \mathscr{C}$ almost fibers. Then X is relaxed, and when one cuts X open to produce a relative h-cobordism* $c = (W; V, V')$ (see 4.2) *one finds that* $\mathscr{F}(X) = i_* \tau(c) \in \mathrm{Wh}\,\pi_1 X$, *where* $i: W \to X$ *is the quotient map.*

Proof of 4.4.: Since an infinite product of h-cobordisms is a collar [23], there exist homeomorphisms $f_-: U_- \to Y$, $f_+: U_+ \to Y$ where $U_- \cap U_+ = \emptyset$; $U_\pm \subset Y$ is $W_\pm = \bigcup_{\pm n \geqslant 1} W_n$ union a (small) open collar, and $f_\pm$ fixes $W_\pm$ pointwise. The corresponding twist-gluing

$$Y_T = Y / \{ f_+^{-1} T f_- (x) = x \mid x \in U_- \}$$

is naturally diffeomorphic to $\qquad W_0 / \{ V_0 \overset{\varphi}{=} V'_0 \} = X$.

Now by a simple chase of definitions in § 5, one can check that this isomorphism $g: Y_T \to X$ lies in the preferred homotopy class (defined in § 7.7). Hence $\mathscr{F}(X) = \mathscr{F}_0(Y_T) + \tau(g) = \mathscr{F}_0(Y_T)$. Now let's examine $\mathscr{F}_0(Y_T)$. For its definition (§ 1) we can choose the pair (Y, W_-). Then all homology modules appearing in the definition are zero. To obtain $\mathscr{F}(Y_T)$ we find a stable isomorphism ψ from the zero module summand in $\mathbf{C}_4 = \mathbf{C}_4(T\tilde{W}_-, \tilde{W}_-)$ to the zero summand in $\mathbf{C}_3$, which gives $\mathbf{C}_*$ zero torsion. This stable isomorphism is therefore an isomorphism $\psi: Z[H]^n \to Z[H]^n$ which has torsion $\tau(TW_-, W_-) = \tau(W_0, V_0) = \tau(c)$. Now the definition says that $\mathscr{F}_0(Y_T) = \tau(0, \psi)$. But clearly $\tau(0, \psi) = i_*[\psi] = i_* \tau(c)$. This establishes 4.4.[†])

We consider next another situation to be encountered in proving 4.1. Let W be an infinite but locally finite, connected C.W. complex of finite dimension[1]) with two ends ε_-, ε_+. Suppose V is a subcomplex such that there is a proper homotopy $h_t: W \to W$, $0 \leqslant t \leqslant 1$, of 1_W, fixing V pointwise, to a proper retraction onto V. In standard terminology V is a strong proper deformation retract of W. In W consider a subcomplex L such that L is a neighborhood of ε_+ and $(W - L) \cup V$ is a neighborhood of ε_-. Let $K = V \cap L$.

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[1]) It suffices to assume instead that W is combinatorially locally finite [15A].

[†]) Readers willing to assume that the ∞-cyclic covering Y of (X, p) in 4.1 is a smooth product with R can pass now directly to 4.9 and page 17. This property is certainly necessary for fibering, and is decided by an obstruction in $\tilde{K}_0 Z[H]$ from [15], cf. page 14.

LEMMA 4.5. *The pair (L, K) is dominated by a pair $(L_0 \cup K, K)$ where L_0 is a finite subcomplex of L.*

Proof of 4.5: Since the homotopy h_t, $0 \leqslant t \leqslant 1$, is proper, there exists a subcomplex $L_1 \subset L$ that is a neighborhood of ε_+ so small that $h_t(L_1) \subset L$ for all t, $0 \leqslant t \leqslant 1$. Extend

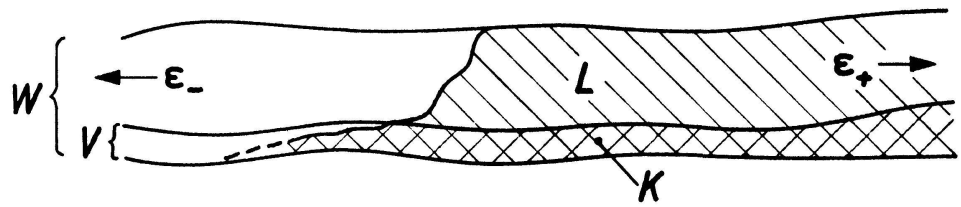

Figure 4.5.1.

$h_t \mid L_1$, $0 \leqslant t \leqslant 1$, to a homotopy k_t, $0 \leqslant t \leqslant 1$, of 1_L that fixes K pointwise. Now $L - L_1 - K$ is finite and has compact closure. So there exists a finite subcomplex $L_0 \subset L$ such that $k_t(L - L_1) \subset L_0 \cup K$ for all t, $0 \leqslant t \leqslant 1$. Now the inclusion $(L_0 \cup K, K) \subsetneq (L, K)$ is a domination because k_t, $0 \leqslant t \leqslant 1$, is a homotopy of $1_{(L,K)}$ to a map into $(L_0 \cup K, K)$.

RECOLLECTIONS 4.6. Let Λ be a ring with unit. We recall some basic results of Wall [28].

1) *A projective (positive) Λ-complex is dominated in the sense of chain homotopy by a (totally) finitely generated projective Λ-complex if and only if it is chain homotopy equivalent to a finitely generated projective Λ-complex.*

2) *If C_* is a projective Λ-complex chain homotopy equivalent to a finitely generated projective Λ-complex P_*, then*

$$\sigma(C_*) = \sum_i (-1)^i [P_i] \in \tilde{K}_0 \Lambda$$

is a well defined chain homotopy invariant of C_*. Note that $\sigma(C_*) = 0$ if P_* can be a free complex. The converse is true.

3) *If $0 \to A_* \to B_* \to C_* \to 0$ is a short exact sequence of projective Λ-complexes of which two are dominated by a finitely generated projective Λ-complex, then so is the third and*

$$\sigma(B_*) = \sigma(A_*) + \sigma(C_*).$$

Return to the situation and notations of 4.5. Let $c: \tilde{W} \to W$ be the universal covering and for any $S \subset W$ set $\tilde{S} = c^{-1}S$. From 4.6 it follows that the chain complex (cellular theory) $\mathbf{C}_*(\tilde{L}, \tilde{K})$, which is a free $Z[\pi_1 W] = \Lambda$-complex, is dominated in the sense of chain homotopy of Λ-complexes by the free Λ-complex $\mathbf{C}_*(\tilde{L}_0 \cup \tilde{K}, \tilde{K})$. We define

$$\sigma(W, V, \varepsilon_+) = \sigma(C_*(\tilde{L}, \tilde{K})) \in K_0 Z[\pi_1 W].$$

This is independent of the choice of K, because if L' is another choice of L, $L' \subset L$, there are only finitely many cells in $L - L'$ outside V. Then 3) applied to

$$0 \to \mathbf{C}_*(\tilde{L}', \tilde{K}') \to \mathbf{C}_*(\tilde{L}, \tilde{K}) \to \mathbf{C}_*(\tilde{L}, \tilde{L}' \cup \tilde{K}) \to 0$$

where $K' = L' \cap V$, shows that $\sigma(\mathbf{C}_*(\tilde{L}', \tilde{K}')) = \sigma(\mathbf{C}_*(\tilde{L}, \tilde{K}))$. Thus $\sigma(W, V, \varepsilon_+)$ is well defined. It is easy to see that $\sigma(W, V, \varepsilon_-) = -\sigma(W, V, \varepsilon_+)$, by applying 3) once again.

Suppose now that W is dominated by a finite complex. Choose neighborhoods L_-, L_+ of ε_-, ε_+ so that $W - L_+$, $W - L_-$ are again neighborhoods of ε_-, ε_+ respectively, and finally $L_- \cup L_+ = W$. Then $L_- \cap L_+$ is finite and fact 3) applied to the Mayer-Vietoris sequence

$$0 \to \mathbf{C}_*(\tilde{L}_- \cap \tilde{L}_+) \to \mathbf{C}_*(\tilde{L}_-) \oplus \mathbf{C}_*(\tilde{L}_+) \to \mathbf{C}_*(\tilde{W}) \to 0$$

shows that $\mathbf{C}_*(\tilde{L}_+)$ is dominated by a finitely generated free complex. Then the same is true for any smaller neighborhood L'_+ of ε_+ and, again by 3), $\sigma(\mathbf{C}_*(\tilde{L}_+)) = \sigma(\mathbf{C}_*(\tilde{L}'_+))$. Thus we can define

$$\sigma(W, \varepsilon_+) = \sigma(\mathbf{C}_*(\tilde{L}_+)) \in \tilde{K}_0 Z[\pi_1 W].$$

Similarly we can define

$$\sigma(V, \varepsilon_+) \in \tilde{K}_0 Z[\pi_1 V] = \tilde{K}_0 Z[\pi_1 W]$$

using for example the neighborhood $K_+ = L_+ \cap V$ of the positive end. Now applying 3) to the short exact chain complex sequence for $(\tilde{L}_+, \tilde{K}_+)$ one gets

$$\sigma(W, \varepsilon_+) = \sigma(W, V, \varepsilon_+) + \sigma(V, \varepsilon_+).$$

For example consider any $(X, p) \in \mathscr{C}$. By assumption the ∞-cyclic covering Y of X is dominated by a finite complex. Also Y has a "positive" end ε_+, corresponding to the positive end of R.

Hence we have $\sigma(Y, \varepsilon_+) \in \tilde{K}_0 Z[\pi_1 Y]$ which we denote $\sigma_+(Y)$ for convenience. It is the obstruction to capping off ε_+ with a boundary [15]. Using the covering translation and the infinite product trick [24], one sees that $\sigma_+(Y)$ is equally the obstruction to expressing Y as a smooth product with the line.

Two-ended pairs (W, V) with nontrivial invariant $\sigma(W, V, \varepsilon_+)$ are encountered as follows. Suppose W_0 is a finite connected complex equipped with a map $f: W_0 \to S^1$ inducing a surjection $\pi_1 W_0 \to \pi_1 S^1 = Z$. Suppose V_0 is a subcomplex of W_0 such that $V_0 \hookrightarrow W_0$ is a homotopy equivalence. Then we let (W, V) be the ∞-cyclic covering of (W_0, V_0) induced by f from $R^1 \to S^1$. Since V_0 is a strong deformation retract of W_0, V is a strong *proper* deformation retract of W [17, § 4.7]. Thus we have an invariant

$$\sigma(W, V, \varepsilon_+) \in \tilde{K}_0 Z[\pi_1 W].$$

We propose to express it in terms of the Whitehead torsion $\tau(W_0, V_0) \in \mathrm{Wh}\,\pi_1 W_0$ of the inclusion $V_0 \hookrightarrow W_0$.

Write $H \subsetneq G \to Z$ for the sequence $\mathscr{S}: \pi_1 W \to \pi_1 W_0 \to \pi_1 S^1 = Z$. There is a natural (purely algebraic) homomorphism

$$p: \mathrm{Wh}(G) \to \tilde{K}_0 Z[H]$$

determined by $\mathscr{S}$ as follows: Decompose $Z[G]$ as

$$\cdots \oplus Z[H]\, t^{-1} \oplus Z[H] \oplus Z[H]\, t \oplus Z[H]\, t^2 \oplus \cdots$$

where $t \in G$ goes to $+1 \in Z$. Write B for the subring $Z[H] \oplus Z[H]\, t \oplus \cdots$ and B' for the subring $Z[H] \oplus Z[H]\, t^{-1} \oplus \cdots$. Given $x \in \mathrm{Wh}\, G$, represent x by a (left) $Z[G]$-isomorphism $\varrho: Z[G]^n \to Z[G]^n$. Choose $s > 0$ so large that $\varrho(B^n t^s) \subset B^n$ and $\varrho(B'^n t^s) \supset B'^n$. Then as a left $Z[H]$-module $B^n/\varrho(B^n t^s)$ is finitely generated and projective (see 10.2 below). By definition, it represents $p(x)$. It is known that p is a well defined group homomorphism (see 8.1 below).

With the notations introduced above we state:

PROPOSITION 4.7.* $\sigma(W, V, \varepsilon_+) = p\tau(W_0, V_0)$.

Proof of 4.7: First observe that if $W_0 \searrow W_0'$ is an elementary cellular collapse mod V_0 (i.e. not affecting V_0) cf. [29, § 2] and $W' \subset W$ is the ∞-cyclic covering W_0', then $W \searrow W'$ by an infinite sequence of disjoint independent elementary cellular collapses not affecting V. It follows easily that (with obvious identifications of fundamental groups) $\sigma(W, V, \varepsilon_+) = \sigma(W', V, \varepsilon_+)$. And of course $\tau(W_0, V_0) = \tau(W_0', V_0)$. But W_0 can be altered by finitely many elementary collapses mod V_0 and their inverses so that W_0 is V_0 plus a certain number (say n) of 2-cells and an equal number of 3-cells and no others. For a proof see [29] or [14]. We conclude that it suffices to verify 4.7 in this case.

Let $q: \tilde{W} \to W$ be a universal covering, and for $S \subset W$ set $\tilde{S} = q^{-1} S$. Now $\tilde{W}$ is also the universal covering of W_0 by a composite projection $\tilde{W} \to W \to W_0$. Choose base points corresponding under these projections. Note that if T is the generating covering translation of W over W_0 and if t is the covering translation of $\tilde{W}$ over W_0 corresponding to $t \in G$, $(t \mapsto 1 \in Z)$, then $qt = Tq$.

For each cell in $W_0 - V_0$ choose a cell e over it in $\tilde{W}$, and orient e. Write $\mathbf{C}_* = \mathbf{C}_*(\tilde{W}, \tilde{V})$

$$\mathbf{C}_*: \cdots \to 0 \to \mathbf{C}_3 \xrightarrow{\partial} \mathbf{C}_2 \to 0 \to \cdots$$

$\mathbf{C}_2$ and $\mathbf{C}_3$ are copies of $Z[G]^n$, since preferred bases derive from the lifted 2-cells $e_1, \ldots, e_n$, and the lifted 3-cells $e_1', \ldots, e_n'$. Now $\tau(W_0, V_0)$ is by definition the class of ∂ in $\mathrm{Wh}\, G$ as automorphism of $Z[G]^n$ (see [13]).

*) $\sigma(W, V, \varepsilon_+)$ is an obstruction to $V \subsetneq W$ being an (infinite) simple homotopy equivalence. It is the only one if kernel $\{\pi_1 W \to Z\}$ is finitely presented [34].

To V add all the 2-cells $\{T^a qe_i \mid a \geqslant 0, 0 \leqslant i \leqslant n\}$ of W, and call the result L_2. We can next choose $s \geqslant 0$ so large that for all $a \geqslant 0$ and for all i, $0 \leqslant i \leqslant n$, the 3-cell $T^a q(t^s e_i') = T^{a+s} qe_i'$ of W is attached to L_2. Then the result $L(s)$ of adding these cells to L_2 is a subcomplex of W such that $L(s)$ is a neighborhood of ε_+ and $(W - L(s)) \cup V$ is a neighborhood of ε_-. Thus, by definition,

$$\sigma(W, V, \varepsilon_+) = \sigma(\mathbf{C}_*(\tilde{L}(s), \tilde{V}))$$

for all large s. We now calculate the right hand side. It is evident that as left $Z[H]$ modules

$$\mathbf{C}_2(\tilde{L}(s), \tilde{V}) = B^n \subset Z[G]^n = \mathbf{C}_2$$
$$\mathbf{C}_3(\tilde{L}(s), \tilde{V}) = B^n t^s \subset Z[G^n] = \mathbf{C}_3 .$$

If s is very large, $B^n/\partial B^n t^s = P_s$ is projective over $Z[H]$ and $p\tau(W_0, V_0) = [P_s] \in \tilde{K}_0[H]$. But when P_s is projective $\mathbf{C}_*(\tilde{L}(s), \tilde{V})$ clearly splits and is chain homotopy equivalent to the complex having P_s isolated in dimension 2. Thus $\sigma(\mathbf{C}_*(L(s), \tilde{V})) = [P_s]$ and we have $\sigma(W, V, \varepsilon_+) = [P_s] = p\tau(W_0, V_0)$, as required.

With the data of 4.7, suppose V is dominated by a finite complex. Suppose also that W_0 is the mapping cylinder $M(g)$ of a homotopy equivalence $g: V_0 \to V_0'$ of V_0 with a finite complex V_0'. Identify $\pi_1 V_0'$ naturally with G and $\pi_1 V'$ with H where $V' \subset W$ covers V_0'. Then $\sigma(V, \varepsilon_+)$ and $\sigma(V', \varepsilon_+)$ are defined in $K_0 Z[H]$ and we have:

COROLLARY 4.8. $\sigma(V', \varepsilon_+) = \sigma(V, \varepsilon_+) + p\tau(g)$.

Proof of 4.8.: $\tau(g) = \tau(W_0, V_0)$ by definition, and $\sigma(V', \varepsilon_+) = \sigma(W, \varepsilon_+)$ because $W_0 = M(g)$ collapses to V_0'. Recall $\sigma(W, \varepsilon_+) = \sigma(V, \varepsilon_+) + \sigma(W, V, \varepsilon_+)$. Apply 4.7.

Given $X \equiv (X, f) \in \mathscr{C}$, write $H \overset{i}{\hookrightarrow} G \to Z$ for $\pi_1 Y \hookrightarrow \pi_1 X \xrightarrow{f_*} \pi_1 S^1$, where Y is the ∞-cyclic covering of X.

PROPOSITION 4.9. *Let $X \in \mathscr{C}$ have invariant $\mathscr{F}(X)$ in i_* Wh$H \subset G$. Then X almost fibers.*

Proof of 4.9: Let Y_T be a relaxation of $X \in \mathscr{C}$. Let $g: Y_T \to X$ be a preferred homotopy equivalence (see 7.7 below), and identify $\pi_1 Y_T$ to $\pi_1 X = G$ under g_*. The ∞-cyclic covering*) $\tilde{Y}_T$ of the relaxation Y_T of X is diffeomorphic to Y by a diffeomorphism taking the positive end to the positive end, and covering (up to homotopy) the preferred homotopy equivalence $g: -1\ Y_T \to X$ (see 7.9 and observations (b) before 7.8). Note that g can give a diffeomorphism $bY_T \to bX$. This is because bY is a relaxation of bX, which fibers (use 4.4). It follows that $\sigma_+(\tilde{Y}_T) = \sigma_+(Y) \in \tilde{K}_0 Z[H]$ with natural identification of fundamental groups.

On the other hand, Corollary 4.8 says that $\sigma_+(Y) = \sigma_+(\tilde{Y}_T) + p\tau(g)$. We conclude that $p\tau(g) = 0$. Now we use the hypotheses about $\mathscr{F}(X)$. Applying p to the identity

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*) $\tilde{Y}_T$ was denoted Z in §§ 1, 2.

$\mathscr{F}(X)=\mathscr{F}_0(Y_T)+\tau(g)$, we get $0=\sigma_+(Y)+0$. Thus, as we have remarked, Y splits as a product with R [15].

In view of the construction of Y_T from Y it is then immediate that Y_T almost fibers. Also $\mathscr{F}_0(Y_T)\in \text{Image}(\text{Wh}\,H)$, by 4.4. Then $\tau(g)=\mathscr{F}(X)-\mathscr{F}_0(Y_T)$ also is in Image (Wh H), and so the Farrell-Hsiang splitting theorem [8, Theorem 4 and note (4)] says that a homotopy inverse $g':X\to Y_T$ of g with $g' \mid bX$ the inverse of $g \mid bY_T$ can be homotoped so that the preimage of a pseudo-fiber for Y_T is a pseudo-fiber for X. So X almost fibers as 4.9 claims.

COMPLEMENT TO 4.9. *In the situation of* 4.9, *let* $X=(X,f)\in\mathscr{C}$ *with* $f \mid bX: bX\to$ $\to S^1$ *a smooth fibration. There is a speudo-fiber F for X such that* $bF=F\cap bX$ *is a fiber of* $f \mid bX$.

Proof of complement: One can construct F from any pseudo-fiber F' by cutting off a little collar of bF' in F' and replacing it by an h-cobordism from bF' to a fiber of $f \mid bX$. The h-cobordism is to be fitted into X by making it spiral down towards bX in a way described by Wall [26, p. 662].

Alternatively one can prove the complement along with 4.9 by applying relative versions of the splitting theorems of [15] [8].

Remark 4.10. The obstruction to "splitting" $g':X\to Y_T$ as described is simply the class of $\tau(g')$ in $\text{Wh}\,G/i_*\,\text{Wh}\,H$. For as we note in § 12,

$$\text{Wh}\,G/i_*\text{Wh}\,H = \tilde{C}(Z[H],\theta)\oplus\tilde{C}(Z[H],\theta^{-1})\oplus\tilde{K}_0Z[H]^\theta$$

where $\tilde{K}_0Z[H]^\theta=\{x\in\tilde{K}_0Z[H] \mid \theta_*(x)=x\}$. Now Farrell and Hsiang find that the obstruction to splitting is the projection of $\tau(g')$ to $\tilde{C}(Z[H],\theta)\oplus K_0\tilde{Z}[H]^\theta$. But as g' is a homotopy equivalence n-manifolds which is a simple homotopy equivalence on the boundaries, one can show that $\tau(g')=(-1)^{n-1}\,\bar{\tau}(g')$ where bar is the duality involution of Wh G depending on $w_1(X)$ [13, § 10]. Now "bar" respects $\tilde{K}_0Z[H]^\theta$ and is easily seen to interchange $\tilde{C}(Z[H],\theta)$ and $\tilde{C}(Z[H],\theta^{-1})$. Thus if ξ is the component of the obstruction on $\tilde{C}(Z[H],\theta)$ the "missing" component on $\tilde{C}(Z[H],\theta^{-1})$ is $(-1)^{n-1}\,\xi$. This remark applies to the general case of the splitting theorem [8].

PROPOSITION 4.11. *Suppose that* $X=(X,p)\in\mathscr{C}$ *almost fibers. Then X fibers if and only if* $\mathscr{F}(X)=0$.

Proof of 4.11.: The "only if" part follows from 4.4. So suppose $\mathscr{F}(X)=0$, and let X be split at a pseudo-fiber F to give a h-cobordism $c=(W;V,V')$ with $bW-\text{int}\,V-\text{int}\,V'\cong bV\times[0,1]$. By 4.4 $\mathscr{F}(X)=i_*\tau(c)=0\in\text{Wh}\,\pi_1X$. We identify $\text{Wh}\,\pi_1W=\text{Wh}\,\pi_1Y=\text{Wh}\,H$, where Y the ∞-cyclic covering of X. By 8.1 or an easier geometrical proof $\tau(c)=x-T_*x$ for some x. This means that by choosing a new pseudo-fiber F separated by a relative h-cobordism F to F' (in positive direction) with torsion x we get a *product* cobordism c' since $\tau(c')=\tau(c)-x+T_*x=0$. Thus X fibers over S^1 with fiber F'.

COMPLEMENT TO 4.11. *If in the above proof $p \mid bX$ is a fibration with bF a fiber, the relative version of the s-cobordism theorem* [14] *shows that the fibration found above can extend $p \mid bX$. This implies that p can be homotoped fixing $p \mid bX$ to a smooth fibration.*

Combining 4.9, 4.11 and their complements we have a proof of 4.1.

Chapter II. GLUING AND RELAXATION

§ 5. Topological twist-gluing

The compact Möbius band is formed from the ribbon $[-1, 1] \times R$ by gluing together the ends after a twist of the ribbon about its central thread $\{0\} \times R$. We propose to examine this operation on generalized ribbons that include the infinite cyclic coverings of the candidates (in § 1) for fibering over the circle. Even these most pleasant ribbons are often not topologically a product with the line – see [18].

To emphasize the topological properties used we will first develop the notion in the category of topological spaces and continuous maps. The sign $\approx$ will indicate topological homeomorphism.

Differentiable and piecewise linear treatments are formally the same (see § 6), as are treatments in all three categories for pairs ($=$2-ads) or, more generally, for n-ads (cf. [19]).

DEFINITIONS 5.1. Let $\hat{Y}$ be a topological space and ε_-, ε_+ two distinct points of $\hat{Y}$ having disjoint neighborhoods. Write $Y = \hat{Y} - \{\varepsilon_-, \varepsilon_+\}$. ($Y$ is the "ribbon".) We regard ε_-, ε_+ as ideal points of Y, so that by a neighborhood of $\varepsilon_\pm$ ($=\varepsilon_-$ or ε_+) in Y we mean a set $N \subset Y$ such that $N \cup \{\varepsilon_\pm\}$ is a neighborhood of $\varepsilon_\pm$ in $\hat{Y}$. Suppose that Y is homeomorphic to arbitrarily small neighborhoods of each point $\varepsilon_\pm$ by homeomorphisms $f_\pm : U_\pm \to Y$ that fix pointwise a smaller neighborhood of $\varepsilon_\pm$, and satisfy the "niceness condition":

(**) *If V is any open neighborhood of $\varepsilon_\mp$ ($=$the other ideal point) in Y, then $f_\pm^{-1}(Y - V)$ is closed in Y (not just in $U_\pm$).*

Under these hypotheses twist-gluings of Y relative to ε_- and ε_+ exist.

Remark: The condition (**) on $f_\pm$ is clearly redundant if $\hat{Y}$ is compact Hausdorff, but not redundant if $\hat{Y}$ is the suspension of R^1.

Let $T: \hat{Y} \to \hat{Y}$ be a homeomorphism of $\hat{Y}$ that fixes ε_-, ε_+. (It's restriction to Y, also denoted T, is the "twist".) Given the above hypotheses choose f_- and f_+ so that $U_- \cap U_+ = \emptyset$, and identify U_- to U_+ in Y under $f_+^{-1} T f_-$ to obtain

$$Y_T(f_-, f_+) = Y / \{x = f_+^{-1} T f_-(x) \quad \text{for} \quad x \in U_-\}.$$

This space $Y_T(f_-, f_+)$ with the quotient topology is a *twist-gluing of Y relative to T and the (ordered) ideal points* ε_-, ε_+. If T is the identity Y_T is called simply the *gluing of Y relative to* ε_-, ε_+.

This chapter is built around the following observation:

THEOREM 5.2. *Any two twist-gluings* $Y_T(f_-, f_+)$, $Y_T(f'_-, f'_+)$ *of Y relative to T and* ε_-, ε_+ *are homeomorphic.*

Proof of 5.2.: One easily reduces the question to the case where $U_+ = U'_+$ and $f_+ = f'_+$. Consider this case. We can find $U_0 \subset U_- \cap U'_-$ an open neighborhood of ε_- so small that $f'_- \mid U_0 = f_- \mid U_0$. Indeed if U_0 is small enough both these restrictions are the identity. Let $Z = Y - f_+^{-1}T(Y - f_- U_0)$. By the niceness condition (**) applied to f_+, Z is open in Y.

Now, in Z identify U_0 to $f_+^{-1}Tf_-(U_0)$ under $f_+^{-1}Tf_- \mid U_0 = f_+^{-1}Tf'_- \mid U_0$ to get $\mathcal{Z}$. Note that this identification is the restriction to Z of the identification defining $Y_T(f_-, f_+)$, (or $Y_T(f'_-, f_+)$). Hence we have a $(1-1)$ continuous map $\mathcal{Z} \to Y_T(f_-, f_+)$.

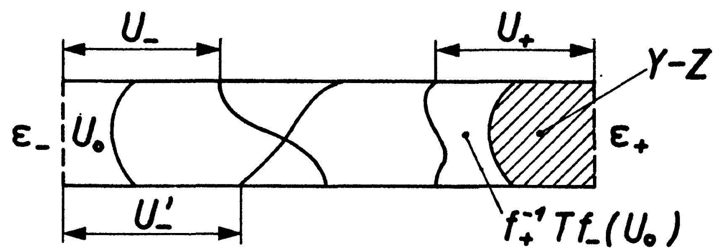

Figure 5.2.1.

It is an open map because Z is open in Y and $Y \to Y_T(f_-, f_+)$ is open. It is onto since $Z \cup U_+ = Y$. Hence $\mathcal{Z} \to Y_T(f_-, f_+)$ is a homeomorphism. Similarly the natural map $\mathcal{Z} \to Y_T(f'_-, f_+)$ is a homeomorphism. This completes the proof.

Complementary Remarks 5.3.

(i) $Y - (U_- \cup U'_- \cup U_+)$ becomes under the quotient maps a common subset of $Y_T(f_-, f_+)$ and $Y_T(f'_-, f_+)$. It is fixed by the above homeomorphism. Hence, more generally, the (composite) homeomorphism $Y_T(f_-, f_+) \approx Y_T(f'_-, f'_+)$ guaranteed by the above proof can fix pointwise the common subset $Y - (U_- \cup U'_- \cup U_+ \cup U'_+)$.

(ii) It will follow from Proposition 5.6 below that the homeomorphism $Y_T(f_-, f_+)$ to $Y_T(f'_-, f'_+)$ lies in a preferred homotopy class, (if Y is a pleasant space). The example $Y = S^2 \times R$, $T = 1$ already shows that (i) does not imply (ii) [9].

A sample application of 5.2 is

COROLLARY 5.4. (folklore of M. Brown cf. [19, § 5]).

If X_1 *and* X_2 *are compacta and* $X_1 \times (-1, 1) \approx X_2 \times (-1, 1)$, *then* $X_1 \times S^1 \approx X_2 \times S^1$.

Proof of 5.4. Set $Y = X_1 \times (-1, 1)$ and let $\hat{Y} \supset Y$ be $X_1 \times [-1, 1]$ with $X_1 \times \{-1\}$

and $X_1 \times \{1\}$ smashed to points $\varepsilon_-, \varepsilon_+$. One gluing of Y is $X_1 \times S^1$. But using the homeomorphism $X_1 \times (-1, 1) \approx X_2 \times (-1, 1)$ one sees that another gluing is $X_2 \times S^1$. Apply 5.3.

EXAMPLE: Let $\hat{Y}$ be the Cantor set and $\varepsilon_-, \varepsilon_+$ two distinct points of $\hat{Y}$. Then there exist gluings of $Y = \hat{Y} - \{\varepsilon_-, \varepsilon_+\}$ relative to $\varepsilon_-, \varepsilon_+$ and they are Cantor sets.

EXAMPLE: Let $\hat{Y} \subset R^1$ be $\varepsilon_- = -1$ and $\varepsilon_+ = +1$ union the points $(1-1/n)$, $-(1-1/n)$ for $n = 1, 2, 3, \ldots$. There do *not* exist gluings of $Y = \hat{Y} - \{\varepsilon_-, \varepsilon_+\}$ relative to $\varepsilon_-, \varepsilon_+$. If hypotheses were weakened so that they did, uniqueness would fail.

PROPOSITION 5.5.

(a) *Y_T is respectively Hausdorff, regular, or satisfies the second axiom of countability provided Y has the corresponding property.*

(b) *Y_T has all the local properties of Y and visa versa.*

(c) *Y_T is compact provided $\hat{Y} = Y \cup \{\varepsilon_-, \varepsilon_+\}$ is compact.*

Proof 5.5: Representing Y_T as $Y_T(f_-, f_+)$ consider the set $Z = Y - (f_-^{-1}U_- \cup \cup f_+^{-1}TU_+)$. It is closed because of the niceness condition (**). If $Y' = Y - (U_- \cup U_+)$, the sets $f_-^{-1}(Y')$ and $f_+^{-1}T(Y')$ are closed, again because of (**).

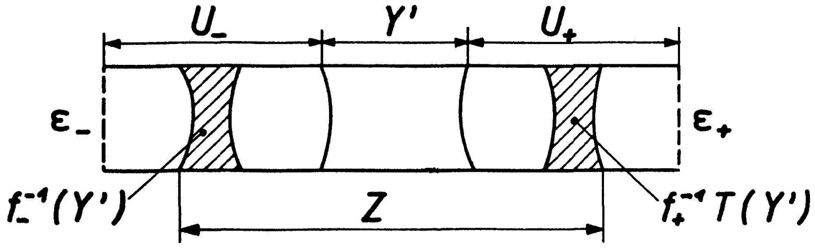

Figure 5.5.1.

The composition $Z \to Y \xrightarrow{q} Y_T(f_-, f_+)$ is clearly onto and identifies the closed set $f_-^{-1}(Y')$ to the disjoint closed set $f_+^{-1}T(Y')$ under $f_+^{-1}Tf_-$. Hence if $\mathcal{Z}$ is the quotient of Z under this identification, we have a natural continuous map $\varphi: \mathcal{Z} \to Y_T(f_-, f_+)$ that is $(1-1)$ and onto. But φ is easily seen to be closed. Hence φ is a homeomorphism. Fact (a) now follows from the fact that the quotient map $Z \to \mathcal{Z} \approx Y_T$ is closed and identifies at most pairs of points – see Kelly [11, p. 148].

As for (b) we can check using $Z \to \mathcal{Z}$ or else $q: Y \to Y_T(f_-, f_+)$ that for any point $y \in Y$, there is a neighborhood N of y in Y and a neighborhood N' of $q(y)$ in $Y_T(f_-, f_+)$ so that q maps N homeomorphically onto N'.

Finally, if $\hat{Y}$ is compact so is Z since Z is closed in $\hat{Y}$. Hence its quotient $\mathcal{Z} \approx Y_T(f_-, f_+)$ is compact as (c) asserts. This completes the proof of 5.5.

PROPOSITION 5.6. *Let $Y_T(f_-, f_+)$ be a twist gluing of Y relative to T and the* *ideal points ε_-, ε_+, and adopt the notations of* 5.1. *Suppose Y is a normal space and Y can be homotoped over itself into any given neighborhood of ε_+ [or of ε_-] by a homotopy that fixes pointwise a smaller neighborhood of ε_+ [respectively of ε_-]. Then there exists a preferred homotopy equivalence from $Y_T(f_-, f_+)$ to the mapping torus $\mathcal{T}_T$ of T.*

Before the proof we insert a lemma

LEMMA 5.7. *Let Y be as in Proposition* 5.6. *Any map $f: Y \to Y$ that fixes pointwise a neighborhood U of $\varepsilon_\pm$ ($=\varepsilon_+$* or *ε_-) is homotopic to 1_Y by a homotopy that fixes pointwise a smaller neighborhood of $\varepsilon_\pm$.*

Proof of 5.7: According to the given assumption about Y there exists g_t, $0 \leqslant t \leqslant 1$, a homotopy of $1_Y = g_0$ to $g_1: Y \to Y$ so that $g_1(Y)$ lies in the set of fixed points of f and g_t, $0 \leqslant t \leqslant 1$, fixes pointwise a small neighborhood of $\varepsilon_\pm$. Then $f \circ g_t$, $0 \leqslant t \leqslant 1$, is a homotopy of f to g_1 while g_{1-t}, $0 \leqslant t \leqslant 1$, is a homotopy of g_1 to 1_Y and both fix pointwise a neighborhood of $\varepsilon_\pm$. Composing we get the required homotopy.

The same argument shows that the entire (semi-simplicial) space of maps $Y \to Y$ fixing a neighborhood of ε is contractible to $1 | Y$. This implies in particular

LEMMA 5.7.2. *If f_t, f'_t, $0 \leqslant t \leqslant 1$, are two homotopies of $1 | Y$ to $f: Y \to Y$, both fixing a neighborhood of ε ($=\varepsilon_+$* or *ε_-), then there is a one-parameter family of such homotopies joining f_t to f'_t.*

Let A be a closed subset of a topological space X such that $A \subsetneq X$ is a cofibration, and hence [25, p. 57, E5, E7] $X \times 0 \cup A \times [0, 1]$ is a deformation retract of $X \times [0, 1]$. Let $f_0: A \to Y$ be a continuous map and form the adjunction space $Y \cup_{f_0} X$ from a disjoint union $Y \cup X$ by identifying each $a \in A$ to $f_0(a) \in Y$. Let f_t, $0 \leqslant t \leqslant 1$, be a homotopy of f_0 to f_1.

ADJUNCTION LEMMA: $Y \cup_{f_0} X \simeq Y \cup_{f_1} X$.

The proof itself is more useful:

CONSTRUCTION 5.7.3. Let $F(t, a) = f_t(a)$ define $F: [0, 1] \times A \to Y$ and consider $Y \cup_F [0, 1] \times X$. For $i=0$ or 1, $Y \cup_{f_i} X$ is naturally a subspace and a deformation retraction

$$r_i: [0, 1] \times X \to [0, 1] \times A \cup \{i\} \times X$$

induces a deformation retraction

$$\varrho_i: Y \cup_F [0, 1] \times X \to Y \cup_{f_i} X.$$

Thus $\varrho_1 | Y \cup_{f_0} X$ is a homotopy equivalence to $Y \cup_{f_1} X$ and $\varrho_0 | Y \cup_{f_1} X$ its homotopy inverse. Clearly the homotopy class of these equivalences depends only on f_t, $0 \leqslant t \leqslant 1$.

We are now ready for

Proof of 5.6: The procedure is to define a mediating space $M=M(f_-,f_+)$ and find natural homotopy equivalences $Y_T\xleftarrow{\theta_1}M\xrightarrow{\theta_2}\mathscr{T}_T$.

$M(f_-,f_+)$ is the adjunction space

$$Y\cup_\varrho Y\times[0,1]$$

where $\varrho: Y\times\{0,1\}\to Y$ maps $(y,0)$ to $f_-^{-1}(y)$ and $(y,1)$ to $f_+^{-1}T(y)$ for y in Y.

θ_1 simply collapses the intervals $y\times[0,1]$, $y\in Y$, to points. A continuous function $Y\to[0,1]$ equal 0 near ε_+ and equal 1 near ε_- permits one to construct an injection $j: Y_T\to M$ with $\theta_1 j=1\,|\,Y_T$ so that $j(Y_T)$ is a strong deformation retract of M. Hence θ_1 is a canonical homotopy equivalence. Draw a picture for $Y=R$!

It remains to give $\theta_2: M\to\mathscr{T}_T$. Now $\mathscr{T}_T$ is

$$Y\cup_{\varrho_1} Y\times[0,1]$$

where $\varrho_1: Y\times\{0,1\}\to Y$ maps $(y,0)$ to y and $(y,1)$ to $T(y)$, for y in Y. Lemma 5.7.1. provides a homotopy ϱ_t, $0\leqslant t\leqslant 1$, from ϱ to ϱ_1. In a standard way (see 5.7.3.) ϱ_t provides a homotopy equivalence $\theta_2(\varrho_t)$. By Lemma 5.7.2. any two of these homotopies ϱ_t, ϱ_t' are joined by a family of such homotopies. This shows that $\theta_2(\varrho_t')$ is homotopic to $\theta_2(\varrho_t)$. Hence θ_2 is a homotopy equivalence in a preferred homotopy class. This completes the proof of Proposition 5.6.

PROPOSITION 5.8. *Under the assumptions of Proposition* 5.6, *any homeomorphism of two twist gluings* $\varphi: Y_T(f_-,f_+)\to Y_T(f_-,f_+)$ *that is constructed by the proof of Theorem* 5.2 *necessarily makes this triangle commute up to homotopy:*

$$\begin{array}{ccc} Y_T(f_-,f_+) & \xrightarrow{\ \varphi\ } & Y_T(f_-',f_+') \\ & \searrow^{\psi} \quad {}^{\psi'}\swarrow & \\ & \mathscr{T}_T & \end{array}$$

where ψ, ψ' *are preferred homotopy equivalences to the mapping torus of* T *provided by Proposition* 5.6. *Hence in particular,* φ *lies in a preferred homotopy class.*

Proof of 5.8: It suffices to prove this for the case where $f_+=f_+': U_+\to Y$, where φ is constructed in one step by the proof of 5.2, and finally where identical homotopies h_t, $0\leqslant t\leqslant 1$, from $Y\xrightarrow{f_+^{-1}}U_+\subset Y$ to $1\,|\,Y$ are used to construct ψ and ψ' for 5.6.

Adopt the notations of the proof of 5.2 and set

$$N=Z\cup_\sigma U_0\times[0,1]$$

where $\sigma: U_0\times\{0,1\}\to Z$ sends $(y,0)$ to y and $(y,1)$ to $f_+T(y)$ for all y in $U_0\subset Y$.

Consider the diagram

$$\begin{array}{ccccc}
Y_T(f'_-,f_+) & \xleftarrow{\approx} & Z & \xrightarrow{\approx} & Y_T(f'_-,f_+) \\
\uparrow{\scriptstyle \theta'_1} & & \uparrow{\scriptstyle \theta} & & \uparrow{\scriptstyle \theta_1} \\
M(f'_-,f_+) & \hookleftarrow & N & \hookrightarrow & M(f_-,f_+) \\
& {\scriptstyle \theta'_2}\searrow & & \swarrow{\scriptstyle \theta_2} & \\
& & M(1,1)=\mathcal{T}_T & &
\end{array}$$

The first row comes from the proof of 5.2, while $\theta_1, \theta_2, \theta'_1, \theta'_2$ come from the proof of 5.6. By definition, θ is $\theta_1|N$ which clearly coincides with $\theta'_1|N$. Now θ is a homotopy equivalence; the argument is like that for θ_1 in 5.6.

Construction of θ_2 in the proof of 5.6 requires h_t (mentioned above) and a homotopy g_t from $1|Y$ to $Y \xrightarrow{f_-^{-1}} U_- \subsetneq Y$ fixing pointwise a neighborhood of ε. Similarly construction of θ'_2 requires h_t and g'_t, a homotopy from $1|Y$ to $Y \xrightarrow{f'^{-1}_-} U_- \subsetneq Y$. Making U_0 smaller if necessary we can assume that g_t *and* g'_t *both fix* U_0 *pointwise.* (This is permissible because making U_0 smaller turns out to have no effect on φ – see 5.2.) It follows that θ_2 and θ'_2 can coincide on N (see 5.6 and 5.7.3.).

Thus the diagram is commutative and consists of homotopy equivalences. Its perimeter yields $\psi \simeq \psi'\varphi$ as asserted by 5.8.

Any twist gluing $Y_T(f_-,f_+)$ has a natural ∞-cyclic covering $\tilde{Y}$ determined by the data Y, f_-, f_+, T, namely the union $\bigcup\{Y \times \{n\};\ n \text{ an integer}\}$ under identification of $U_+ \times \{n\}$ to $U_- \times \{n+1\}$ by (the transport of) $f_-^{-1}T^{-1}f_+$. The covering translation $\tilde{T}$ on $\tilde{Y}$ sends (x, n) to $(x, n+1)$. The orbit space of $\tilde{Y}$ under the action of $\tilde{T}$ is clearly $Y_T(f_-,f_+)$. To $\tilde{Y}$ we can adjoin ideal points $\bar{\varepsilon}_-, \bar{\varepsilon}_+$, giving to $\bar{\varepsilon}_\pm$ the base of neighborhoods in $\tilde{Y}$

$$Y_\pm(n) = \{\text{image in } \tilde{Y} \text{ of } \bigcup_{\pm k \geqslant n} Y \times \{k\}\}, \qquad n = 1, 2, 3, \ldots$$

(As usual $\pm$ denotes consistently $+$ or consistently $-$). In many cases*) one can verify

HYPOTHESIS 5.9. *For each* $n \geqslant 1$, *there exists a homeomorphism of* $Y_\pm(n)$ *onto* $\tilde{Y}$ *fixing* $Y_\pm(n-1)$, *and satisfying the niceness condition* (**) *of* 5.1.

Then in particular there exists a twist-gluing of $\tilde{Y}$ with respect to $\tilde{T}$ and $\bar{\varepsilon}_-, \bar{\varepsilon}_+$.

PROPOSITION 5.10. *If* 5.9 *is verified, then the twist-gluing* $\tilde{Y}_{\tilde{T}}$ *is homeomorphic to* Y_T.

In other words, by repeating the twist-gluing process one gets nothing new; the process is idempotent. The proof uses

LEMMA 5.11. *Let* X *be a topological space and* $f: X \to X$ *be a continuous open map.*

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*) For example, in case $\tilde{Y}$ is compact metric and, for given neighborhoods V_-, V_+ of ε_-, ε_+, there always exists a self-homeomorphism h of $\tilde{Y}$, fixing all points outside a compactum in Y, such that $h(U_-) \cup U_+ \supset Y$; cf. proof of 7.8.

Let Z be an open subset of X. Consider the orbit space of f, $X_f = X/\{x=f(x)\} = X$ mod the least equivalence $\sim$ so that $x \sim f(x)$ for $x \in X$. Consider also the quotient space Z_f of Z defined to be Z divided by the least equivalence relation $\sim$ such that $z \sim f(z)$ whenever z and $f(z)$ belong to Z. In this situation the natural map $\theta : Z_f \to X_f$ is a continuous open map.

Proof of 5.11: Continuity of θ follows from the universal property of quotient topologies. To show θ is open take U open in Z_f. Let $q: X \to X_f$, $q': Z \to Z_f$ be the quotient maps. By definition of Z_f, $q'^{-1}U$ is open in Z, hence also in X. Now

$$q^{-1}\theta U = \bigcup \{f^n(q'^{-1}U);\ n = \dots, -2, -1, 0, 1, 2, \dots\}$$

which is a union of open sets of X, hence is open. This shows that θU is open as required.

Proof of 5.10: It will be convenient to allow the notation $Y_T(f_-, f_+)$ for any space obtained by the twist-gluing process 5.1 applied without the stipulation that the domains U_-, U_+ of f_-, f_+ be disjoint. Then $Y_T(f_-, f_+)$ is Y divided by the least equivalence relation $\sim$ such that $x \sim f_+^{-1} T f_-(x)$ for $x \in U_-$. Note that if U_- meets U_+, $Y_T(f_-, f_+)$ is usually not homeomorphic to a genuine twist-gluing. (Myriad examples exist with $Y = R$ and $T =$ identity.)

Take up the notations of Hypothesis 5.9.

Let $g_\pm : Y_\pm(1) \to \tilde{Y}$ be a homeomorphism fixing $Y_\pm(2)$ hypothesised in 5.9. There results a twist-gluing $\tilde{Y}_{\tilde{T}}(g_-, g_+)$ which we aim to prove homeomorphic to Y_T.

ASSERTION A) *$\tilde{Y}_{\tilde{T}}(g_-, g_+)$ is homeomorphic to $\tilde{Y}_{\tilde{T}}(1_Y, g_+)$.*

To prove this assertion observe that $\tilde{Y}_{\tilde{T}}(g_-, g_+)$ can be formed (cf. the proof of 5.2) from $W = \tilde{Y} - g_+^{-1}\tilde{T}(\tilde{Y} - Y_-(2))$ by the identification

$$Y_-(2) \xrightarrow{g_+{}^{-1}\tilde{T}} g_+^{-1}\tilde{T}\, Y_-(2)$$

or, said in symbols $\tilde{Y}_{\tilde{T}}(g_-, g_+) = W/\sim$. The inclusion $W \subsetneq \tilde{Y}$ clearly induces a map

$$\varphi : W/\sim\ \to \tilde{Y}(1_Y, g_+).$$

By Lemma 5.11, in order to show that φ is a homeomorphism it will suffice to verify it is bijective. This is a routine matter. But to assist the reader we provide a schematic diagram Figure 5.10.1 and an explicit description of $\tilde{Y}_{\tilde{T}}(1_Y, g_+)$.

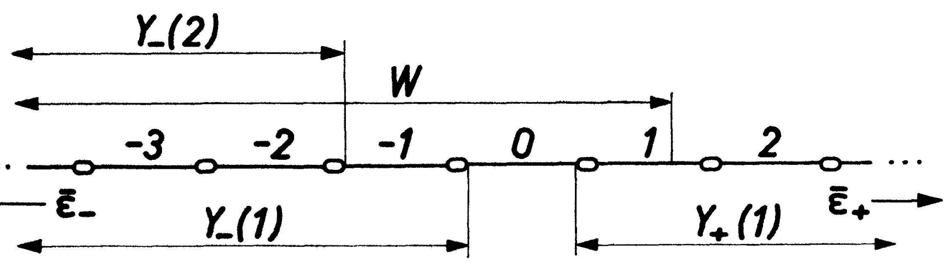

Figure 5.10.1.

In the diagram, the line represents $\bar{Y}$; the segment marked n including its end points represents $Y \times \{n\} \subset \bar{Y}$, and the end points represent overlap of the successive "segments" $Y \times \{n\}$. Observe that $\bar{Y}_{\bar{T}}(1_Y, g_+)$ is the orbit space of the open continuous map $g_+^{-1}\bar{T}: \bar{Y} \to \bar{Y}$ which is translation one segment to the right followed by g_+^{-1}. Thus $\bar{Y}_{\bar{T}}(1_Y, g_+)$ consists of the following mutually disjoint orbits

$$(\bar{T}^{-1} g_+ y, y, \bar{T} y, \bar{T}^2 y, \ldots, \bar{T}^n y, \ldots) \qquad (\#)$$

for y ranging through $Y_+(1) - Y_+(2)$.

ASSERTION B) *$\bar{Y}_{\bar{T}}(1_Y, g_+)$ is homeomorphic to $Y_T(f_-, f_+)$.*

This is proved as follows. Since $g_+^{-1}\bar{T} \mid Y_+(1) = \bar{T} \mid Y_+(1)$ and the orbit space of $Y_+(1) \xrightarrow{\bar{T}} Y_+(1)$ is exactly Y_T, we see that $Y_+(1) \subsetneq \bar{Y}$ induces a map $\psi: Y_T \to \bar{Y}_{\bar{T}}(1_Y, g_+)$ and ψ is a homeomorphism (by Lemma 5.11) provided it is bijective. But (#) shows ψ is bijective. Thus the assertion is established and with it Proposition 5.10.

§ 6. Piecewise-linear and smooth twist-gluings

Consider the categories

PL: Objects are polyhedra – viz. locally compact paracompact spaces equipped with an atlas of piecewise-linearly compatible charts to finite simplicial complexes. Morphisms are continuous maps which are piecewise-linear when expressed via the local charts.

DIFF: Objects are smooth C^∞ finite dimensional paracompact manifolds with boundary (having no corners). Morphisms are smooth C^∞ maps.

DEFINITIONS 6.1. Let CAT be PL or DIFF. In the definition of twist-gluings $Y_T(f_-, f_+)$ suppose that Y (not $\hat{Y}$) is in CAT and that $T: Y \to Y$ is a CAT isomorphism. Elsewhere in the definitions 5.1 read CAT-isomorphism for homeomorphism. Then observe (using 5.5) that $Y_T(f_-, f_+)$ is naturally an object of CAT. It is called a CAT *twist-gluing* of Y with respect to T, and ε_-, ε_+.

Of course a CAT twist-gluing is an ordinary twist gluing if we forget the CAT structure. This means that the results 5.5, 5.6 and 5.8 carry over immediately to CAT-twist-gluings.

On the other hand the CAT versions of 5.2, 5.3, 5.4 and 5.10 assert a CAT isomorphism, hence must be reexamined. But inspection shows that by carrying out the proof, as given, in CAT one automatically arrives at a homeomorphism that is a CAT-isomorphism.

SUMMARY 6.2. *The results proved in* § 5 *for topological twist-gluings can be translated for* CAT *twist-gluings. In particular any two* CAT *twist-gluings of Y relative to T, ε_-, ε_+ are* CAT-*isomorphic. Indeed there is a* CAT-*isomorphism in a preferred* *homotopy class. This preferred homotopy class is (by definition) the same as that for the underlying topological twist-gluing.*

To relieve the dullness of this exposition here is a useful application.

THEOREM 6.3. (CAT = DIFF *or* PL).

Let M be a compactum so that $M\times(-1, 1)$ is a topological n-manifold with $n\geqslant 6$ (or $n\geqslant 5$ if the boundary is empty). Let Σ be a CAT *manifold structure on $M\times(-1, 1)$. Then there exists a* CAT *manifold structure Σ' on $M\times S^1$. In fact if we identify S^1 with $[-1, 1]/\{-1=1\}$ so that $M\times(-1, 1)\subset M\times S^1$, then Σ' can coincide with Σ on $M\times(-\frac{1}{2}, \frac{1}{2})$.*

Proof of 6.3: Let $Y=M\times(-1, 1)$. Let $\hat{Y}$ be $M\times[-1, 1]$ with $M\times\{\pm 1\}$ smashed to points $\varepsilon_{\pm}$. Form a topological gluing Z_1 of Y identifying $M\times(-1, -\frac{1}{2})$ to $M\times(\frac{1}{2}, 1)$ by $(m, t)\to(m, t+\frac{3}{2})$. Then Y_T can be identified to $M\times S^1$ by a homeomorphism fixing the common subset $M\times(-\frac{1}{2}, \frac{1}{2})$.

We assert there exist CAT gluings of Y. In fact for any a, b with $-1<a<b<1$ there exists an isomorphism of $M\times(-1, b)$ onto $M\times(-1, 1)$ fixing $M\times(-1, a)$. This is a standard application of Stallings engulfing theorem and his infinite stretching process. See [23] and [16, Proof of 1.3]. Note that in addition to engulfing in the interior of $M\times(-1, 1)$ one must engulf along the boundary extending each elementary engulfing homeomorphism to the whole of $M\times(-1, 1)$ by using an isotopy of it to the identity in conjunction with a collar of the boundary. It is the engulfing which requires our dimension conditions. Engulfing is a PL method; so in the DIFF case one should triangulate, and then smooth the PL homeomorphism obtained using [10].

Now form a CAT gluing of $Y=M\times(-1, 1)$ which matches $M\times(-1, -\frac{1}{2})$ to $M\times(\frac{1}{2}, 1)$. Call it Z_2. Thinking of Z_2 as a merely topological gluing we appeal to 5.2 and 5.3 to provide a homeomorphism $h:Z_2\to Z_1=M\times S^1$ which fixes the common subset $M\times(-\frac{1}{2}, \frac{1}{2})$. The structure Σ' on $M\times S^1$ transported from Z_2 by h coincides with Σ on $M\times(-\frac{1}{2}, \frac{1}{2})$. This completes the proof.

§ 7. Relaxation

We now present the most interesting examples of twist gluings and explain the notion of relaxation of candidates for fibering over the circle.

Let X be a compact connected manifold in CAT (CAT = PL or DIFF), and let $f:X\to S^1$ be a continuous map such that the covering $Y\to X$ of X, induced by f from the universal covering $R^1\to S^1$, is connected and dominated by a finite complex. If X has non empty boundary bX suppose that $f\mid bX:bX\to S^1$ satisfies the similar condition on each component of bX. Finally suppose that $\dim X\geqslant 6$ (or else $\dim X=5$ and $f\mid bX$ is a CAT fibration over S^1). Thus X is a candidate in CAT for fibration over S^1 slightly more general on bX than in § 1. Note that the ∞-cyclic covering Y of X has positive and negative ends ε_-, ε_+ corresponding to the ends of R^1. Let T be the covering translation of Y corresponding to $+1 \in \pi_1(S^1) = Z$.

PROPOSITION 7.1. *In this situation let* U_-, U_+ *be prescribed open neighborhoods of* ε_-, ε_+. *Then there exists a* CAT *isotopy* h_t, $0 \leqslant t \leqslant 1$, *of* 1_Y *through automorphisms of* Y *such that* $h_1(U_-) \cup U_+ = Y$. *Furthermore this isotopy can be chosen to fix all points outside some compactum in* Y.

To begin the proof of 7.1 we establish:

ASSERTION 7.2. *Under the assumptions of* 7.1, *there exists a closed neighborhood* U *of* ε_+ *such that*

(i) U *is an* n*-submanifold with frontier* ∂U *a locally flat* $(n-1)$*-submanifold meeting* bY *transversely.*

(ii) U *and* ∂U *are connected.*

(iii) $\partial U \subsetneq U$ *and* $U \subsetneq Y$ *induce isomorphisms of fundamental group.*

(iv) $U \subsetneq Y$ *is* $(n-3)$*-connected.*

Proof of 7.2: This is essentially proved in [15, II–V] but this situation is peculiar since bY is present and no explicit assumption about the behavior of π_1 at ε_+ is given. So here is an outline.

As a first approximation try setting $U = g^{-1}[0, \infty)$ where $g: Y \to R^1$ is a proper CAT approximation, regular at 0, to the map covering f. Then (i) holds and U can be modified to satisfy (ii) by abandoning all but the unbounded component of U then connecting up the components of ∂U by tubes having 1-dimensional cores. Here and below we simply operate *away from* bY *only.* Next modify U to satisfy (iii) as follows. First trade 1-handles along ∂U (between U and $Y - U$) to make $\pi_1 \partial U \to \pi_1 Y$ surjective.*) Then one can trade 2-handles along ∂U to make $\pi_1 \partial U \to \pi_1 Y$ an isomorphism. By Van-Kampen's theorem, $\partial U \subsetneq U$, $U \subsetneq Y$ and $\partial U \subsetneq V =$ closure $(Y - U)$ now all induce isomorphisms of fundamental group. To realize (iv) alter U further by adding to U in Y suitable k handles along ∂U, $2 \leqslant k \leqslant n-3$. Thus 7.2 is established.

Proof of 7.1 (continued): Let U be as provided by 7.2. Then by 4.6 and [28] *U is dominated by a finite complex.* Choose n so large that $T^n U \subset U$. Then by (iii) the inclusions

$$U \leftarrow T^n U \leftarrow T^{2n} U \leftarrow T^{3n} U \leftarrow \cdots$$

all induce isomorphisms of fundamental group. Also $\bigcap \{T^{kn} U \mid k = 1, 2, 3, \ldots\} = \emptyset$. The italicized conditions show that ε_+ is a (homotopically) *tame* end in the sense of [15] [15A]. Similarly (or trivially if $\dim Y = 5$) the ends of bY at ε_+ are *tame.* Hence an engulfing argument to be given in [20] establishes:

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*) In fact it always is, cf. [15A].

ASSERTION 7.3 (see [20]). *If* U_1 *is any neighborhood of* ε_+ *there exists a smaller neighborhood* U_2 *of* ε_+ *so that the following holds. If* $U_3 \subset U_2$ *is a prescribed neighborhood of* ε_+*, there exists an isotopy* h_t*,* $0 \leqslant t \leqslant 1$*, of* 1_Y *fixing points outside a compactum in* U_1 *and such that* $h_1(U_2) \subset U_3$.

Using the covering translation T we can now deduce 7.1 as follows. Let $U_1 = Y$. If Assertion 7.3 now applies to U_2 it clearly also applies to $T^{-k}U_2$ for all k. Hence we can assume that $U_- \cup U_2 = Y$. Then let $U_3 = U_+$. The isotopy h_t, $0 \leqslant t \leqslant 1$, provided in this case by 7.3 is the isotopy wanted in 7.1 since

$$h_1(U_-) \cup h_1(U_2) = Y$$

and $h_1 U_2 \subset U_+$.

It is not difficult to establish 7.1 using 7.2 and engulfing, without appealing to [20]. Here is an outline. Using the translation T find a neighborhood U of ε_+ as in 7.2 with $U \subset U_+$. Then find a closed neighborhood $U' \subset U_-$ of ε_- (not ε_+) satisfying (i), (ii), (iii) and (iv) of 7.2. If we proceed by induction on dim Y the proposition is already established for bY. So after a preliminary isotopy along bY (extended to Y using a collar of bY we can assume that $\mathring{U}' \cup U \supset bY$. Engulf now in int Y to find an isotopy k_t, $0 \leqslant t \leqslant 1$, of 1_Y with compact support in int Y such that $k_1(\mathring{U}') \cup \mathring{U} \supset Y$. Most of the necessary argument can be found in [24, § 3].

PROPOSITION 7.4. *With the data of* 7.1, *there exists a neighborhood* $U \subset U_+$ *of* ε_+*, and a* CAT *homotopy* k_t*,* $0 \leqslant t \leqslant 1$*, of* $U \subsetneq Y$ *through* CAT *embeddings* $(U, bU) \to \to (Y, bY)$*, such that*

(a) k_1 *is an isomorphism* $U \to Y$.

(b) *There exists a neighborhood* $V \subset U$ *of* ε_+ *which* k_t *fixes pointwise for* $0 \leqslant t \leqslant 1$.

We abbreviate this statement by saying that U slides onto Y fixing a neighborhood of ε_+*. A similar assertion holds for* ε_-.

Proof of 7.4: This is a consequence of 7.1 and an infinite compression procedure. A full proof will be given in the more general setting, in [20]. With care the reader can construct a proof for himself by applying 7.1 infinitely to push smaller and smaller neighborhoods of ε_- towards ε_+.

COROLLARY 7.5. *With the data of* 7.1, *there exist* CAT *twist-gluings of* Y *relative to T and* ε_-, ε_+.

Let Y be a connected ANR and T a self-homeomorphism of Y. Suppose T has infinite order and that the quotient map $q: Y \to X = Y/\{T(y) = y \mid y \in Y\}$ is a covering projection.

LEMMA 7.6. *There is a natural homotopy equivalence* θ *of* X *with the mapping torus* $\mathcal{T}_T$ *of* T.

Proof of 7.6: By our definition (see 5.6) $\mathcal{T}_T$ is the orbit space of

$$T': Y \times R^1 \to Y \times R^1$$

defined by $T'(y, r) = (Ty, r+1)$. Let $q': Y \times R^1 \to \mathcal{T}_T$ be the quotient map. *The natural homotopy equivalence is induced by the projection* $\theta_0: Y \times R^1 \to Y$. As $\theta_0 T' = T\theta_0$ there is a unique map θ in a commutative square:

$$\begin{array}{ccc} Y \times R^1 & \xrightarrow{\theta_0} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{T}_T & \xrightarrow{\theta} & X \end{array} .$$

θ is the wanted homotopy equivalence*).

PROPOSITION 7.7. *With the data of* 7.1 *let* $Y_T(f_-, f_+)$ *be a twist-gluing of* Y *relative to* $T, \varepsilon_-, \varepsilon_+$. *There exists a natural homotopy equivalence* $g: Y_T(f_-, f_+) \to X$. *If* $Y_T(f'_-, f'_+)$ *is another such twist-gluing and* $g': Y_T(f'_-, f'_+) \to X$ *is the natural homotopy equivalence there is a* CAT *isomorphism* $\varphi: Y_T(f_-, f_+) \to Y_T(f'_-, f'_+)$ *so that* $g'\varphi$ *is homotopic to* g.

Proof of 7.7: By 7.4, 5.6 and 5.8, there exist such natural homotopy equivalences to the mapping torus $\mathcal{T}_T$ of T. But there is canonical homotopy equivalence $\mathcal{T}_T \to X$ as Lemma 7.6 shows. This proves 7.7.

From the compact manifold X in 7.1 equipped with $f: X \to S^1$ we have constructed according to 7.7 an essentially unique twist-gluing Y_T and homotopy equivalence $g: Y_T \to X$. The pair (Y_T, fg) is called a *relaxation* of (X, f). In general Y_T is *not* isomorphic to X (cf. § 1). However if (X, f) is itself a relaxation of another pair (X_0, f_0) then 5.10 applied in CAT shows that $Y_T \cong X$. Actually, to apply 5.10 we need to prove the proposition 7.8 below which in any case was used in § 1 and § 4.

With the data of 7.1 let $Y_T(f_-, f_+)$ be a CAT twist-gluing of Y relative to T, and $\varepsilon_-, \varepsilon_+$. Let $\bar{Y}$ be the natural ∞-cyclic covering of $Y_T(f_-, f_+)$ described before 5.9. The ideal points $\bar{\varepsilon}_-, \bar{\varepsilon}_+$ of $\bar{Y}$ described there are in this case the ends of $\bar{Y}$ [15, Chap. I]. By following the construction of the natural homotopy equivalence $g: Y_T(f_-, f_+) \to X$ one readily observes that there is a (proper) map $g': \bar{Y} \to Y$ covering $g: Y_T(f_-, f_+) \to X$ such that

(a) g' *carries* $\bar{\varepsilon}_+$ to ε_+ *and* $\bar{\varepsilon}_-$ to ε_-.

(b) *The natural inclusion* $Y = Y \times \{0\} \subsetneq \bar{Y}$ *is a (non-proper) homotopy inverse to* g'.

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*) *Added in proof*: In fact, as F. T. Farell has pointed out to me, $\theta: \mathcal{T}_T \to X$ is by its definition an R^1-bundle associated to the covering $q: Y \to X$, which is a principal bundle with infinite cyclic group generated by T; hence $\mathcal{T}_T \cong X \times R$. If X is a candidate for fibering from § 1, one can find its relaxation Y_T embedded in $\mathcal{T}_T \cong X \times R$ in a natural way (up to isotopy) realizing the homotopy equivalence $Y_T \simeq \mathcal{T}_T$ of 5.6. I conjecture that a resulting h-cobordism from Y_T to X has torsion x in a summand $\tilde{C}(Z[H], \theta^{\pm 1})$ of WhG (§ 12), so that x is a projection of $\mathcal{F}(X)$.

Now (a) implies in particular that the ∞-cyclic covering of Y_T corresponding to $fg: Y_T \to S^1$ is $\tilde{Y}$ and that the positive end is indeed $\tilde{\varepsilon}_+$.

Recall from 5.9 that $\tilde{Y}$ is composed of a doubly infinite sequence of copies $Y \times \{n\}$, $n=0, \pm 1, \pm 2, \ldots$ of Y. We now denote these $Y_0, Y_{\pm 1}, Y_{\pm 2}, \cdots$ and let $U_n = Y_n \cap Y_{n+1}$.

PROPOSITION 7.8. *Let $A \subset Y_0$ be closed in $\tilde{Y}_\pm = Y_0 \cup Y_{\pm 1} \cup Y_{\pm 2} \cup \ldots$ Then there exists a* CAT *homotopy $\varphi_t^\pm$, $0 \leqslant t \leqslant 1$, of $Y_0 \subsetneq \tilde{Y}_\pm$ through* CAT *embeddings $(Y_0, bY_0) \to (\tilde{Y}_\pm, b\tilde{Y}_\pm)$ to a* CAT *isomorphism $Y_0 \to \tilde{Y}_\pm$ such that $\varphi_t^\pm \mid A$ is the identity, $0 \leqslant t \leqslant 1$. Thus, in the terminology of* 7.4, *Y_0 slides onto $\tilde{Y}_\pm$ fixing A.*

An immediate consequence is

COROLLARY 7.9. *If $B \subset Y_0$ is closed in $\tilde{Y}$, there exists a* CAT *homotopy, φ_t, $0 \leqslant t \leqslant 1$, of $Y_0 \subsetneq \tilde{Y}$ through* CAT *embeddings $(Y_0, bY_0) \to (\tilde{Y}, b\tilde{Y})$ to a* CAT *isomorphism $Y_0 \to \tilde{Y}$, such that $\varphi_t \mid B =$* identity *for all t, $0 \leqslant t \leqslant 1$.*

Proof of 7.8: As a first step let us construct an isomorphism $\varphi^+: Y_0 \to \tilde{Y}_+$ fixing A. We will stretch U_0 over Y_1 then stretch $U_1 \subset Y_1$ over Y_2, then stretch $U_2 \subset Y_2$ over Y_3 etc. To be sure we converge to an isomorphism we need:

LEMMA 7.10. *Consider the isomorphism $f_-: U_- \to Y$. If $A \subset U_-$ is any set closed in Y there exists a* CAT *isomorphism $g_-: U_- \to Y$ fixing A pointwise.*

The reader can supply the proof using only the conclusion of 7.1. One must think of U_- as a copy of Y and use 7.1 to "push" the points moved by f_- outside A.

Return to the proof of 7.8. Let $A = A_1 \subset A_2 \subset A_3 \subset \cdots$ be closed subsets of $\tilde{Y}_+$ each contained in Y_0, and such that $\bigcup_n \mathring{A}_n = Y_0$. Set $\varphi_1 = 1_{Y_0}$.

Suppose for an inductive construction that we have formed an isomorphism $\varphi_n: Y_0 \to Y_0 \cup \cdots \cup Y_n$ fixing A pointwise. Since $(Y, U_-) \cong (Y_{n+1} U_n)$, Lemma 7.10 provides an isomorphism $g_n: U_n \to Y_{n+1}$ fixing pointwise a neighborhood of the negative end of U_n that contains $\varphi_n(A_n) \cap U_n$. Then g_n extends as the identity outside U_n to an isomorphism

$$g_n': Y_0 \cup \cdots \cup Y_n \to Y_0 \cup \cdots \cup Y_{n+1}.$$

Set $\varphi_{n+1} = g_n' \varphi_n$ to complete the inductive construction of $\varphi_1, \varphi_2, \varphi_3, \ldots$.

Now define $\varphi^+ \mid \mathring{A}_n = \varphi_n \mid \mathring{A}_n$, for all n. Since $\varphi_{n+1} \mid \mathring{A}_n = \varphi_n \mid \mathring{A}_n$ we have here a well defined isomorphism $\varphi_t^+: Y_0 \to \tilde{Y}_+$, that fixes A.

It still remains to find φ_t^+, $0 \leqslant t \leqslant 1$, as 7.8 demands. One way is to construct φ_t^+, $0 \leqslant t \leqslant 1$, by the method used to construct φ^+. Alternatively deduce its existence from that φ^+ as follows. In the terminology of 7.4 we want to show that Y_0 slides onto $\tilde{Y}_+$ fixing A. Since $\tilde{Y} \cong Y$, 7.4 says that certain arbitrarily small neighborhoods of the negative end ε_-' of $\tilde{Y}_+$ slide onto $\tilde{Y}_+$ fixing a neighborhood of ε_-'. Then Y_0

must too by the argument proving 5.7. Next it follows that Y_0 slides onto $\bar{Y}_+$ fixing A by the argument proving 7.10. This concludes the proof of 7.8.

Chapter III. ALGEBRAIC RELAXATION

§ 8. The main result

In this chapter we derive from an exact sequence of groups $\mathscr{S}: 1 \to H \to G \to (Z, +) \to 1$ a sequence

$$\mathrm{Wh}\, H \xrightarrow{1-\theta_*} \mathrm{Wh}\, H \xrightarrow{i_*} \mathrm{Wh}\, G \xrightarrow{p} \tilde{K}_0 Z[H] \xrightarrow{1-\theta_*} \tilde{K}_0 Z[H] \xrightarrow{i_*} \tilde{K}_0 Z[G] \qquad (S)$$

together with certain exactness properties announced in [22]. Recall from § 1 that θ is the automorphism of H given by $\theta(h) = tht^{-1}$ for $h \in H$. Here t is an element of G which maps to $+1 \in Z$. Since an inner automorphism of H induces the identity map of $\mathrm{Wh}\, H$ and $\tilde{K}_0 Z[H]$, neither induced map θ_* depends on the choice of t. The homomorphism p was defined for Proposition 4.7.

THEOREM 8.1. *The sequence* (S) *is exact except at* $\mathrm{Wh}\, G$ *where* $p \circ i = 0$ *and* p *is split modulo the image of* i. *More precisely* (S) *becomes exact if* $\mathrm{Wh}\, G$ *is replaced by the subgroup* $R(G, H)$ *constructed in* § 1; *and there is a natural retraction of* $\mathrm{Wh}\, G$ *onto* $R(G, H)$.

We call the retraction mentioned *relaxation* for reasons mentioned in Remark 2 below. Its existence is probably the most interesting contention of Theorem 8.1.

Remark 1. We will neither use nor prove the main assertions of Farrell and Hsiang in [8]. The information we give is complementary to theirs. Indeed, combining it with theirs we get a simple decomposition of $\mathrm{Wh}\, G$ into three summands which is surely the right analogue of the "classical" analysis [2] for the case $G = Z \times H$. See § 12.

Remark 2. The reader impatient with algebra will find that only the homomorphism p is *logically necessary* in Chapters I and II (namely in § 4). However we contend that the relaxation retraction $\mathrm{Wh}\, G \to R(G, H)$ corresponds by our torsion invariant to the geometrical relaxation in § 1 and § 7 of candidates for fibration over the circle. We expect to prove this in a sequel.

The following sections § 9, § 10, § 11 are devoted to the proof of 8.1. As is natural we work in the K-theory of rings.

§ 9. The exact sequence of a ring endomorphism

By a ring we mean a ring with a multiplicative 1-element. Ring homomorphisms are assumed to carry 1-element to 1-element. By a *module* over a given ring we will mean a left unitary module over that ring, and by a *projective* we will mean a finitely generated left projective module.

For any ring Λ we identify $\mathrm{GL}(\Lambda, n)$ with the automorphisms of Λ^n as left Λ-module. Identifying Λ^n with the 1st n-factors of Λ^∞ we define $\mathrm{GL}(\Lambda) = \bigcup_n \mathrm{GL}(\Lambda, n)$. Then $K_1\Lambda$ is, by definition, $\mathrm{GL}(\Lambda)$ abelianized. If $\mathrm{Aut}_\Lambda(P)$ is the group of automorphisms of a projective P over Λ, a well-defined homomorphism $\mathrm{Aut}_\Lambda(P) \to K_1\Lambda$ is obtained by expressing $P \oplus Q \cong \Lambda^n$ and sending $\alpha \in \mathrm{Aut}_\Lambda(P)$ to the class of $\alpha \oplus 1_Q$. (We do not assume $\Lambda^m \cong \Lambda^n \Rightarrow m = n$.) See [3, § 2.4]. Thus for any Λ-automorphism α of a projective P we can speak of its class $[\alpha] \in K_1\Lambda$.

We say that two isomorphisms $f, g: P \to Q$ of projectives over Λ are *homotopic* if the automorphism $g^{-1}f: P \to P$ determines $[g^{-1}f] = 0 \in K_1\Lambda$. It is equivalent to ask that $[f^{-1}g] = 0 \in K_1\Lambda$.

The projective class group $K_0\Lambda$ is the abelian group generated by isomorphism classes $[P]$ of projectives P over Λ subject to the relations $[P \oplus P'] = [P] + [P']$.

If $\theta: \Lambda \to \Lambda'$ is a ring homomorphism and M is a Λ-module define $\theta_\# M$ to be $\Lambda' \otimes_\Lambda M$ where for the tensor product Λ' is a right Λ-module by the rule $\lambda' \cdot \lambda = \lambda' f(\lambda)$ for $\lambda' \in \Lambda'$, $\lambda \in \Lambda$. If $f: M_1 \to M_2$ is a Λ-module homomorphism, define $\theta_\# f: \theta_\# M_1 \to \theta_\# M_2$ by the rule $\theta_\# f(\lambda' \otimes m) = \theta(\lambda') \otimes f(m)$, for $\lambda' \in \Lambda'$ and $m \in M$. These rules clearly make $\theta_\#$ a functor from the category of Λ-modules to the category of Λ'-modules. $\theta_\#$ sends direct sums to direct sums. Also $\Lambda' \cong \theta_\# \Lambda$ by the standard isomorphism $\Theta(\lambda') = \lambda' \otimes 1 \in \Lambda' \otimes \Lambda$ for $\lambda' \in \Lambda'$. Thus free modules are carried to free modules and projectives to projectives. We conclude that $\theta_\#$ induces homomorphsms $\theta_*: K_i\Lambda \to K_i\Lambda'$, $i = 0,1$.

Now fix a ring A and a ring endomorphism $\theta: A \to A$. Form a category $\mathscr{C}(A, \theta)$ as follows: An object is a pair (P, φ) consisting of a projective P over A and an isomorphism $\varphi: P \to \theta_\# P$ of A-modules. A morphism $(P, \varphi) \to (P', \varphi')$ is an isomorphism of A-modules $g: P \to P'$ so that

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & \theta_\# P \\ g\downarrow & & \downarrow \theta_\# g \\ P' & \xrightarrow{\varphi'} & \theta_\# P' \end{array}$$

is *homotopy commutative* in the sense that $\varphi^{-1}(\theta_\# g)^{-1}\varphi' g: P \to P$ has class $0 \in K_1 A$. (It is equivalent to demand that any other circuit once about the above square give $0 \in K_1 A$.) Notice that every homomorphism is an isomorphism.

From $\mathscr{C}(A, \theta)$ construct an abelian group $K(A, \theta)$ in the following standard way. As generators take all isomorphism classes $[P, \varphi]$ of objects (P, φ) of $\mathscr{C}(A, \theta)$, and admit the relations

$$[P, \varphi] \oplus [P', \varphi'] = [P \oplus P', \varphi \oplus \varphi'] .$$

Let $\Theta_n: A^n \to \theta_\# A^n$ be the direct sum of n copies of the standard isomorphism $\Theta: A \to \theta_\# A$ sending $a \in A$ to $a \otimes 1 \in A \otimes_A A = \theta_\# A$. Then we have standard objects $(A^n, \Theta_n) \in \mathscr{C}(A, \theta)$ representing $n[A, \Theta] \in K(A, \theta)$, $n = 1, 2, 3, \ldots$.

LEMMA 9.1. *Two objects* (P, φ), (P', φ') *of* $\mathscr{C}(A, \theta)$ *represent the same object* $[P, \varphi]=[P', \varphi']\in K(A, \theta)$ *if and only if there exists an integer n and an isomorphism in* $\mathscr{C}(A, \theta)$:

$$(P \oplus A^n, \varphi \oplus \Theta_n) \cong (P' \oplus A^n, \varphi' \oplus \Theta_n).$$

The proof is like the standard proof for K_0 [1, § 2.1]. It is based strictly on the form of the definition of $K(A, \theta)$ plus the fact that given any $(P, \varphi)\in\mathscr{C}(A, \theta)$, there exists $(Q, \psi)\in\mathscr{C}(A, \theta)$ so that for some n

$$(P \oplus Q, \varphi \oplus \psi) \cong (A^n, \Theta_n).$$

From the Whitehead Lemma [13] one deduces that a homomorphism $j: K_1A \to K(A, \theta)$ is defined by associating to $\alpha\in GL(A, n)$ the element $[A^n, \Theta_n\alpha]-$ $-[A^n, \Theta_n]$.

Also a homomorphism $p: K(A, \theta) \to K_0A$ is clearly induced by sending $(P, \varphi)\in\mathscr{C}(A, \theta)$ to $[P]\in K_0A$.

THEOREM 9.2. *The sequence*

$$K_1A \xrightarrow{1-\theta_*} K_1A \xrightarrow{j} K(A, \theta) \xrightarrow{p} K_0A \xrightarrow{1-\theta_*} K_0A$$

is exact.

Proof of 9.2: We will check exactness at K_1A. The rest of the proof is easier.

Suppose that $j[\alpha]=0$, where $\alpha\in GL(A_n)$. This means that $[A^n, \Theta_n\alpha]=[A^n, \Theta_n]$ in $K(A, \theta)$. Now Lemma 9.1 shows that at the cost of increasing n we can arrange that there is in $\mathscr{C}(A, \theta)$, an isomorphism $g: (A^n, \Theta_n\alpha) \to (A^n, \Theta_n)$. Consider the diagram

$$\begin{array}{ccccc} A^n & \xrightarrow{\alpha} & A^n & \xrightarrow{\Theta_n} & \theta_\# A^n \\ \downarrow g & & \downarrow g' & & \downarrow \theta_\# g \\ A^n & \xrightarrow{1} & A^n & \xrightarrow{\Theta_n} & \theta_\# A^n \end{array} \qquad (**)$$

By hypothesis the outside rectangle is homotopy commutative. By definition $[\theta_\# g]=\theta_*[g]\in K_1A$. We define g' so that the right-hand square commutes. Then $[g']=[\theta_\# g]=\theta_*[g]$. And the left-hand square is homotopy commutative, which implies that $[1]=[g]+[g'^{-1}]+[\alpha^{-1}]$, i.e. $[\alpha]=[g]-\theta_*[g]$. We have established that

$$\text{image}(1-\theta_*) \supset \text{kernel}(j).$$

To prove image$(1-\theta_*)=$kernel(j), it remains to show that, for each $g:A^n \to A^n$ in $GL(n, A)$, $j([g]-\theta_*[g])=0$. If we specify this g in the diagram (**) used above, and define $\alpha=g'^{-1}g$ then (**) commutes (strictly) and we conclude that $j[\alpha]=j[1]=0$. But $[\alpha]=[g]-[g']=[g]-\theta_*[g]$, so exactness at K_1A is established.

The rest of the proof is left to the reader.

It is enlightening to consider the analogue of the exact sequence of 9.2 in geometric K-theory from, say, complex vector bundles. Let X be a finite complex and $f:X \to X$ a continuous map. Form a category $\mathscr{C}(X,f)$ as follows: Objects are pairs (ξ, φ) where ξ is a vector bundle over X and $\varphi=\xi \to f^*\xi$ a vector bundle isomorphism of ξ to the induced bundle $f^*\xi$. A morphism $(\xi, \varphi) \to (\xi', \varphi')$ is a vector bundle isomorphism α such that the square

$$\begin{array}{ccc} \xi & \xrightarrow{\varphi} & f^*\xi \\ \alpha\downarrow & & \downarrow f^*\alpha \\ \xi' & \xrightarrow{\varphi'} & f^*\xi' \end{array}$$

commutes up to a homotopy through vector bundle isomorphisms. Then we can form the abelian group $K(X,f)$ generated by isomorphism classes $[\xi, \varphi]$ of elements $(\xi, \varphi) \in \mathscr{C}(X,f)$ subject to the relations $[\xi \oplus \xi', \varphi \oplus \varphi']=[\xi, \varphi]+[\xi', \varphi']$. Now each element (ξ, φ) of $\mathscr{C}(X,f)$ determines a unique vector bundle over the mapping torus $X_f=X \times [0, 1]/\{(f(x), 0)=(x, 1) \mid x \in X\}$, namely the mapping torus of the vector bundle map $\xi \xrightarrow{\varphi} f^*\xi \xrightarrow{\text{nat.}} \xi$ covering f. It is easily seen that this correspondence induces an isomorphism

$$K(X,f) \cong K^0(C_f)$$

with the geometric K-theory of X_f. There is an exact sequence

$$K^{-1}X \xrightarrow{1-f^*} K^{-1}X \xrightarrow{j} K^0(X_f) \xrightarrow{p} K^0X \xrightarrow{1-f^*} K^0X \qquad (\dagger)$$

which can be proved like 9.2 if one recalls that $K^{-1}X$ is the homotopy classes of maps X into $GL(\mathbf{C})=\bigcup_n GL(n, \mathbf{C})$. In fact (†) can be identified with the sequence of 9.2 for $A=\mathbf{C}^X$ and $\theta=f^*:\mathbf{C}^X \to \mathbf{C}^X$, provided X is simply connected (see [3A, Appendix]).*

The following lemma will be useful in the next section:

LEMMA 9.3. *Let P be a projective over A and let $\varphi:P \to tP$ be an isomorphism. Let $\alpha:P \to P$ and $\beta:tP \to tP$ be A-automorphisms. Then*

$$[P, \beta\varphi\alpha]=[P, \varphi]+j[\alpha]+j[\beta].$$

---

*) *Added in proof*: In [8A, Appendix 2] Farrell and Hsiang give an example where (†) does not split at $K^0(X_f)$. Are there such examples for (S) of § 8?

The proof is an exercise with the Whitehead lemma that we again leave to the reader.

§ 10. The relaxation

We fix a ring A with 1-element. (A will take the role of $Z[H]$ in § 8.) And we fix a ring automorphism θ of A. Form the ring of θ-twisted finite Laurent series $A[T]$ over A by adding to A an indeterminate t with an inverse t^{-1} and decreeing the commutation rule

$$ta = \theta(a)\,t \quad \text{or} \quad \theta(a) = tat^{-1} \quad \text{for} \quad a \varepsilon A.$$

Then $A[T]$ decomposes naturally as a direct sum of A-modules

$$A[T] = \cdots \oplus At^{-1} \oplus A \oplus At \oplus At^2 \oplus \cdots$$

and for monomials $a_1 t^{n_1}$, $a_2 t^{n_2}$, $a_i \in A$ one has

$$a_1 t^{n_1} a_2 t^{n_2} = a_1 \theta^{n_1}(a_2)\, t^{n_1+n_2}.$$

Write $A[t]$ for the subring $A \oplus At \oplus At^2 \oplus \cdots$ and $A[t^{-1}]$ for the subring $A \oplus At^{-1} \oplus \oplus At^{-2} \oplus \cdots$.

Warning: Only when we want to *emphasize* that $A[T]$ is formed with θ rather than some other automorphism, do we write $A_\theta[T]$.

For any A-module M and any integer n we can form a new A-module $t^n M$ as follows. As abelian group $t^n M$ is M with t^n written before each element. The action of A is this: for $a \in A$ and $t^n m \in t^n M$

$$at^n m = t^n \theta^{-n}(a)\, m.$$

From a homomorphism of A-modules $f: M_1 \to M_2$ one deduces an A-homomorphism $t^n f: t^n M_1 \to t^n M_2$ by the rule $(t^n f)(t^n m_1) = t^n(f(m_1))$. Thus we have a functor from A-modulus to A-modules. Indeed it is nothing but a concrete way of expressing the usual functor

$$\theta^n_\#(?) = A \otimes_A (?)$$

where, for the tensor product, A is a right A-module by the rule $a \cdot a' = a\theta^n(a')$, for $a, a' \in A$. The natural isomorphism of these functors sends $t^n M$ to $A \otimes_A M$ by $t^n m \mapsto 1 \otimes m$, $m \in M$.

If φ is an automorphism of A and $f: M_1 \to M_2$ is an additive map of A-modules, we say that f is a *φ-skew homomorphism* if for all $a \in A$ and $m_1 \in M_1$, $f(am_1) = \varphi(a) f(m_1)$. Similarly one defines *φ-skew isomorphisms, endomorphisms* and *automorphisms.* For example, $\varphi: A \to A$ is a φ-skew isomorphism of A-modules.

Observe that the natural map $Lt^n: m \mapsto t^n m$ of M to $t^n M$ is a θ^n-skew isomorphism.

Often there is no genuine isomorphism of M with $t^n M$. However, for $M = A^k$, we gave in § 9 a natural isomorphism $\Theta_k : A^k \to \theta_\# A^k \equiv tA^k$, which is now expressible as $\Theta_k(a_1, \ldots, a_k) = t(\theta^{-1}a_1, \ldots, \theta^{-1}a_k) \in tA^k$.

If A is assigned the obvious *right* A-module structure one has $t^n A \otimes_A M \cong \cong t^n(A \otimes_A M) \cong t^n M$. Now

$$A[T] = \cdots \oplus t^{-1}A \oplus A \oplus tA \oplus t^2 A \oplus \cdots$$

so that for any left A-module M

$$A[T] \otimes_A M = \cdots \oplus t^{-1}M \oplus M \oplus tM \oplus t^2 M \oplus \cdots$$

with the obvious left $A[T]$ module structure. Similarly

$$A[t] \otimes_A M = M \oplus tM \oplus t^2 M \oplus \cdots, \quad A[t^{-1}] \otimes_A M = M \oplus t^{-1}M \oplus t^{-2}M \oplus \cdots.$$

We write these $M[T]$, $M[t]$, $M[t^{-1}]$ respectively.

The purpose of this section is to prove:

THEOREM 10.1. *There exist natural group homomorphisms q and i*

$$K(A, \theta) \underset{i}{\overset{q}{\leftrightarrows}} K_1 A[T]$$

such that $qi = 1$. Thus there is a natural direct sum decomposition

$$K_1 A[T] = K(A, \theta) \oplus K^\perp(A, \theta)$$

where $K(A, \theta) = \text{image}(i)$ and $K^\perp(A, \theta) = \text{kernel}(q)$.

Remark: Naturality means that q and i are defined by rules such that if $f : A \to A'$ is a ring homomorphism and θ' is an automorphism of A' with $f\theta = \theta' f$, then $if_* = f_* i$ and $f_* q = qf_*$ in the following diagram:

$$\begin{array}{ccc} K(A, \theta) & \underset{i}{\overset{q}{\leftrightarrows}} & K_1(A[T]) \\ f_* \downarrow & & \downarrow f_* \\ K(A', \theta') & \underset{i}{\overset{q}{\leftrightarrows}} & K_1(A'[T]). \end{array}$$

Here the maps f_* are induced by f, and $A'[T]$ is constructed using θ' just as $A[T]$ was constructed using θ.

We now proceed to prove Theorem 10.1. After, we will deduce Theorem 8.1 from 9.2 and 10.1; and finally we will combine our information with that of Farrell and Hsiang in § 12.

A) *Construction of* $i : K(A, \theta) \to K_1 A[T]$. If P is a projective over A and $\varphi : P \to tP$ is an A-isomorphism, let $\bar{\varphi}$ be the unique $A[T]$-automorphism of

$$A[T] \otimes_A P = \cdots \oplus t^{-1}P \oplus P \oplus tP \oplus t^2 P \oplus \cdots$$

extending φ. It sends t^nP to $t^{n+1}P$ by $t^n\varphi$. Recall that $\tilde{\varphi}$ determines an element $[\tilde{\varphi}]\in K_1A[T]$. We define

$$i[P,\varphi]=[\tilde{\varphi}].$$

The reader will easily verify that i is well defined. Then it is obvious from the definition that i is additive and natural.

B) *Construction of* $q: K_1A[T]\to K(A,\theta)$. It will occupy the next 6 pages. I call q the relaxation homomorphism (see § 8).

Notation: To save space we will write B for $A[t]=A\oplus At\oplus\cdots$; $\bar{B}$ for $A[t^{-1}]\,t^{-1}=$ $=At^{-1}\oplus At^{-2}\oplus\cdots$, and C for $A[T]$. Then $C=\bar{B}\oplus B$ as A-bimodule.

In $\mathrm{GL}(n,C)$ consider the sub-semigroup G_n of automorphisms σ of C^n such that $\sigma B^n\subset B^n$ and $\sigma^{-1}\bar{B}^n\subset\bar{B}^n$. Set $G=\bigcup_{n\geqslant 1}G_n\subset\mathrm{GL}(\infty,C)$.

PROPOSITION 10.2. *For each* $\sigma\in G_n$ *the* A*-module* $B^n/\sigma B^n$ *is finitely generated and projective.*

Proof of 10.2: Since $\sigma\bar{B}^n\supset\bar{B}^n$, the composition $\bar{B}^n\hookrightarrow C^n\xrightarrow{\ \sigma\ }C^n\xrightarrow{\text{projection}}\bar{B}^n$ is onto. Thus its kernel P is a projective A-module. Observe that P consists exactly of those $\bar{b}\in\bar{B}^n$ such that $\sigma(\bar{b})\in B^n$. Then since $\sigma B^n\subset B^n$, we have $P\oplus B^n=\sigma^{-1}B^n$. Now

$$B^n/\sigma B^n\cong\sigma^{-1}B^n/B^n=(P\oplus B^n)/B^n\cong P.$$

Thus $B^n/\sigma B^n$ is projective.

To show $B^n/\sigma B^n$ is finitely generated, find an integer $s\geqslant 0$ so large that

$$\sigma^{-1}B^n\subset B^nt^{-s}. \qquad (*)$$

If $b_1,\ldots,b_n$ are the n free generators of B^n as B-module we can certainly find s so that $\sigma^{-1}(b_i)\subset B^nt^{-s}$ for each i. This s will satisfy (*) since σ^{-1} is a B-module homomorphism. From (*) we get

$$t^sB^n\subset t^s\sigma B^nt^{-s}=\sigma t^sB^nt^{-s}=\sigma B^n.$$

Thus $B^n/\sigma B^n$ is a quotient of $B^n/t^sB^n\cong A^{ns}$.

CONSTRUCTION 10.3. Consider a commutative diagram of *inclusions* of A-modules

$$\begin{array}{ccc} & M & \\ \nearrow & & \nwarrow \\ M_1 & & M_2 \\ \nwarrow & & \nearrow \\ & M_0 & \end{array}$$

Suppose that the quotient for each inclusion is projective. By splitting the sequences

$$0 \to M_i/M_0 \to M/M_0 \to M/M_i \to 0, \quad i = 1, 2$$

we obtain a composite isomorphism

$$M/M_1 \oplus M_1/M_0 \to M/M_0 \to M/M_2 \oplus M_2/M_0$$

Changing the splittings alters the isomorphism by automorphisms which have class zero in K_1A. Hence we have a preferred homotopy class of A-isomorphisms (cf. § 2)

$$M/M_1 \oplus M_1/M_0 \to M/M_2 \oplus M_2/M_0 .$$

CONSTRUCTION 10.4. For an A-isomorphism

$$\varphi : P \oplus A^n \to tP \oplus A^n ,$$

with P a projective we define $[P, \varphi] \in K(A, \theta)$ as follows. Compose with

$$1 \oplus \Theta_n : tP \oplus A^n \to tP \oplus tA^n = t(P \oplus A^n)$$

and define

$$[P, \varphi] = [P \oplus A^n, (1 \oplus \Theta_n) \circ \varphi] - [A^n, \Theta_n] \in K(A, \theta).$$

Clearly free modules with basis can replace A^n. Note that if we increase n (extending φ by the identity) we do not change $[P, \varphi]$. Note also that if $\varphi' : P' \oplus A^{n'} \to tP' + A^{n'}$ is another such A-isomorphism, the isomorphism

$$(P \oplus P') \oplus (A^n \oplus A^{n'}) \cong P + A^n \oplus P' + A^{n'} \xrightarrow{\varphi \oplus \psi} tP \oplus A^n \oplus tP' \oplus A^{n'}$$
$$\cong t(P \oplus P') \oplus (A^n \oplus A^{n'})$$

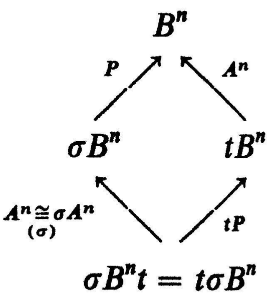
yields $[P, \varphi] + [P', \varphi']$.

Finally, given $\varphi : P \oplus A^m \to A^m \oplus tP$ we define $[P, \varphi] \in K(A, \theta)$ by reordering summands to obtain the above situation.

CONSTRUCTION 10.5. We will now define a map

$$q_n : G_n \to K(A, \theta).$$

Let $\sigma \in G_n$. Note that, as $\sigma \in \mathrm{GL}(n, C)$, $\sigma t^s B^n = t^s \sigma B^n$ for all s. Also $B^n t^s = t^s B^n$. Thus we have the following commutative diagram of inclusions which is the basis of all arguments ahead



.

Beside each inclusion we have indicated a module to which the quotient is naturally identified. P is by definition the projective $B^n/\sigma B^n$.

From 10.3 we get an A-isomorphism

$$\varphi : P \oplus A^n \to tP \oplus A^n$$

determined up to homotopy, and we *define*

$$q_n(\sigma) = [P, \varphi] \in K(A, \theta)$$

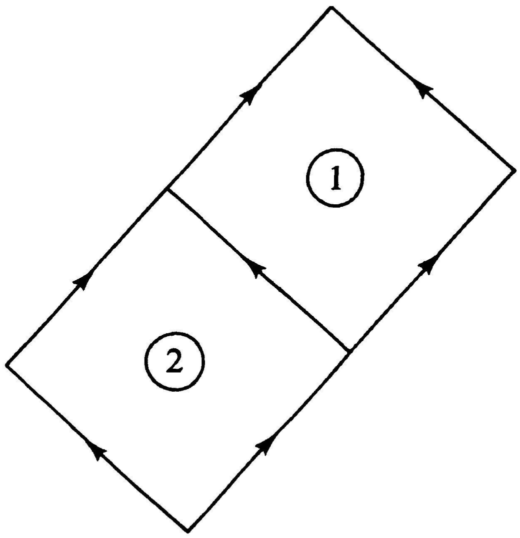
using 10.4.

LEMMA 10.6. *If* $\alpha, \beta \in G_n$, *then* $q_n(\alpha\beta) = q_n(\alpha) + q_n(\beta)$.

Proof of 10.6. Consider the three commutative squares of inclusions

$$\begin{array}{ccc} & B^n & \\ \nearrow & & \nwarrow \\ \alpha B^n & ① & B^n t \\ \nwarrow & & \nearrow \\ & \alpha B^n t & \end{array} \qquad \begin{array}{ccc} & \alpha B^n & \\ \nearrow & & \nwarrow \\ \alpha\beta B^n & ② & \alpha B^n t \\ \nwarrow & & \nearrow \\ & \alpha\beta B^n t & \end{array} \qquad \begin{array}{ccc} & B^n & \\ \nearrow & & \nwarrow \\ \alpha\beta B^n & ③ & B^n t \\ \nwarrow & & \nearrow \\ & \alpha\beta B^n t & \end{array}$$

The square ① defines $q_n(\alpha)$. The square ② is the image under α of the square that defines $q_n(\beta)$; we make identifications under the A isomorphism α. If we unite squares ① and ② thus



the vertices of the outer rectangle form square ③. For convenience we rewrite this diagram:

$$\begin{array}{ccccc} & & X_1 & & \\ & {}^{P}\nearrow & & \nwarrow^{A_1} & \\ & X_2 & ① & Y_1 & \\ {}^{Q}\nearrow & & \nwarrow^{A_2} & & \nearrow_{tP} \\ X_3 & ② & Y_2 & & \\ \nwarrow^{A_3} & & \nearrow_{tQ} & & \\ & Y_3 & & & \end{array}$$

The quotients have been indicated as in 10.5. Each of A_1, A_2, A_3 is a copy of A^n, in other words a free module with a preferred basis of n elements.

Construction 10.3 applied to ① and ② gives isomorphisms φ and ψ determined up to homotopy

$$\varphi: P \oplus A_2 \to A_1 \oplus tP; \qquad \psi: Q \oplus A_3 \to A_2 \oplus tQ$$

which determine $q_n(\alpha)$ and $q_n(\beta)$ by 10.4. We make identifications

$$X_1/X_3 = P \oplus Q, \qquad Y_1/Y_3 = tP \oplus tQ$$

in the preferred homotopy class. Now the isomorphism determined by square ③ is the composition

$$H: P \oplus Q \oplus A_3 \xrightarrow{1_P \oplus \psi} P \oplus A_2 \oplus tQ \xrightarrow{\varphi \oplus 1_{tQ}} A_1 \oplus tP \oplus tQ = A_1 \oplus t(P \oplus Q).$$

Construction 10.4 applied to H yields $q_n(\alpha\beta)$, and applied to

$$\varphi \oplus \psi: P \oplus A_2 \oplus Q \oplus A_3 \cong P \oplus Q \oplus (A_2 \oplus A_3) \to t(P \oplus Q) \oplus (A_1 \oplus A_2)$$
$$\cong A_1 \oplus tP \oplus A_2 \oplus tQ$$

it yields $q_n(\alpha)+q_n(\beta)$. So we want to compare H and $\varphi\oplus\psi$. To the composition H add 1_{A_2} on the right, then between the composants $1_P\oplus\psi\oplus 1_{A_2}$ and $\varphi\oplus 1_{tQ}\oplus 1_{A_2}$ introduce the permutation of the two summands A_2. The result is $\varphi\oplus\psi$ if we forget the order of the summands. But A_2 and A_3 appear (on the left) in reversed order. We conclude, using 9.3, that

$$q_n(\alpha\beta) + j(x) = q_n(\alpha) + q_n(\beta) + j(x) \in K(A, \theta)$$

where $x \in K_1 A$ is represented by the permutation of factors of $A^n \oplus A^n$. This proves Lemma 10.6.

LEMMA 10.7. *$q_{n+1} \mid G_n = q_n$ for all n. Hence we have a well defined homomorphism*

$$q': G = \bigcup_n G_n \to K(A, \theta).$$

Proof: If $\sigma \in G_n$, and the defining diagram in 10.5 for $q_n(\sigma)$ yields

$$\varphi: P \oplus A^n \to tP \oplus A^n,$$

then that for $q_{n+1}(\sigma)$ yields

$$\varphi + 1_A: P \oplus A^{n+1} \to tP \oplus A^{n+1}$$

which proves 10.7.

LEMMA 10.8. *$q': G \to K(A, \theta)$ extends naturally to a mapping $q'': \mathrm{GL}(\infty, C) \to$ $\to K(A, \theta)$.*

Proof: Given $\varrho \in \mathrm{GL}(\infty, C)$, there exists $\tau \in G$ so that $\tau\varrho \in G$. For example τ can be chosen with matrix of the form

$$\begin{pmatrix} t^{s_1} & & & & & 0 \\ & t^{s_2} & & & & \\ & & t^{s_3} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \cdot \end{pmatrix}$$

zero away from the diagonal; almost all $s_i = 1$. We define

$$q''(\varrho) = q'(\tau\varrho) - q'(\tau).$$

To prove $q''(\varrho)$ is independent of the choice of τ let $\tau_1, \tau_2 \in G$ be such that $\tau_1\varrho, \tau_2\varrho \in G$. Then choose $\bar\tau \in G$ so that $\bar\tau\tau_1^{-1}, \bar\tau\tau_2^{-1} \in G$. Then $\bar\tau\varrho \in G$ and

$$q'(\bar\tau\varrho) - q'\bar\tau = q'(\bar\tau\tau_1^{-1}) + q'(\tau_1\varrho) - (q'(\bar\tau\tau_1^{-1}) + q'(\tau_1)) = q'(\tau_1\varrho) - q'(\tau_1).$$

Similarly the left hand side is $q''(\tau_2\varrho) - q'(\tau_2)$. So q'' is well defined.

LEMMA 10.9. *q'': GL$(\infty, C) \to K(A, \theta)$ is a group homomorphism.*

Proof of 10.9.: Consider $\varrho_1, \varrho_2 \in \mathrm{GL}(n, C)$. Find $\tau_1, \tau_2, \sigma_1, \sigma_2$ in G so that $\varrho_1 = \tau_1^{-1}\sigma_1$, $\varrho_2 = \tau_2^{-1}\sigma_1$ and $\tau_2 = R_{t^s}$ right multiplication by t^s, $s \geqslant 0$. Then

$$\varrho_1\varrho_2 = \tau_1^{-1}\tau_2^{-1}(\tau_2\sigma_1\tau_2^{-1})\,\sigma_2.$$

Now

$$\tau_2\sigma_1\tau_2^{-1}B^n = \sigma_1(B^n t^{-s})\,t^s = t^{-s}\sigma_1(B^n)\,t^s \subset t^{-s}B^n t^s = B^n$$

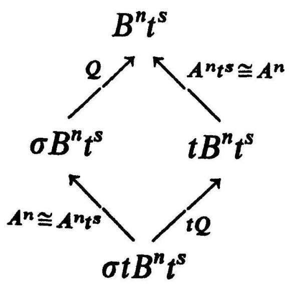
i.e. $\tau_2\sigma_1\tau_2^{-1} \in G$. Thus

$$q''(\varrho_1\varrho_2) = q_n(\tau_2\sigma_1\tau_2^{-1}\sigma_2) - q_n(\tau_2\tau_1) = q_n(\tau_2\sigma_1\tau_2^{-1}) + q_n(\sigma_2) - q_n(\tau_1) - q_n(\tau_2)$$

using the additivity of q_n. To prove additivity of q'' it will now suffice to show that $q'(\tau_2\sigma_1\tau_2^{-1}) = q'(\sigma_1)$. We have a commutative diagram

$$\begin{array}{ccc} B^n & \xrightarrow{\sigma_1} & B^n \\ \cong\downarrow R_{t^s} & & \cong\downarrow R_{t^s} = \tau_2 \\ B^n t^s & \xrightarrow{\sigma} & B^n t^s \end{array}$$

where $\sigma = \tau_2\sigma_1\tau_2^{-1}$. Hence $\tau_2 = R_{t^s}$ carries the defining square for $q_n(\sigma_1)$ isomorphically to the square



So the isomorphism $\varphi : Q \oplus A^n \to tQ \oplus A^n$ it determines represents $q_n(\sigma_1) \in K(A, \theta)$. But this same square is

$$\begin{array}{ccc} & t^s B^n & \\ {\scriptstyle t^sP}\nearrow & & \nwarrow {\scriptstyle t^sA^n \cong A^n} \\ t^s\sigma B^n & & t^s t B^n \\ {\scriptstyle A^n \cong t^s A^n}\nwarrow & & \nearrow {\scriptstyle t^s(tP)} \\ & t^s t\sigma B^n & \end{array}$$

which the image under L_{t^s} of the square that defines $q_n(\sigma)$. It follows with the help of 9.2 and 9.3 that the element of $K(A, \theta)$ determined by this square is $\theta_*^s q_n(\sigma)$ where

$$\theta_* : K(A, \theta) \to K(A, \theta)$$

sends each generator $[R, \psi]$, $\psi : R \to tR$ to $[tR, t\psi]$, $t\psi : tR \to t^2R$. But the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\psi} & tR \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle t\psi} \\ tR & \xrightarrow{t\psi} & t^2R \end{array}$$

shows that θ_* is the identity. Finally we have $q_n(\sigma_1) = q_n(\sigma) \equiv q_n(\tau_2\sigma_1\tau_2^{-1})$ as was required to complete Lemma 10.9.

DEFINITION 10.10. Let $v : K(A, \theta) \to K(A, \theta)$ be the standard involution $v[P, \varphi] = [P, -\varphi]$. Define the homomorphism $q : K_1A[T] \to K(A, \theta)$ of Theorem 10.1 to be the one induced by $vq'' : \mathrm{GL}(\infty, A[T]) \to K(A, \theta)$.

PROBLEM: Calculate v. Is it always the identity?

We have now established the relaxation homomorphism $q : K_1A[T] \to K(A, \theta)$. Its naturality follows from the form of the rule that defines q_n in 10.5.

C) *Proof that* $qi = 1$: If P is a projective over A and $\varphi : P \to tP$ an isomorphism $i[P, \varphi] \in K_1A[T]$ is represented by the unique extension $\tilde{\varphi}$ of φ to an automorphism of

$$P[T] = \cdots \oplus t^{-1}P \oplus P \oplus tP \oplus t^2P \oplus \cdots$$

If we choose Q and an isomorphism $P \oplus Q \cong A^n$ it is also represented by the automorphism $\sigma = \varphi \oplus 1$ of $P[T] \oplus Q[T] = A[T]^n$. Since $\varphi \oplus 1 \in G_n \subset \mathrm{GL}(n, A[T])$ we proceed to calculate $vq[\sigma] = q_n(\sigma)$ using the square (cf. 10.5)

$$\begin{array}{ccc} & B^n & \\ {\scriptstyle P}\nearrow & & \nwarrow {\scriptstyle A^n = P \oplus Q} \\ \sigma B^n & \textcircled{1} & tB^n \\ {\scriptstyle A^n \cong \sigma A^n}\nwarrow & & \nearrow {\scriptstyle tP} \\ & \sigma t B^n & \end{array}$$

This square is the direct sum of two squares

$$
\begin{array}{ccc}
 & P[t] & \\
{}^{P}\nearrow & & \nwarrow{}^{P} \\
\sigma P[t] & ② & tP[t] \\
{}^{P\cong\sigma P}\nwarrow & & \nearrow{}^{tP} \\
 & \sigma tP[t] &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & Q[t] & \\
{}^{\{0\}}\nearrow & & \nwarrow{}^{Q} \\
Q[t] & ③ & tQ[t] \\
{}^{Q}\nwarrow & & \nearrow{}^{\{0\}} \\
 & tQ[t] &
\end{array}
$$

Consider altering the isomorphism determined by ② via 10.3, through premultiplying by the automorphism α of the domain $P\oplus P$ which interchanges factors. This clearly alters the isomorphism determined by ① to $\varphi\oplus 1: P\oplus A^n \to tP\oplus A^n$. Hence $vqi[P, \varphi] = q_n(\sigma) = [P, \varphi] - j[\alpha]$. But $[\alpha]$ is also represented by $-1: P\to P$ and $[\alpha] = -[\alpha]$. Thus, by 9.3,

$$vqi[P, \varphi] = [P, -\varphi]$$

as required.

This ends the proof of Theorem 10.1.

PROPOSITION 10.11.

$$K_0A \xrightarrow{1-\theta_*} K_0A \xrightarrow{j_*} K_0A[T]$$

is an exact sequence. (Here j is inclusion $A \subsetneq A[T]$.*)*

Proof: Theorems 9.2 and 10.1 establish that

$$K_1A \xrightarrow{1-\theta_*} K_1A \xrightarrow{j_*} K_1A[T] \qquad (^*)$$

is exact. We will use (*) substituting for A the *untwisted* finite Laurent series $A[U] = A[u, u^{-1}] = \cdots \oplus Au^{-1} \oplus A \oplus Au \oplus Au^2 \oplus \cdots$ and for θ the unique extension $\theta[U]$ of θ to $A[U]$. Note that there is a natural isomorphism from $(A[U])[T]$ constructed with $\theta[U]$ to $(A[T])[U]$.

For $\theta = 1$ the sequence of Theorem 9.2 yields

$$0 \to K_1A \to K(A, 1) \to K_0A \to 0,$$

and it is split by sending $[P]$ to $[P, 1] \in K(A, 1)$. So in view of 10.1, there is a natural decomposition $K_1A[U] = K_0A \oplus R(A)$ (see [2]) where $R(A)$ is a natural complementary summand.

In the following diagram the vertical arrows are the natural injections and projections

$$\begin{array}{ccccc}
K_0A & \xrightarrow{1-\theta_*} & K_0A & \xrightarrow{j_*} & K_0A[T] \\
\downarrow & & \downarrow & & \downarrow \\
K_1A[U] & \xrightarrow{1-\theta[U]_*} & K_1A[U] & \xrightarrow{j[U]_*} & K_1A[U][T]=K_1A[T][U] \\
\downarrow & & \downarrow & & \downarrow \\
K_0A & \xrightarrow{1-\theta_*} & K_0A & \xrightarrow{j_*} & K_0A[T]
\end{array}$$

Since the diagram commutes the proposition follows.

§ 11. **Proof of the main result** (Theorem 8.1)

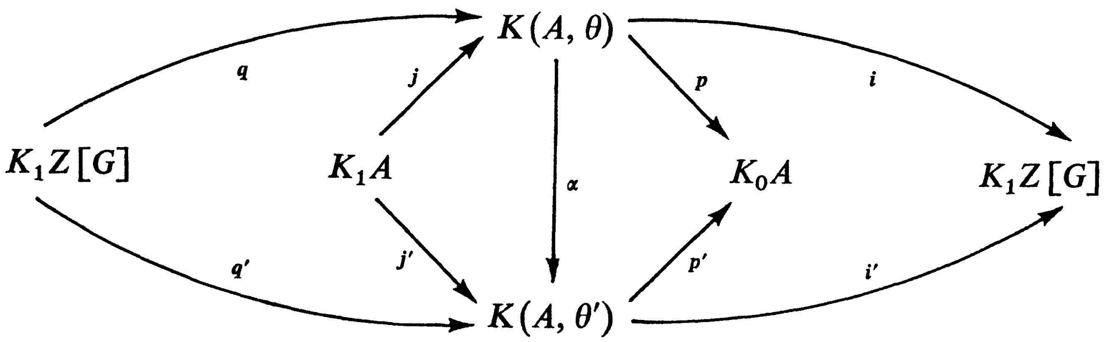
With a view to applying § 9 and § 10, set $A=Z[H]$ and let $\theta: A\to A$ be the automorphism of A induced by the automorphism θ of H, $\theta(h)=tht^{-1}$, for $h\in H$ (see § 8).

If $t'\in G$ like t goes to $1\in Z$, then $t'=tu$, $u\in H$. And if we set $\theta'(h)=(tu)\,h(tu)^{-1}$, for $h\in H$, then there is a natural isomorphism $\alpha: K(A,\theta)\to K(A,\theta')$ as follows.

To any A-isomorphism $\varphi: P\to tP$ associate the isomorphism $\varphi': P\to tuP$ which makes

$$\begin{array}{ccc}
P & \xrightarrow{\varphi} & tP \\
\| & & \uparrow \cong \text{nat} \\
P & \xrightarrow{\varphi'} & (tu)P
\end{array} \qquad (1)$$

commute where the vectical isomorphism maps $(tu)p\mapsto t(up)$ for $p\in P$. Then it is straightforward to verify that



(2)

commutes where j, j' and p, p' come from 9.2; q, q' and i, i' come from 10.1; and we have made the natural identifications $A[t,t^{-1}]=Z[G]=A[t',t'^{-1}]$.

We define $R(\mathscr{S})$ to be $K(A,\theta)$ divided by the subgroup generated by $[\text{A}, \Theta]$ and the images under j of $[(\pm h)]\in K_1A$ for $h\in H$. Here $(\pm h)$ indicates right multiplication of A by $\pm h$. The above paragraph shows that $R(\mathscr{S})$ is independent of the choice of t. i.e. depends only on the sequence $\mathscr{S}$. Also j and p in (2) give maps

$$\text{Wh}\,H \xrightarrow{j} R(\mathscr{S}) \xrightarrow{p} \tilde{K}_0Z[H] \qquad (3)$$

independent of θ. Then from the exactness of the combined sequence of 9.2 and 10.11

$$K_1A \xrightarrow{1-\theta_*} K_1A \xrightarrow{j} K(A,\theta) \xrightarrow{p} K_0A \xrightarrow{1-\theta_*} K_0A \xrightarrow{i^*} K_0A[T] \tag{4}$$

we readily deduce that the quotient sequence

$$\mathrm{Wh}\,H \xrightarrow{1-\theta_*} \mathrm{Wh}\,H \xrightarrow{j} R(\mathscr{S}) \xrightarrow{p} \tilde{K}_0Z[H] \xrightarrow{1-\theta_*} \tilde{K}_0Z[H] \xrightarrow{i_*} \tilde{K}_0Z[G] \tag{5}$$

is exact. The diagram of 10.1

$$K(A,\theta) \underset{i}{\overset{q}{\leftrightarrows}} K_1A[T] \tag{6}$$

with $qi=1$ has as quotient a unique diagram

$$R(\mathscr{S}) \underset{i}{\overset{q}{\leftrightarrows}} \mathrm{Wh}\,G \tag{7}$$

with $qi=1$.

To see this one must note that $i[A,\Theta]=[(t)]\in K_1A[T]$ where (t) indicates right multiplication in $A[T]$ by t. Again (2) shows this diagram depends only on $\mathscr{S}$.

According to the definition of § 1, $R(G,H)=i(R(\mathscr{S}))\subset \mathrm{Wh}\,G$. As our definitions make

$$\begin{array}{ccccc} \mathrm{Wh}\,H & \xrightarrow{j} & R(\mathscr{S}) & \xrightarrow{p} & \tilde{K}_0Z[H] \\ & {\scriptstyle i_*}\searrow & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle p} & \\ & & \mathrm{Wh}\,G & & \end{array}$$

commute (5) and (7) together establish Theorem 8.1.

One last point. As notation indicates $R(G,H)$ depends only on the pair (G,H) not on the preferred generator $+1$ for $G/H=Z$. This is because, if $\varphi: P\to tP$ extends to the isomorphism $\bar{\varphi}: P[T]\to P[T]$, then $t^{-1}\varphi^{-1}: P\to t^{-1}P$ extends to $\bar{\varphi}^{-1}$. Thus if $\mathscr{S}'$ is $\mathscr{S}$ altered by changing the sign of $G\to Z$, then $iR(\mathscr{S})\subset iR(\mathscr{S}')$. By symmetry $iR(\mathscr{S})=iR(\mathscr{S}')$. This justifies the notation $R(G,H)$.

§ 12. Supplementary remarks

Let $\mathscr{N}(A,\theta)$ be the category whose objects are pairs (P,α) where P is a projective over A and $\alpha: P\to P$ is a θ-skew nilpotent endomorphism of P. A morphism $f:(P,\alpha)\to(P',\alpha')$ is a homomorphism $f: P\to P'$ such that $\alpha' f=f\alpha$. Farrell has defined $\tilde{C}(A,\theta)$ to be the abelian group freely generated by isomorphism classes $[P,\varphi]$ of $\mathscr{N}(A,\theta)$ modulo the relations:

(a) $[P,\varphi]=[P',\varphi']+[P'',\varphi'']$ for each exact sequence $0\to(P',\varphi')\to(P,\varphi)\to(P'',\varphi'')\to 0$ of $\mathscr{N}(A,0)$.

(b) $[P,0]=0$ for each zero map.

Farrell and Hsiang [8] give a homomorphism[1])

$$p_+ : K_1 A_\theta [T] \to \tilde{C}(A, \theta)$$

that can be defined as follows. If $\varrho \in \mathrm{GL}(n, A_\theta[T])$ choose s so large that $\sigma = (R_{t^s})\varrho$ satisfies $\sigma B^n \subset B^n$, $\sigma^{-1}(\bar{B}^n) \subset \bar{B}^n$. For notation see 10.2. R_{t^s} has matrix (identity $\times t^s$). Then $(B^n/\sigma B^n, t)$, where t indicates left multiplication by t, is in $\mathcal{N}(A, \theta)$ by 10.2. Define $p_+[\varrho] = [B^n/\sigma B^n, t]$.

There is a natural isomorphism $A_\theta[T] \cong A_{\theta^{-1}}[T]$ sending t to t^{-1}. Let p_- be the composition

$$K_1 A_\theta [T] \cong K_1 A_{\theta^{-1}}[T] \xrightarrow{p_+} C(A, \theta^{-1}).$$

Notice that, for $[P, \varphi] \in K(A, \theta)$, $p_+ i[P, \varphi] = [P, 0] = 0 \in \tilde{C}(A, \theta)$. Thus $iK(A, \theta) \subset$ $\subset K_1 A[T]$ is in the kernel of p_+ and likewise in the kernel of p_-. Restricting attention to $K^\perp(A, \theta) = \ker(q) \subset K_1 A[T]$ we have a map

$$K^\perp(A, \theta) \xrightarrow{(p_-, p_+)} \tilde{C}(A, \theta^{-1}) \oplus \tilde{C}(A, \theta)$$

which is in fact an isomorphism [8] [8A]. Thus the final result is a natural direct sum decomposition

$$K_1 A_\theta [T] \cong K(A, \theta) \oplus \tilde{C}(A, \theta^{-1}) \oplus \tilde{C}(A, \theta). \qquad (\S\S)$$

In [8], $\tilde{C}(A, \theta^{-1})$ was not mentioned because for group rings $\tilde{C}(A, \theta^{-1}) \cong \tilde{C}(A, \theta)$ by a familiar duality cf. [13, p. 52]. Note, in contrast that $K(A, \theta) \cong K(A, \theta^{-1})$ for *any* ring A by a *canonical* isomorphism sending $[P, \varphi]$, $\varphi : P \xrightarrow{\cong} \theta_\# P$, to $[\theta_\# P, \varphi^{-1}]$.

There is also a natural decomposition [8] [8A]

$$K_1 A_\theta [t] \cong K_1 A \oplus \tilde{C}(A, \theta^{-1}). \qquad (\S)$$

The summand $K_1 A$ appears because A is a retract of $A_\theta[t]$ by setting $t = 1$. The projection to the summand $\tilde{C}(A, \theta^{-1})$ is the composition $K_1 A_\theta[t] \to K_1 A_\theta [T] \xrightarrow{p_-}$ $\xrightarrow{p_-} \tilde{C}(A, \theta^{-1})$. Hence the map induced by $A_\theta[t] \subsetneq A_\theta[T]$ from $K_1 A_\theta [t] = K_1 A \oplus$ $\oplus \tilde{C}(A, \theta^{-1})$ to $K_1 A_\theta[T] = K(A, \theta) \oplus \tilde{C}(A, \theta^{-1}) \oplus C(A, \theta)$ is $j \oplus \{\text{inclusion}\}$.

For the group ring situation of § 8 and § 11, formula (§§) gives

$$\mathrm{Wh}\, G = R(G, H) \oplus \tilde{C}(Z[H], \theta^{-1}) \oplus \tilde{C}(Z[H], \theta).$$

This is because the subgroup generated by elements $[(\pm g)] \in K_1 Z[G]$, for $g \in G$, which is killed to produce $\mathrm{Wh}\, G$, is precisely the image of the subgroup of $K(A, \theta)$ killed to produce $R(\mathcal{S}) = R(G, H)$.

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1) *Added in proof*: In [8A], Farrell and Hsiang give in addition, a left inverse to p_+ and thus obtain (§§) – without identifying $K(A, \theta)$.

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